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Two determinants in the universal enveloping algebras of the orthogonal Lie algebras

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Abstract

This paper gives a direct proof for the coincidence of the following two central elements in the universal enveloping algebra of the orthogonal Lie algebra: an element recently given by A. Wachi in terms of the column-determinant in a way similar to the Capelli determinant, and an element given by T. Umeda and the author in terms of the symmetrized determinant. The fact that these two elements actually coincide was shown by A. Wachi, but his observation was based on the following two non-trivial results: (i) the centrality of the first element, and (ii) the calculation of the eigenvalue of the second element. The purpose of this paper is to prove this coincidence of two central elements directly without using these (i) and (ii). Conversely this approach provides us new proofs of (i) and (ii). A similar discussion can be applied to the symplectic Lie algebras.

Keywords: Orthogonal Lie algebras; Central elements of universal enveloping algebras; Capelli identity

Introduction

In this paper, we give a new and direct proof for the coincidence of two central elements in $U(\mathfrak{o}_N)$, the universal enveloping algebra of the orthogonal Lie algebra. One is an element recently given by A. Wachi [W] in terms of the column-determinant: $C_{\det}(u) = \det(F^{\mathfrak{o}(S_0)} + u\mathbf{1}_N + \operatorname{diag}\tilde{\natural}_N)$ (the notation will be given soon). This element is quite similar to the Capelli determinant, a famous central element of $U(\mathfrak{gl}_N)$ which appears in the Capelli identity [Ca1,Ca2].

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The other one is an element given in [IU], and expressed in terms of the symmetrized determinant: $C_{\mathrm{Det}}(u) = \mathrm{Det}(F^{\mathfrak{o}(S_0)} + u\mathbf{1}_N; \tilde{\natural}_N)$. We can also regard this as an analogue of the Capelli determinant. Wachi found that these two elements $C_{\det}(u)$ and $C_{\mathrm{Det}}(u)$ actually coincide [W]:

Theorem A. *We have*

$$\det\bigl(F^{\mathfrak{o}(S_0)} + u\mathbf{1}_N + \mathrm{diag}\,\tilde{\natural}_N\bigr) = \mathrm{Det}\bigl(F^{\mathfrak{o}(S_0)} + u\mathbf{1}_N; \tilde{\natural}_N\bigr).$$

However this observation due to Wachi was obtained by comparing the eigenvalues of these two elements on the irreducible representations. Namely this depends on the following two non-trivial results: (i) the centrality of $C_{\det}(u)$ given in [W], and (ii) the calculation of the eigenvalue of $C_{\mathrm{Det}}(u)$ given in [I1]. The purpose of this paper is to prove Theorem A directly without using these (i) and (ii). Conversely this approach provides us new proofs of (i) and (ii).

Moreover, applying this discussion to the symplectic Lie algebras, we obtain similar central elements in $U(\mathfrak{sp}_N)$. This is discussed elsewhere [I3] (see also Section 7).

Let us explain the main result precisely. Let $S \in \mathrm{Mat}_N(\mathbb{C})$ be a non-degenerate symmetric matrix of size N. We can realize the orthogonal Lie group as the isometry group with respect to the bilinear form determined by S:

$$O(S) = \bigl\{g \in GL_N \bigm| {}^t g S g = S\bigr\}.$$

The corresponding Lie algebra is expressed as follows:

$$\mathfrak{o}(S) = \bigl\{Z \in \mathfrak{gl}_N \bigm| {}^t Z S + S Z = 0\bigr\}.$$

As generators of this $\mathfrak{o}(S)$, we can take $F_{ij}^{\mathfrak{o}(S)} = E_{ij} - S^{-1} E_{ji} S$, where E_{ij} is the standard basis of $\mathfrak{gl}_N$. We introduce the $N \times N$ matrix $F^{\mathfrak{o}(S)}$ whose (i, j)th entry is this generator: $F^{\mathfrak{o}(S)} = (F_{ij}^{\mathfrak{o}(S)})_{1 \leqslant i,j \leqslant N}$. This matrix is an element of $\mathrm{Mat}_N(\mathfrak{o}(S)) \subset \mathrm{Mat}_N(U(\mathfrak{o}(S)))$.

In the representation theory, the case $S = S_0 = (\delta_{i,N+1-j})_{1 \leqslant i,j \leqslant N}$ is important. Indeed, we can take a triangular decomposition of $\mathfrak{o}(S_0)$ simply as follows:

$$\mathfrak{o}(S_0) = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+. \tag{0.1}$$

Here $\mathfrak{n}^-$, $\mathfrak{h}$, and $\mathfrak{n}^+$ are the subalgebras of $\mathfrak{o}(S_0)$ spanned by the elements $F_{ij}^{\mathfrak{o}(S_0)}$ such that $i > j$, $i = j$, and $i < j$, respectively. Namely, the entries in the lower triangular part, in the diagonal part, and in the upper triangular part of the matrix $F^{\mathfrak{o}(S_0)}$ belong to $\mathfrak{n}^-$, $\mathfrak{h}$, and $\mathfrak{n}^+$, respectively. We call this $\mathfrak{o}(S_0)$ the "split realization" of the orthogonal Lie algebra.

The main object of this paper is the following determinant in the universal enveloping algebra $U(\mathfrak{o}(S_0))$ recently defined in [W]:

$$C_{\det}(u) = \det\bigl(F^{\mathfrak{o}(S_0)} + u\mathbf{1}_N + \mathrm{diag}\,\tilde{\natural}_N\bigr).$$

The notation is as follows. First, the symbol "det" means the "column-determinant." Namely we define $\det Z$ for an $N \times N$ matrix $Z = (Z_{ij})$ by

$$\det Z = \sum_{\sigma \in \mathfrak{S}_N} \mathrm{sgn}(\sigma) Z_{\sigma(1)1} Z_{\sigma(2)2} \cdots Z_{\sigma(N)N},$$

even if the entries Z_{ij} are non-commutative. Second, $\mathbf{1}_N$ means the unit matrix of order N. Third, $\tilde{\mathfrak{h}}$ is the sequence defined by

$$\tilde{\mathfrak{h}}_N = \begin{cases} (\frac{N}{2}-1, \frac{N}{2}-2, \dots, 0, 0, \dots, -\frac{N}{2}+1), & N\text{: even}, \\ (\frac{N}{2}-1, \frac{N}{2}-2, \dots, \frac{1}{2}, 0, -\frac{1}{2}, \dots, -\frac{N}{2}+1), & N\text{: odd}. \end{cases}$$

Here the dots mean arithmetic progressions with difference -1. Finally $\operatorname{diag}(a_1, \dots, a_N)$ means the diagonal matrix of size N whose diagonal entries are $a_1, \dots, a_N$.

This definition of $C_{\det}(u)$ is quite similar to that of the Capelli determinant, a famous central element in $U(\mathfrak{gl}_N)$ (see Section 1). Wachi's element $C_{\det}(u)$ is also central in $U(\mathfrak{o}(S_0))$ for any $u \in \mathbb{C}$. Moreover, we can easily calculate its eigenvalue on irreducible representations of $\mathfrak{o}(S_0)$ (see Theorem 4.2). This element is remarkable in this sense.

In addition to $C_{\det}(u)$, we also consider the following determinant given in [IU]:

$$C_{\mathrm{Det}}(u) = \operatorname{Det}\bigl(F^{\mathfrak{o}(S_0)} + u\mathbf{1}_N; \tilde{\mathfrak{h}}_N\bigr).$$

Here, the symbol "Det" means the "symmetrized determinant." Namely, for an $N \times N$ matrix $Z = (Z_{ij})$, we put

$$\operatorname{Det} Z = \frac{1}{N!} \sum_{\sigma, \sigma' \in \mathfrak{S}_N} \operatorname{sgn}(\sigma) \operatorname{sgn}(\sigma') Z_{\sigma(1)\sigma'(1)} Z_{\sigma(2)\sigma'(2)} \cdots Z_{\sigma(N)\sigma'(N)}.$$

Moreover, for N parameters $a_1, \dots, a_N \in \mathbb{C}$, we put

$$\begin{aligned} &\operatorname{Det}(Z; a_1, \dots, a_N) \\ &\quad = \frac{1}{N!} \sum_{\sigma, \sigma' \in \mathfrak{S}_N} \operatorname{sgn}(\sigma) \operatorname{sgn}(\sigma') Z_{\sigma(1)\sigma'(1)}(a_1) Z_{\sigma(2)\sigma'(2)}(a_2) \cdots Z_{\sigma(N)\sigma'(N)}(a_N), \end{aligned}$$

where $Z_{ij}(a) = Z_{ij} + \delta_{ij} a$. This $C_{\mathrm{Det}}(u)$ is also central in $U(\mathfrak{o}(S_0))$ for any $u \in \mathbb{C}$. This central element played an important role in some Capelli type identities as an analogue of the Capelli determinant (see [I2]).

As mentioned in Theorem A above, these $C_{\det}(u)$ and $C_{\mathrm{Det}}(u)$ actually coincide. This was first shown by A. Wachi [W] by comparing the eigenvalues of both sides (recall that any central element in the universal enveloping algebras of semisimple Lie algebras is determined by its eigenvalue). Namely his proof depends on the following four results:

(a) the centrality of $C_{\det}(u)$,
(b) the calculation of the eigenvalue of $C_{\det}(u)$,
(c) the centrality of $C_{\mathrm{Det}}(u)$,
(d) the calculation of the eigenvalue of $C_{\mathrm{Det}}(u)$.

Here (b) and (c) are easy. Indeed (b) is immediate from (a) by noting the triangular decomposition (0.1) and the definition of the column-determinant (Theorem 4.2). The property (c) is also immediate from a more general result (Proposition 2.2) depending on the invariance of the symmetrized determinant (Lemma 1.4).

However (a) and (d) are not trivial. The property (a) was first given by Wachi, and the proof is not so easy as that of (c). The calculation (d) due to [I1] is much more complicated than (b).

The aim of this paper is to prove Theorem A by a direct calculation without using these (a)–(d). Conversely this approach provides us new and simple proofs of (a) and (d). Indeed, these are immediate from Theorem A, because (b) and (c) are obvious.

This paper is organized as follows. In Section 1, we recall the Capelli determinant as the prototype of our main object. The Capelli determinant also has two different expressions corresponding to $C_{\mathrm{det}}(u)$ and $C_{\mathrm{Det}}(u)$, respectively. In Section 2, we construct central elements of $U(\mathfrak{o}(S))$ for general S using the symmetrized determinant. In Section 3, we recall an analogue of the Capelli determinant due to R. Howe and T. Umeda [HU] in the case $S = \mathbf{1}_N$. In Section 4, we state the main result in the case $S = S_0$, and prepare for the proof. We actually give the proof in Sections 5 and 6. The case that N is odd (Section 6) is a bit more complicated than the case that N is even (Section 5). Finally, in Section 7, a development to the symplectic Lie algebra is announced (the details are discussed elsewhere [I3]).

Remarks. (1) The coefficients of $C_{\mathrm{det}}(u)$ as a polynomial in u generate all the invariants in $U(\mathfrak{o}(S_0))$ with respect to the adjoint action of $O(S_0)$ [W].

(2) The element $C_{\mathrm{det}}(u)$ is also equal to the central element given in [M] in terms of the Sklyanin determinant. See [M,MN,MNO,IU,I1,W] for the details.

1. The case of $\mathfrak{gl}_N$

First of all, as the prototype of the main result, we recall the Capelli determinant in $U(\mathfrak{gl}_N)$ and its two different expressions.

1.1. Let E_{ij} be the standard basis of $\mathfrak{gl}_N$, and consider the matrix $E = (E_{ij})_{1 \leqslant i,j \leqslant N}$ in $\mathrm{Mat}_N(\mathfrak{gl}_N) \subset \mathrm{Mat}_N(U(\mathfrak{gl}_N))$. The following "Capelli determinant" in $U(\mathfrak{gl}_N)$ is well known as the key of the Capelli identity [Ca1,H,U1]:

$$C^{\mathfrak{gl}_N}_{\mathrm{det}}(u) = \det(E + u\mathbf{1}_N + \operatorname{diag} \natural_N).$$

Here $\mathbf{1}_N$ means the unit matrix of degree N, and $\natural_N$ means the arithmetic progression $\natural_N = (N-1, N-2, \dots, 0)$. Moreover, the symbol "det" means the "column-determinant." Namely, in general, we define $\det Z$ for $N \times N$ matrix $Z = (Z_{ij})$ by

$$\det Z = \sum_{\sigma \in \mathfrak{S}_N} \operatorname{sgn}(\sigma) Z_{\sigma(1)1} Z_{\sigma(2)2} \cdots Z_{\sigma(N)N}.$$

Here each Z_{ij} is an element of a (non-commutative) associative $\mathbb{C}$-algebra $\mathcal{A}$. This $C^{\mathfrak{gl}_N}_{\mathrm{det}}(u)$ is known to be central:

Theorem 1.1. *The element $C^{\mathfrak{gl}_N}_{\mathrm{det}}(u)$ is central in $U(\mathfrak{gl}_N)$ for any $u \in \mathbb{C}$.*

We will prove this using the "symmetrized determinant" soon.

This Capelli determinant has some good properties. For example, we can easily calculate its eigenvalue on irreducible representations:

Theorem 1.2. *For the irreducible representation $\pi_\lambda^{\mathfrak{gl}_N}$ of $\mathfrak{gl}_N$ determined by the partition $\lambda = (\lambda_1, \ldots, \lambda_N)$, the following relation holds*:

$$\pi_\lambda^{\mathfrak{gl}_N}\bigl(C_{\det}^{\mathfrak{gl}_N}(u)\bigr) = (u + l_1) \cdots (u + l_N).$$

Here we put $l_i = \lambda_i + N - i$.

This is immediate from the definition of the column-determinant and the following triangular decomposition of $\mathfrak{gl}_N$:

$$\mathfrak{gl}_N = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+. \tag{1.1}$$

Here $\mathfrak{n}^-$, $\mathfrak{h}$, and $\mathfrak{n}^+$ are the subalgebras of $\mathfrak{gl}_N$ spanned by the elements E_{ij} such that $i > j$, $i = j$, and $i < j$, respectively. Namely the entries in the lower triangular part, in the diagonal part, and in the upper triangular part of E belong to $\mathfrak{n}^-$, $\mathfrak{h}$, and $\mathfrak{n}^+$, respectively. Considering the action of $C_{\det}^{\mathfrak{gl}_N}(u)$ to the highest weight vector, we can easily check Theorem 1.2. However, Theorem 1.1 is not so trivial.

To show Theorem 1.1, we rewrite the Capelli determinant in terms of the "symmetrized determinant." For an $N \times N$ matrix $Z = (Z_{ij})$, we define $\operatorname{Det} Z$ by

$$\operatorname{Det} Z = \frac{1}{N!} \sum_{\sigma, \sigma' \in \mathfrak{S}_N} \operatorname{sgn}(\sigma) \operatorname{sgn}(\sigma') Z_{\sigma(1)\sigma'(1)} Z_{\sigma(2)\sigma'(2)} \cdots Z_{\sigma(N)\sigma'(N)}.$$

Moreover, for N parameters $a_1, \ldots, a_N \in \mathbb{C}$, we put

$$\begin{aligned} &\operatorname{Det}(Z; a_1, \ldots, a_N) \\ &\quad = \frac{1}{N!} \sum_{\sigma, \sigma' \in \mathfrak{S}_N} \operatorname{sgn}(\sigma) \operatorname{sgn}(\sigma') Z_{\sigma(1)\sigma'(1)}(a_1) Z_{\sigma(2)\sigma'(2)}(a_2) \cdots Z_{\sigma(N)\sigma'(N)}(a_N) \end{aligned}$$

with $Z_{ij}(a) = Z_{ij} + \delta_{ij} a$. We call this "Det" the "symmetrized determinant." This non-commutative determinant is useful to construct central elements in $U(\mathfrak{gl}_N)$. Indeed, we have the following proposition:

Proposition 1.3. *For any $a_1, \ldots, a_N \in \mathbb{C}$, the following is invariant under the adjoint action of GL_N, and hence this is central in $U(\mathfrak{gl}_N)$*:

$$\operatorname{Det}(E; a_1, \ldots, a_N).$$

This is immediate from the following two lemmas:

Lemma 1.4. *The symmetrized determinant is invariant under the conjugation by $g \in GL_N(\mathbb{C})$*:

$$\operatorname{Det}\bigl(gZg^{-1}; a_1, \ldots, a_N\bigr) = \operatorname{Det}(Z; a_1, \ldots, a_N).$$

Here Z is an arbitrary $N \times N$ matrix whose entries are elements of an associative $\mathbb{C}$-algebra $\mathcal{A}$.

Lemma 1.5. *The matrix E satisfies the following relation for any $g \in GL_N$:*

$$\mathrm{Ad}(g)E = {}^t g \cdot E \cdot {}^t g^{-1}.$$

Here $\mathrm{Ad}(g)E$ *means the matrix* $(\mathrm{Ad}(g)E_{ij})_{1\leqslant i,j\leqslant N}$.

Lemma 1.5 can be checked by a direct calculation. Lemma 1.4 is an easy consequence of (1.3) below (see [IU] for the proof).

Using the symmetrized determinant, we put

$$C^{\mathfrak{gl}_N}_{\mathrm{Det}}(u) = \mathrm{Det}(E + u\mathbf{1}_N; \natural_N) = \mathrm{Det}(E; u1_N + \natural_N).$$

Here $u1_N + \natural_N$ means the linear combination of the two vectors $1_N = (1, \dots, 1)$ and $\natural_N$ in $\mathbb{C}^N$. Namely we put $u1_N + \natural_N = (u + N - 1, u + N - 2, \dots, u)$. This $C^{\mathfrak{gl}_N}_{\mathrm{Det}}(u)$ is obviously central in $U(\mathfrak{gl}_N)$ for any $u \in \mathbb{C}$ by Proposition 1.3. However it is not so easy to calculate its eigenvalue directly.

Actually this $C^{\mathfrak{gl}_N}_{\mathrm{Det}}(u)$ coincides with the Capelli determinant $C^{\mathfrak{gl}_N}_{\det}(u)$:

Theorem 1.6. *We have*

$$\det(E + u\mathbf{1}_N + \mathrm{diag}\,\natural_N) = \mathrm{Det}(E + u\mathbf{1}_N; \natural_N).$$

We will prove this soon. Using this proposition, we can easily settle the following two problems at the same time: (i) the centrality of $C^{\mathfrak{gl}_N}_{\det}(u)$, and (ii) the calculation of the eigenvalue of $C^{\mathfrak{gl}_N}_{\mathrm{Det}}(u)$. Indeed, as seen above, the eigenvalue of $C^{\mathfrak{gl}_N}_{\det}(u)$ and the centrality of $C^{\mathfrak{gl}_N}_{\mathrm{Det}}(u)$ are almost obvious.

1.2. Let us recall the proof of Theorem 1.6 given in [IU]. We can regard this proof using the exterior calculus as the prototype of the main calculation of this paper.

Let $Z = (Z_{ij})$ be an $N \times N$ matrix whose entries are elements of a (non-commutative) associative $\mathbb{C}$-algebra $\mathcal{A}$. We can express the column-determinant in the framework of the exterior calculus as follows. Let $e_1, \dots, e_N$ be N anti-commuting formal variables, which generate the exterior algebra $\Lambda_N = \Lambda(\mathbb{C}^N)$. Put $\eta_j(u) = \sum_{i=1}^N e_i Z_{ij}(u)$ as an element in the extended algebra $\Lambda_N \otimes \mathcal{A}$ in which the two subalgebras Λ_N and $\mathcal{A}$ commute with each other. Then, by a direct calculation, we have the following equality [U2]:

$$\eta_1(a_1)\eta_2(a_2)\cdots\eta_N(a_N) = e_1 e_2 \cdots e_N \det\bigl(Z + \mathrm{diag}(a_1, a_2, \dots, a_N)\bigr). \tag{1.2}$$

The symmetrized determinant can be also expressed similarly by doubling the anti-commuting variables. Let $e_1, \dots, e_N, e_1^*, \dots, e_N^*$ be $2N$ anti-commuting formal variables, which generate the exterior algebra $\Lambda_{2N} = \Lambda(\mathbb{C}^N \oplus \mathbb{C}^N)$. We put $\Xi(u) = \sum_{i,j=1}^N e_i e_j^* Z_{ij}(u)$ in $\Lambda_{2N} \otimes \mathcal{A}$. Then, by a direct calculation, we have

$$\Xi(a_1)\Xi(a_2)\cdots\Xi(a_N) = e_1 e_1^* e_2 e_2^* \cdots e_N e_N^* N!\,\mathrm{Det}(Z; a_1, a_2, \dots, a_N). \tag{1.3}$$

Consider the case $Z = E$ to prove Theorem 1.6. Using the relation $[E_{ij}, E_{kl}] = \delta_{kj} E_{il} - \delta_{il} E_{kj}$, we can easily show that $\eta_i(u)$ satisfies the following commutation relation. This is the key of the proof:

Lemma 1.7. *We have*

$$\eta_i(u+1)\eta_j(u) + \eta_j(u+1)\eta_i(u) = 0.$$

In particular we have the following commutation relation for $\tilde{\eta}_i(u) = \eta_i(u)e_i^*$:

Lemma 1.8. *We have*

$$\tilde{\eta}_i(u+1)\tilde{\eta}_j(u) = \tilde{\eta}_j(u+1)\tilde{\eta}_i(u).$$

Proof of Theorem 1.6. By definition we have $\Xi(u) = \sum_{i=1}^N \tilde{\eta}_i(u)$. Hence we have

$$\begin{aligned}
&\Xi(u+N-1)\Xi(u+N-2)\cdots\Xi(u) \\
&\quad= \sum_{1 \leqslant i_1,\dots,i_N \leqslant N} \tilde{\eta}_{i_1}(u+N-1)\tilde{\eta}_{i_2}(u+N-2)\cdots\tilde{\eta}_{i_N}(u).
\end{aligned}$$

Since $\tilde{\eta}_i(u) = \eta_i(u)e_i^*$ contains an anti-commuting element e_i^*, each term in the right-hand side actually vanishes, unless $(i_1, \dots, i_N)$ is disjoint. Namely we can assume that $(i_1, \dots, i_N)$ is a permutation of $(1, 2, \dots, N)$. Moreover, using Lemma 1.8, we can reorder the factors $\tilde{\eta}_i(a)$ as follows:

$$\begin{aligned}
&\Xi(u+N-1)\Xi(u+N-2)\cdots\Xi(u) \\
&\quad= \sum_{\sigma \in \mathfrak{S}_N} \tilde{\eta}_{\sigma(1)}(u+N-1)\tilde{\eta}_{\sigma(2)}(u+N-2)\cdots\tilde{\eta}_{\sigma(N)}(u) \\
&\quad= N!\tilde{\eta}_1(u+N-1)\tilde{\eta}_2(u+N-2)\cdots\tilde{\eta}_N(u) \\
&\quad= (-)^{\frac{N(N-1)}{2}} N!\eta_1(u+N-1)\eta_2(u+N-2)\cdots\eta_N(u)e_1^*e_2^*\cdots e_N^*.
\end{aligned}$$

Compare this equality with (1.2) and (1.3), and we reach to the assertion. □

Remarks. (1) From the expression (1.3), we see that $\mathrm{Det}(Z; a_1, \dots, a_N)$ does not depend on the order of the parameters $a_1, \dots, a_N$. Indeed $\Xi_Z(a_i)$'s commute with each other. Lemma 1.4 is also immediate from this expression (see [IU,I1] for the details).

(2) As seen above, the key point of this proof is the following equality:

$$\tilde{\eta}_{\sigma(1)}(u+N-1)\cdots\tilde{\eta}_{\sigma(N)}(u) = \tilde{\eta}_1(u+N-1)\cdots\tilde{\eta}_N(u).$$

This is equivalent to the following relation between column-determinants:

$$\det(E + u\mathbf{1}_N + \operatorname{diag}\natural_N) = \operatorname{sgn}(\sigma)\det(EI_\sigma + uI_\sigma + I_\sigma \operatorname{diag}\natural_N).$$

Here I_σ means the matrix $I_\sigma = (\delta_{i\sigma(j)})_{1\leqslant i,j\leqslant N}$ determined by $\sigma \in \mathfrak{S}_N$. Theorem 1.3 is immediate from this, because we can express the symmetrized determinant as follows:

$$\mathrm{Det}(Z; a_1, \ldots, a_N) = \frac{1}{N!} \sum_{\sigma\in\mathfrak{S}_N} \mathrm{sgn}(\sigma) \det\bigl(Z I_\sigma + I_\sigma \,\mathrm{diag}(a_1, \ldots, a_N)\bigr).$$

2. General realizations of $\mathfrak{o}_N$

Next let us consider the case of the orthogonal Lie algebra $\mathfrak{o}_N$. Let $S \in \mathrm{Mat}_N(\mathbb{C})$ be a non-degenerate symmetric matrix of size N. We can realize the orthogonal Lie group as the isometry group with respect to the bilinear form determined by S:

$$O(S) = \bigl\{ g \in GL_N \bigm| {}^t g S g = S \bigr\}.$$

The corresponding Lie algebra is expressed as follows:

$$\mathfrak{o}(S) = \bigl\{ Z \in \mathfrak{gl}_N \bigm| {}^t Z S + S Z = 0 \bigr\}.$$

As generators of this $\mathfrak{o}(S)$, we take

$$F_{ij}^{\mathfrak{o}(S)} = E_{ij} - S^{-1} E_{ji} S = E_{ij} - \sum_{1\leqslant a,b\leqslant N} S^{ja} E_{ab} S_{bi}.$$

Here S_{ij} and S^{ij} mean the entries of the matrices S and S^{-1}, respectively. A direct calculation shows the following commutation relation:

$$\bigl[F_{ij}^{\mathfrak{o}(S)}, F_{kl}^{\mathfrak{o}(S)}\bigr] = F_{il}^{\mathfrak{o}(S)} \delta_{kj} - F_{kj}^{\mathfrak{o}(S)} \delta_{il} + \sum_{a=1}^{N} F_{ka}^{\mathfrak{o}(S)} S_{ai} S^{lj} + \sum_{a=1}^{N} S^{la} F_{aj}^{\mathfrak{o}(S)} S_{ki}. \tag{2.1}$$

We arrange the matrix $F^{\mathfrak{o}(S)}$ whose (i,j)th entry is $F_{ij}^{\mathfrak{o}(S)}$. Namely we put $F^{\mathfrak{o}(S)} = (F_{ij}^{\mathfrak{o}(S)})_{1\leqslant i,j\leqslant N}$. By a direct calculation, this $F^{\mathfrak{o}(S)}$ satisfies the following relation:

Lemma 2.1. *For any $g \in O(S)$, we have*

$$\mathrm{Ad}(g) F^{\mathfrak{o}(S)} = {}^t g \cdot F^{\mathfrak{o}(S)} \cdot {}^t g^{-1}.$$

Here $\mathrm{Ad}(g) F^{\mathfrak{o}(S)}$ *means the matrix* $(\mathrm{Ad}(g) F_{ij}^{\mathfrak{o}(S)})_{1\leqslant i,j\leqslant N}$.

Combining this and Lemma 1.4, we have the following proposition:

Proposition 2.2. *The following determinant is invariant under the adjoint action of $O(S)$, and in particular this is central in $U(\mathfrak{o}(S))$:*

$$\mathrm{Det}\bigl(F^{\mathfrak{o}(S)}; a_1, \ldots, a_N\bigr).$$

Thus the symmetrized determinant is useful to obtain central elements of $U(\mathfrak{o}(S))$ as in the case of $\mathfrak{gl}_N$. However, unfortunately, it seems not so easy to construct central elements of $U(\mathfrak{o}(S))$ using the column-determinant at least for general S. Indeed, we do not have any good relation between the column-determinants and the symmetrized determinants of $F^{\mathfrak{o}(S)}$ like Theorem 1.6. To see this, let us try to imitate the proof of Theorem 1.6. Let $e_1, \dots, e_N, e_1^*, \dots, e_N^*$ be the standard generators of the exterior algebra $\Lambda_{2N} = \Lambda(\mathbb{C}^N \oplus \mathbb{C}^N)$, and put $\eta_j(u) = \sum_{i=1}^N e_i F_{ij}^{\mathfrak{o}(S)}(u)$ as an element of the extended algebra $\Lambda_{2N} \otimes U(\mathfrak{o}(S))$. Then, using the relation (2.1), we see the following commutation relation:

Lemma 2.3. *We have*

$$\eta_i(u+1)\eta_j(u) + \eta_j(u+1)\eta_i(u) = -\Theta S^{ij}.$$

Here we put $\Theta = \sum_{i,j,a=1}^N e_i e_j F_{ia}^{\mathfrak{o}(S)} S_{aj}$.

This is similar to Lemma 1.7, but there is an obstacle in the right-hand side. Thus, it is difficult to imitate the proof of Theorem 1.6 any more.

However, for some special S, we have analogues of the Capelli determinant expressed in terms of the column-determinant. We will actually see the case $S = \mathbf{1}_N$ in Section 3, and the case $S = S_0 = (\delta_{i,N+1-j})_{1 \leqslant i,j \leqslant N}$ in Section 4.

Remarks. (1) We can express Θ using $\eta_i(u)$. Indeed, the following relation holds for any $w \in \mathbb{C}$:

$$\Theta = \sum_{j=1}^N \eta_a(w) S_{aj} e_j. \tag{2.2}$$

(2) It is also known that $\mathrm{Det}(F^{\mathfrak{o}(S)}; a_1, \dots, a_N)$ does not depend on S. Namely, for arbitrary non-degenerate symmetric matrices S_1 and S_2, the two determinants $\mathrm{Det}(F^{\mathfrak{o}(S_1)}; a_1, \dots, a_N)$ and $\mathrm{Det}(F^{\mathfrak{o}(S_2)}; a_1, \dots, a_N)$ coincide via the natural isomorphism $\mathfrak{o}(S_1) \simeq \mathfrak{o}(S_2)$ (see [IU]).

(3) When N is even, the following relation holds for any $a \in \mathbb{C}$ [IU]:

$$\mathrm{Det}\bigl(F^{\mathfrak{o}(S)}; \tfrac{N}{2} - 1, \tfrac{N}{2} - 2, \dots, -\tfrac{N}{2} + 1, a\bigr) = \det\bigl(S^{-1}\bigr)\, \mathrm{Pf}\bigl(F^{\mathfrak{o}(S)} S\bigr)^2.$$

Here we define the Pfaffian of $2n \times 2n$ alternating matrix $Z = (Z_{ij})$ by

$$\mathrm{Pf}\, Z = \frac{1}{2^n n!} \sum_{\sigma \in \mathfrak{S}_{2n}} \mathrm{sgn}(\sigma) Z_{\sigma(1)\sigma(2)} Z_{\sigma(3)\sigma(4)} \cdots Z_{\sigma(2n-1)\sigma(2n)}.$$

It is known that $\mathrm{Pf}(F^{\mathfrak{o}(S)} S)$ is also central. Moreover, this $\mathrm{Pf}(F^{\mathfrak{o}(S)} S)$ and the coefficients of $\mathrm{Det}(F^{\mathfrak{o}(S)}; a_1, \dots, a_N)$ as a polynomial in $a_1, \dots, a_N$ generate the center of $U(\mathfrak{o}(S))$ (see [IU] and [I1] for the details).

3. The case of $\mathfrak{o}(\mathbf{1}_N)$

In this section, we consider the case that S is equal to the unit matrix $\mathbf{1}_N$. Namely we consider the Lie algebra consisting of all alternating matrices:

$$\mathfrak{o}(\mathbf{1}_N) = \{Z \in \mathfrak{gl}_N \mid Z + {}^tZ = 0\}.$$

In the universal enveloping algebra $U(\mathfrak{o}(\mathbf{1}_N))$, Howe and Umeda gave an analogue of the Capelli determinant [HU]:

Theorem 3.1. (Howe and Umeda) *The following element is central in $U(\mathfrak{o}(\mathbf{1}_N))$ for any $u \in \mathbb{C}$:*

$$C^{\mathfrak{o}(\mathbf{1}_N)}_{\mathrm{det}}(u) = \det\bigl(F^{\mathfrak{o}(\mathbf{1}_N)} + u\mathbf{1}_N + \operatorname{diag} \natural_N\bigr).$$

This $C^{\mathfrak{o}(\mathbf{1}_N)}_{\mathrm{det}}(u)$ is quite similar to the Capelli determinant $C^{\mathfrak{gl}_N}_{\mathrm{det}}(u)$. However, it is not easy to calculate the eigenvalue of $C^{\mathfrak{o}(\mathbf{1}_N)}_{\mathrm{det}}(u)$. Indeed, for this realization $\mathfrak{o}(\mathbf{1}_N)$, we cannot take its triangular decomposition so simply as (1.1).

As in the case of $\mathfrak{gl}_N$, we can rewrite this in terms of the symmetrized determinant:

Theorem 3.2. *We have*

$$\det\bigl(F^{\mathfrak{o}(\mathbf{1}_N)} + u\mathbf{1}_N + \operatorname{diag} \natural_N\bigr) = \operatorname{Det}\bigl(F^{\mathfrak{o}(\mathbf{1}_N)} + u\mathbf{1}_N; \natural_N\bigr).$$

Theorem 3.1 is immediate from this Theorem 3.2. Indeed, by Proposition 2.2, the following element is central in $U(\mathfrak{o}(\mathbf{1}_N))$ for any $u \in \mathbb{C}$:

$$C^{\mathfrak{o}(\mathbf{1}_N)}_{\mathrm{Det}}(u) = \operatorname{Det}\bigl(F^{\mathfrak{o}(\mathbf{1}_N)} + u\mathbf{1}_N; \natural_N\bigr) = \operatorname{Det}\bigl(F^{\mathfrak{o}(\mathbf{1}_N)}; u1_N + \natural_N\bigr).$$

To show Theorem 3.2 in a way similar to the proof of Theorem 1.6, we put $\eta_j(u) = \sum_{i=1}^N e_i F^{\mathfrak{o}(\mathbf{1}_N)}_{ij}(u)$ in $\Lambda_{2N} \otimes U(\mathfrak{o}(\mathbf{1}_N))$. Then, as a special case of Lemma 2.3, we have the following commutation relation:

Lemma 3.3. *We have*

$$\eta_i(u+1)\eta_j(u) + \eta_j(u+1)\eta_i(u) = -\Theta\delta_{ij}.$$

This is a bit more complicated than Lemma 1.7. However we can remove the obstacle Θ by multiplying the anti-commuting factor e_i^*. Namely $\tilde{\eta}_i(u) = \eta_i(u)e_i^*$ satisfies the following commutation relation:

Lemma 3.4. *We have*

$$\tilde{\eta}_i(u+1)\tilde{\eta}_j(u) = \tilde{\eta}_j(u+1)\tilde{\eta}_i(u).$$

This is equal to Lemma 1.8 in the case of $\mathfrak{gl}_N$. Therefore we can prove Theorem 3.2 in the same way as the proof of Theorem 1.6 (see [IU] for the details).

4. The case of $\mathfrak{o}(S_0)$

Now, we turn to the main subject of this paper. Namely, in this section, we consider the split realization of the orthogonal Lie algebra, the case $S = S_0 = (\delta_{i,N+1-j})$. For convenience we introduce the symbol $i' = N + 1 - i$. Then $\mathfrak{o}(S_0)$ is expressed as follows:

$$\mathfrak{o}(S_0) = \{ Z = (Z_{ij}) \in \mathfrak{gl}_N \mid Z_{ij} + Z_{j'i'} = 0 \}.$$

4.1. A central element of $U(\mathfrak{o}(S_0))$ expressed in terms of the column-determinant was recently given in [W]:

Theorem 4.1. (Wachi) *The following element is central in $U(\mathfrak{o}(S_0))$ for any $u \in \mathbb{C}$:*

$$C^{\mathfrak{o}(S_0)}_{\det}(u) = \det\big(F^{\mathfrak{o}(S_0)} + u\mathbf{1}_N + \operatorname{diag} \tilde{\natural}_N\big).$$

Here we put

$$\tilde{\natural}_N = \begin{cases} (\frac{N}{2} - 1, \frac{N}{2} - 2, \dots, 0, 0, \dots, -\frac{N}{2} + 1), & N\colon \textit{even}, \\ (\frac{N}{2} - 1, \frac{N}{2} - 2, \dots, \frac{1}{2}, 0, -\frac{1}{2}, \dots, -\frac{N}{2} + 1), & N\colon \textit{odd}. \end{cases}$$

The dots mean arithmetic progressions with difference -1.

This element is remarkable. Indeed, the eigenvalue of this $C^{\mathfrak{o}(S_0)}_{\det}(u)$ can be calculated as easily as that of the Capelli determinant $C^{\mathfrak{gl}_N}_{\det}(u)$:

Theorem 4.2. (Wachi) *Let $\pi^{\mathfrak{o}(S_0)}_\lambda$ be the irreducible representation of $\mathfrak{o}(S_0)$ determined by the partition $\lambda = (\lambda_1, \dots, \lambda_{[n]})$, where $[n]$ means the greatest integer not exceeding $n = N/2$. Then the following relation holds*:

$$\pi^{\mathfrak{o}(S_0)}_\lambda\big(C^{\mathfrak{o}(S_0)}_{\det}(u)\big) = \begin{cases} (u^2 - l_1^2) \cdots (u^2 - l_n^2), & N\colon \textit{even}, \\ u(u^2 - l_1^2) \cdots (u^2 - l_{[n]}^2), & N\colon \textit{odd}. \end{cases}$$

Here we put $l_i = \lambda_i + n - i$.

The proof is almost the same as that of Theorem 1.2. Namely this is easy from the definition of the column-determinant and the following triangular decomposition of $\mathfrak{o}(S_0)$:

$$\mathfrak{o}(S_0) = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+.$$

Here $\mathfrak{n}^-$, $\mathfrak{h}$, and $\mathfrak{n}^+$ are the subalgebras of $\mathfrak{o}(S_0)$ spanned by the elements $F^{\mathfrak{o}(S_0)}_{ij}$ such that $i > j$, $i = j$, and $i < j$, respectively. Namely, the entries in the lower triangular part, in the diagonal part, and in the upper triangular part of the matrix $F^{\mathfrak{o}(S_0)}$ belong to $\mathfrak{n}^-$, $\mathfrak{h}$, and $\mathfrak{n}^+$, respectively.

This central element can be rewritten in terms of the symmetrized determinant:

Theorem 4.3. (Wachi) *We have*

$$\det\big(F^{\mathfrak{o}(S_0)} + u\mathbf{1}_N + \operatorname{diag} \tilde{\natural}_N\big) = \operatorname{Det}\big(F^{\mathfrak{o}(S_0)} + u\mathbf{1}_N; \tilde{\natural}_N\big).$$

This theorem was first shown by comparing the eigenvalues of both sides by A. Wachi in [W]. Namely, Theorem 4.3 is immediate from Theorem 4.1 and the following Theorem 4.4 for the eigenvalue of $C^{\mathfrak{o}(S_0)}_{\mathrm{Det}}(u) = \mathrm{Det}(F^{\mathfrak{o}(S_0)} + u\mathbf{1}_N; \tilde{\natural}_N)$. Indeed, the eigenvalue of $C^{\mathfrak{o}(S_0)}_{\det}(u)$ and the centrality of $C^{\mathfrak{o}(S_0)}_{\mathrm{Det}}(u)$ are obvious.

Theorem 4.4. *For the representation $\pi^{\mathfrak{o}(S_0)}_\lambda$, the following relation holds*:

$$\pi^{\mathfrak{o}(S_0)}_\lambda\big(C^{\mathfrak{o}(S_0)}_{\mathrm{Det}}(u)\big) = \begin{cases} (u^2 - l_1^2)\cdots(u^2 - l_{N/2}^2), & N\text{: even},\\ u(u^2 - l_1^2)\cdots(u^2 - l_{[N/2]}^2), & N\text{: odd}. \end{cases}$$

This Theorem 4.4 was given in [I1] through a hard and complicated calculation.

The aim of this paper is to give a new and straight proof for Theorem 4.3. Namely, we will prove this theorem directly not using Theorems 4.1 and 4.4. Conversely these Theorems 4.1 and 4.4 follow from Theorem 4.3.

4.2. To prove Theorem 4.3, we put $\eta_j(u) = \sum_{i=1}^N e_i F^{\mathfrak{o}(S_0)}_{ij}(u)$ in $\Lambda_{2N} \otimes U(\mathfrak{o}(S_0))$ as in the previous sections. This satisfies the following commutation relation as a special case of Lemma 2.3:

Lemma 4.5. *We have*

$$\eta_i(u+1)\eta_j(u) + \eta_j(u+1)\eta_i(u) = -\Theta\delta_{i,j'}$$

with $\Theta = \sum_{i,j=1}^N e_i e_{j'} F^{\mathfrak{o}(S_0)}_{ij}$.

In particular, $\tilde{\eta}_j(u) = \eta_j(u)e_j^*$ satisfies the following relation:

Corollary 4.6. *We have*

$$\tilde{\eta}_i(u+1)\tilde{\eta}_j(u) - \tilde{\eta}_j(u+1)\tilde{\eta}_i(u) = \Theta e_i^* e_j^* \delta_{i,j'}.$$

By (1.2) and (1.3), the main theorem Theorem 4.3 can be rewritten as the following relation for $\tilde{\eta}_j(u)$ and $\Xi(u) = \sum_{i,j=1}^N e_i e_j^* F^{\mathfrak{o}(S_0)}_{ij}(u)$:

Theorem 4.7. *We have*

$$\Xi^N\big(u1_N + \tilde{\natural}_N\big) = N!\tilde{\eta}^N_{1,2,\ldots,N}\big(u1_N + \tilde{\natural}_N\big).$$

Here the symbols $\Xi^k(a_1,\ldots,a_k)$ and $\tilde{\eta}^k_{i_1,\ldots,i_k}(a_1,\ldots,a_k)$ mean

$$\Xi^k(a_1,\ldots,a_k) = \Xi(a_1)\cdots\Xi(a_k), \qquad \tilde{\eta}^k_{i_1,\ldots,i_k}(a_1,\ldots,a_k) = \tilde{\eta}_{i_1}(a_1)\cdots\tilde{\eta}_{i_k}(a_k).$$

By the relation $\Xi(u) = \sum_{j=1}^N \tilde{\eta}_j(u)$, the left-hand side of Theorem 4.7 is equal to

$$\Xi^N\big(u1_N + \tilde{\natural}_N\big) = \sum_{\sigma\in\mathfrak{S}_N} \tilde{\eta}^N_{\sigma(1),\ldots,\sigma(N)}\big(u1_N + \tilde{\natural}_N\big).$$

Indeed $\tilde{\eta}_j(u) = \eta_j(u)e_j^*$ contains the anti-commuting factor e_j^*. Thus we can prove Theorem 4.7 by studying the relations among $\tilde{\eta}^N_{\sigma(1),\dots,\sigma(N)}(u1_N + \tilde{\mathfrak{h}}_N)$. Some of them are obviously equal to each other. For example we have

$$\tilde{\eta}^N_{\sigma(1),\dots,\sigma(N)}\big(u1_N + \tilde{\mathfrak{h}}_N\big) = \tilde{\eta}^N_{1,\dots,N}\big(u1_N + \tilde{\mathfrak{h}}_N\big), \tag{4.1}$$

if $\sigma(i) \leqslant N/2$ for $i \leqslant N/2$, and $\sigma(i) \geqslant N/2+1$ for $i \geqslant N/2+1$. This is easy from Corollary 4.6. This (4.1) can be generalized a bit more (see Lemmas 5.3 and 6.3 below), but does not hold for general $\sigma \in \mathfrak{S}_N$. Indeed, we have the following counterexample, when $N = 4$:

$$\tilde{\eta}_1(u+1)\tilde{\eta}_4(u)\tilde{\eta}_2(u)\tilde{\eta}_3(u-1) \neq \tilde{\eta}_1(u+1)\tilde{\eta}_2(u)\tilde{\eta}_3(u)\tilde{\eta}_4(u-1).$$

Thus, our situation is not so simple.

4.3. As seen in Lemma 4.5 and Corollary 4.6, the commutation relations for $\eta_j(u)$ and $\tilde{\eta}_j(u)$ are a bit more complicated than the cases of $\mathfrak{gl}_N$ and $\mathfrak{o}(\mathbf{1}_N)$. To prove Theorem 4.7, it is convenient to prepare some revised versions of these relations:

Corollary 4.8. *When $k \neq k'$, we have*

$$\eta_k(u+1)\eta_k(u) = 0.$$

Corollary 4.9. *For arbitrary $u, w \in \mathbb{C}$, we have*

$$\eta_k(u)\eta_{k'}(u) + \eta_{k'}(u)\eta_k(u) = -\sum_{a \neq k,k'} \eta_a(w)e_{a'}.$$

Corollary 4.10. *Assume that $i_l \neq i'_l$ for $l = 1,\dots,k$. Then, unless $i_1,\dots,i_k$ are disjoint, we have*

$$\eta_{i_1}(u)\eta_{i_2}(u-1)\cdots\eta_{i_k}(u-k+1) = 0.$$

Corollary 4.11. *For arbitrary $u, w \in \mathbb{C}$, we have*

$$\tilde{\eta}_k(u)\tilde{\eta}_{k'}(u) = \tilde{\eta}_{k'}(u)\tilde{\eta}_k(u) + \sum_{a \neq k,k'} \eta_a(w)e_{a'}e_k^*e_{k'}^*.$$

Here, Corollary 4.8 is immediate from Lemma 4.5. Corollary 4.9 is also clear from Lemma 4.5, because we have the relation $\Theta = \sum_{j=1}^N \eta_j(w)e_{j'}$ as a special case of (2.2). Corollaries 4.10 and 4.11 are easy consequences of Corollaries 4.8 and 4.9, respectively.

Moreover, we consider an element $\Omega_j(u)$ playing as a mediator between $\Xi(u)$ and $\tilde{\eta}_j(u)$:

$$\Omega_j(u) = \tilde{\eta}_j(u) + \tilde{\eta}_{j'}(u). \tag{4.2}$$

Since $\Omega_j(u) = \Omega_{j'}(u)$, we can express $\Xi(u) = \sum_{j=1}^N \tilde{\eta}_j(u)$ as

$$\Xi(u) = \begin{cases} \Omega_1(u) + \cdots + \Omega_n(u), & N\text{: even,} \\ \Omega_1(u) + \cdots + \Omega_{[n]}(u) + \frac{1}{2}\Omega_{[n]+1}(u), & N\text{: odd.} \end{cases}$$

Here we put $n = N/2$. By Corollary 4.6, this $\Omega_j(u)$ satisfies the following commutation relation:

Corollary 4.12. *We have*

$$\Omega_k(u+1)\Omega_l(u) = \Omega_l(u+1)\Omega_k(u).$$

This is as simple as the commutation relations of $\tilde{\eta}_i$'s in the cases of $\mathfrak{gl}_N$ and $\mathfrak{o}(\mathbf{1}_N)$. Theorem 4.7 (and hence the main theorem Theorem 4.3) is proved by combining these simple commutation relations as seen in the following two sections.

5. Proof in the case that N is even

In this section, we prove the main theorem when N is even, namely when $n = N/2$ is an integer. In this case, our goal Theorem 4.7 is immediate from the following two relations:

Proposition 5.1. *We have*

$$\Xi^N\big(u1_N + \tilde{\mathfrak{h}}_N\big) = 2^{-n}N!\,\Omega^N_{1,\dots,N}\big(u1_N + \tilde{\mathfrak{h}}_N\big).$$

Here $\Omega^k_{i_1,\dots,i_k}(a_1,\dots,a_k)$ denotes $\Omega_{i_1}(a_1)\cdots\Omega_{i_k}(a_k)$.

Proposition 5.2. *We have*

$$\Omega^N_{1,\dots,N}\big(u1_N + \tilde{\mathfrak{h}}_N\big) = 2^n\tilde{\eta}^N_{1,\dots,N}\big(u1_N + \tilde{\mathfrak{h}}_N\big).$$

Since $\Omega_j(u) = \Omega_{j'}(u)$, we can rewrite Proposition 5.2 as follows:

$$\Omega^N_{1,2,\dots,n,n,\dots,2,1}\big(u1_N + \tilde{\mathfrak{h}}_N\big) = 2^n\tilde{\eta}^N_{1,\dots,n,n',\dots,1'}\big(u1_N + \tilde{\mathfrak{h}}_N\big).$$

In the remainder of this section, we will prove these two propositions.

5.1. First, let us prove Proposition 5.2 using Corollaries 4.8–4.11. Since the element $\tilde{\eta}_j(u) = \eta_j(u)e^*_j$ contains the anti-commuting factor e^*_j, the following expansion follows from (4.2):

$$\Omega^N_{1,2,\dots,n,n,\dots,2,1}\big(u1_N + \tilde{\mathfrak{h}}_N\big) = \sum_{\varepsilon_1,\dots,\varepsilon_n} \tilde{\eta}^N_{\varepsilon_1,\dots,\varepsilon_n,\varepsilon'_n,\dots,\varepsilon'_1}\big(u1_N + \tilde{\mathfrak{h}}_N\big). \tag{5.1}$$

Here the right-hand side is the sum of $\tilde{\eta}^N_{\varepsilon_1,\dots,\varepsilon_n,\varepsilon'_n,\dots,\varepsilon'_1}(u1_N + \tilde{\mathfrak{h}}_N)$ over $\varepsilon_1 \in \{1,1'\}$, $\varepsilon_2 \in \{2,2'\}$, $\dots$, $\varepsilon_n \in \{n,n'\}$ (recall that ε'_i means $\varepsilon'_i = N+1-\varepsilon_i$). For example, when $N = 4$, this means

$$\begin{aligned}&\Omega^4_{1,2,3,4}\big(u1_4 + \tilde{\mathfrak{h}}_4\big)\\ &\quad= \tilde{\eta}^4_{1,2,3,4}\big(u1_4 + \tilde{\mathfrak{h}}_4\big) + \tilde{\eta}^4_{1,3,2,4}\big(u1_4 + \tilde{\mathfrak{h}}_4\big) + \tilde{\eta}^4_{4,2,3,1}\big(u1_4 + \tilde{\mathfrak{h}}_4\big) + \tilde{\eta}^4_{4,3,2,1}\big(u1_4 + \tilde{\mathfrak{h}}_4\big).\end{aligned}$$

Thus, to prove Proposition 5.2, it is suffices to show the following relation:

Lemma 5.3. *For any $\varepsilon_1 \in \{1, 1'\}, \ldots, \varepsilon_n \in \{n, n'\}$, we have*

$$\tilde{\eta}^N_{\varepsilon_1,\ldots,\varepsilon_n,\varepsilon'_n,\ldots,\varepsilon'_1}\big(u1_N + \tilde{\mathfrak{h}}_N\big) = \tilde{\eta}^N_{1,\ldots,n,n',\ldots,1'}\big(u1_N + \tilde{\mathfrak{h}}_N\big).$$

Namely $\tilde{\eta}^N_{\varepsilon_1,\ldots,\varepsilon_n,\varepsilon'_n,\ldots,\varepsilon'_1}(u1_N + \tilde{\mathfrak{h}}_N)$ does not depend on $\varepsilon_1, \ldots, \varepsilon_n$.

Proof. The left-hand side of the assertion is expressed as

$$\tilde{\eta}^N_{\varepsilon_1,\ldots,\varepsilon_n,\varepsilon'_n,\ldots,\varepsilon'_1}\big(u1_N + \tilde{\mathfrak{h}}_N\big) = P_{\varepsilon_1,\ldots,\varepsilon_n}(u) Q_{\varepsilon'_n,\ldots,\varepsilon'_1}(u),$$

where $P_{i_1,\ldots,i_n}(u)$ and $Q_{i_1,\ldots,i_n}(u)$ mean

$$P_{i_1,\ldots,i_n}(u) = \tilde{\eta}^n_{i_1,\ldots,i_n}(u+n-1, \ldots, u), \qquad Q_{i_1,\ldots,i_n}(u) = \tilde{\eta}^n_{i_1,\ldots,i_n}(u, \ldots, u-n+1).$$

Here the dots in the parentheses mean arithmetic progressions with difference -1. Our aim is to prove that $P_{\varepsilon_1,\ldots,\varepsilon_n}(u) Q_{\varepsilon'_n,\ldots,\varepsilon'_1}(u)$ does not depend on $\varepsilon_1, \ldots, \varepsilon_n$. Namely it is sufficient to show that we can exchange ε_k and ε'_k for any k:

$$P_{\varepsilon_1,\ldots,\varepsilon_n}(u) Q_{\varepsilon'_n,\ldots,\varepsilon'_1}(u) = P_{\varepsilon_1,\ldots,\varepsilon_{k-1},\varepsilon'_k,\varepsilon_{k+1},\ldots,\varepsilon_n}(u) Q_{\varepsilon'_n,\ldots,\varepsilon'_{k+1},\varepsilon_k,\varepsilon'_{k-1},\ldots,\varepsilon'_1}(u). \tag{5.2}$$

To show this, we put

$$P^\dagger_{i_1,\ldots,i_{n-1}}(u) = \tilde{\eta}^{n-1}_{i_1,\ldots,i_{n-1}}(u+n-1, \ldots, u+1),$$
$$Q^\dagger_{i_1,\ldots,i_{n-1}}(u) = \tilde{\eta}^{n-1}_{i_1,\ldots,i_{n-1}}(u-1, \ldots, u-n+1).$$

Then, by Corollary 4.6, we have

$$P_{\varepsilon_1,\ldots,\varepsilon_n}(u) = P^\dagger_{\varepsilon_1,\ldots,\varepsilon_{k-1},\varepsilon_{k+1},\ldots,\varepsilon_n}(u)\tilde{\eta}_{\varepsilon_k}(u),$$
$$P_{\varepsilon_1,\ldots,\varepsilon_{k-1},\varepsilon'_k,\varepsilon_{k+1},\ldots,\varepsilon_n}(u) = P^\dagger_{\varepsilon_1,\ldots,\varepsilon_{k-1},\varepsilon_{k+1},\ldots,\varepsilon_n}(u)\tilde{\eta}_{\varepsilon'_k}(u),$$
$$Q_{\varepsilon'_n,\ldots,\varepsilon'_1}(u) = \tilde{\eta}_{\varepsilon'_k}(u) Q^\dagger_{\varepsilon'_n,\ldots,\varepsilon'_{k+1},\varepsilon'_{k-1},\ldots,\varepsilon'_1}(u),$$
$$Q_{\varepsilon'_n,\ldots,\varepsilon'_{k+1},\varepsilon_k,\varepsilon'_{k-1},\ldots,\varepsilon'_1}(u) = \tilde{\eta}_{\varepsilon_k}(u) Q^\dagger_{\varepsilon'_n,\ldots,\varepsilon'_{k+1},\varepsilon'_{k-1},\ldots,\varepsilon'_1}(u).$$

Noting this, we consider the difference between both sides of (5.2):

$$\begin{aligned} &P_{\varepsilon_1,\ldots,\varepsilon_n}(u) Q_{\varepsilon'_n,\ldots,\varepsilon'_1}(u) - P_{\varepsilon_1,\ldots,\varepsilon_{k-1},\varepsilon'_k,\varepsilon_{k+1},\ldots,\varepsilon_n}(u) Q_{\varepsilon'_n,\ldots,\varepsilon'_{k+1},\varepsilon_k,\varepsilon'_{k-1},\ldots,\varepsilon'_1}(u) \\ &\quad = P^\dagger_{\varepsilon_1,\ldots,\varepsilon_{k-1},\varepsilon_{k+1},\ldots,\varepsilon_n}(u)\big\{\tilde{\eta}_{\varepsilon_k}(u)\tilde{\eta}_{\varepsilon'_k}(u) - \tilde{\eta}_{\varepsilon'_k}(u)\tilde{\eta}_{\varepsilon_k}(u)\big\} Q^\dagger_{\varepsilon'_n,\ldots,\varepsilon'_{k+1},\varepsilon'_{k-1},\ldots,\varepsilon'_1}(u). \end{aligned}$$

By Corollary 4.11, this is equal to

$$\sum_{a \neq k,k'} P^\dagger_{\varepsilon_1,\ldots,\varepsilon_{k-1},\varepsilon_{k+1},\ldots,\varepsilon_n}(u)\eta_a(u) e_{a'} e^*_{\varepsilon_k} e^*_{\varepsilon'_k} Q^\dagger_{\varepsilon'_n,\ldots,\varepsilon'_{k+1},\varepsilon'_{k-1},\ldots,\varepsilon'_1}(u).$$

This is equal to zero, because each term vanishes by Corollary 4.10. Indeed $a \neq k, k'$ must be equal to some element of the following sequence:

$$\varepsilon_1, \dots, \varepsilon_{k-1}, \varepsilon_{k+1}, \dots, \varepsilon_n, \varepsilon'_n, \dots, \varepsilon'_{k+1}, \varepsilon'_{k-1}, \dots, \varepsilon'_1.$$

This means (5.2), and hence Lemma 5.3. □

Remark. We have a similar relation in the case that N is odd (Lemma 6.3). From these Lemmas 5.3 and 6.3, we see that the equality (4.1) also holds, when σ is generated by the transpositions of the form $(i\ i')$.

5.2. Next let us prove Proposition 5.1. This is deduced from Corollary 4.12 by a flat calculation as follows. We put

$$\varXi_k(u) = \varXi(u) - \{\varOmega_1(u) + \cdots + \varOmega_k(u)\} = \varOmega_{k+1}(u) + \cdots + \varOmega_n(u),$$

so that $\varXi_0(u) = \varXi(u)$. The following commutation relation is immediate from Corollary 4.12:

Lemma 5.4. *We have*

$$\varXi_k(u+1)\varOmega_l(u) = \varOmega_l(u+1)\varXi_k(u).$$

Moreover, we have the following lemma:

Lemma 5.5. *We have*

$$\begin{aligned}&\varXi_k(u+1)\varXi_k(u)\varOmega_l(u)\varOmega_l(u-1) + \varOmega_l(u+1)\varOmega_l(u)\varXi_k(u)\varXi_k(u-1)\\&\quad = 2\varOmega_l(u+1)\varXi_k(u)\varXi_k(u)\varOmega_l(u-1).\end{aligned}$$

Proof. We consider the following central elements in $\Lambda_{2N} \otimes U(\mathfrak{o}(S_0))$:

$$\xi_k = e_{k+1}e^*_{k+1} + \cdots + e_{(k+1)'}e^*_{(k+1)'}, \qquad \omega_l = e_l e^*_l + e_{l'}e^*_{l'}.$$

For these, the following relations hold:

$$\varXi_k(u+v) = \varXi_k(u) + v\xi_k, \qquad \varOmega_l(u+v) = \varOmega_l(u) + v\omega_l.$$

In particular, we have $\varOmega_l(u) = \varOmega_l(u \pm 1) \mp \omega_l$, so that

$$\begin{aligned}&\varXi_k(u+1)\varXi_k(u)\varOmega_l(u)\varOmega_l(u-1)\\&\quad = \varXi_k(u+1)\varXi_k(u)\varOmega_l(u-1)\varOmega_l(u-1) + \varXi_k(u+1)\varXi_k(u)\omega_l\varOmega_l(u-1),\end{aligned}$$

$$\begin{aligned}&\varOmega_l(u+1)\varOmega_l(u)\varXi_k(u)\varXi_k(u-1)\\&\quad = \varOmega_l(u+1)\varOmega_l(u+1)\varXi_k(u)\varXi_k(u-1) - \varOmega_l(u+1)\omega_l\varXi_k(u)\varXi_k(u-1).\end{aligned}$$

Here, by Lemma 5.4, the second terms of the right-hand sides are equal:

$$\Xi_k(u+1)\Xi_k(u)\omega_l\Omega_l(u-1) = \Omega_l(u+1)\omega_l\Xi_k(u)\Xi_k(u-1).$$

Thus we have

$$\begin{aligned}
&\Xi_k(u+1)\Xi_k(u)\Omega_l(u)\Omega_l(u-1) + \Omega_l(u+1)\Omega_l(u)\Xi_k(u)\Xi_k(u-1) \\
&\quad = \Xi_k(u+1)\Xi_k(u)\Omega_l(u-1)\Omega_l(u-1) + \Omega_l(u+1)\Omega_l(u+1)\Xi_k(u)\Xi_k(u-1).
\end{aligned}$$

By using Lemma 5.4 again, this is equal to

$$\Omega_l(u+1)\Xi_k(u+1)\Xi_k(u)\Omega_l(u-1) + \Omega_l(u+1)\Xi_k(u)\Xi_k(u-1)\Omega_l(u-1).$$

Moreover, since $\Xi_k(u \pm 1) = \Xi_k(u) \pm \xi_k$, this is equal to

$$\begin{aligned}
&2\Omega_l(u+1)\Xi_k(u)\Xi_k(u)\Omega_l(u-1) \\
&\qquad + \Omega_l(u+1)\xi_k\Xi_k(u)\Omega_l(u-1) - \Omega_l(u+1)\Xi_k(u)\xi_k\Omega_l(u-1) \\
&\quad = 2\Omega_l(u+1)\Xi_k(u)\Xi_k(u)\Omega_l(u-1).
\end{aligned}$$

This means the assertion. □

As a consequence of Lemmas 5.4 and 5.5, we have the following lemma:

Lemma 5.6. *We have*

$$\begin{aligned}
\Xi_k^{N-2k}\big(u1_{N-2k} + \tilde{\natural}_{N-2k}\big) &= \binom{N-2k}{2}\Omega_{k+1}(u+n-k-1) \\
&\quad \cdot \Xi_{k+1}^{N-2k-2}\big(u1_{N-2k-2} + \tilde{\natural}_{N-2k-2}\big) \cdot \Omega_{k+1}(u-n+k+1).
\end{aligned}$$

Here $\Xi_k^l(a_1, \ldots, a_l)$ denotes $\Xi_k(a_1) \cdots \Xi_k(a_l)$.

Before proving this lemma, we note two easy facts. First we have

$$\Omega_k(u_1)\varphi_1\Omega_k(u_2)\varphi_2\Omega_k(u_3) = 0 \tag{5.3}$$

for any $\varphi_1, \varphi_2 \in \Lambda_{2N} \otimes U(\mathfrak{o}(S_0))$. This is easy, because $\Omega_k(u)$ is an element of the ideal generated by two anti-commuting variables e_k^* and $e_{k'}^*$. Similarly, when $l > N-2k$, we have

$$\Xi_k(u_1)\varphi_1\Xi_k(u_2)\varphi_2 \cdots \Xi_k(u_{l-1})\varphi_{l-1}\Xi_k(u_l) = 0 \tag{5.4}$$

for any $\varphi_1, \ldots, \varphi_{l-1} \in \Lambda_{2N} \otimes U(\mathfrak{o}(S_0))$. Indeed $\Xi_k(u)$ is an element of the ideal generated by $N-2k$ anti-commuting variables $e_{k+1}^*, \ldots, e_{(k+1)'}^*$.

Proof of Lemma 5.6. By definition, we have $\Xi_k(u) = \Xi_{k+1}(u) + \Omega_{k+1}(u)$. Noting this and using Lemma 5.4, we have the following binomial expansion:

$$\begin{aligned}
&\Xi_k^{n-k}(u+n-k-1,\dots,u) = \Xi_k(u+n-k-1)\cdots\Xi_k(u)\\
&\quad = \big(\Xi_{k+1}(u+n-k-1)+\Omega_{k+1}(u+n-k-1)\big)\cdots\big(\Xi_{k+1}(u)+\Omega_{k+1}(u)\big)\\
&\quad = \sum_{l=0}^{n-k}\binom{n-k}{l}\Xi_{k+1}^{n-k-l}(u+n-k-1,\dots,u+l)\Omega_{k+1}^{l}(u+l-1,\dots,u).
\end{aligned}$$

Here, $\Omega_k^l(a_1,\dots,a_l)$ means $\Omega_k(a_1)\cdots\Omega_k(a_l)$, and the parameters form arithmetic progressions with difference -1. By (5.3), this is equal to

$$\begin{aligned}
&\binom{n-k}{0}\Xi_{k+1}^{n-k}(u+n-k-1,\dots,u)\\
&\qquad + \binom{n-k}{1}\Xi_{k+1}^{n-k-1}(u+n-k-1,\dots,u+1)\Omega_{k+1}(u)\\
&\qquad + \binom{n-k}{2}\Xi_{k+1}^{n-k-2}(u+n-k-1,\dots,u+2)\Omega_{k+1}(u+1)\Omega_{k+1}(u)\\
&\quad = \Xi_{k+1}^{n-k-2}(u+n-k-1,\dots,u+2)\cdot\bigg\{\binom{n-k}{0}\Xi_{k+1}(u+1)\Xi_{k+1}(u)\\
&\qquad + \binom{n-k}{1}\Xi_{k+1}(u+1)\Omega_{k+1}(u) + \binom{n-k}{2}\Omega_{k+1}(u+1)\Omega_{k+1}(u)\bigg\}.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\Xi_k^{n-k}(u,\dots,u-n+k+1)\\
&\quad = \bigg\{\binom{n-k}{0}\Xi_{k+1}(u)\Xi_{k+1}(u-1) + \binom{n-k}{1}\Omega_{k+1}(u)\Xi_{k+1}(u-1)\\
&\qquad + \binom{n-k}{2}\Omega_{k+1}(u)\Omega_{k+1}(u-1)\bigg\}\cdot\Xi_{k+1}^{n-k-2}(u-2,\dots,u-n+k+1).
\end{aligned}$$

Multiplying both sides of these two equalities, we have

$$\begin{aligned}
&\Xi_k^{N-2k}\big(u1_{N-2k}+\tilde{\natural}_{N-2k}\big)\\
&\quad = \Xi_k^{n-k}(u+n-k-1,\dots,u)\cdot\Xi_k^{n-k}(u,\dots,u-n+k+1)\\
&\quad = \Xi_{k+1}^{n-k-2}(u+n-k-1,\dots,u+2)\\
&\qquad \cdot\bigg\{\binom{n-k}{1}^2\Xi_{k+1}(u+1)\Omega_{k+1}(u)\Omega_{k+1}(u)\Xi_{k+1}(u-1)\\
&\qquad + \binom{n-k}{2}\Omega_{k+1}(u+1)\Omega_{k+1}(u)\Xi_{k+1}(u)\Xi_{k+1}(u-1)\\
&\qquad + \binom{n-k}{2}\Xi_{k+1}(u+1)\Xi_{k+1}(u)\Omega_{k+1}(u)\Omega_{k+1}(u-1)\bigg\}\\
&\qquad \cdot\Xi_{k+1}^{n-k-2}(u-2,\dots,u-n+k+1).
\end{aligned}$$

Indeed, by (5.3) and (5.4), the sum of the exponents of Ξ_{k+1} must be equal to $N-2k-2$, and the degree of Ω_{k+1} must be equal to 2. By Lemma 5.4, the first term in the braces is equal to

$$\binom{n-k}{1}^2 \Omega_{k+1}(u+1)\Xi_{k+1}(u)\Xi_{k+1}(u)\Omega_{k+1}(u-1).$$

Moreover, using Lemma 5.5, we can rewrite the second term and the third term as

$$2\binom{n-k}{2}\Omega_{k+1}(u+1)\Xi_{k+1}(u)\Xi_{k+1}(u)\Omega_{k+1}(u-1).$$

Since $\binom{N-2k}{2} = \binom{n-k}{1}^2 + 2\binom{n-k}{2}$, we have

$$\begin{aligned}
&\Xi_k^{N-2k}\big(u1_{N-2k} + \tilde{\mathfrak{t}}_{N-2k}\big) \\
&\quad = \binom{N-2k}{2}\Xi_{k+1}^{n-k-2}(u+n-k-1,\dots,u+2) \\
&\qquad \cdot \Omega_{k+1}(u+1)\Xi_{k+1}(u)\Xi_{k+1}(u)\Omega_{k+1}(u-1) \\
&\qquad \cdot \Xi_{k+1}^{n-k-2}(u-2,\dots,u-n+k+1) \\
&\quad = \binom{N-2k}{2}\Omega_{k+1}(u+n-k-1)\Xi_{k+1}^{n-k-1}(u+n-k-2,\dots,u) \\
&\qquad \cdot \Xi_{k+1}^{n-k-1}(u,\dots,u-n+k+2)\Omega_{k+1}(u-n+k+1).
\end{aligned}$$

Here we used Lemma 5.4 for the second equality. This means the assertion. □

Using Lemma 5.6 repeatedly, we have

$$\begin{aligned}
\Xi_0^N\big(u1_N + \tilde{\mathfrak{t}}_N\big) &= \binom{N}{2}\Omega_1(u+n-1)\cdot \Xi_1^{N-2}\big(u1_{N-2} + \tilde{\mathfrak{t}}_{N-2}\big)\cdot \Omega_1(u-n+1) \\
&= \binom{N}{2}\binom{N-2}{2}\Omega_1(u+n-1)\Omega_2(u+n-2)\cdot \Xi_2^{N-4}\big(u1_{N-4} + \tilde{\mathfrak{t}}_{N-4}\big) \\
&\quad \cdot \Omega_2(u-n+2)\Omega_1(u-n+1) \\
&= \cdots \\
&= \binom{N}{2}\binom{N-2}{2}\cdots\binom{2}{2}\Omega_1(u+n-1)\Omega_2(u+n-2)\cdots\Omega_n(u) \\
&\quad \cdot \Omega_n(u)\cdots\Omega_2(u-n+2)\Omega_1(u-n+1).
\end{aligned}$$

This means Proposition 5.1.

Thus we have proved Theorem 4.7, and hence Theorem 4.3 in the case that N is even.

6. Proof in the case that N is odd

Next we consider the case that N is odd, namely the case that $n = N/2$ is a half integer (hence $N = 2[n] + 1$). The proof in this case is almost the same as the previous section, but we need some more discussion to deal with the special index $[n] + 1$, which satisfies the relation $([n] + 1)' = [n] + 1$. Also in this case, we aim to prove the following two relations. Our goal Theorem 4.7 is immediate from them.

Proposition 6.1. *We have*

$$\Xi^N\bigl(u1_N + \tilde{\natural}_N\bigr) = 2^{-[n]-1} N!\,\Omega^N_{1,\dots,N}\bigl(u1_N + \tilde{\natural}_N\bigr).$$

Proposition 6.2. *We have*

$$\Omega^N_{1,\dots,N}\bigl(u1_N + \tilde{\natural}_N\bigr) = 2^{[n]+1}\tilde{\eta}^N_{1,\dots,N}\bigl(u1_N + \tilde{\natural}_N\bigr).$$

6.1. First we prove Proposition 6.2 using Corollaries 4.8–4.11. We put

$$P_{i_1,\dots,i_{[n]}}(u) = \tilde{\eta}^{[n]}_{i_1,\dots,i_{[n]}}\bigl(u + n - 1, \dots, u + \tfrac{1}{2}\bigr),$$
$$Q_{i_1,\dots,i_{[n]}}(u) = \tilde{\eta}^{[n]}_{i_1,\dots,i_{[n]}}\bigl(u - \tfrac{1}{2}, \dots, u - n + 1\bigr).$$

Here the dots in the parentheses mean arithmetic progressions with difference -1. Then, we have the following relation as the counterpart of (5.1):

$$\begin{aligned}
&\Omega^N_{1,\dots,N}\bigl(u1_N + \tilde{\natural}_N\bigr)\\
&\quad = \Omega^{[n]}_{1,\dots,[n]}\bigl(u + n - 1, \dots, u + \tfrac{1}{2}\bigr) \cdot 2\tilde{\eta}_{[n]+1}(u) \cdot \Omega^{[n]}_{[n]',\dots,1'}\bigl(u - \tfrac{1}{2}, \dots, u - n + 1\bigr)\\
&\quad = 2\sum_{\varepsilon_1,\dots,\varepsilon_{[n]}} P_{\varepsilon_1,\dots,\varepsilon_{[n]}}(u) \cdot \tilde{\eta}_{[n]+1}(u) \cdot Q_{\varepsilon'_{[n]},\dots,\varepsilon'_1}(u).
\end{aligned} \tag{6.1}$$

Here the summation is taken over $\varepsilon_1 \in \{1, 1'\}, \varepsilon_2 \in \{2, 2'\}, \dots, \varepsilon_{[n]} \in \{[n], [n]'\}$. Hence, to prove Proposition 6.2, it suffices to show the following lemma:

Lemma 6.3. *When $\varepsilon_1 \in \{1, 1'\}, \dots, \varepsilon_{[n]} \in \{[n], [n]'\}$, we have*

$$\begin{aligned}
P_{\varepsilon_1,\dots,\varepsilon_{[n]}}(u) \cdot \tilde{\eta}_{[n]+1}(u) \cdot Q_{\varepsilon'_{[n]},\dots,\varepsilon'_1}(u) &= P_{1,\dots,[n]}(u) \cdot \tilde{\eta}_{[n]+1}(u) \cdot Q_{[n]',\dots,1'}(u)\\
&= \tilde{\eta}^N_{1,\dots,N}\bigl(u1_N + \tilde{\natural}_N\bigr).
\end{aligned}$$

Namely $P_{\varepsilon_1,\dots,\varepsilon_{[n]}}(u) \cdot \tilde{\eta}_{[n]+1}(u) \cdot Q_{\varepsilon'_{[n]},\dots,\varepsilon'_1}(u)$ does not depend on $\varepsilon_1, \dots, \varepsilon_{[n]}$.

Proof. It is enough to show the following relation:

$$\begin{aligned} &P_{\varepsilon_1,\dots,\varepsilon_{[n]}}(u)\cdot\tilde{\eta}_{[n]+1}(u)\cdot Q_{\varepsilon'_{[n]},\dots,\varepsilon'_1}(u)\\ &\quad= P_{\varepsilon_1,\dots,\varepsilon_{k-1},\varepsilon'_k,\varepsilon_{k+1},\dots,\varepsilon_{[n]}}(u)\cdot\tilde{\eta}_{[n]+1}(u)\cdot Q_{\varepsilon'_{[n]},\dots,\varepsilon'_{k+1},\varepsilon_k,\varepsilon'_{k-1},\dots,\varepsilon'_1}(u). \end{aligned}\tag{6.2}$$

Since we have

$$\tilde{\eta}_{[n]+1}(u)=\frac{1}{2}\bigl\{\tilde{\eta}_{[n]+1}\bigl(u+\tfrac{1}{2}\bigr)+\tilde{\eta}_{[n]+1}\bigl(u-\tfrac{1}{2}\bigr)\bigr\},$$

this (6.2) can be deduced from the following two relations:

$$\begin{aligned} &P_{\varepsilon_1,\dots,\varepsilon_{[n]}}(u)\cdot\tilde{\eta}_{[n]+1}\bigl(u+\tfrac{1}{2}\bigr)\cdot Q_{\varepsilon'_{[n]},\dots,\varepsilon'_1}(u)\\ &\quad= P_{\varepsilon_1,\dots,\varepsilon_{k-1},\varepsilon'_k,\varepsilon_{k+1},\dots,\varepsilon_{[n]}}(u)\cdot\tilde{\eta}_{[n]+1}\bigl(u+\tfrac{1}{2}\bigr)\cdot Q_{\varepsilon'_{[n]},\dots,\varepsilon'_{k+1},\varepsilon_k,\varepsilon'_{k-1},\dots,\varepsilon'_1}(u), \end{aligned}\tag{6.3}$$

$$\begin{aligned} &P_{\varepsilon_1,\dots,\varepsilon_{[n]}}(u)\cdot\tilde{\eta}_{[n]+1}\bigl(u-\tfrac{1}{2}\bigr)\cdot Q_{\varepsilon'_{[n]},\dots,\varepsilon'_1}(u)\\ &\quad= P_{\varepsilon_1,\dots,\varepsilon_{k-1},\varepsilon'_k,\varepsilon_{k+1},\dots,\varepsilon_{[n]}}(u)\cdot\tilde{\eta}_{[n]+1}\bigl(u-\tfrac{1}{2}\bigr)\cdot Q_{\varepsilon'_{[n]},\dots,\varepsilon'_{k+1},\varepsilon_k,\varepsilon'_{k-1},\dots,\varepsilon'_1}(u). \end{aligned}\tag{6.4}$$

To show these, we put

$$\begin{aligned} P^{\dagger}_{i_1,\dots,i_{[n]-1}}(u)&=\tilde{\eta}^{[n]-1}_{i_1,\dots,i_{[n]-1}}\bigl(u+n-1,\dots,u+\tfrac{3}{2}\bigr),\\ Q^{\dagger}_{i_1,\dots,i_{[n]-1}}(u)&=\tilde{\eta}^{[n]-1}_{i_1,\dots,i_{[n]-1}}\bigl(u-\tfrac{3}{2},\dots,u-n+1\bigr). \end{aligned}$$

Then, we have the following by Corollary 4.6:

$$\begin{aligned} P_{\varepsilon_1,\dots,\varepsilon_{[n]}}(u)&=P^{\dagger}_{\varepsilon_1,\dots,\varepsilon_{k-1},\varepsilon_{k+1},\dots,\varepsilon_{[n]}}(u)\tilde{\eta}_{\varepsilon_k}\bigl(u+\tfrac{1}{2}\bigr),\\ P_{\varepsilon_1,\dots,\varepsilon_{k-1},\varepsilon'_k,\varepsilon_{k+1},\dots,\varepsilon_{[n]}}(u)&=P^{\dagger}_{\varepsilon_1,\dots,\varepsilon_{k-1},\varepsilon_{k+1},\dots,\varepsilon_{[n]}}(u)\tilde{\eta}_{\varepsilon'_k}\bigl(u+\tfrac{1}{2}\bigr),\\ Q_{\varepsilon'_{[n]},\dots,\varepsilon'_1}(u)&=\tilde{\eta}_{\varepsilon'_k}\bigl(u-\tfrac{1}{2}\bigr)Q^{\dagger}_{\varepsilon'_{[n]},\dots,\varepsilon'_{k+1},\varepsilon'_{k-1},\dots,\varepsilon'_1}(u),\\ Q_{\varepsilon'_{[n]},\dots,\varepsilon'_{k+1},\varepsilon_k,\varepsilon'_{k-1},\dots,\varepsilon'_1}(u)&=\tilde{\eta}_{\varepsilon_k}\bigl(u-\tfrac{1}{2}\bigr)Q^{\dagger}_{\varepsilon'_{[n]},\dots,\varepsilon'_{k+1},\varepsilon'_{k-1},\dots,\varepsilon'_1}(u). \end{aligned}$$

Hence the difference between both sides of (6.3) is equal to

$$\begin{aligned} &P_{\varepsilon_1,\dots,\varepsilon_{[n]}}(u)\cdot\tilde{\eta}_{[n]+1}\bigl(u+\tfrac{1}{2}\bigr)\cdot Q_{\varepsilon'_{[n]},\dots,\varepsilon'_1}(u)\\ &\qquad-P_{\varepsilon_1,\dots,\varepsilon_{k-1},\varepsilon'_k,\varepsilon_{k+1},\dots,\varepsilon_{[n]}}(u)\cdot\tilde{\eta}_{[n]+1}\bigl(u+\tfrac{1}{2}\bigr)\cdot Q_{\varepsilon'_{[n]},\dots,\varepsilon'_{k+1},\varepsilon_k,\varepsilon'_{k-1},\dots,\varepsilon'_1}(u)\\ &\quad=P^{\dagger}_{\varepsilon_1,\dots,\varepsilon_{k-1},\varepsilon_{k+1},\dots,\varepsilon_{[n]}}(u)\\ &\qquad\cdot\bigl\{\tilde{\eta}_{\varepsilon_k}\bigl(u+\tfrac{1}{2}\bigr)\cdot\tilde{\eta}_{[n]+1}\bigl(u+\tfrac{1}{2}\bigr)\cdot\tilde{\eta}_{\varepsilon'_k}\bigl(u-\tfrac{1}{2}\bigr)\\ &\qquad-\tilde{\eta}_{\varepsilon'_k}\bigl(u+\tfrac{1}{2}\bigr)\cdot\tilde{\eta}_{[n]+1}\bigl(u+\tfrac{1}{2}\bigr)\cdot\tilde{\eta}_{\varepsilon_k}\bigl(u-\tfrac{1}{2}\bigr)\bigr\}\\ &\qquad\cdot Q^{\dagger}_{\varepsilon'_{[n]},\dots,\varepsilon'_{k+1},\varepsilon'_{k-1},\dots,\varepsilon'_1}(u). \end{aligned}\tag{6.5}$$

By Corollary 4.6, the quantity in the braces is equal to

$$\begin{aligned}
&\tilde{\eta}_{\varepsilon_k}\big(u+\tfrac{1}{2}\big)\tilde{\eta}_{\varepsilon'_k}\big(u+\tfrac{1}{2}\big)\tilde{\eta}_{[n]+1}\big(u-\tfrac{1}{2}\big)-\tilde{\eta}_{\varepsilon'_k}\big(u+\tfrac{1}{2}\big)\tilde{\eta}_{\varepsilon_k}\big(u+\tfrac{1}{2}\big)\tilde{\eta}_{[n]+1}\big(u-\tfrac{1}{2}\big)\\
&\quad=\big\{\tilde{\eta}_{\varepsilon_k}\big(u+\tfrac{1}{2}\big)\tilde{\eta}_{\varepsilon'_k}\big(u+\tfrac{1}{2}\big)-\tilde{\eta}_{\varepsilon'_k}\big(u+\tfrac{1}{2}\big)\tilde{\eta}_{\varepsilon_k}\big(u+\tfrac{1}{2}\big)\big\}\tilde{\eta}_{[n]+1}\big(u-\tfrac{1}{2}\big)\\
&\quad=\sum_{a\neq k,k'}\eta_a\big(u+\tfrac{1}{2}\big)e_{a'}e^*_{\varepsilon_k}e^*_{\varepsilon'_k}\tilde{\eta}_{[n]+1}\big(u-\tfrac{1}{2}\big).
\end{aligned}$$

Here the last equality is a consequence of Corollary 4.11. Thus (6.5) is equal to

$$P^{\dagger}_{\varepsilon_1,\ldots,\varepsilon_{k-1},\varepsilon_{k+1},\ldots,\varepsilon_{[n]}}(u)\sum_{a\neq k,k'}\eta_a\big(u+\tfrac{1}{2}\big)e_{a'}e^*_{\varepsilon_k}e^*_{\varepsilon'_k}\tilde{\eta}_{[n]+1}\big(u-\tfrac{1}{2}\big)Q^{\dagger}_{\varepsilon'_{[n]},\ldots,\varepsilon'_{k+1},\varepsilon'_{k-1},\ldots,\varepsilon'_1}(u).$$

This is equal to zero by Corollary 4.10, because $a\neq k,k'$ is equal to some element of the following sequence:

$$\varepsilon_1,\ldots,\varepsilon_{k-1},\varepsilon_{k+1},\ldots,\varepsilon_{[n]},\varepsilon_{[n]+1},\varepsilon'_{[n]},\ldots,\varepsilon'_{k+1},\varepsilon'_{k-1},\ldots,\varepsilon'_1.$$

Thus we proved (6.3). We can prove (6.4) similarly. Hence the assertion holds. □

6.2. Next, let us prove Proposition 6.1. We put

$$\varXi_k(u)=\varXi(u)-\big\{\varOmega_1(u)+\cdots+\varOmega_k(u)\big\}=\varOmega_{k+1}(u)+\cdots+\varOmega_{[n]}(u)+\frac{1}{2}\varOmega_{[n]+1}(u).$$

This satisfies the following commutation relation by Corollary 4.12:

Lemma 6.4. *We have*

$$\varXi_k(u+1)\varOmega_l(u)=\varOmega_l(u+1)\varXi_k(u).$$

Moreover, we have the following relations:

Lemma 6.5. *We have*

$$\begin{aligned}
&\varXi_k(u+1)\varXi_k(u)\varOmega_l(u)\varOmega_l(u-1)+\varOmega_l(u+1)\varOmega_l(u)\varXi_k(u)\varXi_k(u-1)\\
&\quad=2\varOmega_l(u+1)\varXi_k(u)\varXi_k(u)\varOmega_l(u-1).
\end{aligned}$$

Lemma 6.6. *We have*

$$\begin{aligned}
&\varXi_k\big(u+\tfrac{3}{2}\big)\varXi_k\big(u+\tfrac{1}{2}\big)\varXi_k(u)\varOmega_l\big(u-\tfrac{1}{2}\big)\varOmega_l\big(u-\tfrac{3}{2}\big)\\
&\qquad+\varOmega_l\big(u+\tfrac{3}{2}\big)\varOmega_l\big(u+\tfrac{1}{2}\big)\varXi_k(u)\varXi_k\big(u-\tfrac{1}{2}\big)\varXi_k\big(u-\tfrac{3}{2}\big)\\
&\quad=2\varOmega_l\big(u+\tfrac{3}{2}\big)\varXi_k\big(u+\tfrac{1}{2}\big)\varXi_k(u)\varXi_k\big(u-\tfrac{1}{2}\big)\varOmega_l\big(u-\tfrac{3}{2}\big).
\end{aligned}$$

Lemma 6.7. *We have*

$$\begin{aligned}&\Xi_l\big(u+\tfrac{1}{2}\big)\Omega_k(u)\Omega_k\big(u-\tfrac{1}{2}\big)+\Omega_k\big(u+\tfrac{1}{2}\big)\Omega_k(u)\Xi_l\big(u-\tfrac{1}{2}\big)\\&\quad=2\Omega_k\big(u+\tfrac{1}{2}\big)\Xi_l(u)\Omega_k\big(u-\tfrac{1}{2}\big).\end{aligned}$$

Proof of Lemmas 6.5, 6.6, and 6.7. These lemmas are all consequences of Lemma 6.4 and the centrality of ξ_k and ω_l in $\Lambda_{2N}\otimes U(\mathfrak{o}(S_0))$. Here we put

$$\xi_k=e_{k+1}e^*_{k+1}+\cdots+e_{(k+1)'}e^*_{(k+1)'},\qquad \omega_l=e_le^*_l+e_{l'}e^*_{l'},$$

so that

$$\Xi_k(u+v)=\Xi_k(u)+v\xi_k,\qquad \Omega_l(u+v)=\Omega_l(u)+v\omega_l.$$

First, the proof of Lemma 6.5 is exactly the same as that of Lemma 5.5.

Next, Lemma 6.6 is proved as follows. Using Lemmas 6.4 and 6.5, we have

$$\begin{aligned}&\Xi_k\big(u+\tfrac{3}{2}\big)\Xi_k\big(u+\tfrac{1}{2}\big)\Xi_k\big(u+\tfrac{1}{2}\big)\Omega_l\big(u-\tfrac{1}{2}\big)\Omega_l\big(u-\tfrac{3}{2}\big)\\&\qquad+\Omega_l\big(u+\tfrac{3}{2}\big)\Omega_l\big(u+\tfrac{1}{2}\big)\Xi_k\big(u+\tfrac{1}{2}\big)\Xi_k\big(u-\tfrac{1}{2}\big)\Xi_k\big(u-\tfrac{3}{2}\big)\\&\quad=\Xi_k\big(u+\tfrac{3}{2}\big)\Xi_k\big(u+\tfrac{1}{2}\big)\Omega_l\big(u+\tfrac{1}{2}\big)\Omega_l\big(u-\tfrac{1}{2}\big)\Xi_k\big(u-\tfrac{3}{2}\big)\\&\qquad+\Omega_l\big(u+\tfrac{3}{2}\big)\Omega_l\big(u+\tfrac{1}{2}\big)\Xi_k\big(u+\tfrac{1}{2}\big)\Xi_k\big(u-\tfrac{1}{2}\big)\Xi_k\big(u-\tfrac{3}{2}\big)\\&\quad=2\Omega_l\big(u+\tfrac{3}{2}\big)\Xi_k\big(u+\tfrac{1}{2}\big)\Xi_k\big(u+\tfrac{1}{2}\big)\Omega_l\big(u-\tfrac{1}{2}\big)\Xi_k\big(u-\tfrac{3}{2}\big)\\&\quad=2\Omega_l\big(u+\tfrac{3}{2}\big)\Xi_k\big(u+\tfrac{1}{2}\big)\Xi_k\big(u+\tfrac{1}{2}\big)\Xi_k\big(u-\tfrac{1}{2}\big)\Omega_l\big(u-\tfrac{3}{2}\big).\end{aligned}$$

Similarly we have

$$\begin{aligned}&\Xi_k\big(u+\tfrac{3}{2}\big)\Xi_k\big(u+\tfrac{1}{2}\big)\Xi_k\big(u-\tfrac{1}{2}\big)\Omega_l\big(u-\tfrac{1}{2}\big)\Omega_l\big(u-\tfrac{3}{2}\big)\\&\qquad+\Omega_l\big(u+\tfrac{3}{2}\big)\Omega_l\big(u+\tfrac{1}{2}\big)\Xi_k\big(u-\tfrac{1}{2}\big)\Xi_k\big(u-\tfrac{1}{2}\big)\Xi_k\big(u-\tfrac{3}{2}\big)\\&\quad=2\Omega_l\big(u+\tfrac{3}{2}\big)\Xi_k\big(u+\tfrac{1}{2}\big)\Xi_k\big(u-\tfrac{1}{2}\big)\Xi_k\big(u-\tfrac{1}{2}\big)\Omega_l\big(u-\tfrac{3}{2}\big).\end{aligned}$$

Adding both sides of these two relations and dividing by 2, we obtain Lemma 6.6, because $\Xi_k(u)=\frac{1}{2}\{\Xi_k(u+\frac{1}{2})+\Xi_k(u-\frac{1}{2})\}$.

Finally Lemma 6.7 is shown as follows. Since $\Omega_l(u)=\Omega_l(u\pm\frac{1}{2})\mp\frac{1}{2}\omega_l$, we have

$$\begin{aligned}&\Xi_k\big(u+\tfrac{1}{2}\big)\Omega_l(u)\Omega_l\big(u-\tfrac{1}{2}\big)\\&\quad=\Xi_k\big(u+\tfrac{1}{2}\big)\Omega_l\big(u-\tfrac{1}{2}\big)\Omega_l\big(u-\tfrac{1}{2}\big)+\frac{1}{2}\Xi_k\big(u+\tfrac{1}{2}\big)\omega_l\Omega_l\big(u-\tfrac{1}{2}\big),\\&\Omega_l\big(u+\tfrac{1}{2}\big)\Omega_l(u)\Xi_k\big(u-\tfrac{1}{2}\big)\\&\quad=\Omega_l\big(u+\tfrac{1}{2}\big)\Omega_l\big(u+\tfrac{1}{2}\big)\Xi_k\big(u-\tfrac{1}{2}\big)-\frac{1}{2}\Omega_l\big(u+\tfrac{1}{2}\big)\omega_l\Xi_k\big(u-\tfrac{1}{2}\big).\end{aligned}$$

By Lemma 6.4, we have $\Xi_k(u+\frac{1}{2})\omega_l\Omega_l(u-\frac{1}{2}) = \Omega_l(u+\frac{1}{2})\omega_l\Xi_k(u-\frac{1}{2})$. Thus, we have

$$\begin{aligned}&\Xi_k\bigl(u+\tfrac{1}{2}\bigr)\Omega_l(u)\Omega_l\bigl(u-\tfrac{1}{2}\bigr)+\Omega_l\bigl(u+\tfrac{1}{2}\bigr)\Omega_l(u)\Xi_k\bigl(u-\tfrac{1}{2}\bigr)\\&\quad=\Xi_k\bigl(u+\tfrac{1}{2}\bigr)\Omega_l\bigl(u-\tfrac{1}{2}\bigr)\Omega_l\bigl(u-\tfrac{1}{2}\bigr)+\Omega_l\bigl(u+\tfrac{1}{2}\bigr)\Omega_l\bigl(u+\tfrac{1}{2}\bigr)\Xi_k\bigl(u-\tfrac{1}{2}\bigr).\end{aligned}$$

By using Lemma 6.4 again, this is equal to

$$\Omega_l\bigl(u+\tfrac{1}{2}\bigr)\Xi_k\bigl(u-\tfrac{1}{2}\bigr)\Omega_l\bigl(u-\tfrac{1}{2}\bigr)+\Omega_l\bigl(u+\tfrac{1}{2}\bigr)\Xi_k\bigl(u+\tfrac{1}{2}\bigr)\Omega_l\bigl(u-\tfrac{1}{2}\bigr).$$

Moreover, since $\Xi_k(u-\frac{1}{2})+\Xi_k(u+\frac{1}{2}) = 2\Xi_k(u)$, this is equal to

$$2\Omega_l\bigl(u+\tfrac{1}{2}\bigr)\Xi_k(u)\Omega_l\bigl(u-\tfrac{1}{2}\bigr). \qquad \square$$

As a consequence of these lemmas, we have the following relation:

Lemma 6.8. *We have*

$$\begin{aligned}\Xi_k^{N-2k}\bigl(u1_{N-2k}+\tilde{\mathfrak{h}}_{N-2k}\bigr) = \binom{N-2k}{2}&\Omega_{k+1}(u+n-k-1)\\&\cdot\Xi_{k+1}^{N-2k-2}\bigl(u1_{N-2k-2}+\tilde{\mathfrak{h}}_{N-2k-2}\bigr)\cdot\Omega_{k+1}(u-n+k+1).\end{aligned}$$

Here $\Xi_k^l(a_1,\dots,a_l)$ *denotes* $\Xi_k(a_1)\cdots\Xi_k(a_l)$.

Before proving this lemma, we note that (5.3) and (5.4) also hold in this case. Namely we have

$$\Omega_k(u_1)\varphi_1\Omega_k(u_2)\varphi_2\Omega_k(u_3) = 0 \tag{6.6}$$

for any $\varphi_1,\varphi_2\in\Lambda_{2N}\otimes U(\mathfrak{o}(S_0))$. Similarly, when $l>N-2k$, we have

$$\Xi_k(u_1)\varphi_1\Xi_k(u_2)\varphi_2\cdots\Xi_k(u_{l-1})\varphi_{l-1}\Xi_k(u_l) = 0 \tag{6.7}$$

for any $\varphi_1,\dots,\varphi_{l-1}\in\Lambda_{2N}\otimes U(\mathfrak{o}(S_0))$.

Proof of Lemma 6.8. The proof is almost the same as that of Lemma 5.6. By definition we have $\Xi_k(u) = \Xi_{k+1}(u)+\Omega_{k+1}(u)$. Hence, we have the following binomial expansion using Lemma 6.4 and (6.6):

$$\begin{aligned}&\Xi_k^{[n]-k}\bigl(u+n-k-1,\dots,u+\tfrac{1}{2}\bigr)\\&\quad=\Xi_{k+1}^{[n]-k-2}\bigl(u+n-k-1,\dots,u+\tfrac{5}{2}\bigr)\cdot\biggl\{\binom{[n]-k}{0}\Xi_{k+1}\bigl(u+\tfrac{3}{2}\bigr)\Xi_{k+1}\bigl(u+\tfrac{1}{2}\bigr)\\&\qquad+\binom{[n]-k}{1}\Xi_{k+1}\bigl(u+\tfrac{3}{2}\bigr)\Omega_{k+1}\bigl(u+\tfrac{1}{2}\bigr)+\binom{[n]-k}{2}\Omega_{k+1}\bigl(u+\tfrac{3}{2}\bigr)\Omega_{k+1}\bigl(u+\tfrac{1}{2}\bigr)\biggr\}.\end{aligned}$$

Similarly we have

$$\begin{aligned}
&\Xi_k^{[n]-k}\big(u-\tfrac{1}{2},\dots,u-n+k+1\big)\\
&\quad=\left\{\binom{[n]-k}{0}\Xi_{k+1}\big(u-\tfrac{1}{2}\big)\Xi_{k+1}\big(u-\tfrac{3}{2}\big)+\binom{[n]-k}{1}\Xi_{k+1}\big(u-\tfrac{1}{2}\big)\Omega_{k+1}\big(u-\tfrac{3}{2}\big)\right.\\
&\qquad\left.+\binom{[n]-k}{2}\Omega_{k+1}\big(u-\tfrac{1}{2}\big)\Omega_{k+1}\big(u-\tfrac{3}{2}\big)\right\}\cdot\Xi_{k+1}^{[n]-k-2}\big(u-\tfrac{5}{2},\dots,u-n+k+1\big).
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&\Xi_k^{N-2k}\big(u1_{N-2k}+\tilde{\mathfrak{h}}_{N-2k}\big)\\
&\quad=\Xi_k^{[n]-k}\big(u+n-k-1,\dots,u+\tfrac{1}{2}\big)\cdot\Xi_k(u)\cdot\Xi_k^{[n]-k}\big(u-\tfrac{1}{2},\dots,u-n+k+1\big)\\
&\quad=\Xi_{k+1}^{[n]-k-2}\big(u+n-k-1,\dots,u+\tfrac{5}{2}\big)\\
&\qquad\cdot\left\{\binom{[n]-k}{0}\Xi_{k+1}\big(u+\tfrac{3}{2}\big)\Xi_{k+1}\big(u+\tfrac{1}{2}\big)\right.\\
&\qquad+\binom{[n]-k}{1}\Xi_{k+1}\big(u+\tfrac{3}{2}\big)\Omega_{k+1}\big(u+\tfrac{1}{2}\big)\\
&\qquad\left.+\binom{[n]-k}{2}\Omega_{k+1}\big(u+\tfrac{3}{2}\big)\Omega_{k+1}\big(u+\tfrac{1}{2}\big)\right\}\\
&\qquad\cdot\big\{\Xi_{k+1}(u)+\Omega_{k+1}(u)\big\}\\
&\qquad\cdot\left\{\binom{[n]-k}{0}\Xi_{k+1}\big(u-\tfrac{1}{2}\big)\Xi_{k+1}\big(u-\tfrac{3}{2}\big)\right.\\
&\qquad+\binom{[n]-k}{1}\Xi_{k+1}\big(u-\tfrac{1}{2}\big)\Omega_{k+1}\big(u-\tfrac{3}{2}\big)\\
&\qquad\left.+\binom{[n]-k}{2}\Omega_{k+1}\big(u-\tfrac{1}{2}\big)\Omega_{k+1}\big(u-\tfrac{3}{2}\big)\right\}\\
&\qquad\cdot\Xi_{k+1}^{[n]-k-2}\big(u-\tfrac{5}{2},\dots,u-n+k+1\big)\\
&\quad=\Xi_{k+1}^{[n]-k-2}\big(u+n-k-1,\dots,u+\tfrac{5}{2}\big)\\
&\qquad\cdot\left\{\binom{[n]-k}{1}^2\Xi_{k+1}\big(u+\tfrac{3}{2}\big)\Omega_{k+1}\big(u+\tfrac{1}{2}\big)\Xi_{k+1}(u)\Omega_{k+1}\big(u-\tfrac{1}{2}\big)\Xi_{k+1}\big(u-\tfrac{3}{2}\big)\right.\\
&\qquad+\binom{[n]-k}{2}\Omega_{k+1}\big(u+\tfrac{3}{2}\big)\Omega_{k+1}\big(u+\tfrac{1}{2}\big)\Xi_{k+1}(u)\Xi_{k+1}\big(u-\tfrac{1}{2}\big)\Xi_{k+1}\big(u-\tfrac{3}{2}\big)\\
&\qquad+\binom{[n]-k}{2}\Xi_{k+1}\big(u+\tfrac{3}{2}\big)\Xi_{k+1}\big(u+\tfrac{1}{2}\big)\Xi_{k+1}(u)\Omega_{k+1}\big(u-\tfrac{1}{2}\big)\Omega_{k+1}\big(u-\tfrac{3}{2}\big)\\
&\qquad+\binom{[n]-k}{1}\Xi_{k+1}\big(u+\tfrac{3}{2}\big)\Omega_{k+1}\big(u+\tfrac{1}{2}\big)\Omega_{k+1}(u)\Xi_{k+1}\big(u-\tfrac{1}{2}\big)\Xi_{k+1}\big(u-\tfrac{3}{2}\big)
\end{aligned}$$

$$+\binom{[n]-k}{1}\Xi_{k+1}\big(u+\tfrac{3}{2}\big)\Xi_{k+1}\big(u+\tfrac{1}{2}\big)\Omega_{k+1}(u)\Omega_{k+1}\big(u-\tfrac{1}{2}\big)\Xi_{k+1}\big(u-\tfrac{3}{2}\big)\Big\}$$
$$\cdot\,\Xi_{k+1}^{[n]-k-2}\big(u-\tfrac{5}{2},\dots,u-n+k+1\big).$$

Indeed, the sum of the exponents of Ξ_{k+1} must be equal to $N-2k-2$, and the degree of Ω_{k+1} must be equal to 2. By Lemma 6.4, the first term in the braces is equal to

$$\binom{[n]-k}{1}^2\Omega_{k+1}\big(u+\tfrac{3}{2}\big)\Xi_{k+1}\big(u+\tfrac{1}{2}\big)\Xi_{k+1}(u)\Xi_{k+1}\big(u-\tfrac{1}{2}\big)\Omega_{k+1}\big(u-\tfrac{3}{2}\big).$$

Moreover, by Lemma 6.6, the second and third terms are equal to

$$2\binom{[n]-k}{2}\Omega_{k+1}\big(u+\tfrac{3}{2}\big)\Xi_{k+1}\big(u+\tfrac{1}{2}\big)\Xi_{k+1}(u)\Xi_{k+1}\big(u-\tfrac{1}{2}\big)\Omega_{k+1}\big(u-\tfrac{3}{2}\big).$$

Similarly, by Lemmas 6.7 and 6.4, the forth and fifth terms are equal to

$$\begin{aligned}&2\binom{[n]-k}{1}\Xi_{k+1}\big(u+\tfrac{3}{2}\big)\Omega_{k+1}\big(u+\tfrac{1}{2}\big)\Xi_{k+1}(u)\Omega_{k+1}\big(u-\tfrac{1}{2}\big)\Xi_{k+1}\big(u-\tfrac{3}{2}\big)\\&\quad=2\binom{[n]-k}{1}\Omega_{k+1}\big(u+\tfrac{3}{2}\big)\Xi_{k+1}\big(u+\tfrac{1}{2}\big)\Xi_{k+1}(u)\Xi_{k+1}\big(u-\tfrac{1}{2}\big)\Omega_{k+1}\big(u-\tfrac{3}{2}\big).\end{aligned}$$

Since $\binom{N-2k}{2}=\binom{[n]-k}{1}^2+2\binom{[n]-k}{2}+2\binom{[n]-k}{1}$, we have

$$\begin{aligned}&\Xi_k^{N-2k}\big(u1_{N-2k}+\tilde{\mathfrak{h}}_{N-2k}\big)\\&\quad=\binom{N-2k}{2}\Xi_{k+1}^{[n]-k-2}\big(u+n-k-1,\dots,u+\tfrac{5}{2}\big)\\&\qquad\cdot\Omega_{k+1}\big(u+\tfrac{3}{2}\big)\Xi_{k+1}\big(u+\tfrac{1}{2}\big)\Xi_{k+1}(u)\Xi_{k+1}\big(u-\tfrac{1}{2}\big)\Omega_{k+1}\big(u-\tfrac{3}{2}\big)\\&\qquad\cdot\Xi_{k+1}^{[n]-k-2}\big(u-\tfrac{5}{2},\dots,u-n+k+1\big)\\&\quad=\binom{N-2k}{2}\Omega_{k+1}(u+n-k-1)\Xi_{k+1}^{[n]-k-1}\big(u+n-k-2,\dots,u+\tfrac{1}{2}\big)\\&\qquad\cdot\Xi_{k+1}(u)\cdot\Xi_{k+1}^{[n]-k-1}\big(u-\tfrac{1}{2},\dots,u-n+k+1\big)\Omega_{k+1}(u-n+k+1).\end{aligned}$$

Here we used Lemma 6.4 for the second equality. This means the assertion. □

Applying Lemma 6.8 repeatedly, we have

$$\begin{aligned}&\Xi_0^N\big(u1_N+\tilde{\mathfrak{h}}_N\big)\\&\quad=\binom{N}{2}\Omega_1(u+n-1)\cdot\Xi_1^{N-2}\big(u1_{N-2}+\tilde{\mathfrak{h}}_{N-2}\big)\cdot\Omega_1(u-n+1)\end{aligned}$$

$$= \binom{N}{2}\binom{N-2}{2}\Omega_1(u+n-1)\Omega_2(u+n-2)\cdot \Xi_2^{N-4}\big(u1_{N-4}+\tilde{\mathfrak{J}}_{N-4}\big)$$
$$\cdot\,\Omega_2(u-n+2)\Omega_1(u-n+1)$$
$$=\cdots$$
$$= \binom{N}{2}\binom{N-2}{2}\cdots\binom{3}{2}\Omega_1(u+n-1)\Omega_2(u+n-2)\cdots\Omega_n(u)\cdot \Xi_{[n]}(u)$$
$$\cdot\,\Omega_n(u)\cdots\Omega_2(u-n+2)\Omega_1(u-n+1).$$

Since $\Xi_{[n]}(u) = \frac{1}{2}\Omega_{[n]+1}$, this means Proposition 6.1.

Thus we have proved Theorem 4.7, and hence Theorem 4.3 in the case that N is odd.

7. The case of the symplectic Lie algebras

Finally, we announce an analogue in the universal enveloping algebras of the symplectic Lie algebras. Throughout this paper, we have studied a relation between two kind of non-commutative determinants in $U(\mathfrak{o}_N)$. Applying a similar discussion to the symplectic case, we obtain generators for the center of $U(\mathfrak{sp}_N)$. These generators are expressed in terms of the "column-permanent," and we can easily calculate their eigenvalues on irreducible representations. We can also express these generators in terms of the "symmetrized permanent."

The proof of this symplectic case is similar to that of the orthogonal case in this paper, but a bit more difficult. Indeed, to prove Theorem 4.3, we only used commutation relations in $\Lambda_{2N} \otimes U(\mathfrak{o}(S_0))$. However, to prove this symplectic case, we also need a "variable transformation" in addition to similar commutation relations.

The details are discussed elsewhere [I3].

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