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Tropical types and associated cellular resolutions

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ABSTRACT

An arrangement of finitely many tropical hyperplanes in the tropical torus $\mathbb{T}^{d-1}$ leads to a notion of 'type' data for points in $\mathbb{T}^{d-1}$, with the underlying unlabeled arrangement giving rise to 'coarse type'. It is shown that the decomposition of $\mathbb{T}^{d-1}$ induced by types gives rise to minimal cocellular resolutions of certain associated monomial ideals. Via the Cayley trick from geometric combinatorics this also yields cellular resolutions supported on mixed subdivisions of dilated simplices, extending previously known constructions. Moreover, the methods developed lead to an algebraic algorithm for computing the facial structure of arbitrary tropical complexes from point data.

1. Introduction

The study of convexity over the tropical semi-ring has been an area of active research in recent years. Fundamental properties of tropical convexity, in particular from a combinatorial perspective, were established by Develin and Sturmfels in [6]. There the notion of a tropical polytope was defined as the tropical convex hull of a finite set of points in the tropical torus $\mathbb{T}^{d-1}$. Fixing the set of generating points yields a decomposition of the tropical polytope called the *tropical complex*, and in [6] it was shown that the collection of such complexes are in bijection with the regular subdivisions of a product of simplices.

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A tropical complex can be realized as the subcomplex of bounded cells of the polyhedral complex arising from an arrangement of tropical hyperplanes, and it is this perspective that we adopt in this paper. We study combinatorial properties of arrangements of tropical hyperplanes, and in particular their relation to algebraic properties of associated monomial ideals. A tropical hyperplane in $\mathbb{T}^{d-1}$ is defined as the locus of 'tropical vanishing' of a linear form (with full support), and can be regarded as a fan polar to a $(d-1)$-dimensional simplex. In this way each tropical hyperplane divides the ambient space $\mathbb{T}^{d-1}$ into *d sectors*. Given an arrangement $\mathcal{A}$ of tropical hyperplanes and a point $p \in \mathbb{T}^{d-1}$, one can record the position of p with respect to each sector of each hyperplane. This tropical analog of the covector data of an oriented matroid is called the *type* data, and the combinatorial approach to tropical convexity taken in [6] is based on this concept. Here we consider a coarsening of the type data (which we call *coarse type*) arising from an arrangement $\mathcal{A}$ of tropical hyperplanes, amounting to neglecting the labels on the individual hyperplanes.

The connection between tropical polytopes/complexes and resolutions of monomial ideals was first exploited by Block and Yu in [4]. There the authors associate a monomial ideal to a tropical polytope with generators in general position, and use algebraic properties of its minimal resolution to determine the facial structure of the bounded subcomplex. The primary tool employed in this context is that of a *cellular resolution* of a monomial ideal. The ideals from [4] are squarefree monomials ideals generated by the *cotype* data, i.e. the complements of the tropical covectors, arising from the associated arrangement of hyperplanes, and hence can be seen as a tropical analog of the (*oriented*) *matroid ideals* studied by Novik, Postnikov and Sturmfels in [15].

In this paper we study the polyhedral complex $\mathcal{C}_{\mathcal{A}}$ and its bounded subcomplex $\mathcal{B}_{\mathcal{A}}$ induced by the type data of an arrangement $\mathcal{A}$ of n hyperplanes in $\mathbb{T}^{d-1}$. Both complexes are naturally labeled by fine and coarse type and cotype data, and we show how the resulting labeled complexes support minimal (co)cellular resolutions of associated monomial ideals. We pay special attention to labels given by coarse type. For instance, we show that $\mathcal{C}_{\mathcal{A}}$ supports a minimal *cocellular* resolution of the ideal $I_{\mathbf{t}(\mathcal{A})}$ generated by monomials corresponding to the set of all coarse types. The proof involves a consideration of the topology of certain subsets of $\mathcal{C}_{\mathcal{A}}$ as well as the combinatorial properties of the coarse type labelings. When the arrangement $\mathcal{A}$ is *sufficiently generic* we show that the resulting ideal is always given by $\langle x_1, \dots, x_d \rangle^n$, the n-th power of the maximal homogeneous ideal; in general, $I_{\mathbf{t}(\mathcal{A})}$ is some Artinian subideal. Our results in this area are all independent of the characteristic of the coefficient field.

Via the connection to products of simplices and the Cayley trick we interpret these results in the context of mixed subdivisions of dilated simplices. In particular, we obtain a minimal *cellular* resolution of $I_{\mathbf{t}(\mathcal{A})}$ supported on a subcomplex of the dilated simplex $n\Delta_{d-1}$. One other direct consequence is that *any* regular fine mixed subdivision of $n\Delta_{d-1}$ supports a minimal resolution of $\langle x_1, \dots, x_d \rangle^n$. This extends a result of Sinefakopoulos from [18] where a particular subdivision is considered (although much less explicitly), and also complements a construction of Engström and the first author from [7] where such complexes are applied to resolutions of hypergraph edge ideals. The duality between tropical complexes and mixed subdivisions of dilated simplices was established in [6], and we show how this extends to the algebraic level in terms of Alexander duality of our resolutions of the coarse type and cotype ideals.

Finally, we show how these algebraic results lead to observations regarding the combinatorics of tropical polytopes/complexes and mixed subdivisions of dilated simplices. We obtain a formula for the f-vector of the bounded subcomplex of an arbitrary tropical hyperplane arrangement in terms of the Betti numbers of the associated coarse type ideal. The uniqueness of minimal resolutions also implies that for any sufficiently generic arrangement $\mathcal{A}$, the multiset of coarse types is independent of the arrangement. Furthermore, we present an algorithm for determining the incidence face structure of a tropical complex from the coordinates of an arbitrary set of vertices, utilizing the fact that such a complex supports a minimal resolution of the square-free monomial cotype ideal. This approach was first introduced by Block and Yu in [4] for the case of sufficiently generic arrangements, and we extend the algorithm to the general case.

The rest of the paper is organized as follows. In Section 2 we review the basic notions of tropical convexity including tropical hyperplanes and type data, and discuss the polytopal complex that arises from an arrangement of tropical hyperplanes. We introduce the notion of *coarse type* and establish some of the basic properties that will be used later in the paper. In Section 3 we introduce the monomial ideals that arise from an arrangement of hyperplanes and show that the polytopal complexes labeled by fine and coarse (co)type support cocellular (and cellular) resolutions. In Section 4 we interpret our results in terms of mixed subdivisions of dilated simplices. We briefly discuss the *staircase triangulation* and recover a result of Sinefakopoulos [18]. In Section 5 we show our results give rise to certain consequences for the combinatorics (e.g., the f-vector) of the bounded subcomplexes of tropical hyperplane arrangements, and also describe an algorithm for determining the entire face poset from the coordinates of the arrangement. This strengthens a result from [4]. Finally, we end in Section 6 with some concluding remarks and open questions.

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2. Tropical convexity and coarse types

In order to fix our notation we begin in this section with a brief review of the foundations of tropical convexity as layed out by Develin and Sturmfels in [6]. We then define ‘coarse types’ and establish some combinatorial and topological results regarding the type decomposition of the tropical torus induced by a finite set of points. While some of these observations may be worthwhile in their own right, their main interest for us will be their applications to subsequent constructions of (co)cellular resolutions.

2.1. Tropical convexity and tropical hyperplane arrangements

Tropical convexity is concerned with linear algebra over the *tropical semi-ring* $(\mathbb{R}, \oplus, \odot)$, where

$$x \oplus y := \min(x, y) \quad \text{and} \quad x \odot y := x + y.$$

We will sometimes replace the operation min with max, and although the two resulting semi-rings are isomorphic via $-\max(x, y) = \min(-x, -y)$, it will be useful for us to consider both structures on the set $\mathbb{R}$ simultaneously. To avoid confusion we will therefore use the terms min-*tropical semi-ring* and max-*tropical semi-ring*, respectively. Componentwise tropical addition and tropical scalar multiplication turn $\mathbb{R}^d$ into a semi-module. The *tropical torus* $\mathbb{T}^{d-1}$ is the quotient of Euclidean space $\mathbb{R}^d$ by the linear subspace $\mathbb{R}\mathbf{1}$, where $\mathbf{1} \in \mathbb{R}^d$ is the all-ones vector. By interpreting this quotient in the category of topological spaces, $\mathbb{T}^{d-1}$ inherits a natural topology which is homeomorphic to the usual topology on $\mathbb{R}^{d-1}$. A set $S \subset \mathbb{T}^{d-1}$ is *tropically convex* if it contains $(\lambda \odot x) \oplus (\mu \odot y)$ for all $x, y \in S$ and $\lambda, \mu \in \mathbb{R}$. For an arbitrary set $S \subset \mathbb{T}^{d-1}$ the *tropical convex hull* $\operatorname{tconv}(S)$ is defined as the smallest tropically convex set containing S. If the set S is finite, then $\operatorname{tconv}(S)$ is called a *tropical polytope*. In this paper, tropical convexity and related notions will be studied with respect to both min and max, and hence we will also talk about max-*tropically convex* sets and the like.

The *tropical hyperplane* with *apex* $-a \in \mathbb{T}^{d-1}$ is the set

$$H(-a) := \bigl\{p \in \mathbb{T}^{d-1}\colon\ (a_1 \odot p_1) \oplus (a_2 \odot p_2) \oplus \cdots \oplus (a_d \odot p_d) \text{ is attained at least twice}\bigr\}.$$

That is, a tropical hyperplane is the tropical vanishing locus of a polynomial homogeneous of degree 1 with real coefficients. We write $H^{\min}(-a)$ and $H^{\max}(-a)$ to explicitly distinguish between the min- and max-versions. Any two min-tropical (respectively max-tropical) hyperplanes are related by an ordinary translation, and hence a tropical hyperplane is completely determined by its apex. The complement of any tropical hyperplane in $\mathbb{T}^{d-1}$ consists of precisely d connected components, its *open sectors*. Each open sector is convex, in both the tropical and ordinary sense. The *k-th* (*closed*) *sector* of the max-tropical hyperplane with apex a is the set

$$S_k^{\max}(a) := \bigl\{p \in \mathbb{T}^{d-1}\colon\ a_k - p_k \leqslant a_i - p_i \text{ for all } i \in [d]\bigr\}.$$

Similarly we have

$$S_k^{\min}(a) := \{p \in \mathbb{T}^{d-1}\colon a_k - p_k \geqslant a_i - p_i \text{ for all } i \in [d]\}$$

for the min-version. Notice that $x \in S_k^{\max}(y)$ if and only if $y \in S_k^{\min}(x)$; this fact can be read as a kind of duality. Each closed sector is the topological closure of an open sector. Again the closed sectors are tropically and ordinarily convex. A sequence of points $V = (v_1, v_2, \ldots, v_n)$ in $\mathbb{T}^{d-1}$ gives rise to the arrangement

$$\mathcal{A}(V) := (H^{\max}(v_1), H^{\max}(v_2), \ldots, H^{\max}(v_n))$$

of n labeled max-tropical hyperplanes. The position of points in $\mathbb{T}^{d-1}$ relative to each hyperplane in the arrangement furnishes combinatorial data and leads to the following definition.

Definition 2.1 *(Fine (co)type).* Let $\mathcal{A} = \mathcal{A}(V)$ be the arrangement of max-tropical hyperplanes given by $V = (v_1, v_2, \ldots, v_n)$ in $\mathbb{T}^{d-1}$. The *fine type* of a point $p \in \mathbb{T}^{d-1}$ with respect to $\mathcal{A}$ is the table $T_{\mathcal{A}}(p) \in \{0,1\}^{n\times d}$ with

$$T_{\mathcal{A}}(p)_{ik} = 1 \quad \text{if and only if} \quad p \in S_k^{\max}(v_i)$$

for $i \in [n]$ and $k \in [d]$. The *fine cotype* $\overline{T}_{\mathcal{A}}(p) \in \{0,1\}^{n\times d}$ is defined by $\overline{T}_{\mathcal{A}}(p)_{ik} = 1 - T_{\mathcal{A}}(p)_{ik}$.

Let us now fix an arrangement $\mathcal{A} = \mathcal{A}(V)$ in $\mathbb{T}^{d-1}$. We write $T(p)$ instead of $T_{\mathcal{A}}(p)$ when no confusion arises. For a fixed type $T = T(p)$, the points in $\mathbb{T}^{d-1}$ with fixed type $T(p) = T$ form a relatively open subset of $\mathbb{T}^{d-1} = \mathbb{R}^{d-1}$, which is convex in both the tropical and ordinary sense. Their closures give a polyhedral subdivision $\mathcal{C}_{\mathcal{A}}$ of the tropical torus $\mathbb{T}^{d-1}$. The bounded closed cells are the *polytropes* studied in [10]; they form the *bounded subcomplex* $\mathcal{B} = \mathcal{B}_{\mathcal{A}}$. The collection of types with the componentwise order is anti-isomorphic to the face lattice of $\mathcal{C}_{\mathcal{A}}$, that is, $T_{\mathcal{A}}(D) \leqslant T_{\mathcal{A}}(C)$ whenever $C \subseteq D$ are closed cells of $\mathcal{C}_{\mathcal{A}}$. Following Ardila and Develin [1], a fine type should be thought of as the tropical equivalent of a covector in the setting of tropical oriented matroids, with the cells of maximal dimension playing the role of the topes. The following result highlights the relation of min-tropical polytopes and max-tropical hyperplane arrangements.

Theorem 2.2. *(See [6, Theorem 15 and Proposition 16].) The* min-*tropical polytope* $\operatorname{tconv}(V)$ *is the union of cells in the bounded subcomplex* $\mathcal{B}_{\mathcal{A}}$ *of the cell decomposition of* $\mathbb{T}^{d-1}$ *induced by the* max-*tropical hyperplane arrangement* $\mathcal{A} = \mathcal{A}(V)$.

The polytopal complex $\mathcal{B}_{\mathcal{A}}$ induced by types is a subdivision of the tropical polytope $\operatorname{tconv}(P)$ called the *tropical complex* generated by V. The points V are in tropically *general position* if the combinatorial type of $\mathcal{C}_{\mathcal{A}}$ (or equivalently $\mathcal{B}_{\mathcal{A}}$) is invariant under small perturbations.

At this point we introduce our running example, borrowed from [4, Example 10].

Example 2.3. The points $v_1 = (0,3,6)$, $v_2 = (0,5,2)$, $v_3 = (0,0,1)$, and $v_4 = (1,5,0)$ give rise to the max-tropical hyperplane arrangement shown in Fig. 1. It decomposes the tropical torus $\mathbb{T}^2$ into 15 two-dimensional cells, three of which are bounded. Note that there is precisely one bounded cell of dimension one (incident with the 0-cell v_1) which is maximal with respect to inclusion. This shows that the polytopal complex $\mathcal{B}_{\mathcal{A}}$ need not be pure. Moreover, one can check that the four points are in general position.

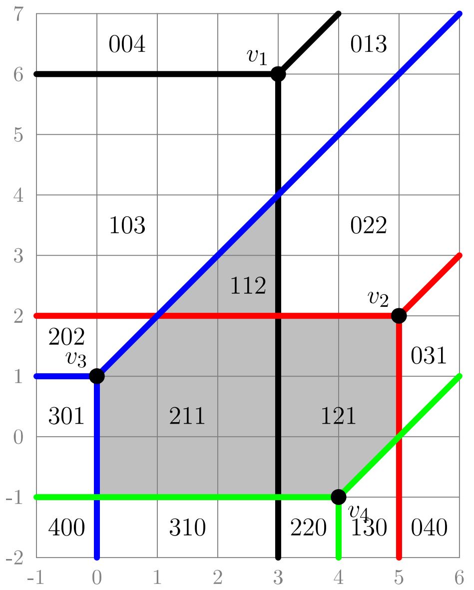

Fig. 1. Coarsely labeled type decomposition of $\mathbb{T}^2$ induced by four max-tropical lines. Bounded cells are shaded.

2.2. Coarse types

As we have seen, the fine type records the position of a point relative to a labeled tropical hyperplane arrangement. Neglecting the labels on the hyperplanes leads to the following coarsening of the type information.

Definition 2.4 *(Coarse (co)type).* Let $\mathcal{A} = \mathcal{A}(V)$ be an arrangement of n max-tropical hyperplanes in $\mathbb{T}^{d-1}$. The *coarse type* of a point $p \in \mathbb{T}^{d-1}$ with respect to $\mathcal{A}$ is given by $\mathbf{t}_{\mathcal{A}}(p) = (t_1, t_2, \ldots, t_d) \in \mathbb{N}^d$ with

$$t_k = \sum_{i=1}^{n} T_{\mathcal{A}}(p)_{ik}$$

for $k \in [d]$. The *coarse cotype* $\bar{\mathbf{t}}_{\mathcal{A}}(p) \in \mathbb{N}^d$ is given by $\bar{\mathbf{t}}_{\mathcal{A}}(p)_k = n - \mathbf{t}_{\mathcal{A}}(p)_k$ for $k = 1, \ldots, d$.

The coarse type entry $\mathbf{t}(p)_k$ records for how many hyperplanes in $\mathcal{A}$ the point p lies in the k-th closed sector. Again we will write $\mathbf{t}(p)$ when no confusion can arise. The map which assigns a vector $\mathbb{N}^d$ to each cell of constant coarse type is also denoted by $\mathbf{t}$.

Let us give an alternative interpretation of coarse types. The tropical hyperplane arrangement $\mathcal{A}$ in $\mathbb{T}^{d-1}$ is the tropicalization of a product of linear forms $h = h_1 \cdot h_2 \cdots h_n$ over the field of Puiseux series [8, Section 2]. The linear forms can be taken to be $h_i = z^{-v_{i1}} x_1 + z^{-v_{i2}} x_2 + \cdots + z^{-v_{id}} x_d \in \mathbb{C}\{\{z\}\}[x_1, \ldots, x_d]$ but there is no canonical choice. As stated, this only works for tropical hyperplanes with *rational* apices. For the general case, it would be necessary to work over a field of generalized Puiseux series [12].

According to [8, Theorem 2.1.1], the tropical hypersurface trop(h) associated to the Laurent polynomial $h = \sum_\alpha \gamma_\alpha x^\alpha$ is the orthogonal projection of the codimension-2-skeleton of the unbounded ordinary polyhedron

$$P_{\mathcal{A}} = \left\{(x, s) \in \mathbb{R}^d \times \mathbb{R}\colon\ s \geqslant \operatorname{val}(\gamma_\alpha) + \langle x, \alpha\rangle \text{ for all } \alpha \text{ with } \gamma_\alpha \neq 0\right\}$$

and the facets of $P_{\mathcal{A}}$ correspond to the monomials of h.

Proposition 2.5. *Let $p \in \mathbb{T}^{d-1} \setminus \mathcal{A}$ be a generic point. Then its coarse type $\mathbf{t}_{\mathcal{A}}(p)$ is the exponent of the monomial in h which defines the unique facet of $P_{\mathcal{A}}$ above p.*

Recall that a *composition* of n into d parts is a sequence $(t_1, t_2, \dots, t_d) \in \mathbb{N}^d$ of nonnegative integers such that $t_1 + t_2 + \dots + t_d = n$. Such compositions bijectively correspond to monomials of total degree n in d variables, of which there are exactly $\binom{n+d-1}{n} = \binom{n+d-1}{d-1}$.

Remark 2.6. From Proposition 4.2 and the interpretation of tropical complexes in terms of mixed subdivisions, we see the map $\mathbf{t}$ from the set of cells in $\mathcal{C}_{\mathcal{A}}$ of maximal dimension $d-1$ to the set of compositions of n into d parts is injective. Moreover, if the points V are sufficiently generic then the map $\mathbf{t}$ is bijective.

Corollary 2.7. *For an arrangement $\mathcal{A} = \mathcal{A}(V)$, the number of cells in $\mathcal{C}_{\mathcal{A}}$ of maximal dimension $d-1$ does not exceed $\binom{n+d-1}{n}$. If the points V are sufficiently generic then this bound is attained.*

Note that the injectivity of $\mathbf{t}$ does not extend to the lower-dimensional cells, even in the sufficiently generic case. For instance, in the arrangement considered in Examples 2.3 the points $(0, 2, 3)$ and $(0, 3, 5)$ lie in the relative interiors of two distinct 1-cells; yet they share the same coarse type $(1, 1, 3)$. However coarse types are *locally* distinct in the following sense.

Proposition 2.8. *Let C and D be two distinct cells in $\mathcal{C}_{\mathcal{A}}$. If C is contained in the closure of D then C and D have distinct coarse types.*

Proof. By [6, Corollary 13] we have $T_{\mathcal{A}}(D) \leqslant T_{\mathcal{A}}(C)$ and $T_{\mathcal{A}}(D) \neq T_{\mathcal{A}}(C)$. This in particular implies that C is contained in more closed sectors with respect to $\mathcal{A}$ and hence C and D cannot have the same coarse type. □

The boundedness of a cell in $\mathcal{C}_{\mathcal{A}}$ can also be read off from its coarse type.

Proposition 2.9. *(See [6, Corollary 12].) Let $\mathcal{A} = \mathcal{A}(V)$ be a tropical hyperplane arrangement and let $C \in \mathcal{C}_{\mathcal{A}}$ be a cell in the induced decomposition. Then C is bounded if and only if $\mathbf{t}(C)_i > 0$ for all $i = 1, \dots, n$.*

Proof. We noted previously that for two points $p, q \in \mathbb{T}^{d-1}$ we have $p \in S_k^{\max}(q)$ if and only if $q \in S_k^{\min}(p)$. Now, the point p is contained in an unbounded cell of $\mathcal{C}_{\mathcal{A}}$ if there is a k such that $S_k^{\min}(p) \cap V = \emptyset$. This is the case if and only if $t_{\mathcal{A}}(p)_k = 0$, that is, there is no hyperplane H_i for which p is contained in the k-th sector. □

2.3. Topology of types

We next investigate topological properties of certain subsets of $\mathbb{T}^{d-1}$ induced by a tropical hyperplane arrangement. The motivation for such a study comes from our applications to (co)cellular resolutions as described in Section 3. However, since the methods used to establish the desired properties are based on notions from (coarse) tropical convexity, we include them here. In particular, we show that certain subsets of $\mathbb{T}^{d-1}$ are contractible by proving that they are, in fact, tropically convex. Let us emphasize that none of the results that follow require the hyperplanes to be in general position.

We begin with the following observation, which was established in [6, Theorem 2].

Proposition 2.10. *A tropically convex set is contractible.*

Recall that for two types $T, T' \in \{0, 1\}^{n\times d}$ we write $T \leqslant T'$ for the componentwise induced partial order. We let $\min(T, T')$ and $\max(T, T')$ denote the tables with entries given by the componentwise minimum and maximum, respectively.

Proposition 2.11. *Let $\mathcal{A} = \mathcal{A}(V)$ be a* max*-tropical hyperplane arrangement in $\mathbb{T}^{d-1}$ and $p, q \in \mathbb{T}^{d-1}$. Then*

$$\min\bigl(T_{\mathcal{A}}(p), T_{\mathcal{A}}(q)\bigr) \leqslant T_{\mathcal{A}}(r) \leqslant \max\bigl(T_{\mathcal{A}}(p), T_{\mathcal{A}}(q)\bigr)$$

for every point $r \in \operatorname{tconv}^{\max}\{p, q\}$ on the max*-tropical line segment between p and q.*

Proof. Let $r = (\lambda \odot p) \oplus (\mu \odot q) = \max\{\lambda\mathbf{1} + p, \mu\mathbf{1} + q\}$ for $\lambda, \mu \in \mathbb{R}$ and let $k \in [d]$ be arbitrary but fixed. We treat each inequality separately but in each case we assume without loss of generality that $r_k = \lambda + p_k \geqslant \mu + q_k$.

For the first inequality suppose that both p and q are in the k-th sector of some hyperplane $H(u)$, so that $p_k - p_i \geqslant u_k - u_i$ and $q_k - q_i \geqslant u_k - u_i$ for all $i \in [d]$. Now, for $j \in [d]$ we distinguish two cases. If $r_j = \lambda + p_j \geqslant \mu + q_j$, then $r_j - r_k = p_j - p_k$, so that r is in the k-th sector of $H(u)$. If $r_j = \mu + q_j \geqslant \lambda + p_j$, then $r_k - r_j \geqslant \mu + q_k - r_j = \mu + q_k - (\mu + q_j) = q_k - q_j$. Hence $r_k - r_j \geqslant u_k - u_j$ for all $j \in [d]$, and again we conclude that r is in the k-th sector of $H(u)$.

For the second inequality suppose that r is contained in the k-th sector of some hyperplane $H(u)$, so that $r_k - r_j \geqslant u_k - u_j$ for all $j \in [d]$. Since $r_k = \lambda + p_k \geqslant \mu + q_k$ we have that $\lambda + p_k \geqslant u_k - u_j + r_j$ for all $j \in [d]$. Also, $r_j \geqslant \lambda + p_j$ and hence $\lambda + p_k \geqslant u_k - u_j + \lambda + p_j$. We conclude $p_k - p_j \geqslant u_k - u_j$ for all $j \in [d]$. Hence p is in the k-th sector of $H(u)$, as desired. $\square$

From the definition of coarse types we obtain the following statement.

Corollary 2.12. *Let $\mathcal{A} = \mathcal{A}(V)$ be a* max*-tropical hyperplane arrangement in $\mathbb{T}^{d-1}$ and $p, q \in \mathbb{T}^{d-1}$. Then*

$$\mathbf{t}_{\mathcal{A}}(r) \leqslant \max\bigl(\mathbf{t}_{\mathcal{A}}(p), \mathbf{t}_{\mathcal{A}}(q)\bigr)$$

for $r \in \operatorname{tconv}^{\max}\{p, q\}$.

The following observation on the topology of regions of bounded fine and coarse (co)type will be the main tool for establishing results regarding (co)cellular resolutions in Section 3.

Corollary 2.13. *Let $\mathcal{A} = \mathcal{A}(V)$ be an arrangement of n* max*-tropical hyperplanes in $\mathbb{T}^{d-1}$, and let $B \in \{0,1\}^{n \times d}$ and $\mathbf{b} \in \mathbb{N}^d$. With labels determined by fine (respectively, coarse) type the following subsets of $\mathbb{T}^{d-1}$ are* max*-tropically convex and hence contractible:*

$$(\mathcal{C}_{\mathcal{A}}, T)_{\leqslant B} := \bigl\{p \in \mathbb{T}^{d-1}\colon\ T_{\mathcal{A}}(p) \leqslant B\bigr\} = \bigcup\bigl\{C \in \mathcal{C}_{\mathcal{A}}\colon\ T_{\mathcal{A}}(C) \leqslant B\bigr\},$$

$$(\mathcal{C}_{\mathcal{A}}, \mathbf{t})_{\leqslant \mathbf{b}} := \bigl\{p \in \mathbb{T}^{d-1}\colon\ \mathbf{t}_{\mathcal{A}}(p) \leqslant \mathbf{b}\bigr\} = \bigcup\bigl\{C \in \mathcal{C}_{\mathcal{A}}\colon\ \mathbf{t}_{\mathcal{A}}(C) \leqslant \mathbf{b}\bigr\}.$$

Similarly, with labels determined by fine (respectively, coarse) co*type the following subsets of $\mathbb{T}^{d-1}$ are* min*-tropically convex and hence contractible:*

$$(\mathcal{C}_{\mathcal{A}}, \overline{T})_{\leqslant B} := \bigl\{p \in \mathbb{T}^{d-1}\colon\ \overline{T}_{\mathcal{A}}(p) \leqslant B\bigr\} = \bigcup\bigl\{C \in \mathcal{C}_{\mathcal{A}}\colon\ \overline{T}_{\mathcal{A}}(C) \leqslant B\bigr\},$$

$$(\mathcal{C}_{\mathcal{A}}, \bar{\mathbf{t}})_{\leqslant \mathbf{b}} := \bigl\{p \in \mathbb{T}^{d-1}\colon\ \bar{\mathbf{t}}_{\mathcal{A}}(p) \leqslant \mathbf{b}\bigr\} = \bigcup\bigl\{C \in \mathcal{C}_{\mathcal{A}}\colon\ \bar{\mathbf{t}}_{\mathcal{A}}(C) \leqslant \mathbf{b}\bigr\}.$$

As a consequence, the two subsets of $\mathbb{T}^{d-1}$ obtained by replacing the complex $\mathcal{C}_{\mathcal{A}}$ in the above pair of formulas with the bounded complex $\mathcal{B}_{\mathcal{A}}$ are min*-tropically convex and hence contractible.*

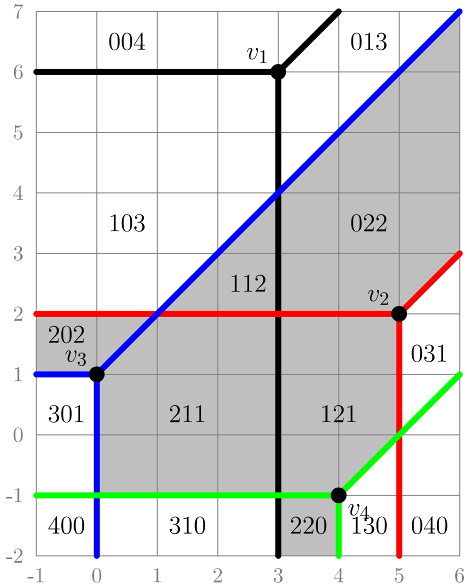

Fig. 2. Coarse down-set $(C_{\mathcal{A}})_{\leqslant(2,2,2)}$ for the arrangement $\mathcal{A}$ from Example 2.3. This is an open subset of $\mathbb{T}^2$.

Proof. The max-tropical convexity of $(\mathcal{C}_{\mathcal{A}}, T)_{\leqslant B}$ follows from Proposition 2.11, and Corollary 2.12 establishes the same property for the coarse variant $(\mathcal{C}_{\mathcal{A}}, \mathbf{t})_{\leqslant \mathbf{b}}$.

If r is a point in the min-tropical line segment between p and q a reasoning similar to the proof of Proposition 2.11 shows that $T_{\mathcal{A}}(r) \geqslant \min(T_{\mathcal{A}}(p), T_{\mathcal{A}}(q))$. Passing to complements yields $\overline{T}_{\mathcal{A}}(r) \leqslant \max(\overline{T}_{\mathcal{A}}(p), \overline{T}_{\mathcal{A}}(q))$ and thus $\bar{\mathbf{t}}_{\mathcal{A}}(r) \leqslant \max(\bar{\mathbf{t}}_{\mathcal{A}}(p), \bar{\mathbf{t}}_{\mathcal{A}}(q))$ for the coarse types. We conclude that the sets $(\mathcal{C}_{\mathcal{A}}, \overline{T})_{\leqslant B}$ and $(\mathcal{C}_{\mathcal{A}}, \bar{\mathbf{t}})_{\leqslant \mathbf{b}}$ are min-tropically convex. For the last claim, we note that by Proposition 2.9,

$$(\mathcal{B}_{\mathcal{A}}, \mathbf{t})_{\leqslant \mathbf{b}} = (\mathcal{C}_{\mathcal{A}}, \mathbf{t})_{\leqslant \widehat{\mathbf{b}}},$$

where $\widehat{\mathbf{b}}_i = \min\{\mathbf{b}_i, n-1\}$. Hence both $(\mathcal{B}_{\mathcal{A}}, \overline{T})_{\leqslant B}$ and $(\mathcal{B}_{\mathcal{A}}, \bar{\mathbf{t}})_{\leqslant \mathbf{b}}$ are intersections of min-tropically convex sets. Such sets are contractible by Proposition 2.10. $\square$

Note that the down-sets of bounded (coarse) type need not be closed or bounded. In Fig. 2 we illustrate an example of a coarse down-set from our running example.

3. Resolutions

In this section we show how the polyhedral complexes $\mathcal{C}_{\mathcal{A}}$ and $\mathcal{B}_{\mathcal{A}}$ arising from a tropical hyperplane arrangement $\mathcal{A} = \mathcal{A}(V)$ support resolutions for associated monomial ideals. We begin with a few definitions.

Definition 3.1. Let $\mathcal{A} = \mathcal{A}(V)$ be an arrangement of n tropical hyperplanes in $\mathbb{T}^{d-1}$. The *fine type* and *fine cotype ideal* associated to $\mathcal{A}$ are the squarefree monomial ideals

$$I_{T(\mathcal{A})} = \left\langle \mathbf{x}^{T(p)}\colon\ p \in \mathbb{T}^{d-1} \right\rangle \subset \mathbb{F}[x_{11}, x_{12}, \ldots, x_{nd}],$$

$$I_{\overline{T}(\mathcal{A})} = \left\langle \mathbf{x}^{\overline{T}(p)}\colon\ p \in \mathbb{T}^{d-1} \right\rangle \subset \mathbb{F}[x_{11}, x_{12}, \ldots, x_{nd}]$$

where $\mathbf{x}^{T(p)} = \prod\{x_{ij}\colon T(p)_{ij} = 1\}$. Analogously, the *coarse type* and *coarse cotype ideal* associated to $\mathcal{A}$ are given by

$$I_{\mathbf{t}(\mathcal{A})} = \langle \mathbf{x}^{\mathbf{t}(p)}\colon p \in \mathbb{T}^{d-1} \rangle \subset \mathbb{F}[x_1, x_2, \ldots, x_d],$$
$$I_{\bar{\mathbf{t}}(\mathcal{A})} = \langle \mathbf{x}^{\bar{\mathbf{t}}(p)}\colon p \in \mathbb{T}^{d-1} \rangle \subset \mathbb{F}[x_1, x_2, \ldots, x_d],$$

where $\mathbf{x}^{\mathbf{t}(p)} = x_1^{t_1} x_2^{t_2} \cdots x_d^{t_d}$ with $\mathbf{t}(p) = (t_1, t_2, \ldots, t_d)$.

An analogous construction of such ideals for classical hyperplane arrangements appears in Novik et al. [15] in the form of *oriented matroid ideals*, where monomial generators are given by (complements of) covector data.

3.1. Cellular and cocellular resolutions

The relation between the decompositions of $\mathbb{T}^{d-1}$ and the various ideals of Definition 3.1 is given via (co)cellular resolutions. Although cellular resolutions are by now a standard tool in (combinatorial) commutative algebra, *co*cellular resolutions are less frequently used and so we take the opportunity to discuss them here in some detail. Our presentation is based on the book [14] of Miller and Sturmfels.

For a fixed field $\mathbb{F}$ we let $S = \mathbb{F}[x_1, \ldots, x_m]$ be the polynomial ring equipped with the $\mathbb{Z}^m$-grading given by $\deg \mathbf{x}^{\mathbf{a}} = \mathbf{a} \in \mathbb{Z}^m$. A free $\mathbb{Z}^m$-graded resolution $\mathcal{F}_\bullet$ of a $\mathbb{Z}^m$-graded module M is a complex of $\mathbb{Z}^m$-graded S-modules

$$\mathcal{F}_\bullet\colon \quad \cdots \xrightarrow{\phi_{k+1}} F_k \xrightarrow{\phi_k} \cdots \xrightarrow{\phi_2} F_1 \xrightarrow{\phi_1} F_0 \to 0$$

where $F_i \cong \bigoplus_{\mathbf{a} \in \mathbb{Z}^m} S(-\mathbf{a})^{\beta_{i,a}}$ are free $\mathbb{Z}^m$-graded S-modules, the maps ϕ_i are homogeneous, and such that the complex is exact except for $\operatorname{coker} \phi_1 \cong M$. The resolution is called *minimal* exactly when $\beta_{i,a} = \dim_{\mathbb{F}} \operatorname{Tor}_i^S(S/I, \mathbb{F})_a$ and the numbers $\beta_{i,a}$ are called the *fine graded Betti numbers*.

Certain $\mathbb{Z}^m$-graded resolutions have a particularly efficient encoding as *cellular* and *cocellular resolutions*, introduced in [2] and [13], respectively. Let $\mathcal{P}$ be an oriented polyhedral complex and let $(\mathbf{a}_H)_{H \in \mathcal{P}} \in \mathbb{Z}^m$ be a labeling of the cells of $\mathcal{P}$ such that

$$\mathbf{a}_H = \max\{\mathbf{a}_G\colon \text{for } G \subset H \text{ a face}\}.$$

The labeled complex $(\mathcal{P}, \mathbf{a})$ gives rise to a complex of free $\mathbb{Z}^m$-graded S-modules in the following way: Let $(C_\bullet, \partial_\bullet)$ be the cellular chain complex for $\mathcal{P}$ and for two cells $G, H \in \mathcal{P}$ with $\dim H = \dim G + 1$ denote by $\varepsilon(H, G) \in \{0, \pm 1\}$ the coefficient of G in the cellular boundary of H. Now define free modules

$$F_i := \bigoplus_{H \in \mathcal{P},\, \dim H = i+1} S(-\mathbf{a}_H).$$

The differentials $\phi_i : F_i \to F_{i-1}$ are given on generators by

$$\phi_i(e_H) := \sum_{\dim G = \dim H - 1} \varepsilon(H, G) \mathbf{x}^{\mathbf{a}_H - \mathbf{a}_G} e_G.$$

It can be verified that this defines a complex $\mathcal{F}_\bullet^{\mathcal{P}}$. For $\mathbf{b} \in \mathbb{Z}^m$ denote by $\mathcal{P}_{\leqslant \mathbf{b}}$ the subcomplex given by all cells $H \in \mathcal{P}$ with $\mathbf{a}_H \leqslant \mathbf{b}$, that is $(\mathbf{a}_H)_i \leqslant b_i$ for all $i \in [m]$.

Lemma 3.2. *(See [14, Proposition 4.5].) Let $\mathcal{F}_{\bullet}^{\mathcal{P}}$ be the complex obtained from the labeled polyhedral complex $(\mathcal{P}, \mathbf{a})$. If for every $\mathbf{b} \in \mathbb{Z}^n$ the subcomplex $\mathcal{P}_{\leqslant \mathbf{b}}$ is acyclic over $\mathbb{F}$, then $\mathcal{F}_{\bullet}^{\mathcal{P}}$ resolves the quotient of S by the ideal $\langle \mathbf{x}^{\mathbf{a}_v}\colon v \in P$ vertex$\rangle$. Furthermore, the resolution is minimal if $\mathbf{a}_H \neq \mathbf{a}_G$ for any two faces $G \subset H$ with* $\dim H = \dim G + 1$.

The complex $\mathcal{F}_{\bullet}^{\mathcal{P}}$ is called a *cellular resolution* if it meets the criterion above, and we say that the polyhedral complex $\mathcal{P}$ *supports* the resolution. If the labeling is such that

$$\mathbf{a}_H = \max\{\mathbf{a}_G\colon \text{ for } G \supset H \text{ a face}\}$$

then $\mathcal{P}$ is said to be *colabeled* and gives rise to a complex utilizing the cellular *cochain* complex of $\mathcal{P}$. For this, let $\mathcal{F}_{\mathcal{P}}^{\bullet}$ denote the complex with free S-modules $F^i := F_i$ as defined above and differentials $\phi^i : F^{i-1} \to F^i$ with

$$\phi^i(e_H) := \sum_{\dim G = \dim H + 1} \delta(H, G)\mathbf{x}^{\mathbf{a}_H - \mathbf{a}_G} e_G$$

where $\delta(H, G)$ records the corresponding coefficient in the coboundary map for $\mathcal{P}$. If the complex is acyclic, the resulting resolution is called *cocellular*. For $\mathbf{b} \in \mathbb{Z}^m$ the collection $\mathcal{P}_{\leqslant \mathbf{b}}$ of relatively open cells H with $\mathbf{a}_H \leqslant \mathbf{b}$ is *not* a subcomplex. However, as a topological space it is the union of the relatively open stars of cells G for which $\mathbf{a}_G \leqslant \mathbf{b}$ is minimal and the cochain complex of the nerve is isomorphic to the degree $\mathbf{b}$ component of $\mathcal{F}_{\mathcal{P}}^{\bullet}$. This yields an analogous criterion regarding the exactness of $\mathcal{F}_{\mathcal{P}}^{\bullet}$. The proof of Lemma 3.2 given in [14] essentially establishes the following criterion.

Lemma 3.3. *If $\mathcal{P}_{\leqslant \mathbf{b}}$ is acyclic over $\mathbb{F}$ for every $\mathbf{b} \in \mathbb{Z}^n$ then $\mathcal{F}_{\mathcal{P}}^{\bullet}$ resolves S/I where $I = \langle \mathbf{x}^{\mathbf{a}_H}\colon H \in \mathcal{P}$ maximal cell$\rangle$. The resolution is minimal if $\mathbf{a}_H \neq \mathbf{a}_G$ for any two faces $G \subset H$ with* $\dim H = \dim G + 1$.

3.2. Resolutions from the arrangement

As usual let $\mathcal{A} = \mathcal{A}(V)$ be a max-tropical hyperplane arrangement in $\mathbb{T}^{d-1}$ and let $\mathcal{C}_{\mathcal{A}}$ be the induced polyhedral decomposition of $\mathbb{T}^{d-1}$. As we have seen, every cell in $\mathcal{C}_{\mathcal{A}}$ is naturally assigned a matrix and a vector determined by its fine and coarse type, respectively. The next result states that these assignments are actually *colabelings* in the sense of Section 3.1.

Proposition 3.4. *Let $\mathcal{A} = \mathcal{A}(V)$ be an arrangement of tropical hyperplanes in $\mathbb{T}^{d-1}$. For every cell $C \in \mathcal{C}_{\mathcal{A}}$ of codimension $\geqslant 1$ we have*

$$T_{\mathcal{A}}(C) = \max\{T_{\mathcal{A}}(D)\colon C \subset D\}, \quad \textit{and}$$

$$\mathbf{t}_{\mathcal{A}}(C) = \max\{\mathbf{t}_{\mathcal{A}}(D)\colon C \subset D\}.$$

Thus, both fine and coarse type yield a colabeling for the complex $\mathcal{C}_{\mathcal{A}}$.

Proof. As we remarked before $C \subseteq D$ implies $T_{\mathcal{A}}(D) \leqslant T_{\mathcal{A}}(C)$ and thus we have to show that $T_{\mathcal{A}}(C)$ is not strictly larger. For $k \in [d]$ consider the set $S_k(0) \subset \mathbb{R}^d$ given by all points $x \in \mathbb{R}^d$ such that $x_k \leqslant x_i$ for all $i \neq k$. The sector $S_k(0)$ is an ordinary polyhedron of dimension d and it can be checked that for $p \in \mathbb{T}^{d-1}$ we have that $p \in S_k^{\max}(v_i)$ implies $p + S_k(0) \subseteq S_k(v_i)$. Now, since $\operatorname{codim}(C) \geqslant 1$ we get that $C + S_k(0)$ meets the star of C, thereby showing that if $T_{\mathcal{A}}(C)_{ik} = 1$ for some i, then there is a cell $D \supset C$ with $T_{\mathcal{A}}(D)_{ik} = 1$. The same argument proves the purported equality for the coarse type. □

As an immediate consequence we obtain sets of generators for the fine and the coarse type ideals.

Corollary 3.5. *For a tropical hyperplane arrangement $\mathcal{A}$ both the fine and coarse type ideals are generated by monomials corresponding to the respective types on the inclusion-maximal cells of $\mathcal{C}_{\mathcal{A}}$.*

We now have all the ingredients to establish our first main result linking the fine/coarse type ideals with the polyhedral decomposition $\mathcal{C}_{\mathcal{A}}$.

Theorem 3.6. *Let $\mathcal{A} = \mathcal{A}(V)$ be a tropical hyperplane arrangement, and let $\mathcal{C}_{\mathcal{A}}$ be the decomposition of the tropical torus $\mathbb{T}^{d-1}$ induced by $\mathcal{A}$. Then with labels given by fine type (respectively, coarse type) the labeled complex $\mathcal{C}_{\mathcal{A}}$ supports a minimal cocellular resolution of the fine type ideal $I_{T(\mathcal{A})}$ (respectively, the coarse type ideal $I_{\mathbf{t}(\mathcal{A})}$).*

Proof. By Proposition 3.4 the polyhedral complex $\mathcal{C}_{\mathcal{A}}$ is colabeled by both fine and coarse type. It follows from Lemma 3.3 and Corollary 2.13 that this yields a cellular resolution of the respective type ideal. The minimality is a consequence of Proposition 2.8. □

A key point is that all of the above is valid for point configurations V which are not necessarily in general position. In the case of hyperplanes in general position, the coarse type ideal is well known and, in particular, is independent of the choice of hyperplanes.

Corollary 3.7. *Let $\mathcal{A} = \mathcal{A}(V)$ be a* sufficiently generic *arrangement of n tropical hyperplanes in $\mathbb{T}^{d-1}$. Then $\mathcal{C}_{\mathcal{A}}$ supports a minimal cocellular resolution of*

$$I_{\mathbf{t}(\mathcal{A})} = \langle x_1, \dots, x_d \rangle^n$$

the n-th power of the homogeneous maximal ideal.

Since the minimal resolution of an ideal is unique up to isomorphism, we have that the resolution arising from Corollary 3.7 is always isomorphic (as a chain complex), to the well-known Eliahou–Kervaire resolution [14, Section 2.3]. However, Corollary 3.7 shows that there is a multitude of colabeled complexes coming from tropical hyperplane arrangements that give rise to a cellular description of the minimal resolution of $\langle x_1, \dots, x_n \rangle^d$.

We next turn to the other class of ideals introduced in the beginning of this section, namely the fine and coarse *co*type ideals. As was the case with the type ideals, we first consider the labeling of the relevant complexes.

Proposition 3.8. *Let $\mathcal{A} = \mathcal{A}(V)$ be an arrangement of tropical hyperplanes. For every cell $D \in \mathcal{C}_{\mathcal{A}}$ of dimension $\geqslant 1$ we have*

$$\overline{T}_{\mathcal{A}}(D) = \max\{\overline{T}_{\mathcal{A}}(C)\colon C \subset D\} \quad \text{and}$$
$$\bar{\mathbf{t}}_{\mathcal{A}}(D) = \max\{\bar{\mathbf{t}}_{\mathcal{A}}(C) : C \subset D\}.$$

Thus, both the fine and coarse types yield labelings for the complex $\mathcal{C}_{\mathcal{A}}$.

Proof. The assertion follows by showing that

$$T_{\mathcal{A}}(D) \geqslant \min\{T_{\mathcal{A}}(C)\colon C \subset D\}.$$

We mimic the argument in the proof of Proposition 3.4. It follows from the full-dimensionality of Q_k that $C + Q_k$ intersects D for every cell $C \subset D$ and thus $T(D)_{ik} \geqslant T(C)_{ik}$ for all C in the boundary of D. □

The previous proposition implies that the cotype ideals are generated by the inclusion-*minimal* faces (namely, the 0-dimensional cells) of $\mathcal{C}_{\mathcal{A}}$. As a consequence (together with the last part of Corollary 2.13) we see that resolutions of these ideals are supported on the collection of bounded faces of $\mathcal{C}_{\mathcal{A}}$, and we obtain our next main result. This generalizes [4, Theorem 1].

Theorem 3.9. *Let $\mathcal{A} = \mathcal{A}(V)$ be an arrangement of tropical hyperplanes and let $\mathcal{B}_{\mathcal{A}}$ be the subcomplex of bounded cells of $\mathcal{C}_{\mathcal{A}}$. Then $\mathcal{B}_{\mathcal{A}}$, with labels given by fine cotype (respectively, coarse cotype) supports a minimal resolution of the fine cotype ideal $I_{\overline{T}(\mathcal{A})}$ (respectively, coarse cotype ideal $I_{\bar{\mathbf{t}}(\mathcal{A})}$).*

It is tempting to see the fine cotype ideal as some kind of polarization of the coarse cotype ideal in the sense of [14, Section 3.2]. But we are not aware of a precise result in this direction. Notice that not every square-free monomial ideal I in $\mathbb{F}[x_{ij}]$ is the polarization of the ideal J in $\mathbb{F}[x_i]$ obtained via the map $\pi = x_{ij} \mapsto x_i$. An example is $I = \langle x_{11}x_{22}x_{31}, x_{12}x_{21}x_{41}\rangle$. The image is $J = \langle x_1x_2x_3, x_1x_2x_4\rangle$. Both ideals are resolved by the 1-simplex but taking the 1-simplex labeled with the generators of I and "contracting" the labels via π yields ${x_1}^2{x_2}^2x_3x_4$ as the label for the full 1-simplex, and hence the downset for $(1, 1, 1, 1)$ is not contractible.

Example 3.10. The *tropical hypersimplex* $\Delta^{\mathrm{trop}}(k,n)$ is defined as the tropical convex hull of all 0/1-vectors of length n with exactly k zeros. Notice that we have the strict inclusions

$$\Delta^{\mathrm{trop}}(1,n) \supsetneq \Delta^{\mathrm{trop}}(2,n) \supsetneq \cdots \supsetneq \Delta^{\mathrm{trop}}(n-1,n).$$

We wish to determine the coarse type ideal corresponding to the configuration of $\binom{d}{k}$ points in $\mathbb{T}^{d-1}$ given by the tropical vertices of $\Delta^{\mathrm{trop}}(k,d)$. The d generators of $\Delta^{\mathrm{trop}}(1,d)$ are in general position, and hence in this case the coarse type ideal is the homogeneous maximal ideal $\langle x_1, x_2, \ldots, x_d\rangle$. The second tropical hypersimplex $\Delta^{\mathrm{trop}}(2,d)$ is contained in the min-tropical hyperplane with the origin as its apex. In particular, $\Delta^{\mathrm{trop}}(2,d)$, seen as a polytopal complex in $\mathbb{R}^{d-1} = \mathbb{T}^{d-1}$, is of dimension $d-2$. This implies that all maximal cells in the type decomposition of $\mathbb{T}^{d-1}$ induced by the tropical vertices of $\Delta^{\mathrm{trop}}(k,d)$ are unbounded if $2 \leqslant k < d$. Equivalently, no generator of the coarse type ideal is divisible by $x_1x_2\cdots x_d$. The following is a special case of [11, Theorem 11].

Proposition 3.11. *Let $2 \leqslant k < d$. Then, up to action of the symmetric group, the coarse types induced by the k-th tropical hypersimplex in $\mathbb{T}^{d-1}$ are given by*

$$\left(\binom{d-\alpha}{k} + \binom{d-1}{k-1}, \binom{d-2}{k-1}, \ldots, \binom{d-\alpha}{k-1}, 0, \ldots, 0\right) \in \mathbb{N}^d$$

where $1 \leqslant \alpha \leqslant d-k+1$.

4. The mixed subdivision picture

As mentioned in Section 2.2, an arrangement $\mathcal{A} = \mathcal{A}(V)$ of n hyperplanes in $\mathbb{T}^{d-1}$ gives rise to a regular subdivision of the product of two simplices $\Delta_{n-1} \times \Delta_{d-1}$ and, via the Cayley trick, a mixed subdivision of a dilated simplex (see Fig. 3). We refer to the recent book [5] for the relevant details and here only outline the main ideas. Let $\Delta_{k-1} = \operatorname{conv}\{e_1, e_2, \ldots, e_k\} \subset \mathbb{R}^k$ be the standard $(k-1)$-simplex. The ordered apices $V = (v_1, \ldots, v_n)$ induce a height function on the vertices of $\Delta_{n-1} \times \Delta_{d-1}$ by $(e_i, e_j) \mapsto (v_i)_j$. The lower envelope of the convex hull of the lifted points yields a regular (or coherent) subdivision of $\Delta_{n-1} \times \Delta_{d-1}$ and it is shown in [6] that these regular subdivisions are in bijection with the tropical complexes of n tropical hyperplanes in $\mathbb{T}^{d-1}$. The polytope $\Delta_{n-1} \times \Delta_{d-1}$ can be viewed as a Cayley polytope and hence, via the Cayley trick, tropical hyperplane arrangements are in bijection with regular mixed subdivisions of $n\Delta_{d-1}$. A mixed subdivision of $n\Delta_{d-1}$ is a polyhedral subdivision such that every cell is of the form $\tau = \Delta_{I_1} + \Delta_{I_2} + \cdots + \Delta_{I_n}$ where $I_j \subseteq [d]$ and $\Delta_{I_j} =$

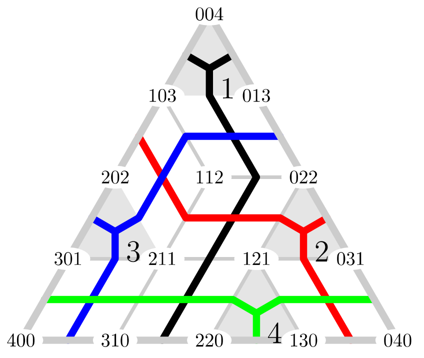

Fig. 3. Mixed subdivision of $4\Delta_2$ corresponding to Example 2.3.

$\operatorname{conv}\{e_i\colon i \in I_j\}$ is a face of Δ_{d-1}. The mixed subdivision is called *fine* if $\dim \tau = \sum_j \dim \Delta_{I_j}$. The aim of this section is to interpret our results from the previous sections in terms of mixed subdivisions of dilated simplices. For this it will be convenient to have the following notion of coarse type of a cell defined purely in terms of the mixed subdivision.

Definition 4.1. Let $\tau = \Delta_{I_1} + \Delta_{I_2} + \cdots + \Delta_{I_n}$ be a cell in a mixed subdivision of $n\Delta_{d-1}$. The *coarse type* $\mathbf{t}(\tau) \in \mathbb{N}^d$ of τ is the vector whose i-th coordinate equals $\#\{j \in [n]\colon i \in I_j\}$, the number of occurrences of i in the decomposition of τ. The *dual coarse type* $\mathbf{d}(\tau) \in \mathbb{N}^n$ is a vector whose i-th coordinate is given by $\#I_i$, the number of elements of I_i.

The following proposition relates to Remark 2.6 made above. Here we provide a complete proof which does appeal to tropical geometry and hence applies to the more general situation of not necessarily regular subdivisions.

Proposition 4.2. *In any fine mixed subdivision Σ of $n\Delta_{d-1}$, the set of* 0*-dimensional cells are precisely the lattice points $n\Delta_{d-1} \cap \mathbb{Z}^d$, and the collection of coarse types of these cells are in bijection with the set of compositions of n into d parts.*

Proof. Let $\tau = \Delta_{I_1} + \Delta_{I_2} + \cdots + \Delta_{I_n}$ be a fine mixed cell of Σ. It follows from the definition of fine mixed cell that if τ is zero-dimensional, then so is Δ_{I_j} for all j. Hence $\tau = e_{i_1} + e_{i_2} + \cdots + e_{i_n}$ is a lattice point in $n\Delta_{d-1}$ and its coarse type is a composition of n into d parts. In order to show that all such lattice points arise as 0-dimensional cells, let $t \in n\Delta_{d-1} \cap \mathbb{Z}^d$ and let $\tau = \sum_{j=1}^n \Delta_{I_j}$ be the inclusion-minimal mixed cell containing t. By [16, Prop. 14.12] every lattice point of τ is of the form $\sum_{j=1}^n \{e_{i_j}\}$ with $i_j \in I_j$ for all j. However, as the mixed subdivision is fine, τ is combinatorially isomorphic to the product $\prod_{j=1}^n \Delta_{I_j}$ and hence every sum of vertices is a vertex. Therefore, τ is a vertex of the mixed subdivision Σ. $\square$

4.1. Resolutions supported by mixed subdivisions

In this section we discuss our results regarding cellular resolutions in the context of mixed subdivisions of dilated simplices. Although these results are more or less translations of the above via the Cayley trick, we find it useful to make this transition explicit. It seems that the mixed subdivision picture allows for more natural statements whereas the tropical convexity picture allows for more natural proofs. We refer the reader back to Definition 4.1 for the definition of coarse type of a mixed cell.

Corollary 4.3. *Let Σ be any regular mixed subdivision of $n\Delta_{d-1}$. Consider Σ to be a labeled polytopal complex with each face σ labeled by the least common multiple of the vertices that it contains. Then for any field $\mathbb{F}$, the* *complex $\Sigma_{\mathcal{A}}$ supports a minimal cellular resolution of the coarse type ideal $I_{\mathbf{t}(\mathcal{A})} = \langle \mathbf{x}^{\mathbf{t}(p)}\colon\ p \in \mathbb{T}^{d-1}\rangle$ in $\mathbb{F}[x_1,\ldots,x_d]$.*

Proof. The fact that $\Sigma_{\mathcal{A}}$ is a labeled complex follows from Proposition 3.4. As a poset $\Sigma_{\mathcal{A}}$ is isomorphic to the corresponding regular subdivision of $\Delta_{n-1} \times \Delta_{d-1}$ via the Cayley trick. By [6, Lemma 22] this regular subdivision is dual to the cell decomposition $\mathcal{C}_{\mathcal{A}}$ of $\mathbb{T}^{d-1}$. In this way the labeling of $\Sigma_{\mathcal{A}}$ turns into the colabeling of $\mathcal{B}_{\mathcal{A}}$ by coarse types. Theorem 3.6 now establishes the claim. $\square$

It is now straightforward to derive the mixed subdivision result corresponding to Corollary 3.7.

Corollary 4.4. *Let Σ be any regular* fine *mixed subdivision of $n\Delta_{d-1}$. Then $\Sigma_{\mathcal{A}}$, as a labeled polyhedral complex, supports a minimal cellular resolution of $\langle x_1,\ldots,x_d\rangle^n$.*

Proof. This follows from Corollary 4.3 and Proposition 4.2. $\square$

Question 4.5. Is there a good characterization of the monomial ideals $I \subset \mathbb{F}[x_1,\ldots,x_d]$ arising as $I_{\mathbf{t}(\mathcal{A})}$ for some arrangement $\mathcal{A}$ of tropical hyperplanes in $\mathbb{T}^{d-1}$ in terms of commutative algebra?

Some necessary conditions are obvious, e.g., I should be homogeneous of some degree n, and that x_i^d should be contained in I for all i. In particular, this means that any coarse type ideal is necessarily Artinian.

4.2. Alexander duality of ideals and resolutions

We have seen how the bounded subcomplexes of tropical hyperplane arrangements are related to mixed subdivisions of dilated simplices in terms of a geometric duality. This duality extends to the algebraic level of our resolutions in the context of *Alexander duality of resolutions*. For this we will need the following notion of the Alexander dual of a (not necessarily square-free) monomial ideal.

Definition 4.6. Suppose I is a monomial ideal in the polynomial ring $\mathbb{F}[x_1,\ldots,x_d]$ and let $\mathbf{a} \in \mathbb{N}^d$. The *Alexander dual of I with respect to* $\mathbf{a}$ is given by the intersection

$$I^{[\mathbf{a}]} = \bigcap \bigl\{\mathfrak{m}^{\mathbf{a}\setminus\mathbf{b}}\colon\ \mathbf{x}^{\mathbf{b}} \text{ is a minimal generator of } I\bigr\},$$

where $\mathbf{a}\setminus\mathbf{b}$ denotes the vector whose i-th coordinate is $a_i + 1 - b_i$ if $b_i \geqslant 1$, and is 0 if $b_i = 0$. Here we borrow the notation $\mathfrak{m}^{\mathbf{a}} := \langle x_i^{a_i}\colon\ a_i \geqslant 1\rangle$.

Note that if I is a square-free monomial ideal (and hence the Stanley–Reisner ring of some simplicial complex) and $\mathbf{a} = \mathbf{1}$ is taken to be the all-ones vector, then this notion recovers the more familiar notion of Alexander duality of simplicial complexes. The main result concerning duality of resolutions, relevant for us, is the following [14, Theorem 5.37].

Theorem 4.7. *Suppose I is a monomial ideal in degrees preceding some $\mathbf{a} \in \mathbb{N}^d$ and suppose $\mathcal{F}_{\bullet}^{\mathcal{P}}$ is a minimal cellular resolution of $S/(I + \mathfrak{m}^{\mathbf{a}+\mathbf{1}})$ such that all face labels on $\mathcal{P}$ precede $\mathbf{a}+\mathbf{1}$. Let $\mathcal{Q}$ denote the labeled complex with the same underlying complex $\mathcal{P}$ but with labels $\bar{\mathbf{t}}_F = \mathbf{a} + \mathbf{1} - \mathbf{t}_F$. Then $\mathcal{F}_{\bullet}^{\mathcal{Q}_{\leqslant \mathbf{a}}}$ is a minimal cocellular resolution of $I^{[\mathbf{a}]}$.*

Applying this theorem we obtain the following dual resolution of the coarse cotype ideal. For σ a face of a mixed subdivision Σ, the *coarse cotype* of σ is defined to be $n\mathbf{1} - \mathbf{t}(\sigma)$, where $\mathbf{t}(\sigma)$ is the coarse type of σ defined in Definition 4.1.

Proposition 4.8. *Given any arrangement $\mathcal{A}$ of n tropical hyperplanes in $\mathbb{T}^{d-1}$, let $\overline{\Sigma}_{\mathcal{A}}$ denote the associated mixed subdivision of $n\Delta_{d-1}$ with labels given by coarse cotype. Then $\overline{\Sigma}_{\mathcal{A}}$ supports a minimal cocellular resolution of the coarse cotype ideal $I_{\bar{\mathbf{t}}(\mathcal{A})}$ in $\mathbb{F}[x_1,\dots,x_d]$. Consequently the associated tropical complex $\mathcal{B}_{\mathcal{A}}$, with labels given by coarse cotype, supports a minimal* cellular *resolution of $I_{\bar{\mathbf{t}}(\mathcal{A})}$.*

Proof. We apply Theorem 4.7 with $\mathbf{a} := (n-1)\mathbf{1}$, and with I defined to be the ideal generated by all monomials f of $I_{\mathbf{t}(\mathcal{A})}$ with $f \leqslant \mathbf{a}$ (in other words, throw out all generators of the form x_i^n for $1 \leqslant i \leqslant d$, all of which show up in $I_{\mathbf{t}(\mathcal{A})}$ regardless of the arrangement $\mathcal{A}$). We then have from Corollary 4.3 that $\Sigma_{\mathcal{A}}$ supports a minimal cellular resolution of $I_{\mathbf{t}(\mathcal{A})} = I + \mathfrak{m}^{\mathbf{a}+1}$. The conditions of Theorem 4.7 are met and we conclude that $(\overline{\Sigma}_{\mathcal{A}})_{\leqslant \mathbf{a}}$ supports a minimal cocellular resolution of $I^{[\mathbf{a}]}$.

We next determine $I^{[\mathbf{a}]}$, the Alexander dual of I with respect to $\mathbf{a} = (n-1)\mathbf{1}$. In [14] it is shown that if $\mathbf{b} \leqslant \mathbf{a}$ then $\mathbf{x}^{\mathbf{b}}$ lies outside I if and only if $\mathbf{x}^{\mathbf{a}-\mathbf{b}}$ lies inside $I^{[\mathbf{a}]}$. Hence to find a set of generators for $I^{[\mathbf{a}]}$ it suffices to determine the maximal monomials which lie outside I. But these monomials correspond to the minimal cotypes that arise in the complex $\mathcal{C}_{\mathcal{A}}$, and these are given by the collection of monomials $\bar{\mathbf{t}}(x)$, for x a 0-dimensional cell in $\mathcal{C}_{\mathcal{A}}$. Hence we conclude that $I^{[\mathbf{a}]} = I_{\bar{\mathbf{t}}(\mathcal{A})}$.

For the second part of the claim, we note that a face $\sigma \in \overline{\Sigma}_{\mathcal{A}}$ with label $\mathbf{b}$ satisfies $\mathbf{b} \leqslant \mathbf{a}$ if and only if $n\mathbf{1} - \mathbf{t}(\sigma) \leqslant (n-1)\mathbf{1}$ for the coarse type label $\mathbf{t}(\sigma)$ in $\Sigma_{\mathcal{A}}$. But this occurs exactly when $\mathbf{t}(\sigma) \geqslant \mathbf{1}$, which happens if and only if the face σ is not contained in the boundary of $\Sigma_{\mathcal{A}}$. Hence the cocellular resolution of $I^{[\mathbf{a}]}$ that we obtain, supported on $(\overline{\Sigma}_{\mathcal{A}})_{\leqslant \mathbf{a}}$, is given by the relative cocellular complex of $(\Sigma_{\mathcal{A}}, \partial\Sigma_{\mathcal{A}})$. By duality, the relative cochain complex $C^*(\Sigma_{\mathcal{A}}, \partial\Sigma_{\mathcal{A}})$ is isomorphic to the chain complex $C_*(\mathcal{B}_{\mathcal{A}})$ of the tropical complex $\mathcal{B}_{\mathcal{A}}$ (the bounded subcomplex of the decomposition of $\mathbb{T}^{d-1}$ induced by $\mathcal{A}$). Hence $\mathcal{B}_{\mathcal{A}}$, with labels given by coarse *cotype*, supports a minimal *cellular* resolution of the ideal $I^{[\mathbf{a}]} = I_{\bar{\mathbf{t}}(\mathcal{A})}$. □

Let us illustrate our results with one particularly well-behaved fine mixed subdivision of $n\Delta_{d-1}$ that arises from applying the Cayley trick to the *staircase triangulation* of $\Delta_{n-1} \times \Delta_{d-1}$. The name derives from the fact that the maximal cells correspond to monotone lattice path from $(1,1)$ to (n,d). Here, the vertices of $\Delta_{n-1} \times \Delta_{d-1}$ are indexed by the nodes of an $n \times d$-grid. To every such lattice path given by $1 = b_1 \leqslant b_2 \leqslant \cdots \leqslant b_n \leqslant b_{n+1} = d$ the corresponding cell is $\Delta_{B_1} + \Delta_{B_2} + \cdots + \Delta_{B_n}$ with $B_i := \{e_{b_i}, e_{b_i+1}, \dots, e_{b_{i+1}}\}$ for $1 \leqslant i \leqslant n$. The collection of these cells forms a regular fine mixed subdivision of $n\Delta_{d-1}$. The corresponding tropical hyperplane arrangement is the so-called *cyclic arrangement* [4] given by $V = (i+j)_{ij}$. The case of $n = d = 3$ is hinted at in [14, Example 2.20]. In [18] Sinefakopoulos constructs a cellular resolution of $\langle x_1,\dots,x_d\rangle^n$ as the basis for minimal cellular resolutions of Borel fixed ideals generated in one degree, and shows that this complex can be realized as a subdivision of a simplex. It can be checked that this complex is isomorphic (as a labeled complex) to the staircase mixed subdivision. From our results we see that *any* regular fine mixed subdivision of $n\Delta_{d-1}$ supports a minimal cellular resolution of the homogeneous ideal $\langle x_1,\dots,x_d\rangle^n$.

5. Face counting and incidence structure of the tropical complex

In Section 3 we saw how the polyhedral complexes $\mathcal{C}_{\mathcal{A}}$ and $\mathcal{B}_{\mathcal{A}}$ associated to an arrangement $\mathcal{A}$ gave rise to resolutions of the coarse type ideal $I_{\mathbf{t}(\mathcal{A})}$ and the fine cotype ideal $I_{\overline{T}(\mathcal{A})}$, respectively. The *minimality* of our resolution also leads to some important implications regarding the combinatorics of $\mathcal{C}_{\mathcal{A}}$ and $\mathcal{B}_{\mathcal{A}}$ themselves. In this section we discuss face numbers of tropical complexes, as well as an algorithm for determining the facial structure of $\mathcal{B}_{\mathcal{A}}$ given the arrangement $\mathcal{A} = \mathcal{A}(V)$. The latter generalizes a result of [4], where a similar algorithm for the case of sufficiently generic arrangements was provided.

5.1. Counting faces

As a first application we point out that the f-vector of $\mathcal{C}_{\mathcal{A}}$ can be determined from the $\mathbb{Z}$-graded ('coarse') Betti numbers of $I_{\mathbf{t}(\mathcal{A})}$. We noted in Proposition 2.9 that from the coarse type it is possible to distinguish bounded from unbounded cells in $\mathcal{C}_{\mathcal{A}}$. Thus, we can also recover the numerical behavior of the bounded complex $\mathcal{B}_{\mathcal{A}}$.

Corollary 5.1. *Let $\mathcal{A} = \mathcal{A}(V)$ be a tropical hyperplane arrangement in $\mathbb{T}^{d-1}$ and let $I_{\mathbf{t}(\mathcal{A})}$ be its coarse type ideal. Then the number of cells in $\mathcal{C}_{\mathcal{A}}$ of dimension k is*

$$f_k(\mathcal{C}_{\mathcal{A}}) = \beta_{d-1-k}(I_{\mathbf{t}(\mathcal{A})}) = \sum_{\mathbf{b}\in\mathbb{Z}^d} \beta_{d-1-k,\mathbf{b}}(I_{\mathbf{t}(\mathcal{A})}).$$

The number of bounded k-cells in $\mathcal{C}_{\mathcal{A}}$ is given by as the sum of Betti numbers $\beta_{d-1-k,\mathbf{b}}(I_{\mathbf{t}(\mathcal{A})})$ for which $\mathbf{b} > 0$.

Note that for the k-cells we need to consider the $(d-1-k)$-th Betti numbers. This is due to the fact that $\mathcal{C}_{\mathcal{A}}$ supports a *co*cellular resolution.

Furthermore, we can use the *uniqueness* of minimal resolutions to derive further results regarding face numbers of arrangements. For this, suppose $\mathcal{A}$ is a sufficiently generic arrangement of n tropical hyperplanes in $\mathbb{T}^{d-1}$, and let $\mathcal{C}_{\mathcal{A}}$ denote the induced polyhedral subdivision of $\mathbb{T}^{d-1}$ determined by type. In Corollary 3.7 we saw that $\mathcal{C}_{\mathcal{A}}$, with labels given by coarse type, supports a minimal cocellular resolution of the ideal $\langle x_1, \dots, x_d\rangle^n$. Any two such resolutions are isomorphic as chain complexes, and in particular the finely graded Betti numbers $\beta_{i,\sigma}$ do not depend on the resolution. By construction, $\beta_{i,\sigma}$ is precisely the number of cells in $\mathcal{C}_{\mathcal{A}}$ with monomial label σ. But the monomial labels are given by the coarse types, and hence we obtain the following.

Corollary 5.2. *Let $\mathcal{A}$ be a sufficiently generic arrangement of n hyperplanes in $\mathbb{T}^{d-1}$. For every $0 \leqslant k \leqslant d-1$ the collection of coarse types $\mathbf{t}(C_T)$ for $\dim C_T = k$, counted with multiplicities, is independent of the arrangement.*

Putting the above result in perspective with the second statement of Corollary 5.1, this proves the following result without appealing to the equidecomposability of $\Delta_{n-1} \times \Delta_{d-1}$.

Corollary 5.3. *The number of cells of $\mathcal{C}_{\mathcal{A}}$ for a tropical hyperplane arrangement $\mathcal{A}$ in general position is independent of the choice of hyperplanes. More precisely, the number of k-dimensional cells induced by the arrangement of n tropical hyperplanes in $\mathbb{T}^{d-1}$ equals*

$$f_k(\mathcal{C}_A) = \sum_{\ell=0}^{k} \binom{n+d-2-\ell}{n-1}\binom{d-1-\ell}{d-1-k}.$$

Proof. By Corollaries 5.1 and 3.7 we have $f_k(\mathcal{C}_{\mathcal{A}}) = \beta_{d-1-k}(\langle x_1, \dots, x_n\rangle^n)$. The Betti numbers of the power of the homogeneous maximal ideal are well known. For example, they can be determined as follows.

The ideal $\mathfrak{m}^n = \langle x_1, \dots, x_d\rangle^n$ is *strongly stable*, that is, $\frac{x_r}{x_s}\mathbf{x}^{\mathbf{b}} \in \mathfrak{m}^n$ for every $\mathbf{x}^{\mathbf{b}}$ monomial divisible by x_s and $r < s$. In particular, $\mathfrak{m}^n$ is Borel fixed, and hence the Betti numbers are given by [14, Theorem 2.18], and we have

$$\beta_i(\mathfrak{m}^n) = \sum_{\mathbf{a}\in\mathbb{N}^d, |\mathbf{a}|=n} \binom{\max(\mathbf{x}^{\mathbf{a}})-1}{i}$$

where $\max(\mathbf{x}^{\mathbf{a}}) = \max\{i\colon\ \mathbf{a}_i > 0\}$. Now if $\max(\mathbf{x}^{\mathbf{a}}) = \ell$, then

$$\mathbf{a} = (a_1, a_2, \dots, a_\ell + 1, 0, \dots, 0)$$

with $a_1, \dots, a_\ell \geqslant 0$ and $\sum_i a_i = n-1$. Hence, the number of generators $\mathbf{x}^{\mathbf{a}}$ with $\max(\mathbf{x}^{a}) = \ell$ is the number of monomials in ℓ variables of total degree $n-1$. This yields

$$\beta_i(\mathfrak{m}^n) = \sum_{\ell=1}^{d} \binom{n-2+\ell}{n-1}\binom{\ell-1}{i}.$$

Now substitute i in the formula with $d-1-k$. $\square$

Using the connection to mixed subdivisions of $n\Delta_{d-1}$ outlined in Section 4, the above result states that all regular fine mixed subdivisions of $n\Delta_{d-1}$ have the same face numbers. For the case of vertices, Corollary 5.3 was already established in 4.2, where the it was shown that the 0-dimensional cells in any fine mixed subdivision correspond to the lattice points of $n\Delta_{d-1}$. The case of facets can also be proven directly using mixed volume calculations, but we know of no such interpretation for the other faces.

5.2. Incidence structure of the bounded complex from the fine cotype ideal

In [4] Block and Yu develop an algorithm to compute the bounded complex of a generic tropical hyperplane arrangement $\mathcal{A} = \mathcal{A}(V)$ that employs methods from computational commutative algebra. There it is shown that the bounded complex supports a minimal cellular resolution of a monomial ideal which we have called the fine cotype ideal $I_{\overline{T}(\mathcal{A})}$ associated to $\mathcal{A}$. It turns out that this ideal is an initial ideal of the toric ideal $I_{n,d}$ for the vertices of $\Delta_{n-1} \times \Delta_{d-1}$. The results in [4] rely on the genericity of the arrangement in two ways: i) the fact that the bounded complex supports a resolution of the fine cotype ideal is proved by appealing to the *polyhedral detour* to tropical convexity presented in [6], and ii) the genericity is needed to guarantee that the initial ideal is indeed monomial.

In this section we extend their algorithm to the case of non-generic hyperplane arrangements. We bypass the two mentioned dependences on genericity as follows. In Section 3 we have already shown that the bounded complex resolves the fine cotype ideal for an *arbitrary* arrangement, using first principles in tropical convexity. As for ii), we next show that the fine cotype ideal is the maximal monomial ideal contained in the initial ideal associated to the weights V. In terms of polyhedral geometry, this corresponds to the passage from the polyhedral subdivision Σ_V of $\Delta_{n-1} \times \Delta_{d-1}$ induced by V to the *crosscut complex*.

The results of the previous sections cannot directly be turned into a practical algorithm as the fine cotype ideal can only be determined *after* computing the bounded complex or, at least, the vertices of $\mathcal{C}_{\mathcal{A}}$. In the case of a generic arrangement of tropical hyperplanes this problem is resolved in [4] as follows. The ideal

$$I_{n,d} = \langle x_{ik}x_{jl} - x_{il}x_{jk} \colon i, j \in [n],\ k, l \in [d] \rangle$$

of 2×2-minors of a general $n \times d$-matrix in $\mathbb{F}[x_{11}, \ldots, x_{nd}]$ is the toric ideal associated to the vertices of the ordinary lattice polytope $\Delta_{n-1} \times \Delta_{d-1}$. The initial ideal $\mathrm{in}_V(I_{n,d})$ with respect to the weights V is a squarefree monomial ideal and, by a celebrated result of Sturmfels [19, Theorem 8.3] is equal to the Stanley–Reisner ideal I_Σ of the triangulation Σ of $\Delta_{n-1} \times \Delta_{d-1}$ induced by V; see Section 4. The face poset of the bounded complex $\mathcal{B}_{\mathcal{A}}$ is anti-isomorphic to the subposet of *interior* cells of Σ. In short, there is a bijection between i-cells of $\mathcal{C}_{\mathcal{A}}$ and $(n+d-2-i)$-cells of Σ which lie in the interior of $\Delta_{n-1} \times \Delta_{d-1}$. The bijection takes a cell of fine type T to the cell of Σ with vertices (e_i, e_j) for $T_{ij} = 1$. Finally, the Alexander dual of I_Σ is the monomial ideal generated by the fine cotypes of the vertices of $\mathcal{C}_{\mathcal{A}}$.

If the placement V of the hyperplanes is not generic, then the induced subdivision Σ is not a triangulation. However, the above bijection is unaffected and, in particular, the fine type of a vertex of $\mathcal{C}_{\mathcal{A}}$ can be read off the corresponding facet of Σ. We define the *crosscut complex* $\mathrm{CrossCut}(\Sigma) \subseteq 2^{[n]\times[d]}$ to be the unique simplicial complex with the same vertices-in-facets incidences as the polyhedral complex Σ. The crosscut complex is a standard notion in combinatorial topology and can be defined in more generality (see Björner [3, pp. 1850ff]).

Proposition 5.4. *For V an ordered sequence of n points in $\mathbb{T}^{d-1}$ let $\mathcal{A} = \mathcal{A}(V)$ be the corresponding tropical arrangement and let Σ be the regular subdivision of $\Delta_{n-1} \times \Delta_{d-1}$ induced by V. Then the fine cotype ideal $I_{\overline{T}(\mathcal{A})}$ is Alexander dual to the Stanley–Reisner ideal of* $\mathrm{CrossCut}(\Sigma)$.

Proof. Both ideals in question are squarefree monomial ideals and hence we can identify their monomial generators with subsets of $[n] \times [d]$. Let $\overline{T} \subseteq [n] \times [d]$. This is a fine cotype of a minimal (bounded) cell of $\mathcal{C}_{\mathcal{A}}$ if and only if the complement $T = [n] \times [d] \setminus \overline{T}$ corresponds to the vertex set of a maximal cell of Σ. This in turn holds if and only if T is a facet of $\mathrm{CrossCut}(\Sigma)$ if and only if $\overline{T}$ is a minimal non-face of $\mathrm{CrossCut}(\Sigma)$, and hence a generator of the Alexander dual. □

The crosscut complex encodes the information of which collections of vertices lie in a common face. Hence, the crosscut complex is a purely combinatorial object and does not see the affine structure of the underlying polyhedral complex.

Algebraically, the initial ideal $\mathrm{in}_V(I_{n,d})$ is a coherent A-graded ideal in the sense of [19, Chapter 10] and encodes the corresponding polyhedral subdivision Σ induced by V (see [19, Theorem 10.10]). The ideal $\mathrm{in}_V(I)$ of a toric ideal I is generated by monomials and binomials. Intuitively, the binomials encode the affine structure within cells of Σ, that is, the affine dependencies, while the monomial generators encode the Stanley–Reisner data for the crosscut complex. For an arbitrary ideal J, denote by $M(J)$ the largest monomial ideal contained in J.

Lemma 5.5. *Let I_A be the toric ideal for $A \in \mathbb{N}^{d\times n}$ and for $\omega \in \mathbb{R}^n$ let $J = \mathrm{in}_\omega(I_A)$ and Σ the regular subdivision of A induced by ω. Then the radical of $M(J)$ is the Stanley–Reisner ideal of the crosscut complex of Σ_ω, that is*

$$I_{\mathrm{Rad}(M(J))} = I_{\mathrm{CrossCut}(\Sigma)}.$$

Proof. By Theorem 10.10 of [19], the ideal J is the intersection of ideals J_σ indexed by the faces $\sigma \in \Sigma$ and J_σ is torus isomorphic to $I_\sigma = I_A + \langle x_i\colon\ i \notin \sigma\rangle$. Hence, we have

$$M(J) = \bigcap_{\sigma\in\Sigma} M(I_\sigma).$$

Under the projection map $\mathbb{F}[x_1, \dots, x_n] \to \mathbb{F}[x_i\colon\ i \in \sigma]$ that takes $x_i \mapsto 0$ for $i \notin \sigma$, I_σ is isomorphic to the toric ideal corresponding to the columns of A indexed by $\sigma \subseteq [n]$. Hence, a monomial $\mathbf{x}^{\mathbf{a}}$ is contained in I_σ if and only if $\mathrm{supp}(\mathbf{x}^a) \nsubseteq \sigma$. Therefore $M(I_\sigma) = \langle x_i\colon\ i \notin \sigma\rangle$ and for $\tau \subseteq [n]$ we have that $\mathbf{x}^\tau \in \mathrm{Rad}(M(J))$ if and only if τ is not contained in any cell of Σ. □

A special case of this lemma appears in [17, Lemma 4.5.4]. Since the toric ideal $I_{n,d}$ is unimodular, the ideal $M(\mathrm{in}_V(I_{n,d}))$ is automatically squarefree and we obtain the following corollary.

Corollary 5.6. *Let $\mathcal{A} = \mathcal{A}(V)$ be a tropical hyperplane arrangement and let $J = \mathrm{in}_V(I_{n,d})$ the initial ideal of $I_{n,d}$ for the weights V. Then the Alexander dual of the squarefree monomial ideal $M(J)$ is the fine cotype ideal of $\mathcal{A}$.*

In the case that $\mathcal{A}$ is generic, the ideal $M(J)$ coincides with the initial ideal $\mathrm{in}_V(I_{n,d})$ and hence recovers the main result of [4]. Moreover, it entails the modification of the algorithm in [4] by replacing the ideal $\mathrm{in}_V(I_{n,d})$. We describe our Algorithm A below.

An algorithm for calculating $M(\cdot)$ for an ideal is given in [17, Algorithm 4.4.2]. All necessary algorithms are implemented in the computer algebra system `Macaulay2` [9] with which we computed the following example that illustrates the results of this section.

input : matrix $V \in \mathbb{R}^{n\times d}$
output : face poset of $\mathcal{B}_{\mathcal{A}(V)}$
1 calculate the initial ideal $J = \mathrm{in}_V(I_{n,d})$
2 calculate $M(J) = \langle \mathbf{x}^{\mathbf{a}} \colon \mathbf{x}^{\mathbf{a}} \in J\rangle$
3 calculate the Alexander dual $M(J)^*$
4 find a minimal fine graded resolution to determine the face poset of $\mathcal{B}_{\mathcal{A}(V)}$

Algorithm A: Computing the face poset of the tropical complex.

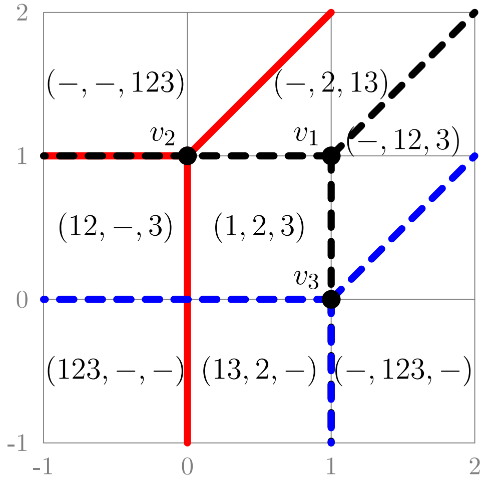

Fig. 4. Non-generic arrangement in $\mathbb{T}^2$ with labeling by fine type.

Example 5.7. Consider the points $v_1 = (0, 1, 1)$, $v_2 = (0, 0, 1)$, and $v_3 = (0, 1, 0)$ in $\mathbb{T}^2$. The induced type decomposition is shown in Fig. 4. We have

$$\begin{aligned} I_{3,3} = \langle & x_{11}x_{22} - \underline{x_{12}x_{21}}, x_{11}x_{33} - \underline{x_{13}x_{31}}, \underline{x_{12}x_{31} - x_{11}x_{32}}, \\ & x_{12}x_{33} - \underline{x_{13}x_{32}}, \underline{x_{13}x_{21} - x_{11}x_{23}}, x_{13}x_{22} - \underline{x_{12}x_{23}}, \\ & x_{21}x_{33} - \underline{x_{23}x_{31}}, x_{22}x_{31} - \underline{x_{21}x_{32}}, x_{22}x_{33} - \underline{x_{23}x_{32}}\rangle, \end{aligned}$$

and the underlined initial forms with respect to the weights $\deg(x_{ij}) = (v_i)_j$ generate the ideal $J = \mathrm{in}_V(I)$. The largest monomial is

$$M(J) = \langle x_{12}x_{21}, x_{12}x_{23},\ x_{13}x_{31},\ x_{23}x_{31}, x_{13}x_{32},\ x_{21}x_{32}, x_{23}x_{32}\rangle,$$

and this is squarefree. Its Alexander dual is

$$M(J)^* = \langle x_{13}x_{21}x_{23}, x_{12}x_{13}x_{23}x_{32}, x_{12}x_{31}x_{32}, x_{21}x_{23}x_{31}x_{32}\rangle.$$

These four generators of $M(J)^*$ encode the fine cotypes of the points v_3, $(0, 0, 0)$, v_2, and v_1, respectively. Setting $S = \mathbb{F}[x_{11}, x_{12}, \ldots, x_{33}]$ we obtain the minimal free resolution

$$0 \to S \xrightarrow{\phi_3} S^4 \xrightarrow{\phi_2} S^4 \xrightarrow{\phi_1} I \to 0,$$

where the non-trivial differentials ϕ_i are given by the matrices

$$\phi_1 = \begin{pmatrix} x_{13}x_{21}x_{23} & x_{12}x_{31}x_{32} & x_{21}x_{23}x_{31}x_{32} & x_{12}x_{13}x_{23}x_{32} \end{pmatrix},$$

$$\phi_2 = \begin{pmatrix} 0 & -x_{31}x_{32} & -x_{12}x_{32} & 0 \\ -x_{21}x_{23} & 0 & 0 & -x_{13}x_{23} \\ x_{12} & x_{13} & 0 & 0 \\ 0 & 0 & x_{21} & x_{31} \end{pmatrix},$$

$$\phi_3 = \begin{pmatrix} -x_{13} \\ x_{12} \\ -x_{31} \\ x_{21} \end{pmatrix}.$$

These matrices are to be multiplied to column vectors from the left. The non-zero finely graded Betti numbers are

$$\beta_{0,(12,13,23,32)} = \beta_{0,(12,31,32)} = \beta_{0,(21,23,31,32)} = \beta_{0,(13,21,23)} = 1,$$

$$\beta_{1,(12,13,23,31,32)} = \beta_{1,(12,21,23,31,32)} = \beta_{1,(13,21,23,31,32)} = \beta_{1,(12,13,21,23,32)} = 1,$$

$$\beta_{2,(12,13,21,23,31,32)} = 1,$$

where, for example, $(12, 13, 23, 32)$ is the squarefree monomial $x_{12}x_{13}x_{23}x_{32}$ corresponding to the point $(0, 0, 0)$. Note that we are resolving the ideal $M(J)^*$ rather than the quotient $S/M(J)^*$ (which would yield a shift of +1 in the first coordinate of each Betti number). The non-zero coarsely graded Betti numbers are then

$$\beta_{0,3} = \beta_{0,4} = 2, \qquad \beta_{1,5} = 4, \qquad \beta_{2,6} = 1.$$

6. Further remarks and open questions

Having constructed cellular resolutions of ideals arising from regular mixed subdivisions of dilated simplices, a natural question to ask is if the assumption of regularity is really necessary. The relevant properties of our subdivisions were established by considering them as induced by arrangements of tropical hyperplanes, and hence these subdivisions were always regular. However, the construction of a labeled complex from an arbitrary mixed subdivision of $n\Delta_{d-1}$ still makes sense, and it is an open question (as far as we know) whether these also support cellular resolutions. As a special case, in light of Proposition 4.2 we can ask whether *any* fine mixed subdivision of $n\Delta_{d-1}$ supports a minimal cellular resolution of $\langle x_1, \dots, x_d\rangle^n$.

A connection to tropical geometry is provided by the *tropical oriented matroids* of Ardila and Develin from [1]. There the authors introduce an axiomatic approach to the study of (fine) types, with a list of properties which they show are satisfied by the collection of fine types arising from an arrangement of tropical hyperplanes. It is conjectured that all abstract oriented matroids are realized by arbitrary subdivisions, and if this were the case we might think of the ideals described in the previous paragraph as 'tropical oriented matroid ideals'. A further task would be to relate the algebraic properties of these ideals with the combinatorial properties of the underlying matroid, in the spirit of [15].

As mentioned above, another unresolved question is to characterize which monomial ideals arise as coarse type ideals for some tropical hyperplane arrangement. We have seen that certain necessary properties are easy to deduce but it seems difficult to provide a complete classification. Do these ideals fit into some other well-known class?

References

[1] F. Ardila, M. Develin, Tropical hyperplane arrangements and oriented matroids, Math. Z. 262 (2009) 795–816.

[2] D. Bayer, B. Sturmfels, Cellular resolutions of monomial modules, J. Reine Angew. Math. 502 (1998) 123–140.

[3] A. Björner, Topological methods, in: Handbook of Combinatorics, vols. 1, 2, Elsevier, Amsterdam, 1995, pp. 1819–1872.

[4] F. Block, J. Yu, Tropical convexity via cellular resolutions, J. Algebraic Combin. 24 (2006) 103–114.

[5] J.A. De Loera, J. Rambau, F. Santos, Triangulations. Structures for Algorithms and Applications, Algorithms Comput. Math., vol. 25, Springer-Verlag, Berlin, 2010.

[6] M. Develin, B. Sturmfels, Tropical convexity, Doc. Math. 9 (2004) 1–27 (electronic).

[7] A. Dochtermann, A. Engström, Cellular resolutions of cointerval ideals, Math Z. 270 (1) (2012) 145–163, doi:10.1007/s00209-010-0789-z.

[8] M. Einsiedler, M. Kapranov, D. Lind, Non-Archimedean amoebas and tropical varieties, J. Reine Angew. Math. 601 (2006) 139–157.

[9] D.R. Grayson, M.E. Stillman, Macaulay2, a software system for research in algebraic geometry, available at http://www.math.uiuc.edu/Macaulay2/.

[10] M. Joswig, K. Kulas, Tropical and ordinary convexity combined, Adv. Geom. 10 (2010) 333–352.

[11] K. Kulas, Coarse types of tropical matroid polytopes, preprint, arXiv:1012.3053, 2010.

[12] T. Markwig, A field of generalised Puiseux series for tropical geometry, preprint, arXiv:0709.3784, 2007.

[13] E. Miller, Alexander duality for monomial ideals and their resolutions, Rejecta Math. 1 (2009) 18–57.

[14] E. Miller, B. Sturmfels, Combinatorial Commutative Algebra, Grad. Texts in Math., vol. 227, Springer-Verlag, New York, 2005.

[15] I. Novik, A. Postnikov, B. Sturmfels, Syzygies of oriented matroids, Duke Math. J. 111 (2002) 287–317.

[16] A. Postnikov, Permutohedra, associahedra, and beyond, Int. Math. Res. Not. IMRN (2009) 1026–1106.

[17] M. Saito, B. Sturmfels, N. Takayama, Gröbner Deformations of Hypergeometric Differential Equations, Algorithms Comput. Math., vol. 6, Springer-Verlag, Berlin, 2000.

[18] A. Sinefakopoulos, On Borel fixed ideals generated in one degree, J. Algebra 319 (2008) 2739–2760.

[19] B. Sturmfels, Gröbner Bases and Convex Polytopes, Univ. Lecture Ser., vol. 8, Amer. Math. Soc., Providence, RI, 1996.