Asymptotica of the Solutions to 

\[ (r y'')' - p y' + q y = \sigma y^* \]

**Philip W. Walker**

*Virginia Polytechnic Institute, Blacksburg, Virginia 24061*

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1. INTRODUCTION

In this paper we shall be concerned with singular formally self-adjoint real coefficient differential operators of the form

\[ l(y) = [(r y'')' - p y']' + q y, \]  

(1.1)

where each of \(1/r, p, \) and \(q\) is a real-valued continuous function defined on a ray \([a, \infty)\). Our primary goal is the development of formulae which yield information about the asymptotic behavior at infinity of the solutions and their quasiderivatives to

\[ l(y) = \sigma y, \]  

(1.2)

where \(\sigma\) is a complex number. In order to establish these formulae it will be necessary to impose certain restrictions on the regularity and relative rates of growth of the functions \(r, p, \) and \(q\); and usually we shall require that one of \(p\) and \(q\) be of one sign on \([a, \infty)\). As will be shown in subsequent papers these results will be particularly useful in analyzing the self-adjoint operators induced by (1.1) in \(L_2(a, \infty)\).

Hinton [1], Naimark [2], and Fedorjuk [3] have established formulae for certain \(n\)-th order equations which, in some instances, give asymptotic behavior of the solutions to Eq. (1.2). In so far as they are applicable to this equation the results of Hinton are confined to the case where \(\sigma = 0\) and \(p \equiv 0\) and are directly extended by our Theorem 2.3 below. The results of Naimark [2, Theorem 9, p. 185] require that \(r(t)\) approach a constant as \(t \to \infty\), and that \(p\) be so small in comparison with \(q\) that it does not enter into the asymptotic behavior of the solutions of Eq. (1.2). The results of Fedorjuk

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generalize those of Naimark and apply to roughly the class of coefficient functions covered by our Theorems 2.3 and 2.5 below. Our results have an advantage in that we show (2.2, 2.4, and 2.6) how to make them explicit in \( p, q, \) and \( r \). In short, the collection of formulae which we shall develop will apply to a much wider class of coefficient functions than is covered by combination of these previous results. Of special interest is our Theorem 2.8 which is believed to be the first result on the asymptotic behavior of the solutions of 1.2 when \( rq \) is small in comparison with \( p \).

We shall adopt the following notation: \( \mathbb{C} \) denotes the complex numbers, \( I \) denotes the identity function, \( I_n \) denotes the \( n \times n \) identity matrix, \( L^p(a, \infty) \) denotes the space of all complex-valued functions defined on \([a, \infty)\) whose \( p \)-th power is absolutely integrable on \([a, \infty)\), \( L(a, \infty) \) is the same as \( L^1(a, \infty) \), \( M^n \) denotes the space of all complex \( n \times n \) matrices, \( \mathbb{R} \) denotes the real numbers, and \( | \cdot | \) denotes the modulus of a complex number or the norm of a matrix.

We begin by considering the following vector-matrix formulation of Eq. (1.2). (This is the same formulation which appears in Chapter V of [2].)

\[
Y' = \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1/r & 0 \\
0 & p & 0 & -1 \\
q - \alpha & 0 & 0 & 0
\end{bmatrix} Y. \tag{1.3}
\]

It is easily verified that if

\[
Y = \begin{bmatrix}
y_1 \\
y_2 \\
y_3 \\
y_4
\end{bmatrix}
\]

is a solution to Eq. (1.3) then \( y_1 \) is a solution to Eq. 1.2, \( y_2 = y_1' \), \( y_3 = ry_1' \), and \( y_4 = py_1' - (ry_1')' \). Conversely, if \( y \) is a solution to Eq. 1.2, then

\[
Y = \begin{bmatrix}
y \\
y' \\
ry'' \\
py' - (ry'')'
\end{bmatrix}
\]

is easily seen to be a solution to Eq. (1.3), the components of \( Y \) being the so-called quasiderivatives associated with Eq. (1.1).

By certain transformations we shall reduce Eq. (1.3) to a vector-matrix differential equation whose coefficient matrix is either a small perturbation of a diagonal matrix or a small perturbation of a constant matrix. We shall then make use of certain results of Levinson ([4, 5]) in order to obtain
information about the asymptotic behavior of the solutions of Eq. (1.3) and hence of Eq. (1.2). In applying Levinson's perturbation theorems it will be convenient to make the following definition:

**Definition 1.1.** Let \( b \) be a real number, and let \( A \) be a real-valued continuous function defined on \([b, \infty)\). The statement that \( A \) satisfies condition * means that either

1. \( \int_b^t A \to \infty \) as \( t \to \infty \) and there exists a real number \( K \) such that if \( b \leq t_1 < t_2 \), then \( \int_{t_1}^{t_2} A > K \), or

2. there exists a real number \( K \) such that if \( b < t_1 < t_2 \), then \( \int_{t_1}^{t_2} A < K \).

Note that \( K \) need not be positive.

The following lemma which we shall use frequently should shed some light on the nature of condition *.

**Lemma 1.2.** Let each of \( b_0 \) and \( b_1 \) be a real number \( b_0 < b_1 \) and let \( A \) be a continuous real-valued function defined on \([b_0, \infty)\). Each of the following conditions implies that \( A \) satisfies condition *:

- The restriction of \( A \) to \([b_1, \infty)\) satisfies condition *. (1.2.1)
- \( A(t) \geq 0 \) for all \( t \geq b_1 \), \( A(t) < 0 \) for all \( t \geq b_1 \), or \( A(t) \neq 0 \) for all \( t \geq b_1 \). (1.2.2)
- There exists a monotone function \( m \) and a bounded function \( w \) such that for \( b_1 < t_1 < t_2 \),\( \int_{t_1}^{t_2} A = [m(\cdot) + w(\cdot)]t_2 - t_1 \). (1.2.3)
- \( A(t) = A_1(t) + A_2(t) \) for \( t \geq b_1 \) where \( A_1 \) satisfies condition * and \( A_2 \) is in \( L(b_1, \infty) \). (1.2.4)

The proof is elementary and will be omitted.

The two theorems of Levinson to which we refer can be found on p. 92–97 of Ref. [4] and are listed as Theorem 1, p. 88 and Theorem 11, p. 114 of Ref. [6]. It should be noted that Theorem 1 p. 88 of [6] remains valid if the condition that \( \text{Re}(\lambda_j - \lambda_k) \) does not change sign be replaced by the weaker condition that \( \text{Re}(\lambda_j - \lambda_k) \) satisfy condition *. (This stronger version is proved in the proof of Theorem 8.1 p. 92 of [4]).

In reducing Eq. 1.3 to a more workable form we shall make frequent use of the following lemma:

**Lemma 1.3.** Let each of \( \mathcal{I} \) and \( \mathcal{G} \) be a nondegenerate connected subset of \( \mathbb{R} \). Suppose there exists a continuously differentiable homeomorphism \( h \) from \( \mathcal{I} \)
onto $\mathcal{F}$ such that $h'(t) \neq 0$ for each $t \in \mathcal{F}$. Let $g$ be the function inverse to $h$ i.e., $h(g(s)) = s$ for each $s \in \mathcal{G}$. Let $A : \mathcal{F} \to M^n$ be continuous. Let $Q : \mathcal{F} \to M^n$ be continuously differentiable and have the property that $Q(t)$ is nonsingular for each $t \in \mathcal{F}$. It then follows that if $Y_0 : \mathcal{F} \to M^n$ is a fundamental matrix for

$$Y' = AY,$$  \hspace{1cm} (1.4)

then $Z_0 = Q(g(\cdot))Y_0(g(\cdot)) : \mathcal{G} \to M^n$ is a fundamental matrix for

$$Z' = \left(1/h'(g(\cdot))\right)\left[Q(g(\cdot))A(g(\cdot))Q^{-1}(g(\cdot)) + Q'(g(\cdot))Q^{-1}(g(\cdot))\right]Z. \hspace{1cm} (1.5)$$

That $Z_0$ satisfies Eq. (1.5) follows from elementary differentiation rules for matrix valued functions. That $Z_0(s)$ is nonsingular (and hence $Z_0$ a fundamental matrix) follows from the fact that each of $Q(g(s))$ and $Y_0(g(s))$ is nonsingular. It might also be noted that if $Z_0$ is a fundamental matrix for Eq. (1.5), then $Y_0 = Q^{-1}Z(h(\cdot))$ is a fundamental matrix for Eq. (1.4). We shall in certain instances be concerned with the special cases where $h = I$ or $Q$ is a constant matrix, and the simpler forms which Eq. (1.5) assumes in these cases should be noted.

2. Basic Results Concerning the Asymptotic Behavior of the Solutions to $[(ry')' - py']' + qy = ay$

In this section we present several theorems which give asymptotic behavior of the solutions to the vector matrix Eq. (1.3). We are motivated to consider the different cases reflected in the various theorems which follow by examining the characteristic roots of the coefficient matrix in Eq. (1.3). In Theorem 2.3 these roots are roughly determined by $(q/r)^{1/4}$, and $(q/r)^{1/4} \notin L(a, \infty)$; and in Theorem 2.8 they are roughly determined by $(p/r)^{1/2}$, and $(p/r)^{1/2} \notin L(a, \infty)$. Theorems 2.1 and 2.7 reflect the complementary cases where, respectively, $(q/r)^{1/4} \in L(a, \infty)$ and $(p/r)^{1/2} \in L(a, \infty)$. Theorem 2.5 fills a gap not covered by these other theorems.

While the hypotheses of some of these theorems are rather complicated it should be noted that they do apply to a large class of frequently encountered coefficient functions. We shall return to this point in more detail in Section 3.

Throughout this section each of $r$, $p$, and $q$ will denote a real-valued continuous function defined on $[a, \infty)$, and $\sigma$ will denote an arbitrary complex number. We shall always require that $r$ be positive-valued and to effect the requirement that $q$ be of one sign we shall consider $(-1)^lq$, $l = 1, 2$, and require $q$ to be positive-valued (similarly for $p$). Unless otherwise stated the integral indices $j$ and $k$ will run from 1 to 4 inclusively.
THEOREM 2.1. Let each of \( q \) and \( r \) be positive-valued and continuously differentiable, and let \( Q : [a, \infty) \to M^4 \) be given by

\[
Q = \text{diag}[(q^3r)^{1/8}, (qr^3)^{1/8}, (q^3r)^{-1/8}, (q^3r)^{-1/8}].
\]

Let \( l = 1 \) or \( l = 2 \). Let \( A : [a, \infty) \to M^4 \) be given by

\[
A = \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1/r & 0 \\
0 & 0 & p & -1 \\
(-1)^l q - \sigma & 0 & 0 & 0
\end{bmatrix}.
\]

Suppose each of \((q/r)^{1/4}, p/(r^9q)^{1/4}, \) and \( \sigma/(q^3r)^{1/4} \) is in \( L(a, \infty) \); and suppose each of \((q/r)^{1/2}, (q^3r)^{1/4}, \) and \( (q^3r)^{-1/4} \) can be expressed as the product of a monotone function and a bounded function which is bounded below by a positive number.

It then follows that there is a fundamental matrix \( Y_0 \) for

\[
Y' = AY
\] (2.1)

with the property that

\[
Q(t) Y_0(t) Q^{-1}(t) \to I_4 \quad \text{as} \quad t \to \infty.
\]

Note, in particular, this implies

\[
Y_{0j}(t) = 1 + o(1)
\]

for each \( j \) (where \( Y_0(t) = [Y_{0j}(t)] \)).

Proof. Let \( Y_1 \) be a fundamental matrix for Eq. 2.1. Let \( Z_1 \) be defined by \( Z_1(t) = Q(t) Y_1(t) \) for each \( t \geq a \). It follows from Lemma 1.3, letting \( h = g = \text{identity on } [a, \infty) \), that \( Z_1 \) is a fundamental matrix for

\[
Z' = [(q'/q) D_1 + (r'/r) D_2 + C]Z.
\] (2.2)

Where \( D_1 = \text{diag}[3/8, 1/8, -1/8, -3/8], D_2 = \text{diag}[1/8, 3/8, -3/8, -1/8] \) and

\[
C = (q/r)^{1/4} \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & p/(qr)^{1/2} & 0 & -1 \\
(-1)^l - \sigma/q & 0 & 0 & 0
\end{bmatrix}.
\]

\( C \) results from simplifying \( QAQ^{-1} \) while \((q'/q) D_1 + (r'/r) D_2 \) comes from \( Q'Q^{-1} \). Note \( |C(t)| \in L(a, \infty) \). Our hypotheses are sufficient to allow applic-
tation of Theorem 1 p. 88 of [6] (see remarks in Section 1 about this theorem) with \( A = \left( \frac{q'}{q} \right) D_1 + \left( \frac{r'}{r} \right) D_2 \). For example,

\[
\int_{t_1}^{t_2} A_{12} = \int_{t_1}^{t_2} \left[ (3/8)\left( \frac{q'}{q} \right) + (1/8)\left( \frac{r'}{r} \right) \right] - \left[ (1/8)\left( \frac{q'}{q} \right) + (3/8)\left( \frac{r'}{r} \right) \right]
\]

\[
= \left[ \ln\left( \frac{q}{r} \right)^{1/4} \right]_{t_1}^{t_2}
\]

\[
= \left[ \ln m(\cdot) + \ln w(\cdot) \right]_{t_1}^{t_2},
\]

where \( m \) is a monotone function and \( w \) is a bounded function which is bounded below by a positive number. Thus \( A_{12} \) satisfies condition \(*\) (Lemma 1.2.3).

Similarly, we find that \( A_{jk} \) for each \( j, k \) satisfies condition \(*\). From Theorem 1 p. 88 of [6] we conclude that there exists a fundamental matrix \( Z_0 \) for Eq. 2.2 with the property that

\[
Z_0(t) \exp \left[ -\int_a^t \left( \left( \frac{q'}{q} \right) D_1 + \left( \frac{r'}{r} \right) D_2 \right) \right] \to I_4
\]

as \( t \to \infty \). Evaluation of this integral shows that

\[
Z_0(t) DQ^{-1}(t) \to I_4 \quad \text{as} \quad t \to \infty,
\]

where \( D \) is a constant nonsingular \( 4 \times 4 \) diagonal matrix. Since each of \( Z_1 \) and \( Z_0 D \) is a fundamental matrix for Eq. 2.2 there is a nonsingular constant \( 4 \times 4 \) matrix \( H \) such that \( Z_0 D = Z_1 H \). Thus if we let \( Y_0 \) be \( Y_1 H \), remembering \( Z_1 = QY_1 \), the theorem is proved.**

The following lemma will be crucial in establishing Theorems 2.3 and 2.5, and Corollaries 2.4 and 2.6.

**Lemma 2.2.** Let each of \( \gamma \) and \( \delta \) be a continuous function defined on \([0, \infty)\). Let \( \gamma \) be real-valued and satisfy \( \gamma(s) \to 0 \) as \( s \to \infty \). Suppose \( \delta(s) = ku(s) \) where \( k \) is a constant, \( u(s) \) is positive, and \( u(s) \to 0 \) as \( s \to \infty \). Let \( \epsilon \) be a nonzero complex number. If \( \epsilon > 0 \) let \( \mu_j \) be such that \( \mu_1 = -i\mu_2 = -\mu_3 = i\mu_4 \) is the positive fourth root of \( \epsilon \); otherwise let \( \mu_j \) be the fourth root of \( \epsilon \) in the \( j \)-th quadrant. Let \( \mathcal{P} : C \times [0, \infty) \to C \) be defined by

\[
\mathcal{P}(z, s) = z^4 - \gamma(s) z^2 - \epsilon - \delta(s)
\]

for each \( (z, s) \in C \times [0, \infty) \). Let \( b_1 \geq 0 \) be such that for \( s \geq b_1 \), \( \mathcal{P}(\cdot, s) \) has four distinct roots. For each \( j \) let \( \lambda_j \) be the continuous function on \([b_1, \infty)\) such that \( \lambda_j(s) \to \mu_j \) as \( s \to \infty \) and \( \mathcal{P}(\lambda_j(s), s) = 0 \) for each \( s \geq b_1 \). For each \( (j, k) \) let \( \Delta_{jk} : [b_1, \infty) \to R \) be defined by \( \Delta_{jk} = \text{Re}(\lambda_j - \lambda_k) \).

It then follows that each \( \Delta_{jk} \) satisfies condition\(*\) (See Definition 1.1). Moreover
there exists a $b_2 \geqslant b_1$ and four doubly indexed complex number sequences $a_j$ such that for each $s \geqslant b_2$

$$\lambda_j(s) = \sum_{m=0}^{\infty} a_j(m,n) \gamma^m(s) \delta^n(s).$$

The $a_j$ are calculable recursively.

**Proof.** In the event that $\text{Im } \epsilon \neq 0$ we see that $\text{Re } \mu_j = \text{Re } \mu_k$ for $j \neq k$; so either $A_{jk}(s) = 0$ for all $s \in [b_1, \infty)$ (when $j = k$) or (by continuity of the $\lambda_j$) $A_{jk}(s)$ approaches a nonzero real number as $s \to \infty$. Thus by Lemma 1.2.2 $A_{jk}$ satisfies condition $\ast$.

To handle the case when $\text{Im } \epsilon = 0$ and $\epsilon > 0$ we make the following observations: First, $z$ is a root of $\mathcal{P}(\cdot, s)$ if and only if $-z$ is a root. And second, our hypotheses on $\epsilon$ imply that either $\delta(s)$ is real for all $s$ in which case (keeping in mind $\text{Im } \epsilon = 0$) $z$ is a root of $\mathcal{P}(\cdot, s)$ if and only if $-z$ is a root; or $\text{Im } \delta(s) \neq 0$ for all $s$ in which case (since $\delta(s) = \lambda^4(s) - \gamma(s) \lambda^2(s) - \epsilon$ and $\gamma(s)$ and $\epsilon$ are real) $\text{Re } \lambda_j(s) \neq 0$ for all $s$. From the limiting values of the $\lambda_j$ we see that $\lambda_j(s) = \lambda_k(s)$ for all large $s$, so $\text{Re } (\lambda_j(s) - \lambda_k(s)) = 2\text{Re } \lambda_j(s)$. Hence if $\text{Im } \delta(s) \neq 0$, $\text{Re } (\lambda_j(s) - \lambda_k(s)) \neq 0$ for all large $s$ implying (Lemma 1.2.2) $A_{jk}$ (and $A_{kj}$) satisfies condition $\ast$. If $\delta(s)$ is real for all $s$ we see that $\lambda_j(s) = \lambda_k(s)$ for all large $s$, implying each of $A_{jk}$ and $A_{kj}$ satisfies condition $\ast$. It is easily verified from the limiting values of the $\lambda_j$ that when $\epsilon > 0$, $A_{jk}$ satisfies condition $\ast$ for $(2, 4) \neq (j, k) \neq (4, 2)$.

In order to handle the remaining case ($\epsilon < 0$) we first verify our assertion concerning the power series representation for the $\lambda_j$. Let $F : \mathbb{C}^2 \to \mathbb{C}$ be defined by

$$F(x, y, z) = z^4 - xz^2 - \epsilon - y$$

for each $(x, y, z) \in \mathbb{C}^2$. $F$ is clearly analytic and since $F(\mu_j, 0, 0) = 0$ and $F(\mu_k, 0, 0) \neq 0$, the implicit function theorem for analytic functions of several variables (see for example Theorem 10.2.4 p. 268 of [7]) assures the existence of a function $u_j$ analytic in a neighborhood of $(0, 0)$ in $\mathbb{C}^2$ with the properties that $u_j(0, 0) = \mu_j$ and $F(u_j(x, y), x, y) = 0$ for all $(x, y)$ in this neighborhood. From analyticity of $u_j$ we have existence of a sequence $a_j$ such that for all $(x, y)$ sufficiently close to $(0, 0)$,

$$u_j(x, y) = \sum_{m=0}^{\infty} a_j(m,n) x^m y^n.$$
coefficients on like powers" enable us to calculate the terms of \( a_j \) recursively. For example \( a_2(0, 0) = \mu_j, a_2(1, 0) = 1/4\mu_j, \) and \( a_2(0, 1) = \mu_j/4\epsilon. \) By the uniqueness of \( \lambda_j \) we have that for all large \( \epsilon, \lambda_j(\epsilon) = u_j(\gamma(\epsilon), \delta(\epsilon)). (u_j(\gamma(\cdot), \delta(\cdot)) \) satisfies all of the conditions imposed on \( \lambda_j). \) Hence the power series representation for \( \lambda_j \) holds as asserted.

We return now to the problem of showing each \( \Delta_{jk} \) satisfies condition * when \( \epsilon \) is real and negative. Rearranging our power series representation for \( u_j \) we have for \( (x, y) \) sufficiently close to \((0, 0)\)

\[
u_j(x, y) = \sum_{m=0}^{\infty} a_j(m, 0) x^{m} + \sum_{n=0}^{\infty} a_j(m, n) x^{m} y^{n}
\]

\[
u_j(x, y) = u_j(x, 0) + y[u_j(0, 1) + \Sigma_j(x, y)],
\]

where

\[
\Sigma_j(x, y) = x \sum_{m=1}^{\infty} a_j(m, 1) x^{m-1} + y \sum_{n=2}^{\infty} a_j(m, n) x^{m} y^{n-2}.
\]

Note each of the last two sums represents a bounded function in a neighborhood of \((0, 0)\); so \( \Sigma_j(x, y) \to 0 \) as \((x, y) \to (0, 0)\). Inspection of \( F(\cdot, x, 0) \) shows that it is a polynomial with real coefficients when \( x \) and \( \epsilon \) are real. Moreover, in the special case we are considering \((\epsilon < 0)\) we see from the continuity of the \( u_j \) and the facts that \( \mu_1 = \mu_4 \) and \( \mu_2 = \mu_3 \) that for \( x \) real and sufficiently close to 0, \( u_1(x, 0) = u_4(x, 0) \) and \( u_2(x, 0) = u_3(x, 0) \). So remembering \( \lambda_j(\epsilon) = u_j(\gamma(\epsilon), \delta(\epsilon)) \) we see from Eq. (2.3) that

\[
\text{Re}(\lambda_1(\epsilon) - \lambda_4(\epsilon)) = \text{Re}(\delta(\epsilon)[(a_1(0, 1) - a_4(0, 1)) + o(1))]
\]

\[
\text{Re}(\lambda_2(\epsilon) - \lambda_3(\epsilon)) = \text{Re}(\delta(\epsilon)[(a_2(0, 1) - a_3(0, 1)) + o(1))].
\]

Remembering our calculations we have that \( a_4(0, 1) - a_4(0, 1) = (1/4\epsilon)(\mu_1 - \mu_4) = \xi_14; \ a_4(0, 1) - a_3(0, 1) = (1/4\epsilon)(\mu_2 - \mu_3) = \xi_23 \) and see that each of these is a complex number with nonzero imaginary and zero real part when \( \epsilon < 0 \). So if \( \delta \) is not real-valued (implying the constant \( \rho \) such that \( \rho\delta \) is positive-valued has nonzero imaginary part) we see that

\[
\text{Re}(\lambda_1(\epsilon) - \lambda_4(\epsilon)) = \rho\delta(\epsilon)[\text{Re}(\xi_14/\rho) + o(1)] \quad \text{and}
\]

\[
\text{Re}(\lambda_2(\epsilon) - \lambda_3(\epsilon)) = \rho\delta(\epsilon)[\text{Re}(\xi_23/\rho) + o(1)].
\]

Since each of \( \text{Re}(\xi_14/\rho) \) and \( \text{Re}(\xi_23/\rho) \) is not zero and \( \rho\delta(\epsilon) > 0 \) we see that each of \( \Delta_{14}(\epsilon) \) and \( \Delta_{23}(\epsilon) \) is eventually of one sign. Hence each of \( \Delta_{14}, \Delta_{41}, \Delta_{23}, \) and \( \Delta_{32} \) satisfies condition *. In case \( \delta(\epsilon) \) is real-valued (and \( \epsilon < 0 \)) inspection of \( \mathcal{P}(\cdot, \epsilon) \) shows it to have real coefficients and from the limiting
values of the $\lambda_j$ we see that $\lambda_1(s) = \overline{\lambda_4(s)}$ and $\lambda_2(s) = \overline{\lambda_3(s)}$ for all large $s$ so that each of $A_{14}, A_{14}, A_{36}$ is eventually identically zero.

To see that the other $A_{jk}$ satisfy condition * when $\epsilon < 0$, we merely need to observe that as $s \to \infty$, $A_{jk}(s) \to \Re(\mu_j - \mu_k) \neq 0$ for $(j, k)$ taking the values $(1, 2), (2, 1), (1, 3), (3, 1), (2, 4),$ and $(4, 2)$.

**Theorem 2.3.** Let each of $r$ and $q$ be positive valued and twice continuously differentiable. Let $l = 1$ or $l = 2$. When $l = 1$ let $\mu_j$ be given by $\mu_1 = i\mu_2 = -\mu_3 = i\mu_4 = 1$, and when $l = 2$ let $\mu_j$ be the fourth root of $-1$ in the $j$-th quadrant. Let $h : [a, \infty) \to [0, \infty)$ be given by $h(t) = \int_0^t (q/r)^{1/4}$ and suppose $h(t) \to \infty$ as $t \to \infty$. Let $g$ be the function inverse to $h$ (i.e., $h(g(s)) = s$ for each $s \geq 0$). Let each of the functions $\alpha, \beta, \gamma,$ and $\delta$ be defined as follows: For each $s \geq 0$

$$
\alpha(s) = \left[\left(q/r\right)^{-1/4}r'/q\right](g(s)),
$$

$$
\beta(s) = \left[\left(q/r\right)^{-1/4}r'/r\right](g(s)),
$$

$$
\gamma(s) = \left[p/q(r)^{1/2}\right](g(s)),
$$

and

$$
\delta(s) = \sigma/q(g(s)).
$$

Suppose $\gamma(s) \to 0$ and $\delta(s) \to 0$ as $s \to \infty$. Let $P : \mathbb{C} \times [0, \infty) \to \mathbb{C}$ be given by

$$
P(z, s) = z^4 - \gamma(s) z^2 + (-1)^s - \delta(s)
$$

for each $(z, s) \in \mathbb{C} \times [0, \infty)$. Let $b \geq 0$ be such that for each $s \geq b$ $P(\cdot, s)$ has four distinct roots. For each $j$, let $\lambda_j$ be the continuous function defined on $[b, \infty)$ such that $P(\lambda_j, s) = 0$ for each $s \geq b$ and $\lambda(s) \to \mu_j$ as $s \to \infty$ (See Lemma 2.2). Let $E : [b, \infty) \to M^b$ be defined by

$$
E(s) = \exp \left\{ -\int_b^s \text{diag}[\lambda_1, \ldots, \lambda_4] \right\}
$$

for each $s \geq b$. Let each of $Q$ and $A$ be defined as in Theorem 2.1. Suppose in addition to the conditions stated above that each of $\alpha', \beta', \gamma', \delta', \alpha^2,$ and $\beta^2$ is in $L(0, \infty)$.

It then follows that there is a fundamental matrix $Y_0$ for

$$
Y' = AY
$$

(2.4)

with the property that

$$
Q(t) Y_0(t) E(h(t)) \to K \quad \text{as} \quad t \to \infty,
$$

where $K$ is a constant matrix.
where
\[ K = \begin{bmatrix}
1 & 1 & 1 & 1 \\
\mu_1 & \mu_2 & \mu_3 & \mu_4 \\
\mu_2 & \mu_3 & \mu_3 & \mu_4 \\
-\mu_1 & -\mu_2 & -\mu_3 & -\mu_4
\end{bmatrix}. \]

Proof. Let the conditions and definitions stated above hold. Let \( Y_1 \) be a fundamental matrix for Eq. (2.4). Let \( Z_1 \) be defined by \( Z_1(s) = Q(g(s)) Y_1(g(s)) \) for each \( s \geq 0 \). It is easily verified (See Lemma 1.3) that \( Z_1 \) is a fundamental matrix for
\[ Z' = [A_0 + V]Z, \quad (2.5) \]
where
\[ A_0 = \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1 \\
(-1)^t & 0 & 0 & 0
\end{bmatrix} \]
and \( V : [0, \infty) \rightarrow M^4 \) is given by
\[ V = \begin{bmatrix}
(1/8)(3\alpha + \beta) & 0 & 0 & 0 \\
0 & (1/8)(\alpha + 3\beta) & 0 & 0 \\
0 & \gamma & (1/8)(\alpha + 3\beta) & 0 \\
-\delta & 0 & 0 & (1/8)(3\alpha + \beta)
\end{bmatrix}. \]

\( A_0 + V \) results from simplifying
\[ (1/h'(g(\cdot))[Q(g(\cdot)) A(g(\cdot)) Q^{-1}(g(\cdot)) + Q'(g(\cdot)) Q^{-1}(g(\cdot)))] \]
and breaking it into its constant and variable parts. We shall show that our hypotheses are sufficient to enable us to apply Theorem 8.1, p. 92 of [4] to Eq. (2.5). It is easily verified that \( A_0 \) has distinct characteristic roots \( \mu_1, \mu_2, \mu_3, \) and \( \mu_4 \) and that the \( j \)-th column of \( K \) is an eigenvector of \( A_0 \) associated with \( \mu_j \). Also the conditions imposed on \( \alpha, \beta, \gamma, \) and \( \delta \) (note \( \alpha', \beta', \alpha, \beta \in L(0, \infty) \) implies \( \alpha(s) \rightarrow 0 \) and \( \beta(s) \rightarrow 0 \) as \( s \rightarrow \infty \)) imply \( |V(\cdot)| \) is in \( L(0, \infty) \) and \( V(s) \rightarrow 0 \) as \( s \rightarrow \infty \). Direct computation shows that the family \( \mathcal{P} \) of characteristic polynomials for \( A_0 + V \) is given by \( \det(A_0 + V(s) - zI_4) = \mathcal{P}(z, s) = z^4 + (d_1d_4 + d_2d_3)(s^2 - \gamma(s))^2 + d_1d_2d_3d_4 - (d_1d_4\gamma)(s) + (-1)^t - \delta(s) \) for each \( (\alpha, s) \in \mathcal{C} \times [0, \infty) \), where
\[ d_1 = -d_4 = (1/8)(3\alpha + \beta) \]
and
\[ d_2 = -d_3 = (1/8)(\alpha + 3\beta). \]
Let \( b \geq b \) be such that \( s \geq b \), \( \mathcal{P}(\cdot, s) \) has four distinct characteristic roots. For each \( j \) let \( \lambda_j \) be the continuous function on \([b, \infty)\) such that \( \mathcal{P}(\lambda_j(s), s) = 0 \) for each \( s \geq b \) and \( \lambda_j(s) \to \mu_j \) as \( s \to \infty \). We shall now show that for each \( j \), \( \lambda_j - \lambda_j \) is in \( L(b, \infty) \). Then from Lemma 1.2.4 (since from Lemma 2.2 each \( \text{Re}(\lambda_j - \lambda_k) \) satisfies condition \(*\)) we shall be able to conclude that for each \( j, k \), \( \text{Re}(\lambda_j - \lambda_k) \) satisfies condition \(*\).

For \( s \geq b \) we have, from \( \mathcal{P}(\lambda_j(s), s) - \mathcal{P}(\lambda_j(s), s) = 0 \), that

\[
(\lambda_j(s) - \lambda_j(s))[(\lambda_j(s) + \lambda_j(s))(\lambda_j(s) + \lambda_j(s)) - \gamma(s)(\lambda_j(s) + \lambda_j(s))] = (d_1d_4\gamma)(s) - ((d_1d_4)(s) + (d_2d_4)(s)) \lambda_j(s) - d_1d_2d_2d_4.
\]

Since each \( d_k, \gamma \) and \( \lambda_j \) is bounded and each \( d_k \in L^2(0, \infty) \), the right side of the above equation represents a function in \( L(b, \infty) \). Since the term in brackets approaches a nonzero limit as \( s \to \infty \), we see that \( \lambda_j - \lambda_j \) is eventually dominated by a constant multiple of a function in \( L(b, \infty) \). Hence \( \lambda_j - \lambda_j \) is in \( L(b, \infty) \).

Applying Theorem 8.1 p. 92 of [4] to Eq. (2.5), we conclude that there is a fundamental matrix \( Z_0 \) for Eq. (2.5) with the property that

\[
Z_0(s) \exp \left\{ \int_b^s \text{diag}[\lambda_1, \ldots, \lambda_4] \right\} \to K
\]
as \( s \to \infty \). Letting \( D \) be the constant nonsingular diagonal matrix whose \( j \)-th diagonal entry is

\[
\exp \left\{ \int_b^s \lambda_j - \int_b^\infty (\lambda_j - \lambda_j) \right\},
\]

we see that \( Z_0D \) is a fundamental matrix for Eq. (2.5) with the property that for \( s \geq b \),

\[
Z_0(s) \, DE(s) = Z_0(s) \exp \left\{ -\int_b^s \text{diag}[\lambda_1, \ldots, \lambda_4] \right\} \cdot \exp \left\{ -\int_s^\infty \text{diag}[\lambda - \lambda_1, \ldots, \lambda_4 - \lambda_4] \right\}
\]

Since the last exponential term approaches \( I_4 \) as \( s \to \infty \) we have \( Z_0(s) \, DE(s) \to K \) as \( s \to \infty \). Since each of \( Z_0D \) and \( Z_1 \) is a fundamental matrix for Eq. (2.3), there exists a nonsingular constant matrix \( H \) such that \( Z_0D = Z_1H \). Letting \( Y_0 \) be \( Y_1H \) and remembering \( Z_1(h(t)) = Q(t) \, Y_1(t) \) for \( t \geq a \), we see that \( Y_0 \) is a fundamental matrix for Eq. (2.4) and

\[
Q(t) \, Y_0(t) \, E(h(t)) = Z_1(h(t)) \, HE(h(t)) = Z_1(h(t)) \, DE(h(t)) \to K \text{ as } t \to \infty.
\]
It should be noted that the conditions imposed on the functions $\alpha$, $\beta$, $\gamma$, and $\delta$ in Theorem 2.3 are readily translatable into conditions on the coefficient functions $p$, $q$, and $r$. $\gamma(s) \to 0$ and $\delta(s) \to 0$ as $s \to \infty$ are equivalent, respectively, to $(p/(q^{1/2}r))(t) \to 0$ and $(\sigma/q(t)) \to 0$ as $t \to \infty$. $\alpha'$, $\beta'$, $\alpha^2$, $\beta^2$, $\gamma'$, $\delta' \in L(0, \infty)$ are equivalent, respectively, to $[(q/r)^{-1/4}q'/q]'$, $[(q/r)^{-1/4}r'/r]'$, $[(q/r)^{-1/4}(q'/q)^2]$, $[(q/r)^{-1/4}(r'/r)^2]$, $[p/(q^{1/2}r)]$, $[\sigma/q'(t)] \in L(a, \infty)$.

It should also be noted that in the special case that $r \equiv 1$ the single condition $(q'/q^{1/2}) \in L(a, \infty)$ implies each of the following: $h(t) \to \infty$ as $t \to \infty$, $\alpha' \in L(a, \infty)$ and $\alpha^2 \in L(a, \infty)$. For a proof of this assertion see the corollary, p. 594 of [1]. Since $\beta \equiv 0$ in this case the only other conditions to check in order to apply Theorem 2.3 are: $(p/q^{1/2})(t) \to 0$ and $(\sigma/q(t)) \to 0$ as $t \to \infty$, and $[\sigma/q'] \in L(a, \infty)$.

By requiring very little more we can make our results in Theorem 2.3 considerably more precise in that we can replace the functions $\lambda_j$ by ones which can be calculated in a finite number of steps from the coefficient functions $p$, $q$, and $r$. This situation is reflected in the following corollary:

**Corollary 2.4.** Let the conditions and definitions stated in Theorem 2.3 hold. For each $j$ let $\alpha_j$ be the sequence with the property that for all sufficiently large $s$

$$\lambda_j(s) = \sum_{m=0}^{\infty} a_j(m, n) \gamma^m(s) \sigma^n(s)$$

(See Lemma 2.2). If for some nonnegative integer $n_0$ each of $\gamma$ and $\delta$ is in $L_{n_0+1}(0, \infty)$ it follows that Eq. (2.4) has a fundamental matrix $Y_2$ with the property that

$$Q(t) Y_2(t) \exp \left\{-\int_0^{h(t)} \text{diag} [\theta_1, \ldots, \theta_n] \right\} \to K$$

as $t \to \infty$. Where for each $j$, $\theta_j$ is given by

$$\theta_j(s) = \sum_{m=0}^{n_0} a_j(m, n) \gamma^m(s) \delta^n(s)$$

**Proof.** From Lemma 2.2 we have for $s$ large

$$\lambda_j(s) = \theta_j(s) + \gamma^{n_0+2}(s) \sum_{m=n_0+1}^{\infty} a_j(m, n) \gamma^{m-(n_0+1)}(s) \delta^n(s)$$

$$+ \delta^{n_0+1}(s) \sum_{m=n_0+1}^{\infty} a_j(m, n) \gamma^m(s) \delta^{n-(n_0+1)}(s).$$
Since each of the last two sums represents a bounded function for all large values of $s$, $\lambda_j - \theta_j$ is in $L(b, \infty)$. Letting $Y_0$ satisfy the conclusion to Theorem 2.3 and letting $Y_2$ be $Y_0D$ where $D$ is that nonsingular constant diagonal matrix whose $j$-th diagonal entry is

$$\exp \left\{ \int_0^b \lambda_j - \int_0^b (\lambda_j - \theta_j) \right\},$$

we have for large $t$

$$Q(t) Y_2(t) \exp \left\{ - \int_0^{h(t)} \text{diag} [\theta_1, \ldots, \theta_4] \right\}$$

$$= Q(t) Y_0(t) \exp \left\{ - \int_0^{h(t)} \text{diag} [\lambda_1, \ldots, \lambda_4] \right\}$$

$$\cdot \exp \left\{ - \int_0^{h(t)} \text{diag} [\lambda_1 - \theta_1, \ldots, \lambda_4 - \theta_4] \right\}$$

Noting that the last exponential factor approaches $I_4$ as $t \to \infty$ and remembering the conclusion of Theorem 2.3 for $Y_0$, we see that $Y_2$ satisfies the conclusion of this corollary. 

In order to put things back in terms of the coefficient functions, note

$$\int_0^{h(t)} \theta_j = \sum_{m=0}^{m=n_0} \sum_{n=n_0} a_j(m,n) \int_a^t \sigma^{p(m+1/2)/2} q^{n+1/2 - 1/4}$$

$\gamma^n \in L(0, \infty)$ if and only if $[p^{m+1/2}/2] q^{n+1/2 - 1/4}$ is in $L(a, \infty)$ and $\delta^n \in L(a, \infty)$ if and only if $[\sigma^{q} q^{n+1/4} r^{1/4}]$ is in $L(a, \infty)$. In the special case of $\gamma^2$ and $\delta^2$ is in $L(a, \infty)$, we find

$$\int_0^{h(t)} \theta_j = \mu_j h(t) + (1/4 \mu_j) \int_a^t \sigma^{1/4} (r^2 q)^{1/4} + (1/4 \mu_j) \int_a^t \sigma^{1/4} (r^2 q)^{1/4}$$

**Theorem 2.5.** Suppose $r$ is positive-valued and twice continuously differentiable. Let $a$ be a nonzero complex number. Let $h : [a, \infty) \to [0, \infty)$ be defined by

$$h(t) = \int_a^t \left( \frac{1}{r} \right)^{1/4}$$

for each $t \in [a, \infty)$. Suppose $h(t) \to \infty$ as $t \to \infty$. Let $g : [0, \infty) \to [a, \infty)$ be the function inverse to $h$ (i.e. $h(g(s)) = s$ for each $s \in [0, \infty)$). Let each of $\beta, \gamma, \delta$ be defined as follows: For each $s \in [0, \infty)$

$$\beta(s) = \left[ r^{1/4} r' / r \right] (g(s))$$

$$\gamma(s) = \left[ p / r^{1/2} \right] (g(s))$$

$$\delta(s) = \left[ q \right] (g(s)).$$
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Suppose $\gamma(s) \to 0$ and $\delta(s) \to 0$ as $s \to \infty$. Let $\mathcal{P} : C^r \times [0, \infty) \to \mathbb{R}$ be defined by

$$\mathcal{P}(z, s) = z^4 - \gamma(s) z^2 - \sigma + \delta(s).$$

If $\sigma$ is real and positive, let $\mu_j$ be defined by $\mu_1 = -i\mu_2 = -\mu_3 = i\mu_4$ is the positive fourth root of $\sigma$; otherwise let $\mu_j$ be the fourth root of $\sigma$ in the $j$-th quadrant. Let $b \geq 0$ be such that for $s \geq b$ $\mathcal{P}(\cdot, s)$ has four distinct roots. For each $j$ let the continuous function $\lambda_j : [b, \infty) \to C$ be such that $\lambda_j(s) \to \mu_j$ as $s \to \infty$ and $\mathcal{P}(\lambda_j(s), s) = 0$ for each $s \geq b$. In addition to the above condition imposed on $\gamma$ and $\delta$, suppose each of $\beta'$, $\gamma'$, $\delta'$, and $\beta^2$ is in $L(0, \infty)$. Let $A : [a, \infty) \to \mathbb{R}$ be given by

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1/r & 0 \\ 0 & p & 0 & -1 \\ q - \sigma & 0 & 0 & 0 \end{bmatrix}.$$

Let $Q : [a, \infty) \to M^4$ be defined by

$$Q = \text{diag}[r^{1/8}, r^{3/8}, r^{-3/8}, r^{-1/8}],$$

and let $E : [b, \infty) \to M^4$ be defined by

$$E(s) = \exp \left\{ -\int_b^s \text{diag}[\lambda_1, \ldots, \lambda_4] \right\}.$$

It then follows that there exists a fundamental matrix $Y_0$ for

$$Y' = AY$$

with the property that $Q(t) Y_0(t) E(h(t)) \to K$ as $t \to \infty$ where

$$K = \begin{bmatrix} 1 & 1 & 1 & 1 \\ \mu_1 & \mu_2 & \mu_3 & \mu_4 \\ \mu_1^2 & \mu_2^2 & \mu_3^2 & \mu_4^2 \\ -\mu_1^3 & -\mu_2^3 & -\mu_3^3 & -\mu_4^3 \end{bmatrix}.$$

The proof of this theorem closely parallels that of Theorem 2.3; hence some of the more repetitious details will be omitted.

Proof. Let the conditions and definitions stated in the theorem hold. Let $Y_1$ be a fundamental matrix for Eq. (2.6). Let $Z_1$ be defined by $Z_1(s) = Q(g(s)) Y_1(g(s))$ for each $s \geq 0$. It is easily verified (See Lemma 1.3) that $Z_1$ is a fundamental matrix for

$$Z' = [A_0 + V]Z,$$
where

\[ A_0 = \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1 \\
-\sigma & 0 & 0 & 0
\end{bmatrix} \]

and

\[ V = \begin{bmatrix}
(1/8)\beta & 0 & 0 & 0 \\
0 & (3/8)\beta & 0 & 0 \\
0 & \gamma & (-3/8)\beta & 0 \\
\delta & 0 & 0 & (-1/8)\beta
\end{bmatrix}. \]

Let \( \delta \geq b \) be such that \( A_0 + V(s) \) has four distinct characteristic roots for each \( s \geq \delta \). For each \( j \), let \( \lambda_j(s) \) be the continuous function defined on \([\delta, \infty)\) such that \( \lambda_j(s) \to \mu_j \) as \( s \to \infty \) and \( \lambda_j(s) \) is a characteristic root of \( A_0 + V(s) \) for each \( s \geq \delta \). By the same procedure used in the proof of Theorem 2.3 we see that for each \( j \), \( \lambda_j \) is in \( L(\delta, \infty) \), and from Lemmas 2.2 and 1.2.4 we conclude that \( \text{Re}(\lambda_j - \lambda_k) \) satisfies condition * for each \( (j, k) \). It is easily verified that \( V(s) \to 0 \) as \( s \to \infty \), that \( |V'(s)| \in L(0, \infty) \), and that the \( j \)-th column of \( K \) is an eigenvector of \( A_0 \) associated with \( \mu_j \).

Hence applying Theorem 8.1, p. 92 of Ref. [4] we conclude that there is a fundamental matrix \( Z_0 \) for Eq. (2.7) with the property that

\[
Z_0(s) \exp \left\{ -\int_{\delta}^{s} \text{diag}(\lambda_1, ..., \lambda_4) \right\} \to K
\]

as \( s \to \infty \). Letting \( D \) be the constant nonsingular diagonal matrix whose \( j \)-th diagonal entry is

\[
\exp \left\{ \int_{\delta}^{s} \lambda_j - \int_{\delta}^{\infty} (\lambda_j - \lambda_j) \right\}.
\]

We see that \( Z_0D \) is a fundamental matrix for Eq. (2.7) with the property that \( Z_0(s) DE(s) \to K \) as \( s \to \infty \). Letting \( H \) be the nonsingular matrix such that \( Z_0D = Z_1H \) and letting \( Y_0 \) be \( Y_1H \) we find that \( Y_0 \) satisfies the conclusion to the theorem.

It should be noted that the requirements (in Theorem 2.5) that \( \gamma(s) \to 0 \) and \( \delta(s) \to 0 \) as \( s \to \infty \) are equivalent to \( (p/r^{1/2})(t) \to 0 \) and \( q(t) \to 0 \) as \( t \to \infty \). Also \( \beta', \gamma', \delta', \beta' \in L(0, \infty) \) is equivalent to \( (r'/r^{3/4})', (p/r^{1/2})', q', [r^{1/4}(r'/r)^2] \in L(a, \infty) \).

As was the case with Theorem 2.3, we find that a slightly stronger hypothesis for Theorem 2.5 yields considerably sharper results. This situation is reflected in the following corollary:
Corollary 2.6. Let the conditions and definitions stated in Theorem 2.5 hold. For each \( j \) let \( a_j \) be the sequence with the property that for all sufficiently large \( s \)

\[
\lambda_j(s) = \sum_{m=0}^{\infty} a_j(m, n) \gamma^m(s) \delta^n(s)
\]

(see Lemma 2.2). If for some nonnegative integer \( n_0 \) each of \( \gamma \) and \( \delta \) is in \( L_{n_0+1}(0, \infty) \) it follows that Eq. 2.6 has fundamental matrix \( Y_2 \) with the property that

\[
Q(t) Y_2(t) \exp \left\{ -\int_0^t \text{diag}[\theta_1, \ldots, \theta_4] \right\} \to K
\]

as \( t \to \infty \), where for each \( j \), \( \theta_j \) is given by

\[
\theta_j(s) = \sum_{m=0}^{m=n_0} a_j(m, n) \gamma^m(s) \delta^n(s).
\]

The proof of this corollary is exactly like that of Corollary 2.4 and will be omitted.

It should be noted that \( \gamma^m \) is in \( L(0, \infty) \) if and only if \( (pm/r^{1/4}+(m/2)) \) is in \( L(a, \infty) \) and \( \delta^n \) is in \( L(0, \infty) \) if and only if \( (qn/r^{1/4}) \) is in \( L(a, \infty) \). Also

\[
\int_0^{h(t)} \theta_j = \sum_{m=0}^{m=n_0} a_j(m, n) \int_a^t \left( pmq/m^{1/4}+(m/2) \right).
\]

Theorem 2.7. Let each of \( p \) and \( r \) be positive valued and twice continuously differentiable, and let \( Q : [a, \infty) \to M^4 \) be given by

\[
Q = \text{diag}\{ (p^2/r)^{1/4}, (pr)^{1/4}, (pr)^{-1/4}, (r/p^2)^{1/4} \}.
\]

Let \( l = 1 \) or \( l = 2 \). Let \( A : [a, \infty) \to M^4 \) be given by

\[
A = \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1/r & 0 \\
0 & (-1)^{1/2} & 0 & -1 \\
q - \sigma & 0 & 0 & 0
\end{bmatrix}.
\]

Suppose each of \( (p/r)^{1/2} \), \( q \cdot (r/p^2)^{1/2} \), and \( \sigma \cdot (r/p^2)^{1/2} \) is in \( L(a, \infty) \), and suppose each of \( p \), \( (pr)^{1/2} \), \( (p/r)^{1/2} \), and \( (p^2/r)^{1/2} \) can be expressed as the product of a monotone function and a bounded function which is bounded below by a positive number.
It then follows that there is a fundamental matrix $Y_0$ for

$$Y' = AY$$

(2.8)

with the property that

$$Q(t) Y_0(t) Q^{-1}(t) \rightarrow I_4 \quad \text{as} \quad t \rightarrow \infty.$$  

Note, in particular this, implies

$$Y_{0j}(t) = 1 + o(1)$$

for each $j$ (where $Y_0(t) = [Y_0j(t)]$).

The proof of this theorem closely parallels that of Theorem 2.1 and repetitious details will be omitted.

Proof. Let $Y_1$ be a fundamental matrix for Eq. (2.8), and let $Z_1$ be defined by $Z_1(t) = Q(t) Y_1(t)$ for each $t \geq a$. From Lemma 1.3 we see that $Z_1$ is a fundamental matrix for

$$Z' = \left((p'/p) D_1 + (r'/r) D_2 + Q AQ^{-1}\right) Z,$$  

(2.9)

where $D_1 = \text{diag}[3/4, 1/4, -1/4, -3/4]$ and $D_2 = \text{diag}[-1/4, 1/4, -1/4, 1/4]$. Computation shows that our hypotheses imply $|Q(\cdot) A(\cdot) Q^{-1}(\cdot)|$ is in $L(a, \infty)$. Also our hypotheses are sufficient to allow application of Theorem 1 p. 88 of Ref. [6] (see remarks in Section 1 concerning this theorem) to Eq. (2.9) with $A = (p'/p) D_1 + (r'/r) D_2$. For example, if $a \leq t_1 \leq t_2$,

$$\int_{t_1}^{t_2} A_{12} = \int_{t_1}^{t_2} \left[\left(\frac{3}{4}(p'/p) + (-1/4)(r'/r) + ((1/4)(p'/p') + (1/4)(r'/r))\right.\right.$$  

$$+ \left.\ln(p/r)^{1/2}\right]^{t_2}_{t_1} = \left[\ln m(\cdot) + \ln w(\cdot)\right]^{t_2}_{t_1},$$

where $m$ is a monotone function and $w$ is a bounded function which is bounded below by a positive number. Thus by Lemma 1.2.3, $A_{12}$ satisfies condition *. Similarly, $A_{jk}$ satisfies condition * for each $(j, k)$. From Theorem 1, p. 88 of Ref. [6] we conclude that there exists a fundamental matrix $Z_0$ for Eq. (2.9) with the property that

$$Z_0(t) \exp \left\{ -\int_a^t (p'/p) D_1 + (r'/r) D_2\right\} \rightarrow I_4$$

as $t \rightarrow \infty$. Evaluation of this integral shows that $Z_0(t) DQ^{-1}(t) \rightarrow I_4$ as
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$t \to \infty$ where $D$ is a constant nonsingular diagonal matrix. Letting $H$ be the nonsingular constant matrix such that $Z_0D = Z_1H$ and letting $Y_0$ be $Y_1H$, we see that the theorem is proved. 

The next theorem represents the situation when $p^2$ dominates the product of $q - \sigma$ and $r$ and $(p/r)^{1/2} \not\in L(a, \infty)$. This case is considerably more difficult than the cases reflected in Theorems 2.3 and 2.5 for here the various characteristic roots of the coefficient matrix in Eq. (1.3) have different rates of growth. Any transformation of Eq. (1.3) which reduces the coefficient matrix to a constant matrix plus a variable matrix approaching zero seems to result in a constant matrix with nondiagonal Jordan canonical form, and the only perturbation theorems available for handling this type of situation require such strong condition on the variable matrix that they are of no use to us. Hence we find it necessary to transform Eq. (1.3) into an equation to which Theorem 1, p. 88 of Ref. [6] is applicable.

**Theorem 2.8.** Suppose each of $p$ and $r$ is positive-valued and twice continuously differentiable. Let $l = 1$ or $l = 2$. Let $\mu_1$ and $\mu_2$ be given by $\mu_1 = -\mu_2 = -(i)^l$. Let $h : [a, \infty) \to [0, \infty)$ be given by $h(t) = \int_a^t \frac{(p/r)^{1/2}}{s}$ and suppose $h(t) \to \infty$ as $t \to \infty$. Let $g$ be the function inverse to $h$ (i.e., $h(g(s)) = s$ for each $s \geq 0$). Let each of the functions $\alpha$, $\beta$, $\gamma$ and $\delta$ be defined as follows: For each $s \geq 0$,

\[
\begin{align*}
\alpha(s) &= \left[(p/r)^{-1/2}p'[\beta](g(s))\right], \\
\beta(s) &= \left[(p/r)^{-1/2}p'[\gamma](g(s))\right], \\
\gamma(s) &= \left[rq/p^2\right](g(s)), \\
\delta(s) &= \alpha \cdot \left[r/p^2\right](g(s)).
\end{align*}
\]

Let $A : [a, \infty) \to M^4$ be defined as in Theorem 2.7. Let $v > 1$ and let $G : [a, \infty) \to M^4$ be given by

\[G = \text{diag}[(p^3/r)^{1/4}, (pr)^{1/4}, (pr)^{-1/4}, h^v \cdot (r/p^3)^{1/4}].\]

Let $E : (0, \infty) \to M^4$ be given by

\[E(s) = \text{diag}[e^{-\mu_1 s}, e^{-\mu_2 s}, s^{-v} \cdot (p^3/r)^{1/4}(g(s)), (r/p^3)^{1/4}(g(s))].\]

Suppose each of $\alpha'$, $\beta'$, $\gamma'$, $\delta'$, $\alpha^2$, $\beta^2$, $I^v \cdot \beta$, and $I^v \cdot \delta$ is in $L(0, \infty)$ ($I(s) = s$ for each $s$), and suppose each of $(r/p^3)^{1/4}$, $h^v \cdot (r/p^3)^{1/4}$, and $h^v \cdot (r/p^3)^{1/2}$ can be expressed as the product of a monotone function and a bounded function which is bounded below by a positive number.

It then follows that there exists a fundamental matrix $Y_0$ for

\[Y' = AY \quad (2.10)\]
with the property that

\[ G(t) Y_0(t) E(h(t)) \to K \quad \text{as} \quad t \to \infty, \]

where

\[
K = \begin{bmatrix}
1 & 1 & 0 & 1 \\
\mu_1 & \mu_2 & 0 & 0 \\
(-1)^t & (-1)^t & 0 & 0 \\
0 & 0 & (-1)^t & 0
\end{bmatrix}.
\]

**Proof.** Let \( Y_1 \) be a fundamental matrix for Eq. (2.10). Let \( Q : [a, \infty) \to M^4 \) be defined as in Theorem 2.7. Let \( Z_1 \) be defined by \( Z_1(s) = Q(g(s)) Y_1(g(s)) \) for each \( s \geq 0 \). It follows from Lemma 1.3 that \( Z_1 \) is a fundamental matrix for

\[ Z' = (1/h'(g(\cdot))) [Q(g(\cdot))A(g(\cdot))Q^{-1}(g(\cdot)) + Q'(g(\cdot))Q^{-1}(g(\cdot))] Z. \quad (2.11) \]

Direct computation shows that Eq. (2.11) is the same as

\[ Z' = [A_0 + V]Z, \]

where

\[
A_0 = \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & (-1)^t & 0 & -1 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

and \( V : [0, \infty) \to M^4 \) is given by

\[
V = \begin{bmatrix}
(1/4)(3\alpha - \beta) & 0 & 0 & 0 \\
0 & (1/4)(\alpha + \beta) & 0 & 0 \\
0 & 0 & (-1/4)(\alpha + \beta) & 0 \\
\gamma - \delta & 0 & 0 & (-1/4)(3\alpha - \beta)
\end{bmatrix}.
\]

Let

\[
J = \begin{bmatrix}
1 & 1 & 0 & 1 \\
\mu_1 & \mu_2 & 1 & 0 \\
(-1)^t & (-1)^t & 0 & 0 \\
0 & 0 & (-1)^t & 0
\end{bmatrix}.
\]

While other choices of \( J \) are possible in order to obtain the same \( J^{-1}A_0 J \) matrix below this one produces a simpler \( J^{-1}VJ \) matrix.

\[
J^{-1} = \begin{bmatrix}
0 & (-1)^t \mu_1/2 & (-1)^t 2/2 & \mu_2/2 \\
0 & (-1)^t \mu_2/2 & (-1)^t 2/2 & \mu_1/2 \\
0 & 0 & 0 & (-1)^t \\
1 & 0 & -(1)^t & 0
\end{bmatrix}
\]
Letting $W_1$ be defined by $W_1(s) = J^{-1}Z_1(s)$ for each $s \geq 0$, we find that $W_1$ is a fundamental matrix for

$$W' = [J^{-1}A_0J + J^{-1}VJ]W.$$  (2.12)

Computations shows: $\mu_1 = -\mu_2$, $(\mu_j)^2 = (-1)^j$ for $j = 1, 2,$

$$J^{-1}A_0J = \begin{bmatrix} \mu_1 & 0 & 0 & 0 \\ 0 & \mu_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

and

$$J^{-1}VJ =
\begin{bmatrix}
(-1/4)(\alpha + \beta) & (\mu_2/2)(\gamma - \delta) & (\mu_2/2)(\gamma - \delta) & (\mu_2/2)(\gamma - \delta) \\
(\mu_2/2)(\gamma - \delta) & (\mu_2/2)(\gamma - \delta) & (\mu_2/2)(\gamma - \delta) & (\mu_2/2)(\gamma - \delta) \\
0 & (\mu_1/2)(\gamma - \delta) & (\mu_1/2)(\gamma - \delta) & (\mu_1/2)(\gamma - \delta) \\
0 & 0 & (\mu_1/2)(\gamma - \delta) & (\mu_1/2)(\gamma - \delta) \\
\end{bmatrix}.$$

Let $P : [1, \infty) \to M^4$ be given by

$$P(s) = \text{diag}[1, 1, s^\alpha, 1],$$

for each $s \geq 1$, and let $X_1$ be given by $X_1(s) = P(s)W_1(s)$ for each $s \geq 1$. It follows that $X_1$ is a fundamental matrix for

$$X' = [PJ^{-1}A_0JP^{-1} + PJ^{-1}VJP^{-1} + P'P^{-1}]X.$$  (2.13)

It is easy to see that Eq. (2.13) is the same as

$$X' = [B + C]X,$$

where $B : [1, \infty) \to M^4$ is given by

$$B = \begin{bmatrix} \mu_1 & (-1/4)(\alpha + \beta) & 0 & 0 \\ (-1/4)(\alpha + \beta) & \mu_2 & 0 & 0 \\ 0 & 0 & (-1/4)(3\alpha - \beta) + (\psi/1) & 0 \\ \alpha & \alpha & 0 & (1/4)(3\alpha - \beta) \end{bmatrix}.$$
and $C : [1, \infty) \to M^4$ is given by

\[
C = \begin{bmatrix}
  \left(\mu_2/2\right)(\gamma - \delta) & \left(\mu_2/2\right)(\gamma - \delta) & (-1)^j\left(\mu_2/2\right)\alpha & \left(\mu_2/2\right)(\gamma - \delta) \\
  \left(\mu_2/2\right)(\gamma - \delta) & \left(\mu_2/2\right)(\gamma - \delta) & (-1)^j\left(\mu_2/2\right)\alpha & \left(\mu_2/2\right)(\gamma - \delta) \\
  (-1)^j\left(\mu_2/2\right)\alpha & (-1)^j\left(\mu_2/2\right)\alpha & 0 & (-1)^j\left(\mu_2/2\right)\alpha \\
  0 & 0 & (1/\gamma) & 0
\end{bmatrix}.
\]

Note that our hypotheses imply $|C(\cdot)| \in L(1, \infty)$. Next, we introduce a transformation which will “continuously diagonalize” $B$. Let $\mathcal{P} : \mathcal{C} \times [1, \infty) \to \mathcal{C}'$ be given by $\mathcal{P}(z, s) = \det(B - zI_4)$ for each $(z, s) \in \mathcal{C} \times [1, \infty)$. Computation shows

\[
\mathcal{P}(z, s) = \{(1/4)[3\alpha(s) - \beta(s)] - z\}((-1/4)[3\alpha(s) - \beta(s)] + (s/s - z) \\
\cdot [z^2 - (-1)^j - (1/16)(\alpha(s) + \beta(s))^2].
\]

Note that our hypotheses imply $\alpha(s) \to 0$ and $\beta(s) \to 0$ as $s \to \infty$. Let $b \geq 1$ be such that for $j = 1, 2$ there is a unique continuous function $\lambda_j : [b, \infty) \to \mathcal{C}'$ with the properties that $\lambda_j(s) \to \mu_j$ as $s \to \infty$ and $\mathcal{P}(\lambda_j(s), s) = 0$ for each $s \geq b$. Let $\lambda_4 : [b, \infty) \to \mathcal{C}'$ be given by $\lambda_4(s) = (-1/4)(3\alpha(s) - \beta(s)) + (s/s)$ and $\lambda_4(s) - (1/4)(3\alpha(s) - \beta(s))$ for each $s \geq b$. Clearly, we then have $\mathcal{P}(\lambda_4(s), s) = 0$ for each $j$ and each $s \geq b$. Let $\mathbf{S} : [b, \infty) \to M^4$ be given by

\[
S = \begin{bmatrix}
  \mu_1 + \lambda_1 & -\eta & 0 & 0 \\
  \eta & \mu_1 - \lambda_2 & 0 & 0 \\
  0 & 0 & 1 & 0 \\
  \alpha \cdot (\mu_1 + \lambda_4 - \eta) & \alpha \cdot (\lambda_4 - \mu_1 - \eta) & 0 & -(1) + \lambda_4^2 - \eta^2
\end{bmatrix},
\]

where $\eta = (1/4)(\alpha + \beta)$. From $\mathcal{P}$ we find that for $j = 1, 2$ and $s$ sufficiently large

\[
\lambda_j^2(s) - (-1)^j - \eta^2(s) = 0.
\]

Use of this identity and direct computation shows

\[
\sum_{n=1}^{4} S_{jn}(s) B_{nk}(s) = \lambda_j(s) \cdot S_{jk}(s)
\]

for all large $s$ and $j, k = 1, \ldots, 4$ (Here, of course, $S_{jn}$ is the $j, n$ coordinate function of $S$, and $B_{nk}$ is the $n, k$ coordinate function of $B$). Inspection of $S$ shows it to be continuously differentiable and shows $|S'(\cdot)| \in L(b, \infty)$. (From our hypotheses in the theorem we find each of $\alpha$ and $\beta$ and hence each $\eta$ and $\lambda_4$ is continuously differentiable. From $\lambda_j^2 = \eta^2 + (-1)^j$, $j = 1, 2$ and the fact that $\eta(s) \to 0$ as $s \to \infty$, we see each of $\lambda_1$ and $\lambda_2$ is continuously differentiable when restricted to large values of its argument. Also, we have
\( \alpha', \beta', \) and hence \( \eta' \) and \( \lambda_4' \) in \( L(b, \infty) \); from this and \( a(s), \eta(s), \lambda_4(s) \to 0 \) as \( s \to \infty \) we see \( S_{jk} \in L(b, \infty) \) except for \( (j, k) = (1, 1) \) or \((2,2)\) and these cases follow from \( \lambda_j' = \eta \eta' / \lambda_j \) \((j = 1, 2)\). From the fact that

\[
S(s) \to \text{diag}[2 \mu_1, 2 \mu_1, 1, -(-1)^1] \quad \text{as} \quad s \to \infty,
\]

we have that \( S(s) \) is nonsingular for all large \( s \). Hence from Lemma 1.3 we see that there is a \( b_1 \geq b \) such that if \( U_1 \) is defined by \( U_1(s) = S(s) X_1(s) \) for each \( S \geq b_1 \), \( U_1 \) is a fundamental matrix for

\[
U' = [SBS^{-1} + SCS^{-1} + S'S^{-1}]U. \quad (2.14)
\]

We have shown \( SB = \text{diag}[\lambda_1, \ldots, \lambda_4]S; \) hence \( SBS^{-1} = \text{diag}[\lambda_1, \ldots, \lambda_4] \). Since each of \( S \) and \( S^{-1} \) is bounded on \([b_1, \infty)\), and since each of \( |C(\cdot)| \) is in \( L(b_1, \infty) \), so is \( |S(\cdot)C(\cdot) S^{-1}(\cdot) + S'(\cdot) S^{-1}(\cdot)| \) in \( L(b_1, \infty) \). From \( \partial^\alpha(\lambda_j(s), s) = 0 \) we see that, for \( j = 1, 2 \), and \( s \) large, 0 = \( \lambda_j^2(s) - (-1)^1 - \eta^2(s) = (\lambda_j(s) - \mu_j)(\lambda_j(s) - \mu_2) - \eta^2(s) \). Hence for all large \( s \), \( (\lambda_j(s) - \mu_1) = \eta^2(s)/(\lambda_j(s) - \mu_2) \), since \( \lambda_j(s) - \mu_2 \to -2 \mu_2 \neq 0 \) as \( s \to \infty \), \( |\lambda_j - \mu_j| \) is eventually dominated by a constant multiple of a function in \( L(b_1, \infty) \) (namely \( \eta^2(s) \)), so \( \lambda_j - \mu_j \) is in \( L(b_1, \infty) \). Similarly, we see \( \lambda_2 - \mu_2 \) is in \( L(b_1, \infty) \). Thus Eq. (2.14) is the same as

\[
U' = [A + F]U, \quad (2.15)
\]

where \( A = \text{diag}[\mu_1, \mu_2, \lambda_3, \lambda_4] \) and

\[
F = \text{diag}[\lambda_1 - \mu_1, \lambda_2 - \mu_2, 0, 0] + SCS^{-1} + S'S^{-1} \text{ is in } L(b_1, \infty).
\]

Our hypotheses imply that if \( b_1 \leq t_1 \leq t_2 \), and \( A_{jk} \) is the real part of the \( j \)-th diagonal entry of \( A \) minus the \( k \)-th, then

\[
\int_{t_1}^{t_2} A_{jk} = \left[ \ln m_{jk}(\cdot) + \ln w_{jk}(\cdot) \right]_{t_1}^{t_2},
\]

where \( m_{jk} \) is a monotone function and \( w_{jk} \) is a bounded function which is bounded below by a positive number. Thus, from Lemma 1.2.3 and Theorem 1, p. 88 of Ref. [6] we conclude that there is a fundamental matrix \( U_0 \) for Eq. (2.15) with the property that

\[
U_0(s) \exp \left\{ - \int_{b_1}^{s} \text{diag}[\mu_1, \mu_2, \lambda_3, \lambda_4] \right\} \to I_4
\]

as \( s \to \infty \).

Evaluation of this integral (using \( g(h(t)) = t \)) shows

\[
U_0(s) D \text{diag}[e^{-\mu_1 s}, e^{-\mu_2 s}, s^{-\nu}, [(p^2 r)^{1/4}] g(s), [(r/p^3)^{1/4}] g(s)] \to I_4
\]

\( 505/9/1-9 \)
as $s \to \infty$, where $D$ is a constant nonsingular diagonal matrix. Since each of $U_1$ and $U_0D$ is a fundamental matrix for Eq. (2.15) there is a nonsingular constant matrix $H$ such that $U_0D = U_1H$. Noting that $U_1(h(t)) = S(h(t)) P(h(t)) J^{-1} Q(t) Y_2(t)$ for $t \geq g(b_1)$ and setting $Y_2 = Y_1H$ on $[g(b_1), \infty)$ and extending $Y_2$ to be a fundamental matrix for Eq. 2.10 on all of $[a, \infty)$ we see that $Y_2$ is a fundamental matrix for Eq. 2.10 and

$$S(h(t)) P(h(t)) J^{-1} Q(t) Y_2(t) E(h(t)) \to I_s$$

as $t \to \infty$. Hence, since $JS^{-1}(s)$ has a limit as $s \to \infty$,

$$JP(h(t)) J^{-1} Q(t) Y_2(t) E(h(t)) \to JS^{-1}(\infty)$$

as $t \to \infty$ where

$$S^{-1}(\infty) = \text{diag} \left[ \frac{1}{2\mu_1}, \frac{1}{2\mu_2}, 1, -(-1)^i \right].$$

Noting $JP(h(t)) J^{-1} = K_1(t) \text{diag}[1, 1, 1, h^q(t)]$, where

$$K_1(t) = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & (-1)^i(1 - h^{-q}(t)) \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix},$$

and noting

$$K_1^{-1}(t) = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & (-1)^i(h^{-q}(t) - 1) \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix},$$

we see that

$$\text{diag}[1, 1, 1, h^q(t)] Q(t) Y_2(t) E(h(t)) \to K_1^{-1}(\infty) J S^{-1}(\infty),$$

where

$$K_1^{-1}(\infty) = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & -(1)^i \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}.$$
as $t \to \infty$. A quick computation shows that $K^{-1}(\infty)I$ is $K$ and the theorem is proved. \|

Translating our hypotheses in Theorem 2.8 into conditions on the coefficient functions $p$, $q$, and $r$ we find that $\alpha'$, $\beta'$, $\gamma'$, $\delta'$, $\alpha^2$, $\beta^2$, $I^\nu\gamma$, and $I^\nu\sigma \in L(0, \infty)$ are equivalent, respectively, to $[(p/r)^{-1/2}p'/p]'$, $[(p/r)^{-1/2}r'/r]'$, $[rq/p^2]'$, $\sigma \cdot [r/p^2]'$, $[(p/r)^{-1/2}(p'/p)^2]'$, $[(p/r)^{-1/2}(r'/r)^2]'$, $[(p/r)^{-1/2}(r/p^3)'^2]'$, and $\sigma \cdot [h^\nu \cdot (r/p^3)'^2]' \in L(a, \infty)$.

It should be noted that in the special case that $r = 1$ the corollary on p. 594 of Ref. [1] show that the single condition $(p'/p^3)^2 \in L(a, \infty)$ implies each of $h(t) \to \infty$, $\alpha' \in L(a, \infty)$, and $\alpha^2 \in L(a, \infty)$.

3. Applicability of the Various Theorems in Section 2 to $[(t^\nu y')' - (\pm t^{2\nu} y')]' + (\pm t^{2\nu} y) = \sigma y$

In this section we test the applicability of the various theorems in Section 2 in the special case $[a, \infty) = [1, \infty)$, $r(t) = t^\nu$, $p(t) = t^\nu$, and $q(t) = t^\nu$ where each of $\nu_1$, $\nu_2$, and $\nu_3$ is a real number. While none of our hypotheses require such smooth functions, this is a convenient way to check the relative rates of growth which they impose on $p$, $q$, and $r$.

When $\sigma = 0$ we find:

(i) Theorem 2.1 holds if and only if $4\nu_2 - 3\nu_1 + 4 < \nu_3 < \nu_1 - 4$.

(ii) Theorem 2.3 holds if and only if $\nu_3 > \nu_1 - 4$ and $\nu_3 > 2\nu_2 - \nu_1$.

(iii) Corollary 2.4 holds for some $n_0$ if and only if Theorem 2.3 holds. It holds for $n_0 = 0$ when $\nu_3 > \nu_1 - 4$ and $\nu_3 > 4\nu_2 - 3\nu_1 + 4$. It holds for $n_0 = 1$ when $\nu_3 > \nu_1 - 4$ and $\nu_3 > (1/3)(8\nu_2 - 5\nu_1 + 4)$.

(iv) Theorem 2.7 holds if and only if $(1/3)(2\nu_3 + \nu_1 + 2) < \nu_2 < \nu_1 - 2$.

(v) Theorem 2.8 holds for some $\nu > 1$ if and only if $\nu_2 > \nu_1 - 2$ and $\nu_3 < \nu_2 - 2$.

When $\sigma \neq 0$ we find:

(vi) Theorem 2.1 holds if and only if $4\nu_2 - 3\nu_1 + 4 < \nu_3 < \nu_1 - 4$ and $\nu_3 > (1/3)(4 - \nu_1)$.

(vii) Theorem 2.3 holds if and only if $\nu_3 > \nu_1 - 4$, $\nu_3 > 2\nu_2 - \nu_1$, and $\nu_3 > 0$.

(viii) Theorem 2.5 holds if and only if $\nu_1 < 4$, $\nu_2 < 0$, and $\nu_2 < (1/2)\nu_1$.

(ix) Theorem 2.7 holds if and only if $(1/3)(2\nu_3 + \nu_1 + 2) < \nu_2 < \nu_1 - 2$ and $\nu_3 > (1/3)(\nu_1 + 2)$.

(x) Theorem 2.8 holds for some $\nu > 1$ if and only if $\nu_2 > \nu_1 - 2$, $\nu_3 < \nu_2 - 2$, and $\nu_3 > 2$. 
Verification of the above assertions is straightforward except perhaps in the case of Theorem 2.8. As an example we show that if \( \sigma = 0 \), \( v_2 > v_1 - 2 \) and \( v_3 < v_2 - 2 \), then Theorem 2.8 holds for some \( v > 1 \).

Suppose \( \sigma = 0 \), \( v_2 > v_1 - 2 \) and \( v_3 < v_2 - 2 \). From \( v_2 > v_1 - 2 \) we have that

\[
\lim_{t \to \infty} h(t) = \int_{1}^{\infty} \left( \frac{p}{r} \right)^{1/2} = \infty
\]

and that each of \( \left[ \left( \frac{p}{r} \right)^{-1/2} \right] ', \left[ \left( \frac{p}{r} \right)^{-1/2} \right] ', \left[ \left( \frac{p}{r} \right)^{-1/2} (p' / p)^2 \right] ', \) and \( \left[ \left( \frac{p}{r} \right)^{-1/2} (r' / r)^2 \right] \) is in \( L(1, \infty) \). Remembering \( g(h(t)) = t \), we have that each of \( \alpha', \beta', \alpha^2 \), and \( \beta^2 \) is in \( L(0, \infty) \). From \( v_3 < v_2 - 2 \), we have

\[
[1/2(v_2 - v_1) + 1] + [v_1 + v_3 - 2v_2] + [1/2(v_3 - v_1)] < -1.
\]

So there is a number \( v > 1 \) such that

\[
v[1/2(v_3 - v_1) + 1] + [v_1 + v_3 - 2v_2] + [1/2(v_3 - v_1)] < -1.
\]

Thus \( f \cdot \left[ \frac{r}{p^2} \right] \cdot h' \in L(1, \infty) \) where \( f(t) = t^{v[1/2(v_3 - v_1) + 1] + 1} \). From \( v_2 > v_1 - 2 \) we have \( (1/2)(v_2 - v_1) + 1 > 0 \). So \( h^v(t) = f(t)(c + o(1)) \) for a positive constant \( c \). Thus \( h^v \cdot \left[ \frac{r}{p^2} \right] \cdot h' \in L(1, \infty) \) or \( I^v \cdot \gamma \in L(0, \infty) \). Since \( \sigma = 0 \), \( I^v \cdot \delta \in L(0, \infty) \), and clearly each of \( (r/p^3)^{1/4} \), \( h^v \cdot (r/p^3)^{1/4} \), and \( h^v \cdot (r/p^3)^{1/2} \) can be expressed as the product of a monotone function and a bounded function which is bounded below by a positive number.

References