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Subdivisions with infinitely supported mask ☆

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Abstract

In this paper we investigate the convergence of subdivision schemes associated with masks being polynomially decay sequences. Two-scale vector refinement equations are the form

$$\phi(x) = \sum_{\alpha \in \mathbb{Z}} a(\alpha)\phi(2x - \alpha), \quad x \in \mathbb{R},$$

where the vector of functions $\phi = (\phi_1, \ldots, \phi_r)^{\mathrm{T}}$ is in $(L_2(\mathbb{R}))^r$ and $a =: (a(\alpha))_{\alpha \in \mathbb{Z}}$ is polynomially decay sequence of $r \times r$ matrices called refinement mask. Associated with the mask a is a linear operator on $(L_2(\mathbb{R}))^r$ given by

$$Q_a f(x) := \sum_{\alpha \in \mathbb{Z}} a(\alpha) f(2x - \alpha), \quad x \in \mathbb{R}, \quad f = (f_1, \ldots, f_r)^{\mathrm{T}} \in (L_2(\mathbb{R}))^r.$$

By using same methods in [B. Han, R. Q. Jia, Characterization of Riesz bases of wavelets generated from multiresolution analysis, manuscript]; [B. Han, Refinable functions and cascade algorithms in weighted spaces with infinitely supported masks, manuscript]; [R.Q. Jia, Q.T. Jiang, Z.W. Shen, Convergence of cascade algorithms associated with nonhomogeneous refinement equations, Proc. Amer. Math. Soc. 129 (2001) 415–427]; [R.Q. Jia, Convergence of vector subdivision schemes and construction of biorthogonal multiple wavelets, in: Advances in Wavelet, Hong Kong,1997, Springer, Singapore, 1998, pp. 199–227], a characterization of convergence of the sequences $(Q_a^n f)_{n=1,2,\ldots}$ in the L_2-norm is given, which extends the main results in [R.Q. Jia, S.D. Riemenschneider, D.X. Zhou, Vector subdivision schemes and multiple wavelets, Math. Comp. 67 (1998) 1533–1563] on convergence of the subdivision schemes associated with a finitely supported mask to the case in which mask a is polynomially decay sequence. As an application, we also obtain a characterization of smoothness of solutions of the refinement equation mentioned above for the case $r = 1$.

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1. Introduction

Two-scale refinement equation with infinitely supported mask a is defined by

$$\varphi(x)=\sum_{\alpha\in\mathbb{Z}}a(\alpha)\varphi(2x-\alpha),\quad x\in\mathbb{R}, \tag{1.1}$$

where a is absolute summable sequence on $\mathbb{Z}$. When mask a is finitely supported sequences, the study of solutions of refinement equation (1.1) called refinable functions have been well understood. However, in some applications of signal proceeding, one needs to find solutions of refinement equation (1.1) with infinitely supported masks, such as the band limited wavelets which are generated by refinable functions that satisfy refinement equation (1.1) with infinitely supported masks [9]. Not much is known about refinement equation with such masks. Cohen and Daubechies [3] gave the first analysis when mask a decays exponentially fast. For the investigation of Riesz bases of wavelet generated from multiresolution analysis, Han and Jia [10] and Han [13] considered refinement equations with masks having exponentially decay sequences, respectively. In this case, the spectral theory of compact operator is involved.

Let $(L_2(\mathbb{R}))^r$ denote the linear space of all square integrable complex-valued functions on $\mathbb{R}$. The norm on $(L_2(\mathbb{R}))^r$ is given by

$$\|f\|_2:=\left(\int_{\mathbb{R}}|f(x)|^2\,\mathrm{d}x\right)^{1/2}.$$

By $(L_2(\mathbb{R}))^r$, we denote the linear space of all vector of functions $f=(f_1,\ldots,f_r)^{\mathrm{T}}$ such that $f_1,\ldots,f_r\in L_2(\mathbb{R})$. The norm on $(L_2(\mathbb{R}))^r$ is defined by

$$\|f\|_2:=\left(\sum_{j=1}^{r}\|f_j\|_2^2\right)^{1/2}.$$

By $L_{2,c}(\mathbb{R})$ we denote the linear space of all compactly supported functions on $L_2(\mathbb{R})$.

The Fourier analysis is an indispensable tool in our study. The Fourier transform of a vector of functions in $(L_1(\mathbb{R}))^r$ is defined by

$$\hat{f}(\xi):=\int_{\mathbb{R}}f(x)\mathrm{e}^{\mathrm{i}x\xi}\,\mathrm{d}x,\quad \xi\in\mathbb{R},$$

where $(L_1(\mathbb{R}))^r$ denotes the space of all Lebesgue integrable vector of functions on $\mathbb{R}$. The Fourier transform can be naturally extended to functions in $L_2(\mathbb{R})$. Similarly, if c is a (complex-valued) summable sequence on $\mathbb{Z}$, then its Fourier series is defined by

$$\hat{c}(\xi):=\sum_{\alpha\in\mathbb{Z}}c(\alpha)\mathrm{e}^{-\mathrm{i}\alpha\cdot\xi},\quad \xi\in\mathbb{R}.$$

Evidently, $\hat{c}$ is a 2π-periodic continuous function on $\mathbb{R}$. When c is finitely supported, $\hat{c}$ is a trigonometric polynomial. We call $\hat{c}$, the symbol of c. We also can define the Fourier series for c to be vector sequences or matrix sequences in similar ways.

Multi-wavelets have been well developed in wavelet analysis since the beginning of 1990's. The corresponding refinement equations are defined by

$$\phi(x)=\sum_{\alpha\in\mathbb{Z}}a(\alpha)\phi(2x-\alpha),\quad x\in\mathbb{R}, \tag{1.2}$$

where $\phi=(\phi_1,\ldots,\phi_r)^{\mathrm{T}}$ is the unknown, a is an infinitely supported refinement mask such that each $a(\alpha)$ is an $r\times r$ complex number matrix. Eq. (1.2) is called a homogeneous vector refinement equation [1,4,11,8,18]. Refinable function vectors with infinitely supported masks have been studied in [26]. To study Riesz bases of wavelet generated by refinable function vectors, Jia [15] also investigated vector refinement equations with exponentially decaying masks.

Suppose φ is compactly supported vector of functions in $(L_2(\mathbb{R}))^r$, satisfies $\hat{\varphi}(0) \neq 0$ and span $\{\hat{\varphi}(2\pi\beta) : \beta \in \mathbb{Z}\} = \mathbb{C}^r$. If φ satisfies refinement equation (1.1) with mask a being finitely supported, then 1 is a simple eigenvalue of $M = \sum_{\alpha\in\mathbb{Z}} a(\alpha)/2$ and other eigenvalues of M are less than 1 in modulus (see [17]). These conditions are called Eigenvalue condition. Throughout this paper we assume that this condition is satisfied. In this case, matrix M has the following form:

$$M = \begin{pmatrix} 1 & 0 \\ 0 & \Lambda \end{pmatrix}, \quad \text{where } \lim_{n\to\infty} \Lambda^n = 0.$$

For $j = 1, \ldots, r$, we use e_j to denote the *jth* column of the $r \times r$ identity matrix. Obviously, $e_1^{\mathrm{T}} M = e_1^{\mathrm{T}}$ and $M e_1 = e_1$.

Suppose $a \in (\ell_1(\mathbb{Z}))^{r\times r}$, the linear space of all sequences of $r \times r$ matrices such that its each entry absolutely converges on $\mathbb{Z}$. Let Q_a be the bounded linear operator on $(L_2(\mathbb{R}))^r$ given by

$$Q_a \phi(x) = \sum_{\alpha\in\mathbb{Z}} a(\alpha)\phi(2x-\alpha), \quad x \in \mathbb{R}, \quad \phi \in (L_p(\mathbb{R}))^r. \tag{1.3}$$

Let ϕ_0 be a vector of compactly supported functions in $(L_2(\mathbb{R}))^r$. Consider the iteration scheme $\phi_n := Q_a^n \phi_0, n = 1, 2, \ldots$. This iteration scheme is called the vector cascade algorithm or vector subdivision schemes associated with mask a. A vector $\phi = (\phi_1, \ldots, \phi_r)^{\mathrm{T}} \in (L_{2,c}(\mathbb{R}))^r$ is said to satisfy Strang–Fix conditions of order 1 if

$$e_1^{\mathrm{T}} \hat{\phi}(0) = 1 \quad \text{and} \quad e_1^{\mathrm{T}} \hat{\phi}(2\pi\beta) = 0, \quad \forall \beta \in \mathbb{Z}\backslash\{0\}. \tag{1.4}$$

By using the Possion summation formula, we see that (1.4) is equivalent to the following condition:

$$e_1^{\mathrm{T}} \sum_{\alpha\in\mathbb{Z}} \phi(\cdot - \alpha) = 1, \tag{1.5}$$

which is also called the moment conditions of order 1.

We say that the (vector) subdivision scheme associated with a converges in the L_2-norm, if there exists a vector $\phi \in (L_2(\mathbb{R}))^r$ such that for any $\phi_0 \in (L_{2,c}(\mathbb{R}))^r$ satisfying the moment conditions of order 1,

$$\lim_{n\to\infty} \|Q_a^n \phi_0 - \phi\|_2 = 0.$$

If this is the case, then ϕ is a solution of the refinement equation (1.2) in $(L_2(\mathbb{R}))^r$.

The convergence of subdivision schemes is fundamental to wavelet theory and subdivision. Subdivision schemes have been studied mainly for the case in which the mask a is finitely supported. The L_2-convergence and L_p-convergence $(1 \leqslant p \leqslant \infty)$ of subdivision schemes were investigated in many papers such as [16,14,17,24]. When $r = 1$, Strang and Nguyen [25] studied L_2-convergence of subdivision schemes, Jia [14] gave a characterization for the L_p-convergence of a subdivision scheme for $1 \leqslant p \leqslant \infty$. In the multidimensional case, Lawton et al. [20] studied the convergence of subdivision schemes in L_2-norm. For the vector case $(r > 1)$, Shen [24] gave a complete characterization on L_2-convergence of subdivision scheme, Jia et al. [17] obtained a characterization for the L_p-convergence of subdivision schemes when $1 \leqslant p \leqslant \infty$. For general setting, Han [8], Zhou [29], and Chen et al. [1] also investigated the convergence of subdivision schemes in Sobolev spaces, respectively.

In electrical engineering, infinitely supported masks are called infinite impulse response filters [9]. Due to some desirable properties, infinitely supported masks, including masks with exponential decay and masks for fractional splines [27], are of interest in the area of digital signal processing in electrical engineering [2,3,5,12,22].

The purpose of this paper is to investigate vector refinement equation with mask being a polynomially decay sequence. Infinitely supported sequences decaying polynomially fast were used in [21] to characterize finitely generated shift-invariant spaces whose generators decay in a polynomial rate. Thus sequences are closely related to certain Banach algebra. In this paper, we will investigate the solutions of refinement equations (1.2) with mask a being polynomial decay rate. To study the L_2-solution of Eq. (1.2), we will provide a necessary and sufficient conditions for the convergence of subdivision schemes with this mask in $(L_2(\mathbb{R}))^r$. Consequently, if this subdivision schemes with $r = 1$ converges in L_2-norm, then its refinable function ϕ must belong to $H^\eta(\mathbb{R})$ for some $\eta > 0$, where $H^\eta(\mathbb{R})$ denotes Sobolev space for $\eta > 0$. Compared with the well-developed smoothness analysis of solutions of (1.2) with mask a being finitely supported, the situation for mask a being infinitely supported is different and their smoothness analysis of solution of (1.2) is much less so far. Therefore, these theories will provide some new choices in wavelets theory and applications.

2. Some elementary notations and lemmas

For some $m \in \mathbb{Z}_+$, we denote B_m the linear space of all sequence u on $\mathbb{Z}$, for which

$$\|u\|_{B_m} := \sum_{\alpha\in\mathbb{Z}} |u(\alpha)|(1+|\alpha|)^m < \infty.$$

Equipped with the norm $\|\cdot\|_{B_m}$, B_m becomes a Banach space. Let A_m be the set of sequence of symbols of B_m, with the norm $\|\cdot\|_{B_m}$ and the usual pointwise operator, A_m becomes a Banach algebra (see [21]). In [21], a similar space was used by Lei et al. to investigate finitely generated shift-invariant spaces whose generators decay in a polynomial rate. By B_m^r we denote the linear space of all vector sequences $u(\alpha) = (u_1(\alpha), \ldots, u_r(\alpha))^{\mathrm{T}}$ such that $u_1, \ldots, u_r \in B_m$. The norm on B_m^r is defined by

$$\|u\|_{B_m^r} := \max_{1\leqslant j\leqslant r} \|u_j\|_{B_m}.$$

By $B_m^{r\times r}$ we denote the linear space of all matrix sequences $u(\alpha) = (u_{j,k}(\alpha))_{1\leqslant j,k\leqslant r}$ such that $u_{j,k} \in B_m$ $j, k = 1, \ldots, r$. The norm on $B_m^{r\times r}$ is defined by

$$\|u\|_{B_m^{r\times r}} := \max_{1\leqslant j,k\leqslant r} \|u_{j,k}\|_{B_m}.$$

In the study of vector refinement equation, the Kronecker product of two matrices is a useful tool (see [18,16,15,28]). The Kronecker product was used by in [6] in the study of spectral radius of a bi-infinite periodic and slanted matrix. It was used in [28] in the work on joint spectral radius of a finite collection of matrices. Let us mention some useful properties of the Kronecker product from [19]. Let $A = (a_{i,j})_{1\leqslant i\leqslant m, 1\leqslant j\leqslant n}$, and $B = (b_{i,j})_{1\leqslant i\leqslant k, 1\leqslant j\leqslant l}$, be two matrices. The (right) Kronecker product of A and B, written $A \otimes B$, is defined to be the block matrix

$$A \otimes B := \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{pmatrix}.$$

For three matrices A, B and C of the same type, we have

$$(A + B) \otimes C = (A \otimes C) + (B \otimes C),$$

$$A \otimes (B + C) = (A \otimes B) + (A \otimes C).$$

If A, B, C, D are four matrices such that the products AC and BD are well defined, then

$$(A \otimes B)(C \otimes D) = (AC) \otimes (BD).$$

If $\lambda_1, \ldots, \lambda_r$ are the eigenvalues of an $r \times r$ matrix A and $\mu_1, \ldots, \mu_r$ are the eigenvalues of an $r \times r$ matrix B, it follows from [19] that the eigenvalues of $A \otimes B$ are $\lambda_j\mu_k$, $j, k = 1, \ldots, r$.

For two functions f, g in $L_2(\mathbb{R})$, $f \odot g$ is defined as follows:

$$f \odot g(x) := \int_R f(x+y)\overline{g(y)}\,\mathrm{d}y, \quad x \in \mathbb{R}.$$

It is easily seen that $f \odot g$ lies in $C_0(\mathbb{R})$, the space of continuous functions on $\mathbb{R}$ which vanish at ∞. Evidently

$$\|f \odot g\|_\infty \leqslant \|f\|_2\|g\|_2. \tag{2.1}$$

Moreover $(f \odot f)(0) = \|f\|_2^2$.

Let $l(\mathbb{Z})$ denote the linear space of all complex-valued sequences on $\mathbb{Z}$ and let $l_0(\mathbb{Z})$ denote the linear space of all finitely supported sequences on $\mathbb{Z}$.

Let $l_1(\mathbb{Z})$ denote the linear space of all sequences u for which $\|u\|_1<\infty$, the norm $\|\cdot\|_1$ on $l_1(\mathbb{Z})$ is defined by

$$\|u\|_1:=\sum_{\alpha\in\mathbb{Z}}|u(\alpha)|$$

and let $l_\infty(\mathbb{Z})$ denote the linear space of all sequences u for which $\|u\|_\infty$ to be the supremum of $|u|$ on $\mathbb{Z}$. It is well known that equipped with the norms $\|\cdot\|_1$ and $\|\cdot\|_\infty$, the linear spaces $l_1(\mathbb{Z})$ and $l_\infty(\mathbb{Z})$ become Banach spaces. By $(l_1(\mathbb{Z}))^r$ and $(l_\infty(\mathbb{Z}))^r$, we denote the linear spaces of all vector sequences such that $\|u\|_1:=\sum_{j=1}^r\|u_j\|_1<\infty$, and $\|u\|_\infty=\max_{1\leqslant j\leqslant r}\|u_j\|_\infty<\infty$, where $u=(u_1,\ldots,u_r)^{\mathrm{T}}$. Similarly, we define $(l_0(\mathbb{Z}))^r$, $(l(\mathbb{Z}))^r$, $(l_0(\mathbb{Z}))^{r\times r}$, $(l(\mathbb{Z}))^{r\times r}$, and $(l_1(\mathbb{Z}))^{r\times r}$.

For a matrix $A=(a_{ij})_{1\leqslant i,j\leqslant r}$, the vector

$$(a_{11},\ldots,a_{r1},a_{12},\ldots,a_{r2},\ldots,a_{1r},\ldots,a_{rr})^{\mathrm{T}}$$

is said to be the vec-function of A and written as $\mathrm{vec}A$. Suppose A, X and B are three $r\times r$ matrices. Then we have (see [13])

$$\mathrm{vec}(AXB)=(B^{\mathrm{T}}\otimes A)\mathrm{vec}X. \tag{2.2}$$

Suppose $\phi=(\phi_1,\ldots,\phi_r)^{\mathrm{T}}$ and $\psi=(\psi_1,\ldots,\psi_r)^{\mathrm{T}}$ belong to $(L_2(\mathbb{R}))^r$, let $\phi\odot\psi^{\mathrm{T}}$ be defined as follows:

$$\phi\odot\psi^{\mathrm{T}}:=\begin{pmatrix}\phi_1\odot\psi_1 & \phi_1\odot\psi_2 & \cdots & \phi_1\odot\psi_r\\ \phi_2\odot\psi_1 & \phi_2\odot\psi_2 & \cdots & \phi_2\odot\psi_r\\ \vdots & \vdots & \ddots & \vdots\\ \phi_r\odot\psi_1 & \phi_r\odot\psi_2 & \cdots & \phi_r\odot\psi_r\end{pmatrix}.$$

By (2.1) we have

$$\|\mathrm{vec}(\phi\odot\psi^{\mathrm{T}})\|_\infty\leqslant\|\phi\|_2\|\psi\|_2 \tag{2.3}$$

and

$$|\mathrm{vec}(\phi\odot\phi^{\mathrm{T}})(0)|=\sum_{j=1}^r\sum_{k=1}^r|\phi_j\odot\phi_k(0)|\geqslant\sum_{j=1}^r|\phi_j\odot\phi_j(0)|=\sum_{j=1}^r\|\phi_j\|_2^2.$$

Consequently

$$|\mathrm{vec}(\phi\odot\phi^{\mathrm{T}})(0)|\geqslant\|\phi\|_2^2. \tag{2.4}$$

Suppose $a\in B_m^{r\times r}$, for some $m\in\mathbb{Z}_+$. Let b be defined as follows:

$$b(\alpha):=\sum_{\alpha\in\mathbb{Z}}\overline{a(\beta)}\otimes a(\alpha+\beta)/2,\quad \alpha\in\mathbb{Z}. \tag{2.5}$$

Then b lies in $B_m^{r^2\times r^2}$. Let T_b be the transition operator on $B_m^{r^2}$ defined by

$$T_bu(\alpha):=\sum_{\beta\in\mathbb{Z}}b(2\alpha-\beta)u(\beta),\quad \alpha\in\mathbb{Z},\quad u\in B_m^{r^2}. \tag{2.6}$$

It is known that the transition operator plays an important role in the study of refinement equation (see [1,7,10,8,18,16,15]).

Following Lemmas 2.1 and 2.2 show that the transition operator T_b is a bounded and compact operators on $B_m^{r^2}$.

Lemma 2.1. *Let $a\in B_m^{r\times r}$, for some $m\in\mathbb{Z}_+$. Then the transition operator T_a is a bounded operator on B_m^r. Moreover,*

$$\|T_au\|_{B_m^r}\leqslant\|a\|_{B_m^{r\times r}}\|u\|_{B_m^r}. \tag{2.7}$$

Proof. First note that for any $\alpha, \beta \in \mathbb{Z}$,

$$1+|\alpha| \leqslant 1+|\alpha+\beta|+|\beta| \leqslant (1+|\alpha+\beta|)(1+|\beta|).$$

Consequently,

$$(1+|\alpha|)^m \leqslant (1+|\alpha+\beta|)^m(1+|\beta|)^m.$$

It follows that

$$\begin{aligned}\|T_a u\|_{B_m^r} &= \sum_{\alpha\in\mathbb{Z}} |T_a u(\alpha)|(1+|\alpha|)^m = \sum_{\alpha\in\mathbb{Z}} \Big|\sum_{\beta\in\mathbb{Z}} a(2\alpha-\beta)u(\beta)\Big|(1+|\alpha|)^m \\ &\leqslant \|a\|_{B_m^{r\times r}}\|u\|_{B_m^r}. \qquad \square\end{aligned}$$

Lemma 2.2. *Let $a \in B_m^{r\times r}$, for some $m \in \mathbb{Z}_+$. Then the transition operator T_a is a compact operator on B_m^r.*

Proof. If a is finitely supported, then T_a is the limit of a sequence of finite-rank operator, hence T_a is a compact operator. In general, for $L = 1, 2, \ldots$, let a_L be the sequences in B_m^r defined by $a_L(\alpha) = a(\alpha)$ for $|\alpha| \leqslant L$, and $a_L(\alpha) = 0$ for $|\alpha| > L$. Each a_L is a finitely supported, then each T_{a_L} is a compact operator for $L = 1, 2, \ldots$. If we prove that

$$\lim_{L\to\infty} \|T_{a_L} - T_a\| = 0.$$

Then T_a is a compact operator. By the definition of a_L,

$$(T_a - T_{a_L})u(\alpha)(1+|\alpha|)^m = \sum_{\beta\in\mathbb{Z}, |2\alpha-\beta|>L} a(2\alpha-\beta)u(\beta)(1+|\alpha|)^m.$$

Hence

$$\begin{aligned}\|(T_a - T_{a_L})u\|_{B_m^r} &\leqslant \sum_{\alpha\in\mathbb{Z}} \sum_{\beta\in\mathbb{Z}, |2\alpha-\beta|>L} |a(2\alpha-\beta)u(\beta)|(1+|\alpha|)^m \\ &\leqslant \sum_{|\alpha|>L} \sum_{j=1}^{r}\sum_{k=1}^{r} |a_{jk}(\alpha)|(1+|\alpha|)^m \|u\|_{B_m^r},\end{aligned}$$

where $a(\alpha) = (a_{jk}(\alpha))_{1\leqslant j,k\leqslant r}$. Therefore

$$\|T_a - T_{a_L}\| \leqslant \sum_{|\alpha|>L} \sum_{j=1}^{r}\sum_{k=1}^{r} |a_{jk}(\alpha)|(1+|\alpha|)^m.$$

Which implies that T_a is a compact operator.

Since T_b is a compact linear operator on $B_m^{r^2\times r^2}$. The Riesz Theory of compact operators (see [23, Chapter 3]) says that the spectrum of T_b is a countable compact set whose only possible limit point is 0. In particular, there exists an eigenvalue σ of T_b such that $\rho(T_b) = |\sigma|$, where $\rho(T_b)$ denotes the spectral radius of T_b. It follows from [10] that if $(T_n)_{n=1,2,\ldots}$ is a sequence of bounded linear operators on Banach space $B_m^{r^2\times r^2}$ such that $\|T_n - T\| \to 0$ as $n \to \infty$, then $\lim_{n\to\infty} \rho(T_n) = \rho(T)$.

By (2.5) we have

$$\sum_{\alpha\in\mathbb{Z}} b(\alpha)/2 = \left(\sum_{\beta\in\mathbb{Z}} \overline{a(\beta)}/2\right) \otimes \left(\sum_{\alpha\in\mathbb{Z}} a(\alpha+\beta)/2\right) = \overline{M} \otimes M$$

and

$$(\overline{e_1^{\mathrm{T}}} \otimes e_1^{\mathrm{T}})(\overline{M} \otimes M) = (\overline{e_1^{\mathrm{T}}}\,\overline{M}) \otimes (e_1^{\mathrm{T}} M) = \overline{e_1^{\mathrm{T}}} \otimes e_1^{\mathrm{T}}.$$

By the above discussions, we know that the matrix $\sum_{\alpha\in\mathbb{Z}} b(\alpha)/2$ has a simple eigenvalue 1, $\overline{e_1^{\mathrm{T}}} \otimes e_1^{\mathrm{T}}$ is a left eigenvector of $\sum_{\alpha\in\mathbb{Z}} b(\alpha)/2$ corresponding to eigenvalue 1, and other eigenvalues of $\sum_{\alpha\in\mathbb{Z}} b(\alpha)/2$ are less than 1 in modulus. Let $a \in B_m^{r\times r}$, we say that a satisfies the basic sum rule, if

$$e_1^{\mathrm{T}} \sum_{\alpha\in\mathbb{Z}} a(2\alpha) = e_1^{\mathrm{T}} \sum_{\alpha\in\mathbb{Z}} a(2\alpha-1) = e_1^{\mathrm{T}}.$$

If a satisfies the basic sum rule, we claim that b also satisfies the basic sum rule. In fact

$$\begin{aligned}(\overline{e_1^{\mathrm{T}}} \otimes e_1^{\mathrm{T}}) \sum_{\alpha\in\mathbb{Z}} b(2\alpha) &= (\overline{e_1^{\mathrm{T}}} \otimes e_1^{\mathrm{T}}) \sum_{\alpha\in\mathbb{Z}} \sum_{\beta\in\mathbb{Z}} \overline{a(\beta)} \otimes a(2\alpha+\beta)/2 \\ &= \overline{e_1^{\mathrm{T}} \sum_{\beta\in\mathbb{Z}} a(\beta)} \otimes e_1^{\mathrm{T}} \sum_{\alpha\in\mathbb{Z}} a(2\alpha+\beta)/2 \\ &= \overline{e_1^{\mathrm{T}} \sum_{\beta\in\mathbb{Z}} a(\beta)} \otimes e_1^{\mathrm{T}}/2 = \overline{e_1^{\mathrm{T}}} \otimes e_1^{\mathrm{T}}.\end{aligned}$$

Similarly, we can prove that

$$(\overline{e_1^{\mathrm{T}}} \otimes e_1^{\mathrm{T}}) \sum_{\alpha\in\mathbb{Z}} b(2\alpha-1) = \overline{e_1^{\mathrm{T}}} \otimes e_1^{\mathrm{T}}.$$

We point out that the converse of this statement is also true. It is easily seen that b satisfies basis sum rule if and only if $(\overline{e_1^{\mathrm{T}}} \otimes e_1^{\mathrm{T}})\hat{b}(\pi) = 0$. By the definition of b, we know that $(\overline{e_1^{\mathrm{T}}} \otimes e_1^{\mathrm{T}})\hat{b}(\pi) = 0$ implies $e_1^{\mathrm{T}}\hat{a}(\pi) = 0$. Hence a satisfies basis sum rule.

Let $a \in B_m^{r\times r}$, b and T_b be given by (2.5) and (2.6), respectively. Consider the subspace V of $B_m^{r^2}$ defined by

$$V := \left\{ v \in B_m^{r^2} : (\overline{e_1^{\mathrm{T}}} \otimes e_1^{\mathrm{T}}) \sum_{\alpha\in\mathbb{Z}} v(\alpha) = 0 \right\}. \qquad \square \tag{2.8}$$

When $r = 1$ and $m = 1$, linear space V was used in [10] to characterize Riesz bases generated from multiresolution analysis in $L_2(\mathbb{R})$.

Theorem 2.3. *Let $b \in B_m^{r^2\times r^2}$, for some $m \in \mathbb{Z}_+$. Then V is invariant under T_b, if and only if b satisfies the basic sum rule.*

Proof. Let b satisfy basic sum role and $v \in V$. Then

$$(\overline{e_1^{\mathrm{T}}} \otimes e_1^{\mathrm{T}}) \sum_{\alpha\in\mathbb{Z}} T_b v(\alpha) = (\overline{e_1^{\mathrm{T}}} \otimes e_1^{\mathrm{T}}) \sum_{\alpha\in\mathbb{Z}} \sum_{\beta\in\mathbb{Z}} b(2\alpha-\beta)v(\beta) = (\overline{e_1^{\mathrm{T}}} \otimes e_1^{\mathrm{T}}) \sum_{\beta\in\mathbb{Z}} v(\beta) = 0.$$

Hence $v \in V$ implies $T_b v \in V$. This shows that V is invariant under T_b.

Since V is invariant under T_b, we have $T_b(\overline{e_k} \otimes e_j \nabla\delta) \in V$, for $j, k = 1, 2, \ldots, r$, where the difference operator ∇ is defined by

$$\nabla u := u - u(\cdot - 1), \quad u \in l(\mathbb{Z}).$$

Hence

$$\sum_{\alpha\in\mathbb{Z}} (\overline{e_1^{\mathrm{T}}} \otimes e_1^{\mathrm{T}})[b(2\alpha) - b(2\alpha-1)](\overline{e_k} \otimes e_j) = (\overline{e_1^{\mathrm{T}}} \otimes e_1^{\mathrm{T}}) \sum_{\alpha\in\mathbb{Z}} T_b(\overline{e_k} \otimes e_j \nabla\delta)(\alpha) = 0.$$

Since the above relation is true for all $j, k = 1, 2, \ldots, r$. It follows that

$$(\overline{e_1^{\mathrm{T}}} \otimes e_1^{\mathrm{T}}) \sum_{\alpha\in\mathbb{Z}} [b(2\alpha) - b(2\alpha-1)] = 0.$$

Note that $e_1^{\mathrm{T}} \sum_{\alpha\in\mathbb{Z}} a(\alpha)/2 = e_1^{\mathrm{T}}$, we have

$$(\overline{e_1^{\mathrm{T}}} \otimes e_1^{\mathrm{T}}) \sum_{\alpha\in\mathbb{Z}} b(\alpha) = \left(\overline{e_1^{\mathrm{T}} \sum_{\beta\in\mathbb{Z}} a(\beta)}\right) \otimes \left(e_1^{\mathrm{T}} \sum_{\beta\in\mathbb{Z}} a(\alpha+\beta)\right) \Big/ 2 = 2(\overline{e_1^{\mathrm{T}}} \otimes e_1^{\mathrm{T}}).$$

The above equalities yield the necessary of the theorem. □

In Section 3, we will show that the vector subdivision schemes associated with a being polynomial decay sequences converges in the L_2-norm if and only if V is invariant under T_b and $\rho(T_b|_V) < 1$.

3. Main theorems and Proof of theorems

Suppose $\phi \in (L_2(\mathbb{R}))^r$ is a solution of the refinement equation (1.1), where the mask a is assumed to be in $(l_1(\mathbb{Z}))^{r\times r}$ for the times being, then

$$\phi \odot \phi^{\mathrm{T}} = \sum_{\alpha\in\mathbb{Z}} \sum_{\beta\in\mathbb{Z}} a(\alpha)\phi(2\cdot-\alpha) \odot \phi^{\mathrm{T}}(2\cdot-\beta)\overline{a(\beta)}^{\mathrm{T}}.$$

Note that

$$\phi(2\cdot-\alpha) \odot \phi^{\mathrm{T}}(2\cdot-\beta) = \tfrac{1}{2}\phi \odot \phi^{\mathrm{T}}(2\cdot-\alpha+\beta).$$

With (2.2), we have

$$\mathrm{vec}(a(\alpha)\phi(2\cdot-\alpha) \odot \phi^{\mathrm{T}}(2\cdot-\beta)\overline{a(\beta)}^{\mathrm{T}}) = \tfrac{1}{2}\overline{a(\beta)} \otimes a(\alpha)\,\mathrm{vec}(\phi \odot \phi^{\mathrm{T}})(2\cdot-\alpha+\beta).$$

Then

$$\mathrm{vec}(\phi \odot \phi^{\mathrm{T}}) = \sum_{\alpha\in\mathbb{Z}} \sum_{\beta\in\mathbb{Z}} \frac{1}{2}\overline{a(\beta)} \otimes a(\alpha)\,\mathrm{vec}(\phi \odot \phi^{\mathrm{T}})(2\cdot-\alpha+\beta). \tag{3.1}$$

Let $f := \mathrm{vec}(\phi \odot \phi^{\mathrm{T}})$, then $f \in (C_0(\mathbb{R}))^{r^2}$, the linear space of $r^2 \times 1$ vectors of functions in $C_0(\mathbb{R})$. We have that f satisfies the refinement equation as follows:

$$f = \sum_{\alpha\in\mathbb{Z}} b(\alpha) f(2\cdot-\alpha),$$

where b is given by (2.5).

For $n = 1, 2, \ldots,$ let $a_1 = a$ and a_n be defined by the following iterative relations:

$$a_n(\alpha) = \sum_{\beta\in\mathbb{Z}} a_{n-1}(\beta)a(\alpha-2\beta), \quad \alpha \in \mathbb{Z}. \tag{3.2}$$

By (1.2), (3.2) and induction on n, it is easily seen that

$$Q_a^n\phi = \sum_{\alpha\in\mathbb{Z}} a_n(\alpha)\phi(2^n\cdot-\alpha). \tag{3.3}$$

Similarly, for $f \in (C_0(\mathbb{R}))^{r^2}$, we have

$$Q_b^n f = \sum_{\alpha\in\mathbb{Z}} b_n(\alpha) f(2^n\cdot-\alpha), \tag{3.4}$$

where $b_n (n = 1, 2, \ldots)$ are defined as follows:

$$b_1 = b \quad \text{and} \quad b_n(\alpha) = \sum_{\beta\in\mathbb{Z}} b_{n-1}(\beta)b(\alpha-2\beta), \quad \alpha \in \mathbb{Z}. \tag{3.5}$$

From the definitions of a_n and b_n, we can obtain

$$b_n(\alpha) = \sum_{\beta\in\mathbb{Z}} \overline{a_n(\beta)} \otimes a_n(\alpha+\beta)/2^n, \quad \alpha \in \mathbb{Z}, n = 1, 2, \ldots . \tag{3.6}$$

In fact, we can prove (3.6) by induction on n. By the definition of b, (3.6) holds true for $n = 1$. Suppose $n > 1$ and (3.6) is valid for $n - 1$. For $\alpha \in \mathbb{Z}$, we have

$$\begin{aligned} b_n(\alpha) &= \sum_{\eta\in\mathbb{Z}} b_{n-1}(\eta) b(\alpha - 2\eta) \\ &= 2^{-n} \sum_{\eta\in\mathbb{Z}} \left(\sum_{\gamma\in\mathbb{Z}} \overline{a_{n-1}(\gamma)} \otimes a_{n-1}(\eta+\gamma) \right) \left(\sum_{\beta\in\mathbb{Z}} \overline{a(\beta)} \otimes a(\alpha - 2\eta + \beta) \right) \\ &= 2^{-n} \sum_{\beta\in\mathbb{Z}} \sum_{\gamma\in\mathbb{Z}} \sum_{\eta\in\mathbb{Z}} \overline{a_{n-1}(\gamma) a(\beta - 2\gamma)} \otimes (a_{n-1}(\eta) a(\alpha + \beta - 2\eta)) \\ &= 2^{-n} \sum_{\beta\in\mathbb{Z}} \overline{a_n(\beta)} \otimes a_n(\alpha+\beta), \end{aligned}$$

which implies that (3.6) is true for all n.

Let ϕ_0 and ψ_0 lie in $(L_2(\mathbb{R}))^r$. It follows from above discussions that:

$$\mathrm{vec}((Q_a^n \phi_0) \odot (Q_a^n \psi_0)^{\mathrm{T}}) = \sum_{\alpha\in\mathbb{Z}} \sum_{\beta\in\mathbb{Z}} 2^{-n} \overline{a_n(\beta)} \otimes a_n(\alpha)\, \mathrm{vec}(\phi_0 \odot \psi_0^{\mathrm{T}})(2^n \cdot - \alpha + \beta).$$

By (3.6), we have, for $n = 1, 2, \ldots,$

$$\mathrm{vec}((Q_a^n \phi_0) \odot (Q_a^n \psi_0)^{\mathrm{T}}) = Q_b^n(\mathrm{vec}(\phi_0 \odot \psi_0^{\mathrm{T}})). \tag{3.7}$$

For $\beta \in \mathbb{Z}$, we denote δ_β the sequence on $\mathbb{Z}$ by

$$\delta_\beta(\alpha) = \begin{cases} 1 & \text{for } \alpha = \beta, \\ 0 & \text{for } \alpha \in \mathbb{Z} \backslash \{\beta\}. \end{cases}$$

If $\beta = 0$, we write δ for δ_0.

Theorem 3.1. *Suppose $a \in B_m^{r\times r}$, for some $m \in \mathbb{Z}_+$, and that $M = \sum_{\alpha\in\mathbb{Z}} a(\alpha)/2$ satisfies Eigenvalue condition. Let b and T_b be given by* (2.5) *and* (2.6), *respectively. Then $\rho(T_b) \geqslant 1$.*

Proof. Some ideas of proof of Theorem 3.1 are from [15]. First, we assume that a is finitely supported. Let ϕ_0 be the characteristic function of the unit cube [0, 1]. By (2.4), (3.4) and (3.7), we have

$$\| Q_a^n(e_1^{\mathrm{T}} \phi_0) \|_2^2 \leqslant |\mathrm{vec}((Q_a^n(e_1^{\mathrm{T}} \phi_0)) \odot (Q_a^n(e_1 \phi_0))^{\mathrm{T}})(0)| = |Q_b^n(\mathrm{vec}((e_1^{\mathrm{T}} \phi_0) \odot (e_1^{\mathrm{T}} \phi_0)^{\mathrm{T}}))(0)|.$$

For $n = 1, 2, \ldots,$ by an induction on n, we have

$$T_b^n v(\alpha) = \sum_{\beta\in\mathbb{Z}} b_n(2^n \alpha - \beta) v(\beta). \tag{3.8}$$

Therefore

$$\| Q_a^n(e_1 \phi_0) \|_2^2 \leqslant |Q_b^n(\mathrm{vec}((e_1 \phi_0) \odot (e_1 \phi_0)^{\mathrm{T}}))(0)| = |T_b^n(\mathrm{vec}((e_1 \phi_0) \odot (e_1 \phi_0)^{\mathrm{T}}))(0)|.$$

If $\rho(T_b) < 1$, it tells that $Q_a^n(e_1 \phi_0)$ would converge to 0 in the L_2-norm. Since $M = \sum_{\alpha\in\mathbb{Z}} a(\alpha)/2$ satisfies Eigenvalue condition, by a simple computation, we have

$$\hat{Q}_a^n(e_1 \phi_0)(0) = M^n e_1 = e_1.$$

This contradiction demonstrates that $\rho(T_b) \geqslant 1$.

In general, suppose $a \in B_m^{r\times r}$ for some $m \in \mathbb{Z}_+$. For $L = 1, 2, \ldots,$ we can find matrix sequences $a_L (L = 1, 2, \ldots)$ such that each a_L is supported on $[-L, L]$, $e_1^{\mathrm{T}} \sum_{\alpha\in\mathbb{Z}} a_L(\alpha)/2 = e_1^{\mathrm{T}}$ and $\|a_L - a\|_{B_m^{r\times r}} \to 0$ as $L \to \infty$, let $b_L(\alpha) = \sum_{\beta\in\mathbb{Z}} \overline{a_L(\beta)} \otimes a_L(\alpha+\beta)/2$, by Lemma 2.1, $\|T_{b_L} - T_b\| \to 0$ as $L \to \infty$. It follows that $\lim_{n\to\infty} \rho(T_{b_L}) = \rho(T_b)$. If $\rho(T_b) < 1$, then $\rho(T_{b_L}) < 1$ for sufficiently large L, which is impossible. Therefore, we have $\rho(T_b) \geqslant 1$.

Following Theorem 3.2 establishes necessary and sufficient conditions for the L_2-convergence of subdivision schemes with mask a being polynomial decaying fast. □

Theorem 3.2. *Let $a \in B_m^{r\times r}$ for some $m \in \mathbb{Z}_+$,* and $M = \sum_{\alpha\in\mathbb{Z}} a(\alpha)/2$ *satisfy Eigenvalue condition. Suppose b is given by* (2.5) *and T_b is defined by* (2.6). *Then the subdivision scheme associated with a converges in the L_2-norm if and only if*

(1) $\lim_{n\to\infty} \|T_b^n v\|_\infty = 0, \quad \forall v \in V.$
(2) *a satisfies the basic sum rule,*

where V is denoted by (2.7).

Proof. We following the lines of [10,15,17]. Suppose that the subdivision scheme associated with a converges in the L_2-norm. We choose χ to be the characteristic function of the unit interval [0, 1). Then $e_1\chi$ satisfies the moment conditions of order 1, and $\mathrm{vec}((e_1\chi) \odot (e_1\chi)^{\mathrm{T}}) = (\overline{e_1} \otimes e_1)h$, where h is the hat function given by $h(x) := \max\{1 - |x|, 0\}$, $x \in \mathbb{R}$. We have that $Q_a^n(e_1\chi)$ converge to the same limit function ϕ in the L_2-norm.

With (2.3), we have

$$\begin{aligned}
&\|\mathrm{vec}(Q_a^n(e_1\chi) \odot (Q_a^n(e_1\chi))^{\mathrm{T}} - \phi \odot \phi^{\mathrm{T}})\|_\infty \\
&\quad\leqslant \|\mathrm{vec}(Q_a^n(e_1\chi) \odot (Q_a^n(e_1\chi) - \phi)^{\mathrm{T}})\|_\infty + \|\mathrm{vec}((Q_a^n(e_1\chi) - \phi) \odot \phi^{\mathrm{T}})\|_\infty \\
&\quad\leqslant \|Q_a^n(e_1\chi)\|_2 \|Q_a^n(e_1\chi) - \phi\|_2 + \|Q_a^n(e_1\chi) - \phi\|_2 \|\phi\|_2.
\end{aligned}$$

Which implies that $\mathrm{vec}(Q_a^n(e_1\chi) \odot (Q_a^n(e_1\chi))^{\mathrm{T}})$ converges to $\mathrm{vec}(\phi \odot \phi^{\mathrm{T}})$ uniformly. By (3.7), we have $Q_b^n(\overline{e_1} \otimes e_1 h)$ converges to $\mathrm{vec}(\phi \odot \phi^{\mathrm{T}})$ uniformly. Since $\mathrm{vec}(\phi \odot \phi^{\mathrm{T}})$ is uniformly continuous, and

$$\begin{aligned}
&\|Q_b^n(\overline{e_1} \otimes e_1 h) - Q_b^n(\overline{e_1} \otimes e_1 h)(\cdot - 2^{-n})\|_\infty \\
&\quad\leqslant \|Q_b^n(\overline{e_1} \otimes e_1 h) - \mathrm{vec}(\phi \odot \phi^{\mathrm{T}})\|_\infty + \|\mathrm{vec}(\phi \odot \phi^{\mathrm{T}}) - \mathrm{vec}(\phi \odot \phi^{\mathrm{T}})(\cdot - 2^{-n})\|_\infty \\
&\qquad + \|Q_b^n(\overline{e_1} \otimes e_1 h)(\cdot - 2^{-n}) - \mathrm{vec}(\phi \odot \phi^{\mathrm{T}})(\cdot - 2^{-n})\|_\infty.
\end{aligned}$$

Consequently,

$$\lim_{n\to\infty} \|Q_b^n(\overline{e_1} \otimes e_1 h) - Q_b^n(\overline{e_1} \otimes e_1 h)(\cdot - 2^{-n})\|_\infty = 0.$$

It follows from (3.4) that

$$Q_b^n(\overline{e_1} \otimes e_1 h) - Q_b^n(\overline{e_1} \otimes e_1 h)(\cdot - 2^{-n}) = \sum_{\alpha\in\mathbb{Z}} \nabla b_n(\alpha)(\overline{e_1} \otimes e_1 h)(2^n \cdot - \alpha).$$

Note that the shifts of h are stable, therefore

$$\lim_{n\to\infty} \|\nabla b_n(\overline{e_1} \otimes e_1)\|_\infty = 0. \tag{3.9}$$

For $j = 2, \ldots, r$, we know that $e_1\chi$ and $(e_1 + e_j)\chi$ both satisfy the moment conditions of order 1, hence, $Q_a^n(e_1\chi)$ and $Q_a^n(e_1 + e_j)\chi$ converge to the same limit ϕ in the L_2-norm. This shows that, for $j = 2, \ldots, r$, $\|Q_a^n(e_j\chi)\|_2 \to 0$ as $n \to \infty$. By (3.7), we have

$$\mathrm{vec}((Q_a^n(e_j\chi)) \odot (Q_a^n(e_k\chi))^{\mathrm{T}}) = Q_b^n((\overline{e_k} \otimes e_j)h), \quad j, k = 1, \ldots, r, \quad n = 1, 2, \ldots .$$

By (2.3), we obtain

$$\lim_{n\to\infty} \|Q_b^n(\overline{e_k} \otimes e_j)h\|_\infty = 0, \quad (j, k) \neq (1, 1).$$

Since

$$Q_b^n(\overline{e_k} \otimes e_j)h = \sum_{\alpha\in\mathbb{Z}} b_n(\alpha)(\overline{e_k} \otimes e_j)h(2^n \cdot -\alpha).$$

It follows that

$$\lim_{n\to\infty} \|b_n(\overline{e_k} \otimes e_j)\|_\infty = 0, \quad (j, k) \neq (1, 1). \tag{3.10}$$

Since $\{e_j, j = 1, \dots, r\}$ is a basis for C^r. Then, $\{\overline{e_k} \otimes e_j, \quad j, k = 1, \dots, r\}$ is a basis for C^{r^2}. Then each $v \in V$ can be expressed as

$$v = \sum_{j=1}^{r}\sum_{k=1}^{r}\sum_{\beta\in\mathbb{Z}} d_{jk}(\beta)(\overline{e_k} \otimes e_j)\delta_\beta,$$

where $d_{jk} \in B_m$, $j, k = 1, \dots, r$. Since $v \in V$, we have

$$0 = (\overline{e_1^{\mathrm{T}}} \otimes e_1^{\mathrm{T}}) \sum_{\alpha\in\mathbb{Z}} v(\alpha) = (\overline{e_1^{\mathrm{T}}} \otimes e_1^{\mathrm{T}}) \sum_{j=1}^{r}\sum_{k=1}^{r}\sum_{\beta\in\mathbb{Z}} d_{jk}(\beta)(\overline{e_1^{\mathrm{T}}} \otimes e_1^{\mathrm{T}}) = \sum_{\beta\in\mathbb{Z}} d_{11}(\beta).$$

Therefore $d_{11} = \sum_{\alpha\in\mathbb{Z}} c(\alpha)\nabla\delta_\alpha$, where

$$c(\alpha) = \sum_{l=0}^{\infty} d_{11}(\alpha - l), \quad \text{for } \alpha \leqslant 1 \quad \text{and} \quad c(\alpha) = -\sum_{l=1}^{\infty} d_{11}(\alpha + l), \quad \text{for } \alpha \geqslant 0.$$

Let us show $c \in \ell_1(\mathbb{Z})$. We have

$$\sum_{\alpha=-\infty}^{1} |c(\alpha)| \leqslant \sum_{\alpha=-\infty}^{1}\sum_{l=0}^{\infty} |d_{11}(\alpha - l)| = \sum_{\alpha=-\infty}^{1}\sum_{\gamma=-\infty}^{\alpha} |d_{11}(\gamma)| \leqslant \sum_{\gamma=-\infty}^{1} |\gamma||d_{11}(\gamma)| \leqslant \|d_{11}\|_{B_m}.$$

Similarly,

$$\sum_{\alpha=0}^{\infty} |c(\alpha)| \leqslant \sum_{\alpha=0}^{\infty}\sum_{l=1}^{\infty} |d_{11}(\alpha + l)| = \sum_{\alpha=0}^{\infty}\sum_{\gamma=\alpha+1}^{\infty} |d_{11}(\gamma)| = \sum_{\gamma=1}^{\infty} |\gamma||d_{11}(\gamma)| \leqslant \|d_{11}\|_{B_m}.$$

It follows from above discussions that c belongs to $l_1(\mathbb{Z})$. By (2.6),(3.9),(3.10) and a simple computation, we have for every $\beta \in \mathbb{Z}$ and $(j, k) \neq (1, 1)$,

$$\lim_{n\to\infty} \|T_b^n(\overline{e_1} \otimes e_1)\nabla\delta_\beta)\|_\infty = 0 \quad \text{and} \quad \lim_{n\to\infty} \|T_b^n(\overline{e_k} \otimes e_j\delta_\beta)\|_\infty = 0.$$

By using the expression of v, we prove that (1) of Theorem 3.2 holds.

To prove (2) of Theorem 3.2, we claim that V is invariant under T_b. Indeed, if not, then there exists $v \in V$ such that $T_b v$ is not in V. Note that the codimension of V in $B_m^{r^2}$ is 1. Hence, any $u \in B_m^{r^2}$ can be expressed as $u = w + c(T_b v)$ for some $w \in V$ and $c \in C$. By (1) of Theorem 3.2, we have

$$\lim_{n\to\infty} \|T_b^n u\|_\infty = 0, \quad \forall u \in B_m^{r^2}.$$

Therefore, $\rho(T_b) < 1$. On the other hand, by Theorem 3.1, we have $\rho(T_b) \geqslant 1$. This contradiction shows that V is invariant under T_b. It follows from Theorem 2.3 that b satisfies the basic sum rule. Hence, a also satisfies the basic sum rule.

Next, we establish the sufficiency part of the theorem. We pick a vector of compactly supported functions ϕ_0 in $(L_2(\mathbb{R}))^r$ such that ϕ_0 satisfies the moment conditions of order 1. Let us consider $Q_a^{n+1}\phi_0 - Q_a^n\phi_0$. We observe that

$$Q_a^{n+1}\phi_0 - Q_a^n\phi_0 = Q_a^n(Q_a\phi_0 - \phi_0) = Q_a^n g_0, \tag{3.11}$$

where $g_0 := Q_a\phi_0 - \phi_0$. Since $e_1^{\mathrm{T}}\sum_{\alpha\in\mathbb{Z}}\phi_0(\cdot-\alpha)=1$ and a satisfies the basic sum rule, we have

$$\begin{aligned} e_1^{\mathrm{T}}\sum_{\alpha\in\mathbb{Z}}(Q_a\phi_0)(\cdot-\alpha) &= e_1^{\mathrm{T}}\sum_{\alpha\in\mathbb{Z}}\sum_{\beta\in\mathbb{Z}}a(\beta)\phi_0(2\cdot-2\alpha-\beta)\\ &= \sum_{\beta\in\mathbb{Z}}e_1^{\mathrm{T}}\left[\sum_{\alpha\in\mathbb{Z}}a(\beta-2\alpha)\right]\phi_0(\cdot-\beta)=e_1^{\mathrm{T}}\sum_{\beta\in\mathbb{Z}}\phi_0(\cdot-\beta)=1.\end{aligned}$$

Therefore, $Q_a\phi_0$ also satisfies the moment conditions of order 1. It follows that for almost every $x\in\mathbb{R}$,

$$e_1^{\mathrm{T}}\sum_{\alpha\in\mathbb{Z}}g_0(x-\alpha)=0.$$

We claim that $\mathrm{vec}(g_0\odot g_0^{\mathrm{T}})$ lies in V. By (2.2), we obtain

$$(\overline{e_1^{\mathrm{T}}}\otimes e_1^{\mathrm{T}})\sum_{\alpha\in\mathbb{Z}}\mathrm{vec}(g_0\odot g_0^{\mathrm{T}})(\alpha)=(\overline{e_1^{\mathrm{T}}}\otimes e_1^{\mathrm{T}})\sum_{\alpha\in\mathbb{Z}}\int_{\mathbb{R}}\mathrm{vec}(g_0(\alpha+x)\overline{g_0(x)}^{\mathrm{T}})\,\mathrm{d}x=0.$$

Since $\|a\|_{B_m^{r\times r}}<\infty$ and ϕ_0 is compactly supported, by a simple computation, we have

$$\|\mathrm{vec}(g_0\odot g_0^{\mathrm{T}})\|_{B_m^{r^2}}<\infty.$$

Hence $\mathrm{vec}(g_0\odot g_0^{\mathrm{T}})$ lies in V. By (2.4), we have

$$\|Q_a^n g_0\|_2^2\leqslant|\mathrm{vec}((Q_a^n g_0)\odot(Q_a^n g_0)^{\mathrm{T}})(0)|.$$

Note that

$$T_b^n(\mathrm{vec}(g_0\odot g_0^{\mathrm{T}}))(\alpha)=\sum_{\beta\in\mathbb{Z}}b_n(2^n\alpha-\beta)\,\mathrm{vec}(g_0\odot g_0^{\mathrm{T}})(\beta).$$

Then

$$\begin{aligned} T_b^n\mathrm{vec}(g_0\odot g_0^{\mathrm{T}})(0) &= \sum_{\beta\in\mathbb{Z}}b_n(-\beta)\,\mathrm{vec}(g_0\odot g_0^{\mathrm{T}})(\beta)=\sum_{\beta\in\mathbb{Z}}b_n(\beta)\,\mathrm{vec}(g_0\odot g_0^{\mathrm{T}})(-\beta)\\ &= Q_b^n\,\mathrm{vec}(g_0\odot g_0^{\mathrm{T}})(0)=\mathrm{vec}((Q_a^n g_0)\odot(Q_a^n g_0)^{\mathrm{T}})(0).\end{aligned}$$

Therefore

$$\begin{aligned}\|Q_a^n g_0\|_2^2 &\leqslant|\mathrm{vec}((Q_a^n g_0)\odot(Q_a^n g_0)^{\mathrm{T}})(0)|\\ &=|T_b^n\,\mathrm{vec}(g_0\odot g_0^{\mathrm{T}})(0)|\leqslant\|T_b^n\,\mathrm{vec}(g_0\odot g_0^{\mathrm{T}})\|_\infty,\quad n=1,2,\ldots\ .\end{aligned}$$

Since T_b is a compact operator, it follows from the discussions in Section 2 that $\rho(T_b|_V)=|\tau|$ for some eigenvalue τ of $T_b|_V$. Suppose $T_b v=\tau v$ for some $v\in V$ with $v\neq 0$. It follows that $T_b^n v=\tau^n v$, for $n=1,2,\ldots$. By (1) of Theorem 3.2, we have $\lim_{n\to\infty}\|T_b^n v\|_\infty=0$. Which implies $\rho(T_b|_V)<1$. Hence there exist positive constants C and $0<\eta<1$, such that

$$\|Q_a^n g_0\|_2^2\leqslant C\eta^n,\quad n=1,2,\ldots,\ .$$

This shows that $Q_a^n\phi_0$ is a Cauchy sequence in $(L_2(\mathbb{R}))^r$. If ψ_0 is another $r\times 1$ vector of $(L_{2,c}(\mathbb{R}))^r$ that satisfies the moment conditions of order 1, then $e_1^{\mathrm{T}}\sum_{\alpha\in\mathbb{Z}}(\phi_0-\psi_0)=0$. By what have been proved, $Q_a^n(\phi_0-\psi_0)$ converges to 0 in the L_2-*norm*. It means that $Q_a^n\phi_0$ and $Q_a^n\psi_0$ converge to the same limit. Therefore, the subdivision scheme associated with a converges in the L_2-norm. □

Following the proof of Theorem 3.2, we have

Theorem 3.3. *Let $a \in B_m^{r\times r}$ for some $m \in \mathbb{Z}_+$ and $M = \sum_{\alpha\in\mathbb{Z}} a(\alpha)/2$ satisfy Eigenvalue condition. Let b be given by* (2.5) *and T_b be defined by* (2.6). *Then the subdivision scheme associated with a converges in the L_2-norm if and only if*

(1) *a satisfies the basic sum rule*, *and*
(2) $\rho(T_b|_V) < 1$,

where V is the linear space denoted by (2.7).

Remark 3.4. We remark that Theorem 3.3 was established in [17] for the case in which mask a is finitely supported. The convergence of subdivision schemes associated with mask a being an exponential decay were investigated, respectively, in [10,9,15].

For $r = 1$, the following Theorem 3.5 establishes a sufficient condition for the convergence of subdivision schemes in $L_2(\mathbb{R})$. This extends well known results on convergence of subdivision schemes associated with a finitely supported mask (see [14]).

Theorem 3.5. *Let $a \in B_m$, for some $m \in \mathbb{Z}_+$, $\sum_{\alpha\in\mathbb{Z}} a(\alpha) = 2$ and suppose $b(\alpha) = \sum_{\beta\in\mathbb{Z}} \overline{a(\beta)} a(\alpha+\beta)/2$. Let ϕ_0 be a function on $\mathbb{R}$ such that the shifts of ϕ_0 are stable in $L_2(\mathbb{R})$. If sequences $(Q_a^n\phi_0)_{n=1,2,\ldots}$ converges to ϕ in L_2-norm, then subdivision schemes with $r = 1$ converges in L_2-norm.*

Proof. The proof of Theorem 3.5 is based in [10, Lemma 2.2]. Since $(Q_a^n\phi_0)_{n=1,2,\ldots}$ converges to ϕ in L_2-norm, there exists a positive constant C_1 such that $\|Q_a^n\phi_0\|_2 \leqslant C_1$ for all n. By (3.3), we have

$$Q_a^n\phi_0(x) = \sum_{\alpha\in\mathbb{Z}} a_n(\alpha)\phi_0(2^n x - \alpha), \quad x \in \mathbb{R}.$$

It follows from the stability of ϕ_0 that there exists a positive constant C_2 such that

$$\|2^{-n/2}a_n\|_2 \leqslant C_2\|\phi_n\|_2 \leqslant C_1C_2, \quad \forall n \in \mathbb{N}.$$

Note that

$$\nabla_{2^{-n}}\phi_n(x) = \sum_{\alpha\in\mathbb{Z}} \nabla a_n(\alpha)\phi_0(2^n x - \alpha), \quad x \in \mathbb{R}.$$

Hence

$$\|2^{-n/2}\nabla a_n\|_2 \leqslant C_2\|\nabla_{2^{-n}}\phi_n\|_2.$$

Since

$$\|\nabla_{2^{-n}}\phi_n\|_2 \leqslant \|\nabla_{2^{-n}}(\phi_n - \phi_0)\|_2 + \|\nabla_{2^{-n}}\phi\|_2.$$

Then

$$\lim_{n\to\infty} \|2^{-n/2}\nabla a_n\|_2 = 0.$$

By the definition of b, we obtain

$$\|\nabla b_n\|_\infty \leqslant \|2^{-n/2}a_n\|_2\|2^{-n/2}\nabla a_n\|_2.$$

Therefore

$$\lim_{n\to\infty} \|\nabla b_n\|_\infty = 0.$$

For $v \in V$, define u to be sequence on $\mathbb{Z}$ by $u(\alpha) = \sum_{\beta=-\infty}^{\alpha} v(\beta)$, $\alpha \in \mathbb{Z}$. It is easily seen that $v = \nabla u$. We claim that $u \in \ell_1(\mathbb{Z})$. In fact

$$\sum_{\alpha=0}^{\infty} |u(\alpha)| = \sum_{\alpha=0}^{\infty} \sum_{\beta=\alpha+1}^{\infty} |v(\beta)| = \sum_{\beta=1}^{\infty} |\beta||v(\beta)| \leqslant \|v\|_{B_m}$$

and

$$\sum_{\alpha=-\infty}^{-1} |u(\alpha)| = \sum_{\alpha=-\infty}^{-1} \sum_{\beta=-\infty}^{\alpha} |v(\beta)| = \sum_{\beta=-\infty}^{-1} |\beta||v(\beta)| \leqslant \|v\|_{B_m}.$$

It follows from (3.8) that

$$T_b^n v(\alpha) = \sum_{\beta\in\mathbb{Z}} b_n(2^n\alpha - \beta)v(\beta) = \sum_{\beta\in\mathbb{Z}} b_n(2^n\alpha - \beta)\nabla u(\beta), \quad \alpha \in \mathbb{Z}.$$

Therefore

$$\|T_b^n v\|_\infty \leqslant \sup_{\alpha\in\mathbb{Z}} \left| \sum_{\beta\in\mathbb{Z}} b_n(2^n\alpha - \beta)\nabla u(\beta) \right| \leqslant \|\nabla b_n\|_\infty \|u\|_1.$$

Which implies that

$$\lim_{n\to\infty} \|T_b^n v\|_\infty = 0, \quad \forall v \in V.$$

By Theorem 3.2, we know that the subdivision schemes with $r = 1$ converges in L_2-norm. □

The characterization of smoothness of refinable function is an important issue in wavelet analysis. For $\eta > 0$, let $H^\eta(\mathbb{R})$ denote the Sobolev space that consists of all functions $f \in L_2(\mathbb{R})$ such that

$$\int_{\mathbb{R}} |\hat{f}(\xi)|^2 (1 + |\xi|^2)^\eta \, \mathrm{d}\xi < \infty. \tag{3.12}$$

When a is finitely supported, there are many papers devoted to studying smoothness of refinable functions. However, when a is infinitely supported, the case is different. The smoothness analysis of refinable functions associated with infinitely supported mask is much less so far. For the case in which mask decays exponentially, Han and Jia [10] obtained a characterization of smoothness of refinable function. By using some ideas of [10], we have

Theorem 3.6. *Let $a \in B_m$, for some $m \in \mathbb{Z}_+$ and $\sum_{\alpha\in\mathbb{Z}} a(\alpha) = 2$. If subdivision scheme associated with $r = 1$ converges in L_2-norm, then there exists some constant $\eta > 0$ such that the limit function ϕ belongs to $H^\eta(\mathbb{R})$.*

Proof. Let $H(\xi) = \sum_{\alpha\in\mathbb{Z}} a(\alpha)\mathrm{e}^{-\mathrm{i}\alpha\xi}/2$, $\xi \in \mathbb{R}$. Since $a \in B_m$ for some $m \in \mathbb{Z}_+$ and $\sum_{\alpha\in\mathbb{Z}} a(\alpha) = 2$. By a simple computation, we know that there exists a positive constant C_3 independent of ξ such that $|H(\xi)| \leqslant 1 + C_3|\xi|$. Therefore, the product $\prod_{k=1}^n H(2^{-k}\xi)$ converges to $\hat{\phi}$ as $n \to \infty$. Furthermore, it is easily seen that the convergence is uniform on compact subsets of $\mathbb{R}$. It follows that $\hat{\phi}$ is a continuous function on $\mathbb{R}$. Let O be an open subset of $\mathbb{R}$ such that $0 \in O \subset 2O \subset [-\pi, \pi]$. Since $\hat{\phi}(\xi)$ is a continuous function, we have

$$\sup_{\xi\in[-\pi,\pi]\backslash O} |\hat{\phi}(\xi)|^2 < \infty.$$

Choose $v = \Delta\delta$, where the difference operator Δ is given by $\Delta v = v(\cdot + 1) + 2v - v(\cdot - 1)$, $v \in \mathbb{Z}$. It is easily seen that $v \in V$, where V is given by (2.7) with $r = 1$ and $\hat{v}(\xi) = 2(1 - \cos\,\xi)$. Hence, there exists a constant $G > 0$ such that

$$|\hat{\phi}(\xi)|^2 \leqslant G\hat{v}(\xi), \quad \forall \xi \in [-\pi, \pi]\backslash O. \tag{3.13}$$

Denote $U = 2O\backslash O$. For a given $\xi \in \mathbb{R}$, let

$$E_0 := \{\beta \in \mathbb{Z} : \xi + 2\pi\beta \in 2O\}$$

and

$$E_n := \{\beta \in \mathbb{Z} : \xi + 2\pi\beta \in 2^n U\}, \quad n \in \mathbb{N}.$$

It follows from the definitions of E_0 and E_n that $\mathbb{Z} = \cup_{n=0}^{\infty}$ and the union is disjoint. Therefore,

$$\sum_{\beta\in\mathbb{Z}}(1+|\xi+2\pi\beta|^2)^\eta|\hat{\phi}(\xi+2\pi\beta)|^2 = \sum_{n=0}^{\infty}\sum_{\beta\in E_n}(1+|\xi+2\pi\beta|^2)^\eta|\hat{\phi}(\xi+2\pi\beta)|^2.$$

It is easily seen that there exists a constant $B > 1$ such that $1 + |\tau|^2 \leqslant B^n$ for all $\tau \in 2^n U$. Hence,

$$\sum_{\beta\in E_n}(1+|\xi+2\pi\beta|^2)^\eta|\hat{\phi}(\xi+2\pi\beta)|^2 \leqslant B^{\eta n}\sum_{\beta\in E_n}|\hat{\phi}(\xi+2\pi\beta)|^2.$$

Since ϕ satisfies refinement equation (1.1), then $\phi = Q_a^n\phi$. By (3.3), we obtain

$$\sum_{\beta\in E_n}|\hat{\phi}(\xi+2\pi\beta)|^2 = \sum_{\beta\in E_n}\frac{1}{2^{2n}}|\hat{a}_n(2^{-n}(\xi+2\pi\beta))|^2|\hat{\phi}(2^{-n}(\xi+2\pi\beta))|^2.$$

Note that $2^{-n}(\xi+2\pi\beta) \in U = 2O\backslash O \subset [-\pi,\pi]\backslash O$ for $\beta \in E_n$. It follows from (3.13) that

$$|\hat{\phi}(2^{-n}(\xi+2\pi\beta))|^2 \leqslant G\hat{v}(2^{-n}(\xi+2\pi\beta)).$$

Therefore,

$$\sum_{\beta\in E_n}|\hat{\phi}(\xi+2\pi\beta)|^2 \leqslant G\sum_{\beta\in E_n}\frac{1}{2^n}\hat{b}_n(2^{-n}(\xi+2\pi\beta))\hat{v}(2^{-n}(\xi+2\pi\beta)).$$

By the proof in [10, Theorem 4.3], we obtain

$$\sum_{\beta\in E_n}\frac{1}{2^n}\hat{b}_n(2^{-n}(\xi+2\pi\beta))\hat{v}(2^{-n}(\xi+2\pi\beta)) \leqslant (\hat{T_b^n} v)(\xi), \quad \forall\xi \in \mathbb{R}.$$

Since the subdivision schemes associated with mask a converges in $L_2(\mathbb{R})$-norm, it follows from Theorem 3.3 that there exist two constants G_1 and $0 < t < 1$ such that

$$(\hat{T_a^n} v)(\xi) \leqslant \|T_b^n v\|_{B_m} \leqslant G_1 t^n, \quad \forall n \in \mathbb{N} \quad \text{and} \quad \xi \in \mathbb{R}.$$

We choose η is small enough such that $B^\eta t < 1$. By above discussions, we have

$$\sum_{\beta\in E_n}(1+|\xi+2\pi\beta|^2)^\eta|\hat{\phi}(\xi+2\pi\beta)|^2 \leqslant GB^{\eta n}G_1 t^n, \quad \forall n \in \mathbb{N}.$$

Note that

$$\sum_{\beta\in E_0}(1+|\xi+2\pi\beta|^2)^\eta|\hat{\phi}(\xi+2\pi\beta)|^2 \leqslant B_1$$

for some constant B_1. The above estimates tell us that there exist two constants $C_4 > 0$ and $\eta > 0$ such that for any $\xi \in \mathbb{R}$

$$\sum_{\beta\in\mathbb{Z}}(1+|\xi+2\pi\beta|^2)^\eta|\hat{\phi}(\xi+2\pi\beta)|^2 \leqslant C_4.$$

Which implies that $\phi \in H^\eta(\mathbb{R})$ for some $\eta > 0$. $\square$

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