Topology and its Applications 157 (2010) 508–515

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

The fibered isomorphism conjecture in L-theory

S.K. Roushon

School of Mathematics, Tata Institute, Homi Bhabha Road, Mumbai 400005, India

ARTICLE INFO

Article history:
Received 13 March 2009
Received in revised form 22 October 2009
Accepted 23 October 2009

MSC:
primary 19G24, 19J25
secondary 55N91

Keywords:
Braid groups
Poly-free groups
Fibered Isomorphism Conjecture
L-theory
Surgery groups

ABSTRACT

This is the first of three articles on the Fibered Isomorphism Conjecture of Farrell and Jones for L-theory. Here we prove the conjecture for several well-known classes of groups. In fact we consider a general class of groups satisfying certain conditions which includes the above classes of groups.

1. Introduction

One of the fundamental conjecture in Geometry and Topology is the Borel Conjecture. The conjecture says that if two closed aspherical manifolds are homotopically equivalent then they are homeomorphic. In other words the fundamental group of a closed aspherical manifold determine the topology of the manifold. The classical approach to this problem needs the study of two sets of obstruction groups, the lower K-groups and the surgery L-groups of the integral group ring of the fundamental group of the manifold. In [4, §1.6, §1.7] Farrell and Jones formulated the (Fibered) Isomorphism Conjectures to compute and understand these obstruction groups. The Isomorphism Conjecture was formulated for three functors: the pseudoisotopy theory, K-theory and $L^{\langle -\infty \rangle}$-theory. The conjectures for the first and the last theories give the computations of the lower K-groups and the surgery L-groups respectively and together they imply the Borel Conjecture. See [9] for some more consequences of the Isomorphism Conjectures.

Our objective is to set up some general methods to prove the Isomorphism Conjectures for all the three theories for groups acting on trees. In our earlier works we developed some machinery and studied the pseudoisotopy case of the conjecture. In the present article we set up this machinery in the $L^{\langle -\infty \rangle}$-theory case of the conjecture using some known facts and by proving some basic results. Finally, we see how this tool can be used to deduce the Fibered Isomorphism Conjecture for $L^{\langle -\infty \rangle}$-theory for several well-known classes of groups.

The Isomorphism Conjecture says that the above three theories can be computed for a group if we can compute them for all its virtually cyclic subgroups. The Fibered Isomorphism Conjecture is stronger and is appropriate for induction arguments. This property is crucial for the method we use here.

E-mail address: roushon@math.tifr.res.in.
URL: http://www.math.tifr.res.in/~roushon/.

0166-8641/$ – see front matter
doi:10.1016/j.topol.2009.10.012

We also consider the Fibered Isomorphism Conjecture for $\underline{L}^{\langle -\infty\rangle}$-theory, where $\underline{L}^{\langle -\infty\rangle} = L^{\langle -\infty\rangle} \otimes \mathbb{Z}[\frac{1}{2}]$. The advantage of taking $\underline{L}^{\langle -\infty\rangle}$-theory is that we can consider the 'finite subgroups' instead of the 'virtually cyclic subgroups'. This will also help to prove the conjecture for $\underline{L}^{\langle -\infty\rangle}$-theory for a larger class of groups.

Now we mention some conventions and definitions we need. Throughout the article by 'group' we mean 'discrete countable group' unless otherwise mentioned. A group is said to have a property P *virtually* if it has a finite index subgroup having the property P and we say that the group is *virtually* P.

Definition 1.1. Let $\underline{\mathcal{FICWF}}$ ($\mathcal{FICWF}$) be the smallest class of groups satisfying the conditions 1 to 5 (i to iv) below.

1. $\underline{\mathcal{FICWF}}$ contains the cocompact discrete subgroups of linear Lie groups with finitely many components.
2. (Subgroup) If $H < G \in \underline{\mathcal{FICWF}}$ then $H \in \underline{\mathcal{FICWF}}$
3. (Free product) If $G_1, G_2 \in \underline{\mathcal{FICWF}}$ then $G_1 * G_2 \in \underline{\mathcal{FICWF}}$.
4. (Direct limit) If $\{G_i\}_{i\in I}$ is a directed system of groups with $G_i \in \underline{\mathcal{FICWF}}$ then the direct limit $\lim_{i\in I} G_i \in \underline{\mathcal{FICWF}}$.
5. (Extension) For an exact sequence of groups $1 \to K \to G \to N \to 1$, if $K, N \in \underline{\mathcal{FICWF}}$ then $G \in \underline{\mathcal{FICWF}}$.

i. $\mathcal{FICWF}$ contains the cocompact discrete subgroups of Lie groups with finitely many components.
ii. 2, 3 and 4 as above after replacing $\underline{\mathcal{FICWF}}$ by $\mathcal{FICWF}$.
iii. (Direct product) If $G_1, G_2 \in \mathcal{FICWF}$ then $G_1 \times G_2 \in \mathcal{FICWF}$.
iv. (Polycyclic extension) For an exact sequence of groups $1 \to K \to G \to N \to 1$, if either K is virtually cyclic and $N \in \mathcal{FICWF}$ or N is finite and $K \in \mathcal{FICWF}$ then $G \in \mathcal{FICWF}$.

The notation $\mathcal{FICWF}$ is the short form of (F)ibered (I)somorphism (C)onjecture (W)reath product with (F)inite groups and $\underline{\mathcal{FICWF}}$ denotes the same when we consider the conjecture tensored with $\mathbb{Z}[\frac{1}{2}]$. See Definition 1.2.

For two groups A and B the *wreath product* $A \wr B$ is the semidirect product $A^B \rtimes B$ with respect to the regular action of B on A^B. Here A^B denotes the group of tuples $\{a_b\}_{b\in B}$ of elements of A indexed by B with respect to coordinate-wise multiplication so that all but finitely many elements of a tuple are the identity element of A. In this article we consider cases when B is a finite group, therefore in this particular case A^B is the direct product of copies of A indexed by the elements of B.

We now recall a general statement of the (Fibered) Isomorphism Conjecture in equivariant homology theory from [1] before we state our main results.

Let $\mathcal{H}^?_*$ be an equivariant homology theory with values in R-modules for R a commutative associative ring with unit.

We always assume that a class of groups $\mathcal{C}$ is closed under isomorphisms, taking subgroups and taking quotients. We call such a class a *family of groups*. We denote by $\mathcal{C}(G)$ the set of subgroups of a group G which belong to $\mathcal{C}$. Then $\mathcal{C}(G)$ is a family of subgroups of G.

Given a group homomorphism $\phi : G \to H$ and $\mathcal{C}$ a family of subgroups of H define $\phi^*\mathcal{C}$ by the family of subgroups $\{K < G \mid \phi(K) \in \mathcal{C}\}$ of G. For a family $\mathcal{C}$ of subgroups of a group G there is a G-CW complex $E_{\mathcal{C}}(G)$ which is unique up to G-equivalence satisfying the property that for each $H \in \mathcal{C}$ the fixpoint set $E_{\mathcal{C}}(G)^H$ is contractible and $E_{\mathcal{C}}(G)^H = \emptyset$ for H not in $\mathcal{C}$.

(Fibered) Isomorphism Conjecture: ([1, Definition 1.1]) Let G be a group and $\mathcal{C}$ be a family of subgroups of G. Then the *Isomorphism Conjecture* for the pair $(G, \mathcal{C})$ states that the projection $p : E_{\mathcal{C}}(G) \to pt$ to the point pt induces an isomorphism

$$\mathcal{H}^G_n(p) : \mathcal{H}^G_n\big(E_{\mathcal{C}}(G)\big) \simeq \mathcal{H}^G_n(pt)$$

for $n \in \mathbb{Z}$.

And the *Fibered Isomorphism Conjecture* for the pair $(G, \mathcal{C})$ states that for any group homomorphism $\phi : K \to G$ the Isomorphism Conjecture is true for the pair $(K, \phi^*\mathcal{C})$.

Let $\mathcal{VC}$ and $\mathcal{FIN}$ denote the family of virtually cyclic groups and the family of finite groups respectively.

The notation for the equivariant homology theory associated to the $\underline{L}^{\langle -\infty\rangle}$-theory ($L^{\langle -\infty\rangle}$-theory) is $\mathcal{H}^?_n(-, \underline{\mathbf{L}}^{\langle -\infty\rangle})$ ($\mathcal{H}^?_n(-, \mathbf{L}^{\langle -\infty\rangle})$), where $\underline{\mathbf{L}}^{\langle -\infty\rangle}$ ($\mathbf{L}^{\langle -\infty\rangle}$) denotes the spectrum whose homotopy groups are the surgery groups $\underline{L}^{\langle -\infty\rangle}_*$ ($L^{\langle -\infty\rangle}_*$). See [9, Section 6.2] for details. For the equivariant homology theory $\mathcal{H}^?_n(-, \mathbf{L}^{\langle -\infty\rangle})$ and for $\mathcal{C} = \mathcal{VC}$ and $R = \mathbb{Z}$ the (Fibered) Isomorphism Conjecture is equivalent to the $L^{\langle -\infty\rangle}$-theory case of the conjecture in [4, §1.6, §1.7]. Also it is known that the (Fibered) Isomorphism Conjectures for the $\underline{L}^{\langle -\infty\rangle}$-theory for the two family of groups $\mathcal{VC}$ and $\mathcal{FIN}$ are equivalent.

Notational convention: For the notations defined in Definitions 1.2 and 2.1 we make the following further conventions. In the particular case when $\mathcal{C} = \mathcal{FIN}$ or $\mathcal{VC}$ the same notations will be used for the conjecture in $L^{\langle -\infty\rangle}$-theory and for the $\underline{L}^{\langle -\infty\rangle}$-theory the corresponding notations will have a superscript '–'.

Definition 1.2. ([14, Definition 2.1]) Let $\mathcal{C}$ be a family of groups. If the (Fibered) Isomorphism Conjecture is true for the pair $(G, \mathcal{C}(G))$ we say that the $(F)IC_{\mathcal{C}}$ *is true for* G or simply say $(F)IC_{\mathcal{C}}(G)$ *is satisfied*. Also we say that the $(F)ICwF_{\mathcal{C}}(G)$ *is satisfied* if the $(F)IC_{\mathcal{C}}$ is true for the wreath product $G \wr H$ for any finite group H.

We prove the following theorems.

Theorem 1.1. *Let $\Gamma \in \underline{\mathcal{FICWF}}$ and $\Delta \in \mathcal{FICWF}$. Let G and H be two groups with homomorphisms $\phi : G \to \Gamma \wr F$ and $\psi : H \to \Delta \wr K$ where F and K are finite groups. Then the following assembly maps are isomorphisms for all n.*

$$\mathcal{H}_n^G\big(E_{\phi^*\mathcal{FIN}(\Gamma\wr F)}(G), \underline{\mathbf{L}}^{\langle-\infty\rangle}\big) \to \mathcal{H}_n^G\big(pt, \underline{\mathbf{L}}^{\langle-\infty\rangle}\big) \simeq \underline{L}_n^{\langle-\infty\rangle}(\mathbb{Z}G),$$
$$\mathcal{H}_n^H\big(E_{\psi^*\mathcal{VC}(\Delta\wr K)}(H), \mathbf{L}^{\langle-\infty\rangle}\big) \to \mathcal{H}_n^H\big(pt, \mathbf{L}^{\langle-\infty\rangle}\big) \simeq L_n^{\langle-\infty\rangle}(\mathbb{Z}H).$$

In other words the Fibered Isomorphism Conjecture of Farrell and Jones for the $\underline{L}^{\langle-\infty\rangle}$-theory ($L^{\langle-\infty\rangle}$-theory) is true for the group $\Gamma \wr F$ ($\Delta \wr K$). Equivalently, the FICwF$^-_{\mathcal{FIN}}(\Gamma)$ and the FICwF$_{\mathcal{VC}}(\Delta)$ are satisfied.

Theorem 1.2. *Let $\mathcal{C}$ ($\mathcal{D}$) be the class of groups which satisfy the FICwF$^-_{\mathcal{FIN}}$ (FICwF$_{\mathcal{VC}}$). Then $\mathcal{C}$ ($\mathcal{D}$) has the properties* 2 *to* 5 (ii *to* iv) *after replacing $\underline{\mathcal{FICWF}}$ ($\mathcal{FICWF}$) by $\mathcal{C}$ ($\mathcal{D}$) in Definition* 1.1.

Our next goal is to show that $\underline{\mathcal{FICWF}}$ and $\mathcal{FICWF}$ contain some well-known classes of groups.

Theorem 1.3. *$\underline{\mathcal{FICWF}}$ contains the following groups.*

1. *Virtually cyclic groups.*
2. *Free groups and abelian groups.*
3. *Poly-free groups. A poly-free group G admits a filtration by subgroups*: $1 < G_0 < G_1 < \cdots < G_n = G$ *so that G_i is normal in G_{i+1} and G_{i+1}/G_i is free. Here n is called the index of G.*
4. *Strongly poly-free groups. See* [2, *Definition* 1.1].
5. *Full braid groups.*
6. *Cocompact discrete subgroups of Lie groups with finitely many components.*
7. *Groups whose some derived subgroup belongs to $\underline{\mathcal{FICWF}}$.*

$\mathcal{FICWF}$ contains the following groups.

i. *Virtually cyclic groups.*
ii. *Free groups and abelian groups.*
iii. *Groups appearing in* 6 (*by definition*).
iv. *Virtually polycyclic groups.*

Remark 1.1. Here we should remark that the $IC^-_{\mathcal{VC}}$ for a class of groups including poly-free groups and one-relator groups was proved in [1, Theorem 0.13].

When Γ and Δ are torsion free, $F = K = \{1\}$ and ϕ and ψ are the identity maps, Theorem 1.1 reduces to the isomorphism of the classical assembly map in surgery theory. Therefore we have the following corollary. See [4, 1.6.1] for details.

Corollary 1.1. *Let $\Gamma \in \underline{\mathcal{FICWF}}$ and $\Delta \in \mathcal{FICWF}$ and in addition assume that Γ and Δ are torsion free. Then the following assembly maps are isomorphism for all n.*

$$H_n\big(B\Gamma, \underline{\mathbf{L}}^{\langle-\infty\rangle}\big) \to \underline{L}_n^{\langle-\infty\rangle}(\mathbb{Z}\Gamma),$$
$$H_n\big(B\Delta, \mathbf{L}^{\langle-\infty\rangle}\big) \to L_n^{\langle-\infty\rangle}(\mathbb{Z}\Delta).$$

In other words the surgery groups $\underline{L}_n^{\langle-\infty\rangle}(\mathbb{Z}\Gamma)$ of Γ and the surgery groups $L_n^{\langle-\infty\rangle}(\mathbb{Z}\Delta)$ of Δ form generalized homology theories.

Since surgery groups with different decorations differ by 2-torsions, that is, $\underline{L}_n^{\langle-\infty\rangle}(\mathbb{Z}\Gamma) \simeq L_n^h(\mathbb{Z}\Gamma) \otimes \mathbb{Z}[\frac{1}{2}] \simeq L_n^s(\mathbb{Z}\Gamma) \otimes \mathbb{Z}[\frac{1}{2}]$ for any group Γ (see [6, Section 5, para 1]), Theorem 1.1 is true for the functors $L^h \otimes \mathbb{Z}[\frac{1}{2}]$ and $L^s \otimes \mathbb{Z}[\frac{1}{2}]$ also. It is known that Theorem 1.1 is not true for the L^h- and L^s-theory if we do not tensor with $\mathbb{Z}[\frac{1}{2}]$ ([5]).

Remark 1.2. The main ingredient behind the proof of Theorem 1.1 is [4, Theorem 2.1 and Remark 2.1.3]. This was also used before to prove the $FIC^-_{\mathcal{FIN}}$ and the $FIC^-_{\mathcal{VC}}$ in [6] for elementary amenable groups, and in [10] for computation of K and L-groups of cocompact planar groups. In this regard also see [14, Remark 8.1].

Remark 1.3. We also note here that the $IC^-_{\mathcal{VC}}$ is known for many classes of groups. See [9, 5.3]. In [8] it was proved that the $IC_{\mathcal{VC}}$ is true for the fundamental groups of closed manifolds with a $\widetilde{\mathbb{SL}} \times \mathbb{E}^n$ structure for $n \geqslant 2$.

2. Some basic results on the Isomorphism Conjecture

Given a normal subgroup H of a group G by [7, Algebraic Lemma] G can be embedded as a subgroup in the wreath product $H \wr (G/H)$. We will always use this fact without explicitly mentioning it.

We begin by noting that if $H \in \mathcal{C}$ then the $(F)IC_{\mathcal{C}}(H)$ is satisfied.

The following observation is known as the *hereditary property* of the Fibered Isomorphism Conjecture.

Lemma 2.1. *If the $FIC_{\mathcal{C}}$ ($FICwF_{\mathcal{C}}$) is true for a group G then the $FIC_{\mathcal{C}}$ ($FICwF_{\mathcal{C}}$) is true for any subgroup of G.*

Throughout the article a 'graph' (that is an one-dimensional CW-complex) is assumed to be connected and locally finite. And a *graph of groups* consists of a graph $\mathcal{G}$ and to each vertex v or edge e of $\mathcal{G}$ there is associated a group $\mathcal{G}_v$ (called the *vertex group* of the vertex v) or $\mathcal{G}_e$ (called the *edge group* of the edge e) respectively with the assumption that for each edge e and for its two end vertices v and w ($v = w$ is possible) there are injective group homomorphisms $\mathcal{G}_e \to \mathcal{G}_v$ and $\mathcal{G}_e \to \mathcal{G}_w$. A *subgraph of groups* $\mathcal{H}$ of a graph of groups $\mathcal{G}$ consists of a subgraph of the graph and for each vertex v (or edge e) of $\mathcal{H}$, $\mathcal{H}_v = \mathcal{G}_v$ ($\mathcal{H}_e = \mathcal{G}_e$) with the same homomorphisms coming from $\mathcal{G}$. The *fundamental group* $\pi_1(\mathcal{G})$ of the graph $\mathcal{G}$ can be defined so that in the simple cases of graphs of groups where the graph has two vertices and one edge or one vertex and one edge the fundamental group is the amalgamated free product or the *HNN*-extension respectively. See [3] for some more on this subject.

Definition 2.1. ([14, Definition 2.2]) Let $\mathcal{C}$ be a family of groups. We say that $\mathcal{T}_{\mathcal{C}}$ (${}_w\mathcal{T}_{\mathcal{C}}$) is satisfied if for a graph of groups $\mathcal{G}$ with vertex groups (and hence edge groups) belonging to $\mathcal{C}$, the $FIC_{\mathcal{C}}$ ($FICwF_{\mathcal{C}}$) for $\pi_1(\mathcal{G})$ is true.

We say that ${}_t\mathcal{T}_{\mathcal{C}}$ (${}_{wt}\mathcal{T}_{\mathcal{C}}$) is satisfied if for a graph of groups $\mathcal{G}$ with trivial edge groups and the vertex groups belonging to $\mathcal{C}$, the $FIC_{\mathcal{C}}$ ($FICwF_{\mathcal{C}}$) for $\pi_1(\mathcal{G})$ is true.

And we say that $\mathcal{P}_{\mathcal{C}}$ is satisfied if for $G_1, G_2 \in \mathcal{C}$ the product $G_1 \times G_2$ satisfies the $FIC_{\mathcal{C}}$.

The following general lemma is a combination of [14, Proposition 5.2] and [15, Lemma 3.4].

Lemma 2.2. *Assume that $\mathcal{P}_{\mathcal{C}}$ is satisfied.*

(1) *If the $FIC_{\mathcal{C}}$ ($FICwF_{\mathcal{C}}$) is true for G_1 and G_2 then $G_1 \times G_2$ satisfies the $FIC_{\mathcal{C}}$ ($FICwF_{\mathcal{C}}$).*
(2) *Let G be a finite index subgroup of a group K. If the $FICwF_{\mathcal{C}}$ is true for G then it is also true for K.*
(3) *Let $p : G \to Q$ be a group homomorphism. If the $FICwF_{\mathcal{C}}$ is true for Q and for $p^{-1}(H)$ for all $H \in \mathcal{C}(Q)$ then the $FICwF_{\mathcal{C}}$ is true for G.*

The following is a consequence of the definition of the $\underline{L}^{\langle -\infty \rangle}$-theory.

Lemma 2.3. *For the family of groups $\mathcal{C} = \mathcal{VC}$ or $\mathcal{FIN}$ the $FIC_{\mathcal{C}}$ implies the $FIC^-_{\mathcal{C}}$ and the $FICwF_{\mathcal{C}}$ implies the $FICwF^-_{\mathcal{C}}$.*

If the Isomorphism Conjecture is true for a group with respect to a family $\mathcal{C}$ of subgroups then it is true for the group with respect to a family of subgroups containing $\mathcal{C}$. The following lemma shows that sometimes the converse is also true.

Lemma 2.4. *If a group G satisfies the $FIC^-_{\mathcal{VC}}$ ($FICwF^-_{\mathcal{VC}}$) then it also satisfies the $FIC^-_{\mathcal{FIN}}$ ($FICwF^-_{\mathcal{FIN}}$).*

Proof. See [6, Lemma 5.1] or [9, Proposition 2.18]. □

Lemma 2.5. *Let $\mathcal{C} = \mathcal{VC}$. Let $\{G_i\}_{i \in I}$ be a directed system of groups with direct limit G. If each G_i satisfies J then G also satisfies J where $J = 1, 2, 3$ or 4 are as below.*

1. $FIC^-_{\mathcal{C}}$. 2. $FIC_{\mathcal{C}}$. 3. $FICwF^-_{\mathcal{C}}$. 4. $FICwF_{\mathcal{C}}$.

The above statement is also true for $\mathcal{C} = \mathcal{FIN}$.

Proof. For 1 and 2 the lemma directly follows from [6, Theorem 7.1] and for 3 and 4 note that for a finite group F, $G \wr F$ is the direct limit of the directed system $\{G_i \wr F\}_{i \in I}$ and then apply [6, Theorem 7.1]. □

Before we come to some more results on the Isomorphism Conjecture let us deduce some group theoretic results in the following Lemmas 2.6 to 2.9.

Lemma 2.6. *Let G be a Lie group with finitely many components and let F be a finite group. Then the wreath product $G \wr F$ is again a Lie group with finitely many components with respect to the product topology on $G^F \times F$ where F is given the discrete topology and G^F denotes the $|F|$-times direct product of G.*

Proof. Recall that an element of G^F is of the form $(g_{f_1}, \ldots, g_{f_{|F|}})$ where $f_i \in F$. Now let $f \in F$. Then the regular action of F on G^F is by definition $f(g_{f_1}, \ldots, g_{f_{|F|}}) = (g_{f_1 f^{-1}}, \ldots, g_{f_{|F|} f^{-1}})$. It now follows from the definition of semi-direct product that the product and inverse operations on $G \wr F$ both are smooth. Therefore $G \wr F$ is a Lie group and also it has finitely many components. □

Lemma 2.7. *Let S be a closed orientable surface of genus $\geqslant 1$. Then $\pi_1(S)$ is a discrete cocompact subgroup of a Lie group with finitely many components.*

Proof. If the genus of S is 1 then $\pi_1(S)$ is a discrete cocompact subgroup of the Lie group of isometries of the flat Euclidean plane. And if the genus of S is $\geqslant 2$ then the corresponding Lie group is the group of isometries of the hyperbolic plane. □

Lemma 2.8. *If a graph of groups $\mathcal{G}$ has trivial edge groups then $\pi_1(\mathcal{G})$ is isomorphic to the free product of a free group and the vertex groups of $\mathcal{G}$.*

Proof. Apply [14, Lemma 6.2] and note that $\mathcal{G}$ is the direct limit of the directed system of its finite subgraphs of groups. Here the directed system is obtained by inclusion maps of subgraphs. □

Lemma 2.9. *Let V_1 and V_2 be two virtually free groups then $V_1 * V_2$ is virtually free.*

Proof. We have a surjective homomorphism $p : V_1 * V_2 \to V_1 \times V_2$. Let H_i be a free subgroup of V_i of finite index for $i = 1, 2$. Hence $H = H_1 \times H_2$ has finite index in $V_1 \times V_2$. Note that $V_1 * V_2$ acts on a tree with trivial edge stabilizers and the vertex stabilizers are conjugate to V_1 or V_2. Hence $p^{-1}(H)$ also acts on the same tree. It follows that the edge stabilizers of this restricted action are again trivial and the vertex stabilizers are conjugates of H_1 or H_2 and hence free. Therefore $p^{-1}(H)$ is a free group by Lemma 2.8. This completes the proof. □

Lemma 2.10. *Let Γ be a discrete cocompact subgroup of a Lie group with finitely many connected components. Then Γ satisfies the $FICwF_{\mathcal{VC}}w$, $FIC^-_{\mathcal{VC}}$, $FIC^-_{\mathcal{FIN}}$, $FICwF_{\mathcal{VC}}$, $FICwF^-_{\mathcal{VC}}$ and the $FICwF^-_{\mathcal{FIN}}$.*

Proof. By Lemma 2.3 $FIC_{\mathcal{VC}}$ implies $FIC^-_{\mathcal{VC}}$ and then apply Lemma 2.4 to get $FIC^-_{\mathcal{FIN}}$. Similarly $FICwF_{\mathcal{VC}}$ implies $FICwF^-_{\mathcal{VC}}$ and then applying Lemma 2.4 we get $FICwF^-_{\mathcal{FIN}}$. Therefore we only have to show that Γ satisfies $FIC_{\mathcal{VC}}$ and $FICwF_{\mathcal{VC}}$. For $FIC_{\mathcal{VC}}$ it follows directly from [4, Theorem 2.1 and Remark 2.1.3].

Now if Γ is a discrete cocompact subgroup of G then $\Gamma \wr F$ is a discrete cocompact subgroup of $G \wr F$ for any finite group F. Here the Lie group structure on $G \wr F$ is as described in Lemma 2.6.

Hence we can again use [4, Theorem 2.1 and Remark 2.1.3] to see that the $FICwF_{\mathcal{VC}}$ is satisfied for Γ.

This completes the proof. □

Lemma 2.11. *$\mathcal{P}_{\mathcal{VC}}$, $\mathcal{P}^-_{\mathcal{VC}}$ and $\mathcal{P}^-_{\mathcal{FIN}}$ are satisfied.*

Proof. Recall that $\mathcal{P}_{\mathcal{VC}}$ states that the $FIC_{\mathcal{VC}}$ is true for $V_1 \times V_2$ for any two virtually cyclic groups V_1 and V_2. Let V_1 and V_2 be two such groups then $V_1 \times V_2$ contains a free abelian normal subgroup H (on at most 2 generators) of finite index. Hence $V_1 \times V_2$ is a subgroup of $H \wr ((V_1 \times V_2)/H)$. Therefore by Lemma 2.1 it is enough to prove the $FIC_{\mathcal{VC}}$ for $H \wr ((V_1 \times V_2)/H)$.

If $V_1 \times V_2$ is virtually cyclic then there is nothing to prove. If H has rank 2 then applying Lemmas 2.7 and 2.10 we see that $\mathcal{P}_{\mathcal{VC}}$ is satisfied. Next we apply Lemma 2.3 to see that $\mathcal{P}^-_{\mathcal{VC}}$ is also satisfied. And there is nothing to prove for $\mathcal{P}^-_{\mathcal{FIN}}$ as product of two finite groups is finite. □

Lemma 2.12. *Assume that J is true for two groups G_1 and G_2 then J is true for the direct product $G_1 \times G_2$. Here $J = 1, 2, 3$ or 4 are as below.*

1. $FIC_{\mathcal{VC}}$. 2. $FIC^-_{\mathcal{VC}}$. 3. $FICwF_{\mathcal{VC}}$. 4. $FICwF^-_{\mathcal{VC}}$.

Proof. The proof is a combination of Lemma 2.11 and (1) of Lemma 2.2. □

Lemma 2.13. *The $FICwF_{\mathcal{VC}}$, $FICwF^-_{\mathcal{VC}}$ and $FICwF^-_{\mathcal{FIN}}$ are true for any virtually free group.*

Proof. We only prove the Lemma for the $FICwF_{\mathcal{VC}}$. The other two conclusions will follow using Lemmas 2.3 and 2.4.

Let Γ be a virtually free group and G be a free normal subgroup of Γ with F the finite quotient group. We can assume that G is nontrivial since the lemma is true for finite groups. Let F' be another finite group and denote by C the wreath product $F \wr F'$. Then we have the following inclusions.

$$\Gamma \wr F' < (G \wr F) \wr F' < G^{F\times F'} \wr C < (G \wr C) \times \cdots \times (G \wr C).$$

(See [13, Lemma 5.4] for the second inclusion and there are $|F \times F'|$ factors in the last term.) Therefore using Lemmas 2.1 and 2.12 we see that it is enough to prove the $FIC_{\mathcal{VC}}$ for $G \wr C$ for an arbitrary finite group C. Equivalently, we need to prove the $FICwF_{\mathcal{VC}}$ for G. If G is infinitely generated then let G be the direct limit of a directed system of finitely generated subgroups of G. Here the directed system is defined by the inclusion homomorphisms of subgroups. Hence by Lemma 2.5 we can assume that G is finitely generated. Therefore G is isomorphic to the fundamental group of an orientable 2-manifold M with boundary. Consequently, G is isomorphic to a subgroup of $\pi_1(M \cup_\partial M)$, where $M \cup_\partial M = S$ denotes the double of M. Again using Lemma 2.1 it is enough to prove the $FICwF_{\mathcal{VC}}$ for $\pi_1(S)$, where S is a closed orientable surface. Since G is nontrivial S has genus $\geqslant 1$. Now applying Lemmas 2.7 and 2.10 we complete the proof of the lemma. □

Lemma 2.14. *${}_w\mathcal{T}^-_{\mathcal{FIN}}$, ${}_{wt}\mathcal{T}_{\mathcal{VC}}$ and ${}_{wt}\mathcal{T}^-_{\mathcal{VC}}$ are satisfied.*

Proof. We check ${}_w\mathcal{T}^-_{\mathcal{FIN}}$ first. So let G be a group and $\mathcal{G}$ be a graph of finite groups with $\pi_1(\mathcal{G}) \simeq G$. If $\mathcal{G}$ is an infinite graph then we write $\mathcal{G}$ as the direct limit of the directed system of finite subgraphs $\mathcal{G}_i$ of $\mathcal{G}$. Then $\pi_1(\mathcal{G}) \simeq \lim_{i\to\infty} \pi_1(\mathcal{G}_i)$. Hence using Lemma 2.5 we can assume that $\mathcal{G}$ is finite. It is now well known that $\pi_1(\mathcal{G})$ contains a finitely generated free subgroup of finite index. See [14, Lemma 3.2]. ${}_w\mathcal{T}^-_{\mathcal{FIN}}$ now follows from Lemmas 2.13 and 2.4.

Next we prove ${}_{wt}\mathcal{T}_{\mathcal{VC}}$ and ${}_{wt}\mathcal{T}^-_{\mathcal{VC}}$.

Let $\mathcal{G}$ be a graph of groups with virtually cyclic vertex groups and trivial edge groups. As before we can assume that $\mathcal{G}$ is finite. Hence the group $\pi_1(\mathcal{G})$ is virtually free. This follows from Lemmas 2.8 and 2.9.

Therefore we can apply Lemmas 2.5 and 2.13 to complete the proof of Lemma 2.14. □

Proposition 2.1. *Assume that the J is true for two groups G_1 and G_2 then the J is true for the free product $G_1 * G_2$ also. Here $J = 1$ to 6 are as below*:

1. $FIC_{\mathcal{VC}}$. 2. $FIC^-_{\mathcal{VC}}$. 3. $FIC^-_{\mathcal{FIN}}$. 4. $FICwF_{\mathcal{VC}}$. 5. $FICwF^-_{\mathcal{VC}}$. 6. $FICwF^-_{\mathcal{FIN}}$.

Proof. The proof follows from Lemmas 2.11, 2.14 and [14, Lemma 6.3]. □

Lemma 2.15. *Let $1 \to K \to G \to N \to 1$ be an exact sequence of groups. Then the following hold for any equivariant homology theory and for $\mathcal{C} = \mathcal{FIN}$.*

1. *If the $FICwF_{\mathcal{C}}$ is true for K and the $FIC_{\mathcal{C}}$ is true for N then the $FIC_{\mathcal{C}}$ is true for G.*
2. *If the $FICwF_{\mathcal{C}}$ is true for K and N then the $FICwF_{\mathcal{C}}$ is true for G.*

Proof. Apply (2) and (3) of Lemma 2.2 and note that $\mathcal{P}_{\mathcal{C}}$ is satisfied. □

3. Braid groups

Let $\mathbb{C}^N$ be the N-dimensional complex space. A *hyperplane arrangement* in $\mathbb{C}^N$ is by definition a finite collection $\{V_1, V_2, \ldots, V_n\}$ of $(N-1)$-dimensional linear subspaces of $\mathbb{C}^N$.

Now we recall the definition of a fiber-type hyperplane arrangement from [11, p. 162]. Let us denote by $\mathcal{V}_n$ the arrangement $\{V_1, V_2, \ldots, V_n\}$ in $\mathcal{C}^N$. $\mathcal{V}_n$ is called *strictly linearly fibered* if after a suitable linear change of coordinates, the restriction of the projection of $\mathbb{C}^N - \bigcup_{i=1}^n V_i$ to the first $(N-1)$ coordinates is a fiber bundle projection whose base space is the complement of an arrangement $\mathcal{W}_{n-1}$ in $\mathbb{C}^{N-1}$ and whose fiber is the complex plane minus finitely many points. By definition the arrangement 0 in $\mathbb{C}$ is fiber-type and $\mathcal{V}_n$ is defined to be *fiber-type* if $\mathcal{V}_n$ is strictly linearly fibered and $\mathcal{W}_{n-1}$ is of fiber type. It follows by a repeated application of the homotopy exact sequence for fibration that the complement $\mathbb{C}^N - \bigcup_{i=1}^n V_i$ is aspherical and hence the fundamental group is torsion free.

Lemma 3.1. *([7, Theorem 5.3]) $\pi_1(\mathbb{C}^N - \bigcup_{i=1}^n V_i)$ is a strongly poly-free group.*

Now recall that the pure braid group PB_n on n strings is by definition $\pi_1(\mathbb{C}^{n+1} - \bigcup_{i,j} V_{ij})$ where V_{ij} is the hyperplane $x_i = x_j$ for $i < j$ and x_i's being the coordinates in $\mathbb{C}^{n+1}$. One can show that $\{V_{ij}\}$ is a fiber-type arrangement and hence PB_n is a strongly poly-free group. See [2, Theorem 2.1].

The full braid group B_n is by definition $\pi_1((\mathbb{C}^{n+1} - \bigcup_{i,j} V_{ij})/S_{n+1})$ where the symmetric group S_{n+1} on $(n+1)$-symbols acts on $\mathbb{C}^{n+1} - \bigcup_{i,j} V_{ij}$ by permuting the coordinates. This action is free and therefore PB_n is a normal subgroup of B_n with quotient S_{n+1}.

Recall that in [2] we proved the following.

Theorem 3.1. *([2, Theorem 1.3 and Corollary 1.4]) Let Γ be the fundamental group of a fiber-type hyperplane arrangement complement or more generally a strongly poly-free group. Then $Wh(\Gamma) = \tilde{K}_0(\mathbb{Z}\Gamma) = K_i(\mathbb{Z}\Gamma) = 0$ for $i < 0$.*

Theorem 3.1 and an application of the Rothenberg's exact sequence show the following. See [12, 17.2].

Lemma 3.2. *Let Γ be as in Theorem* 3.1 *then $L_n^{\langle -\infty \rangle}(\mathbb{Z}\Gamma) \simeq L_n^h(\mathbb{Z}\Gamma) \simeq L_n^s(\mathbb{Z}\Gamma)$.*

In the situation of Γ as in Theorem 3.1, Lemma 3.2 shows that the 2-torsions which appear in the three surgery groups are isomorphic.

4. Proof of Theorem 1.3

Proof of Theorem 1.3 (1) and (i). (Virtually cyclic groups). Since $\mathcal{FICWF}$ ($\underline{\mathcal{FICWF}}$) contains the discrete cocompact subgroups of (linear) Lie groups with finitely many components it follows that finite groups and the infinite cyclic group belong to $\mathcal{FICWF}$ ($\underline{\mathcal{FICWF}}$). Next apply the 'polycyclic extension' ('extension') condition to complete the proof of (1) ((i)).

(2) and (ii). (Free groups and abelian groups). At first note that a countable infinitely generated group is the direct limit of the directed system of its finitely generated subgroups.

Now using (1) ((i)) and the 'free product' condition we get that finitely generated free groups belong to $\underline{\mathcal{FICWF}}$ ($\mathcal{FICWF}$) and since $\underline{\mathcal{FICWF}}$ ($\mathcal{FICWF}$) has the property 'direct limit' the proof follows for infinitely generated free groups.

Using (1) ((i)) and the 'extension' ('polycyclic extension') condition we see that finitely generated abelian groups belong to $\underline{\mathcal{FICWF}}$ ($\mathcal{FICWF}$). Therefore countable abelian groups belong to $\underline{\mathcal{FICWF}}$ ($\mathcal{FICWF}$) by the 'direct limit' condition.

(3). (Poly-free groups). The proof is by induction on the index of the poly-free group. If $n = 1$ then G is free and hence $G \in \underline{\mathcal{FICWF}}$ (apply (2)). So assume that the poly-free groups of index $\leqslant n-1$ belong to $\underline{\mathcal{FICWF}}$ and let G has an index n filtration. Note that G_{n-1} is a poly-free group of index $n-1$ and G/G_{n-1} is a free group. Since $\underline{\mathcal{FICWF}}$ is closed under extensions we can now apply (2) and the induction hypothesis to show that $G \in \underline{\mathcal{FICWF}}$.

(4) and (5). (Strongly poly-free and the full braid groups). Recall from Section 3 that pure braid groups are strongly poly-free and strongly poly-free groups are poly-free. Also the full braid group B_n contains the pure braid group PB_n as a subgroup of finite index. Hence using (2) and since $\underline{\mathcal{FICWF}}$ is closed under extensions the proofs of (4) and (5) are complete.

(6). (Cocompact discrete subgroups of Lie groups with finitely many components). Let Γ be a cocompact discrete subgroup of a Lie group with finitely many components. Then following the steps in the proof of [15, 2(a) of Theorem 2.2] or of [4, Theorem 2.1] we have the following three exact sequences.

$$1 \to F \to \Gamma \to \Gamma' \to 1,$$
$$1 \to \Gamma_R \to \Gamma' \to \Gamma_S \to 1,$$
$$1 \to \Gamma_H \to \Gamma_S \to \Gamma_L \to 1.$$

Where F is finite, Γ_R is virtually poly-$\mathbb{Z}$, Γ_H is virtually finitely generated abelian and Γ_L is a cocompact discrete subgroup of a Linear Lie group with finitely many components. Now note that finitely generated free abelian groups and poly-$\mathbb{Z}$ groups (since they are also poly-free) belong to $\underline{\mathcal{FICWF}}$. Therefore, we can again apply the hypothesis that $\underline{\mathcal{FICWF}}$ is closed under extensions and use the above three exact sequences to complete the proof of (6).

(7). (Groups whoes some derived subgroup belongs to $\underline{\mathcal{FICWF}}$). Let Γ be a group so that $\Gamma^{(n)} \in \underline{\mathcal{FICWF}}$ for some n. Using the extension condition it is enough to show that $\Gamma/\Gamma^{(n)} \in \underline{\mathcal{FICWF}}$, that is we need to show that $\underline{\mathcal{FICWF}}$ contains the solvable groups.

So let Γ be a solvable group. Using the 'direct limit' condition in the definition of $\underline{\mathcal{FICWF}}$ we can assume that Γ is finitely generated, for any countable infinitely generated group is the direct limit of the directed system of its finitely generated subgroups.

We say that Γ is n-step solvable if $\Gamma^{(n+1)} = (1)$ and $\Gamma^{(n)} \neq (1)$. The proof is by induction on n. Since countable abelian groups belong to $\underline{\mathcal{FICWF}}$ (by (2)), the induction starts.

So assume that a finitely generated k-step solvable group for $k \leqslant n-1$ belongs to $\underline{\mathcal{FICWF}}$ and Γ is n-step solvable.

We have the following exact sequence.

$$1 \to \Gamma^{(n)} \to \Gamma \to \Gamma/\Gamma^{(n)} \to 1.$$

Note that $\Gamma^{(n)}$ is abelian and $\Gamma/\Gamma^{(n)}$ is $(n-1)$-step solvable. Using the 'extension' condition and the induction hypothesis we complete the proof.

(iii). (Groups appearing in (6)). This follows from the definition of $\mathcal{FICWF}$.

(iv). (Virtually poly-cyclic groups). Using the 'polycyclic extension' condition and the following lemma we complete the proof.

Lemma 4.1. *Let G be a virtually polycyclic group. Then G contains a finite normal subgroup so that the quotient is a discrete cocompact subgroup of a Lie group with finitely many components.*

Proof. See [16, Theorem 3, Remark 4 on p. 200]. □

5. Proofs of Theorems 1.1 and 1.2

Proof of Theorem 1.1. The proof of the theorem follows from the following two observations.

• The *FICwF*$^-_{\mathcal{FIN}}$ (*FICwF*$_{\mathcal{VC}}$) for the groups appearing in (1) ((i)) of Definition 1.1 is true. This follows from Lemma 2.10.
• The statement *'The FICwF*$^-_{\mathcal{FIN}}$ *(FICwF*$_{\mathcal{VC}}$*) is satisfied'* is closed under the operations described in (2) to (5) ((ii) to (iv)) of Definition 1.1. This follows from Theorem 1.2. □

Proof of Theorem 1.2. For $\mathcal{C}$: (2), (3), (4) and (5) follows from Lemma 2.1, Proposition 2.1, Lemma 2.5 and Lemma 2.15 respectively.

For $\mathcal{D}$: (ii) follows from Lemma 2.1, Proposition 2.1 and Lemma 2.5. (iii) follows from Lemma 2.12. (iv) follows using (3) of Lemma 2.2, Lemma 4.1 and Lemma 2.10. To apply (3) of Lemma 2.2 we will need the fact that if a group contains a finite normal subgroup with virtually cyclic quotient then the group is virtually cyclic. This follows from [14, Lemma 6.1]. □

Remark 5.1. We finally remark that in our applications we used a weaker version of the 'direct limit' condition. That is, when the maps in the directed system of groups are injective homomorphism. In fact we needed only a 'filtered system'. But in the following example we need the general 'direct limit' condition. Let M be a noncompact manifold. Assume that the *FICwF*$_{\mathcal{VC}}$ (*FICwF*$^-_{\mathcal{FIN}}$) is true for the fundamental group of any compact submanifold of M, then the 'direct limit' condition imply that the *FICwF*$_{\mathcal{VC}}$ (*FICwF*$^-_{\mathcal{FIN}}$) is true for $\pi_1(M)$. To prove the *FICwF*$_{\mathcal{VC}}$ or *FICwF*$^-_{\mathcal{FIN}}$ for 3-manifold groups we will need the general 'direct limit' condition as mentioned in [15, Theorem 2.2]. This fact was very crucial in [13] even to prove the conjecture for a certain class of compact 3-manifolds.

Acknowledgement

The author is grateful to the referee for his constructive comments and suggestions which helped to improve this article.

References

[1] A. Bartels, W. Lück, Isomorphism Conjecture for homotopy K-theory and groups acting on trees, J. Pure Appl. Algebra 205 (2006) 660–696.
[2] C.S. Aravinda, F.T. Farrell, S.K. Roushon, Algebraic K-theory of pure braid groups, Asian J. Math. 4 (2000) 337–344.
[3] W. Dicks, M.J. Dunwoody, Groups Acting on Graphs, Cambridge Stud. Adv. Math., vol. 17, Cambridge University Press, 1989.
[4] F.T. Farrell, L.E. Jones, Isomorphism conjectures in algebraic K-theory, J. Amer. Math. Soc. 6 (1993) 249–297.
[5] F.T. Farrell, L.E. Jones, W. Lück, A caveat on the isomorphism conjecture in L-theory, Forum Math. 14 (3) (2002) 413–418.
[6] F.T. Farrell, P.A. Linnell, K-theory of solvable groups, Proc. London Math. Soc. (3) 87 (2003) 309–336.
[7] F.T. Farrell, Sayed K. Roushon, The Whitehead groups of braid groups vanish, Int. Math. Res. Not. 10 (2000) 515–526.
[8] J.A. Hillman, S.K. Roushon, Surgery on $\widetilde{\mathbb{SL}} \times \mathbb{E}^n$-manifolds, Canad. Math. Bull., in press, math.GT/0410615.
[9] W. Lück, H. Reich, The Baum–Connes and the Farrell–Jones Conjectures in K- and L-theory, in: E.M. Friedlander, D.R. Grayson (Eds.), Handbook of K-Theory, vol. 2, Springer, 2005, pp. 703–842.
[10] W. Lück, R. Stamm, Computations of K- and L-theory of cocompact planar groups, K-Theory 21 (2000) 249–292.
[11] Peter Orlik, Hiroaki Terao, Arrangements of Hyperplanes, Springer-Verlag, Berlin, New York, 1992.
[12] A. Ranicki, Lower K- and L-Theory, Cambridge University Press, Cambridge, 1992.
[13] S.K. Roushon, The Farrell–Jones isomorphism conjecture for 3-manifold groups, J. K-Theory 1 (2008) 49–82.
[14] S.K. Roushon, On the isomorphism conjecture for groups acting on trees, math.KT/0510297v5.
[15] S.K. Roushon, Algebraic K-theory of groups wreath product with finite groups, Topology Appl. 154 (2007) 1921–1930.
[16] B. Wilking, Rigidity of group actions on solvable Lie groups, Math. Ann. 317 (2) (2000) 195–237.