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The endofinite spectrum of a tame algebra

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Abstract

Let R be a ring. Based on indecomposable endofinite R-modules and characters that were introduced by Crawley-Boevey [Modules of finite length over their endomorphism ring, in: S. Brenner, H. Tachikawa (Eds.), Representations of Algebras and Related Topics, in: London Math. Soc. Lecture Note Ser., vol. 168, 1992, pp. 127–184], we define the endofinite spectrum of the ring R. We compute this spectrum in some examples and study the behaviour of it under certain functors, with the objective of understanding the endofinite spectrum of tame algebras. Furthermore, we show that the endomorphism ring of a minimal point of the endofinite spectrum is a skew field. Hence the minimal points belong to the Cohn spectrum, as studied by Ringel [The spectrum of a finite dimensional algebra, in: Proc. Conf. on Ring Theory, Dekker, New York, 1979, pp. 535–598], which in turn is a subset of the endofinite spectrum. Finally, we introduce the normalised endofinite spectrum.

1. Introduction: basic definitions and main results

Let R be an associative ring with 1. Denote by $\operatorname{Mod} R$ the category of left R-modules and by $\operatorname{mod} R$ the category of finitely presented left R-modules. For an R-module M let $\ell_R(M)$ be its composition length. The *endolength* of an R-module M, denoted by $\operatorname{endol}(M)$, is the length of M considered as module over its endomorphism ring in the canonical way. A module of finite endolength is called an *endofinite* module. A *generic* module is an indecomposable module of finite endolength but of infinite length.

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We recall from [4] that a *character* for the category mod R is a function $\chi : \operatorname{mod} R \to \mathbb{N}_0$ which satisfies the following conditions:

(C1) $\chi(X \amalg Y) = \chi(X) + \chi(Y)$ for all X and Y in mod R.
(C2) $\chi(X) + \chi(Z) \geqslant \chi(Y) \geqslant \chi(Z)$ for every exact sequence $X \to Y \to Z \to 0$ in mod R.

The value $\chi(R)$ is called the *degree* of the character χ. A non-zero character χ is *irreducible* if there do not exist characters χ_1 and χ_2 both non-zero such that $\chi = \chi_1 + \chi_2$. Given an endofinite left R-module M, let

$$\chi_M : \operatorname{mod} R \to \mathbb{N}_0, \quad X \mapsto \chi_M(X) := \ell_{\operatorname{End}_R(M)} \operatorname{Hom}_R(X, M)$$

be the *character of M*, where the $\operatorname{End}_R(M)$-module structure of $\operatorname{Hom}_R(X, M)$ is given in the canonical way. Note that our definition of the character χ_M is dual to that of [4].

In [4] Crawley-Boevey has shown that the assignment $M \mapsto \chi_M$ induces a one-to-one correspondence between the irreducible characters for mod R and the isomorphism classes of indecomposable endofinite R-modules. This motivates the following definition.

Definition 1.1.

(1) We define the following partial order on the characters for mod R. Let χ and χ' be characters,

$$\chi \leqslant \chi' \quad :\Longleftrightarrow \quad \chi(X) \leqslant \chi'(X) \text{ for all finitely presented } R\text{-modules } X.$$

(2) The partial order defined by (1) induces a partial order on the isomorphism classes of indecomposable endofinite R-modules. Let M and M' be indecomposable endofinite R-modules,

$$M \leqslant M' \quad :\Longleftrightarrow \quad \chi_M \leqslant \chi_{M'}.$$

(3) Define the set of all isomorphism classes of indecomposable endofinite R-modules with the partial order given by (2) to be the *endofinite spectrum* of the ring R, denoted by $\operatorname{Spec}_{\mathrm{end}}(R)$. The isomorphism classes of indecomposable endofinite modules are the *points* of the endofinite spectrum.

Having given the main definitions, we now provide a short summary of the paper and its main results. In Section 2 we give dual definitions and compute the endofinite spectrum of the polynomial ring in one variable over a field and of the Kronecker algebra. In Section 3 we are concerned with the behaviour of the endofinite spectrum along functors between module categories and get the following result.

Theorem 1.2. *Let $F : \operatorname{Mod} S \to \operatorname{Mod} R$ be a functor having the following properties*:

(1) *The functor F has a left adjoint G and preserves direct limits.*

(2) *The functor F induces an isomorphism* $\mathrm{End}_S(M)/\mathrm{rad} \to \mathrm{End}_R(FM)/\mathrm{rad}$ *for every indecomposable endofinite S-module M.*

Then F sends every indecomposable endofinite S-module to an indecomposable endofinite R-module. Moreover, given an indecomposable endofinite S-module M, we have $\chi_{FM}(Y) = \chi_M(GY)$ for all finitely presented R-modules Y. Hence $M \leqslant N$ implies $FM \leqslant FN$.

The next two sections are devoted to applying this result and obtaining information about the endofinite spectrum of tame algebras. In Section 4 we determine the endofinite spectrum of a finitely generated localisation of the polynomial ring $k[T]$ over an algebraically closed field k. We consider the restriction of scalars for a ring epimorphism and show that it *preserves* (see Definition 3.5) the endofinite spectrum. In the special case of a localisation of a principal ideal domain R, we find that the endofinite spectrum of the localisation is identified via restriction with a *full subspectrum* (see Definition 3.5) of the endofinite spectrum of R. Finally, we look at a finitely generated localisation of $k[T]$ and find out which points actually belong to this full subspectrum. In Section 5 we consider tame algebras using functors introduced by Crawley-Boevey [3]. These functors are defined between the module category of a finitely generated localisation of the polynomial ring $k[T]$ and the module category of a tame algebra over the algebraically closed field k. We show

Corollary 1.3. *Let Λ be a finite dimensional algebra of tame representation type over an algebraically closed field k. Then for every generic Λ-module G there is a finitely generated localisation R_G of the polynomial ring $k[T]$ such that* $\mathrm{Spec}_{\mathrm{end}}(R_G)$ *can be identified with a subspectrum of* $\mathrm{Spec}_{\mathrm{end}}(\Lambda)$ *that contains G as a point. Moreover, given a natural number d, then for almost all points M of dimension d there exists a generic module G such that* $\mathrm{Spec}_{\mathrm{end}}(R_G)$ *contains M.*

In Section 6 we are concerned with the minimal points of the endofinite spectrum, as well as epimorphisms from R to simple artinian rings. In [7] Ringel gives a one-to-one correspondence between the isomorphism classes of indecomposable endofinite modules with a skew field as endomorphism ring and the equivalence classes of epimorphisms to simple artinian rings. We show that the minimal points of the endofinite spectrum are of this form.

Another interesting analogy is found with respect to degeneration. Here one looks at modules of a fixed dimension and defines an order by degeneration. The degeneration order implies our order on characters, and the inverse implication holds in several cases (see [10]).

In Section 7 we study characters normalised with respect to their degree and introduce the corresponding *normalised endofinite spectrum*. In this way, we recover Schofield's Sylvester module rank functions (see [8]) on which Crawley-Boevey's characters are based.

2. Dual definitions and examples

We recall from [4] that for χ a character for $\operatorname{mod} R$, the *dual character* $D\chi$ is a character for $\operatorname{mod} R^{\mathrm{op}}$ defined as follows. For X a finitely presented left R^{op}-module let $P \xrightarrow{\alpha} Q \to X \to 0$ be a projective presentation of X and define

$$D\chi(X) := \chi(Q^*) - \chi(P^*) + \chi(\operatorname{Coker} \alpha^*), \quad \text{where } (-)^* = \operatorname{Hom}_R(-, R).$$

Then $DD\chi = \chi$ and hence the assignment $\chi \mapsto D\chi$ is a duality between the characters for $\operatorname{mod} R$ and those for $\operatorname{mod} R^{\mathrm{op}}$. This duality induces a one-to-one correspondence between the irreducible characters for $\operatorname{mod} R$ and $\operatorname{mod} R^{\mathrm{op}}$ and hence between the isomorphism classes of indecomposable endofinite R and R^{op}-modules. Furthermore, it holds that $D\chi_M(X) = \ell_{\operatorname{End}_R(M)}(X \otimes_R M)$, where $f \in \operatorname{End}_R(M)$ operates by $f \cdot (x \otimes m) := x \otimes f(m)$.

Using the dual characters, we define another partial order on the isomorphism classes of indecomposable endofinite R-modules.

Definition 2.1.

(1) Let M and M' be indecomposable endofinite R-modules. Define

$$M \leqslant_D M' \quad :\Longleftrightarrow \quad D\chi_M \leqslant D\chi_{M'}.$$

(2) The set of all isomorphism classes of indecomposable endofinite R-modules with the partial order given by (1) is the *dual endofinite spectrum* of the ring R.

Remark 2.2. Because of the duality D between the characters for $\operatorname{mod} R$ and $\operatorname{mod} R^{\mathrm{op}}$ and the above one-to-one correspondence between the isomorphism classes of indecomposable endofinite R and R^{op}-modules, we have that the endofinite spectrum of R and the dual endofinite spectrum of R^{op} are isomorphic as partially ordered sets. In the following, we will always consider the endofinite spectrum. In Example 2.3(3a) and (3b) we will see that the endofinite spectrum of a ring R and the dual endofinite spectrum of R, i.e., the endofinite spectrum of R^{op}, will in general not be isomorphic to each other, although the points are in one-to-one correspondence.

In the following, we illustrate the endofinite spectrum by drawing the corresponding Hasse diagram of the poset. This we will call the *diagram* of the endofinite spectrum.

Example 2.3. Let k be an algebraically closed field.

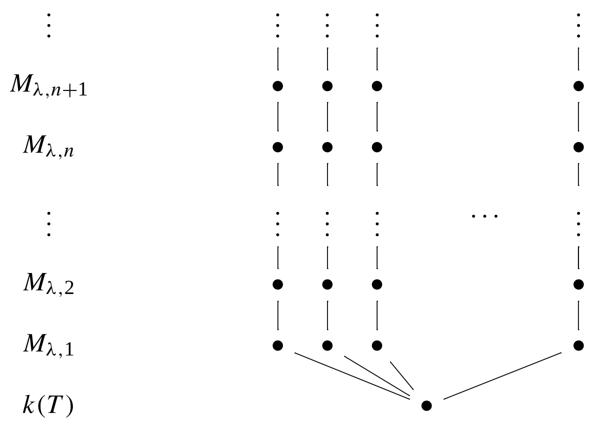
(1) $R = k[T]$. The isomorphism classes of indecomposable finitely presented $k[T]$-modules are represented by the torsion modules $M_{\lambda,n} := k[T]/(T-\lambda)^n$ for $\lambda \in k$ and $n \in \mathbb{N}$, and the torsion free module $k[T]$. By [4, 4.7 Examples (6)], the isomorphism classes of indecomposable endofinite $k[T]$-modules are represented by the finite length

modules $M_{\lambda,n}$ with $\lambda \in k$ and $n \in \mathbb{N}$, and the unique generic module $k(T)$. We have that $\mathrm{End}_{k[T]}(M_{\lambda,n}) \cong k[T]/(T-\lambda)^n =: E_{\lambda,n}$ and $\mathrm{End}_{k[T]}(k(T)) \cong k(T)$.

We compute

$$\chi_{M_{\lambda,n}}\big(k[T]\big) = \ell_{E_{\lambda,n}}(M_{\lambda,n}) = n,$$
$$\chi_{M_{\lambda,n}}(M_{\lambda',n'}) = \begin{cases} \ell_{E_{\lambda,n}}(0) = 0, & \text{if } \lambda \neq \lambda', \\ \ell_{E_{\lambda,n}}(M_{\lambda,\min(n,n')}) = \min(n,n'), & \text{if } \lambda = \lambda', \end{cases}$$
$$\chi_{k(T)}\big(k[T]\big) = \ell_{k(T)}\big(k(T)\big) = 1,$$
$$\chi_{k(T)}(M_{\lambda,n}) = \ell_{k(T)}(0) = 0,$$

and get the following diagram of the endofinite spectrum of $k[T]$:



(2) $R = k(1 \cdot \overset{\alpha}{\underset{\beta}{\rightrightarrows}} \cdot 2)$. The isomorphism classes of the indecomposable finitely presented R-modules are represented by the:

- preprojective modules: $P_n = (k^n \overset{f_\alpha}{\underset{f_\beta}{\rightrightarrows}} k^{n+1})$, $n \in \mathbb{N}_0$ with $f_\alpha = \binom{I}{0}$, $f_\beta = \binom{0}{I}$;
- preinjective modules: $Q_n = (k^{n+1} \overset{g_\alpha}{\underset{g_\beta}{\rightrightarrows}} k^n)$, $n \in \mathbb{N}_0$ with $g_\alpha = (I, 0)$, $g_\beta = (0, I)$;
- regular modules: $R_{p,n}$, $p \in \mathbb{P}^1(k)$, $n \in \mathbb{N}$. These are of the form $(k^n \overset{I}{\underset{J_{\lambda,n}}{\rightrightarrows}} k^n)$ for $p \neq \overline{(0,1)}$ and of the form $(k^n \overset{J_{0,n}}{\underset{I}{\rightrightarrows}} k^n)$ for $p = \overline{(0,1)}$ with $J_{\lambda,n} = \begin{pmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & & 1 \\ & & & \lambda \end{pmatrix}$.

The isomorphism classes of the indecomposable endofinite R-modules are represented by the indecomposable preprojective, preinjective, and regular modules, and in addition the unique generic module $G := (k(T) \overset{\mathrm{id}}{\underset{T\cdot}{\rightrightarrows}} k(T))$.

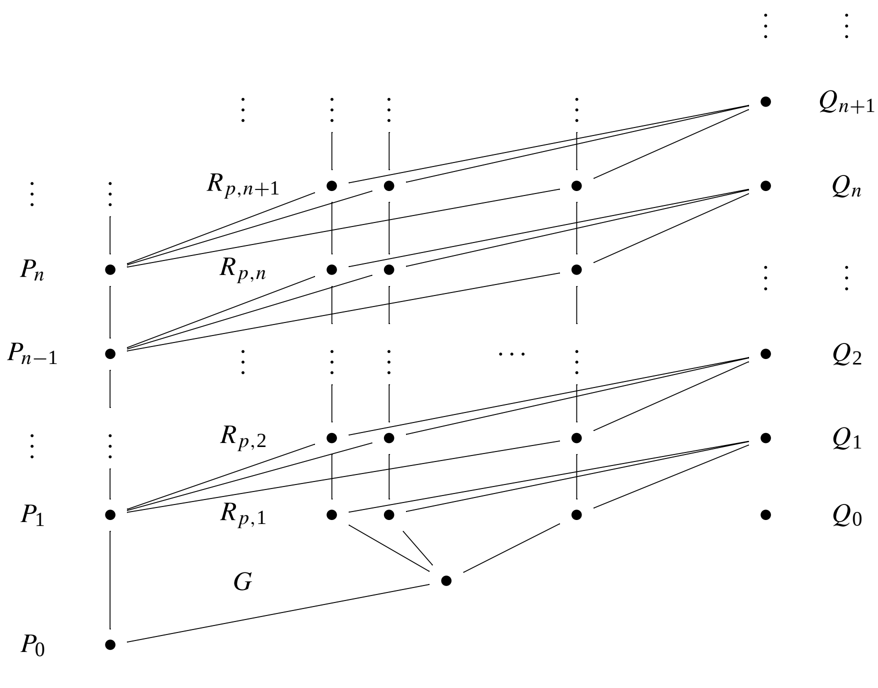
We have

$$\mathrm{End}_R(G) \cong k(T), \qquad \mathrm{End}_R(P_n) \cong k, \qquad \mathrm{End}_R(Q_n) \cong k, \quad \text{and}$$
$$\mathrm{End}_R(R_{p,n}) \cong k[T]/\bigl(T^n\bigr).$$

We get the following values for $\ell_{\mathrm{End}_R(M)} \mathrm{Hom}_R(X, M)$, where X is specified in the left-hand column and M is specified in the top row:

	P_n	$R_{p,n}$	Q_n	G
P_m	$n+m$	$\max\{0, n-m+1\}$	n	1
$R_{q,m}$	0	$\delta_{p,q} \min\{n, m\}$	m	0
Q_m	0	0	$\max\{0, m-n+1\}$	0

Hence we get the following diagram of the endofinite spectrum of $k(1 \cdot \overset{\alpha}{\underset{\beta}{\rightrightarrows}} \cdot 2)$:



(3a) $R = k(2 \to 1 \leftarrow 3)$. The isomorphism classes of indecomposable finitely presented R-modules and the isomorphism classes of indecomposable endofinite R-modules are the isomorphism classes of indecomposable finite length modules. From the AR-quiver we conclude the following dimensions of the homomorphism vector spaces $\mathrm{Hom}_R(X, M)$, where X and M are specified in the table as in (2):

$$\begin{tikzcd}
& P_2 \arrow[dr] \arrow[rr, dash, dashed] & & I_3 \\
P_1 \arrow[ur] \arrow[dr] \arrow[rr, dash, dashed] & & I_1 \arrow[ur] \arrow[dr] & \\
& P_3 \arrow[ur] \arrow[rr, dash, dashed] & & I_2
\end{tikzcd}$$

	P_1	P_2	P_3	I_1	I_2	I_3
P_1	1	1	1	1	0	0
P_2	0	1	0	1	1	0
P_3	0	0	1	1	0	1
I_1	0	0	0	1	1	1
I_2	0	0	0	0	1	0
I_3	0	0	0	0	0	1

We get the diagram of the endofinite spectrum of $k(2 \to 1 \leftarrow 3)$:

$$\begin{tikzcd}
& I_1 & & & \\
P_2 \arrow[ur, dash] & & P_3 \arrow[ul, dash] & & \\
& P_1 \arrow[ul, dash] \arrow[ur, dash] & & I_2 & I_3
\end{tikzcd}$$

(3b) $R^{\mathrm{op}} \cong k(2 \leftarrow 1 \to 3)$. Dually we have the AR-quiver of R^{op} and again therefore the dimensions of the homomorphism vector spaces $\mathrm{Hom}_{R^{\mathrm{op}}}(X, M)$:

$$\begin{tikzcd}
P_2 \arrow[dr] \arrow[rr, dash, dashed] & & I_1 \arrow[dr] & \\
& P_3 \arrow[ur] \arrow[dr] \arrow[rr, dash, dashed] & & I_3 \\
P_1 \arrow[ur] \arrow[rr, dash, dashed] & & I_2 \arrow[ur] &
\end{tikzcd}$$

	P_1	P_2	P_3	I_1	I_2	I_3
P_1	1	0	1	1	0	0
P_2	0	1	1	0	1	0
P_3	0	0	1	1	1	1
I_1	0	0	0	1	0	1
I_2	0	0	0	0	1	1
I_3	0	0	0	0	0	1

We get the diagram of the endofinite spectrum of $k(2 \leftarrow 1 \to 3)$:

$$\begin{tikzcd}
I_1 \arrow[dd, dash] & & I_2 \arrow[dd, dash] & \\
& P_3 & & \\
P_1 \arrow[ur, dash] & & P_2 \arrow[ul, dash] & I_3
\end{tikzcd}$$

We see that the endofinite spectrum of R and the endofinite spectrum of R^{op} (the dual endofinite spectrum of R) are not isomorphic.

3. Change of rings

We are interested in the question for which functors $F : \mathrm{Mod}\, S \to \mathrm{Mod}\, R$ the endofinite spectrum is *preserved* (see Definition 3.5). Before we prove Theorem 1.2, we need some lemmas. In the following we mean by the radical of a ring the Jacobson radical. Let R be a ring and $I \lhd R$ a two sided ideal. Consider the projection $p : R \to R/I$. The restriction functor $p_* : \mathrm{Mod}(R/I) \to \mathrm{Mod}\, R$ induces an isomorphism between $\mathrm{Mod}(R/I)$ and the full subcategory of $\mathrm{Mod}\, R$ of all R-modules such that $I \cdot M = 0$. We clearly have

Lemma 3.1. *The functor p_* induces for each R/I-module M an order isomorphism of submodule lattices.*

In particular, the (semi)simple modules of $\mathrm{Mod}\, R$ and $\mathrm{Mod}(R/\mathrm{rad})$ correspond to each other since $\mathrm{rad}\, R \cdot M = 0$ for each (semi)simple R-module M. Therefore we have

Lemma 3.2. *Let $f : S \to R$ be a ring homomorphism mapping the radical of S to the radical of R and suppose that the induced ring homomorphism $\bar{f} : S/\mathrm{rad} \to R/\mathrm{rad}$ is an isomorphism. If we consider X as an S-module by restriction, then*

$$\ell_S(X) = \ell_R(X) \quad \textit{for every } R\textit{-module } X \textit{ of finite length.}$$

We will also use the following statement about indecomposable endofinite modules.

Lemma 3.3 [4, Proposition 4.4]. *Let M be an indecomposable endofinite R-module. Then $\mathrm{End}_R(M)$ is local.*

Since left adjoints preserve colimits and pseudo-cokernels, and module categories are locally finitely presented, we get directly from [5, Theorem 6.7] the following lemma.

Lemma 3.4. *Let $F : \mathrm{Mod}\, S \to \mathrm{Mod}\, R$ be a functor between module categories with a left adjoint $G : \mathrm{Mod}\, R \to \mathrm{Mod}\, S$. Then the following are equivalent*:

(1) *The functor F preserves direct limits.*
(2) *The functor G preserves finitely presented modules.*

Now we are ready to prove Theorem 1.2 from the introduction.

Proof of Theorem 1.2. The functor F preserves direct limits by assumption, and F also preserves limits since it is a right adjoint. Moreover, F is coherent and therefore preserves endofiniteness (see [6, Section 3]). Indecomposability of endofinite modules is also preserved by F. For if we assume that FM is decomposable, then there is an idempotent e in $\mathrm{End}_R(FM)$ not equal to 0 or 1. By Lemma 3.3, $\mathrm{End}_S(M)$ is local and hence $\mathrm{End}_S(M)/\mathrm{rad} \cong \mathrm{End}_R(FM)/\mathrm{rad}$ is a skew field. Since in a skew field 0 and 1 are the only idempotents, we have that $\bar{e} \in \mathrm{End}_R(FM)/\mathrm{rad}$ is 0 or 1, i.e., one of e or $1 - e$ lies in $\mathrm{rad}(\mathrm{End}_R(FM))$. It follows that $1 - e$ or e is invertible; a contradiction to e not equal to 0 or 1 since 1 is the only invertible idempotent. Let M be an indecomposable endofinite S-module and Y a finitely presented R-module. By Lemma 3.4, GY is a finitely presented S-module. Using Lemma 3.2 and the canonical isomorphism $\mathrm{Hom}_R(Y, FM) \to \mathrm{Hom}_S(GY, M)$, we get $\chi_{FM}(Y) = \ell_{\mathrm{End}_R(FM)} \mathrm{Hom}_R(Y, FM) = \ell_{\mathrm{End}_S(M)} \mathrm{Hom}_S(GY, M) = \chi_M(GY)$. If N is an indecomposable endofinite S-module such that $M \leqslant N$ (i.e., $\chi_M(X) \leqslant \chi_N(X)$ for all finitely presented left S-modules X), then we have $\chi_{FM}(Y) = \chi_M(GY) \leqslant \chi_N(GY) = \chi_{FN}(Y)$, i.e., $FM \leqslant FN$. $\square$

Remark. If every finitely presented S-module X is isomorphic to one GY for some finitely presented R-module Y, then we also have that $FM \leqslant FN$ implies $M \leqslant N$.

Definition 3.5.

(1) We say that a functor $F : \mathrm{Mod}\, S \to \mathrm{Mod}\, R$ *preserves the endofinite spectrum* if, for any indecomposable endofinite S-modules M and N, we have
  (a) FM is indecomposable endofinite.
  (b) $FM \cong FN$ implies $M \cong N$.
  (c) $M \leqslant N$ implies $FM \leqslant FN$.
  In this case the spectrum of S is identified via F with a *subspectrum* of the spectrum of R.
(2) If in addition we have
  (d) $FM \leqslant FN$ implies $M \leqslant N$,
  then the spectrum of S is identified via F with a *full subspectrum* of the spectrum of R.

4. Localisation

In this section we are mainly interested in the endofinite spectrum of a finitely generated localisation of the polynomial ring over an algebraically closed field k. Let us first start with a more general setup. Let $f : R \to S$ be a ring homomorphism. Consider the restriction of scalars

$$f_* : \mathrm{Mod}\, S \to \mathrm{Mod}\, R, \quad M \mapsto M|_R \quad \text{where } r \cdot m := f(r) \cdot m \text{ for } r \in R,\ m \in M,$$

and the extension of scalars

$$f^* : \mathrm{Mod}\, R \to \mathrm{Mod}\, S, \quad M \mapsto S \otimes_R M.$$

The restriction functor f_* is faithful and additive and has f^* as a left adjoint. We begin with some preliminary results.

Lemma 4.1 [9, XI Proposition 1.2]. *The following properties of a ring homomorphism $f : R \to S$ are equivalent*:

(1) *The ring homomorphism f is a ring epimorphism.*

(2) *The composition* $f^* \circ f_*$ *is naturally equivalent to* $\mathrm{id}_{\mathrm{Mod}\,S}$.
(3) *The restriction functor* f_* *is full.*

Lemma 4.2. *Let* $f : R \to S$ *be a ring epimorphism. Then*

(1) *The restriction functor* f_* *preserves the endolength.*
(2) *Let* M *be an* S*-module. Then* f_*M *is indecomposable if and only if* M *is indecomposable.*

Proof. The assertions follow from the isomorphism $\mathrm{End}_S(M) \to \mathrm{End}_R(f_*M)$, which maps α to $f_*\alpha$, and using idempotents. □

Corollary 4.3. *Let* $f : R \to S$ *be a ring epimorphism. Then the restriction of scalars* $f_* : \mathrm{Mod}\,S \to \mathrm{Mod}\,R$ *preserves the endofinite spectrum.*

Proof. Since f_* is fully faithful, it induces an embedding of the sets of isomorphism classes, and we have the canonical isomorphism $\mathrm{End}_S(M) \to \mathrm{End}_R(f_*M)$. Furthermore, f^* is a left adjoint of f_*, and f_* preserves direct limits. Thus by Theorem 1.2, we have that f_* preserves the endofinite spectrum. □

Hence for $f : R \to S$ a ring epimorphism we have that the endofinite spectrum of S is identified via restriction with a subspectrum of $\mathrm{Spec}_{\mathrm{end}}\,R$. One should ask whether the subspectrum is full, and if one knows the endofinite spectrum of R, which isomorphism classes of indecomposable endofinite R-modules are actually in the image of the embedding functor $f_* : \mathrm{Mod}\,S \to \mathrm{Mod}\,R$. We will answer this questions in the case of a finitely generated localisation of the polynomial ring $k[T]$ over an algebraically closed field k.

Let R be a commutative ring and $\Sigma \subseteq R$ be a multiplicatively closed subset. The localisation homomorphism $\varphi : R \to R[\Sigma^{-1}]$ is an epimorphism. We will see that in the case of a localisation $\varphi : R \to R[\Sigma^{-1}]$ of a principal ideal domain R, the endofinite spectrum of $R[\Sigma^{-1}]$ is identified with a full subspectrum of the endofinite spectrum of R.

Lemma 4.4. *Let* $\varphi : R \to R[\Sigma^{-1}]$ *be a localisation of a principal ideal domain* R. *Then* φ_* *preserves simple modules which are torsion.*

Proof. Let M be a simple $R[\Sigma^{-1}]$-module. Consider the injective envelope $M \hookrightarrow I$ of M. Since $\varphi : R \to R[\Sigma^{-1}]$ is a flat ring epimorphism, φ_*I is injective by [9, XI, Proposition 3.11] and indecomposable by Lemma 4.2(2); hence it is a Prüfer module, i.e., artinian and uniserial. The simple socle is the only submodule with a skew field as endomorphism ring. By Schur's lemma, $\mathrm{End}_R(\varphi_*M) \cong \mathrm{End}_{R[\Sigma^{-1}]}(M)$ is a skew field, and so φ_*M is simple. □

Lemma 4.5. *Let* $\varphi : R \to R[\Sigma^{-1}]$ *be a localisation of a principal ideal domain* R. *Then every finitely presented* $R[\Sigma^{-1}]$*-module is up to isomorphism of the form* φ^*X *for some finitely presented* R*-module* X.

Proof. Let Y be an indecomposable finitely presented $R[\Sigma^{-1}]$-module. Note that $R[\Sigma^{-1}]$ is again a p.i.d. If Y is torsion free, then Y is free, and we have $Y \cong R[\Sigma^{-1}] \cong \varphi^* R$. If Y is torsion, then $Y \cong R[\Sigma^{-1}] \cdot y \cong R[\Sigma^{-1}]/\operatorname{ann}(y)$ is artinian and noetherian, hence of finite length. By Lemma 4.4, we have that the restriction $\varphi_* Y$ is again of finite length, hence finitely presented, since R is left noetherian. By Lemma 4.1(2), we have that $\varphi^*\varphi_* Y \cong Y$. $\square$

Remark. It is not clear to us for which rings beside the principal ideal domains the property that the localisation functor $\varphi^* : \operatorname{mod} R \to \operatorname{mod} R[\Sigma^{-1}]$ is dense holds.

By the remark after the proof of Theorem 1.2, we have that for a principal ideal domain R, the endofinite spectrum of a localisation $R[\Sigma^{-1}]$ is identified via restriction with a full subspectrum of the endofinite spectrum of R. If one knows the points in the image of the restriction functor $\varphi_* : \operatorname{Mod} R[\Sigma^{-1}] \to \operatorname{Mod} R$, then one can conclude $\operatorname{Spec}_{\text{end}} R[\Sigma^{-1}]$ from $\operatorname{Spec}_{\text{end}} R$.

For a finitely generated localisation of the polynomial ring $k[T]$ with k algebraically closed, we will now answer the question of which points of the endofinite spectrum of $k[T]$ (computed in Example 2.3(1)) lie in the image of the embedding $\varphi_* : \operatorname{Mod} k[T][\Sigma^{-1}] \to \operatorname{Mod} k[T]$. A finitely generated localisation here means that Σ is finitely generated; hence the localisation is of the form $k[T, f^{-1}]$ for a non-zero $f \in k[T]$. The isomorphism classes of indecomposable endofinite $k[T]$-modules are represented by the finite length modules $M_{\lambda,n} := k[T]/(T-\lambda)^n$ where $\lambda \in k$ and $n \in \mathbb{N}$, and the unique generic module $k(T)$. Let $f = (T-\lambda_1)\cdots(T-\lambda_m)$. From the universal property of the localisation we have that a $k[T]$-module is in the image of φ_* if and only if f is invertible in $\operatorname{End}(M)$. This is the case if and only if $T-\lambda_i$ is invertible in $\operatorname{End}(M)$ for all $i = 1, \ldots, m$. Thus $k(T)$ is clearly in $\operatorname{Im}\varphi_*$, and the modules $M_{\lambda,n}$ with $n \in \mathbb{N}$ are in $\operatorname{Im}\varphi_*$ if and only if $\lambda \neq \lambda_i$ for all $i = 1, \ldots, m$. This is since we can write $T - \lambda_i = (T-\lambda) + (\lambda - \lambda_i) \in \operatorname{End}(M_{\lambda,n}) \cong k[T]/(T-\lambda)^n$. Using that $\operatorname{End}(M_{\lambda,n})$ is a local ring, that $T-\lambda$ is nilpotent and that $\lambda - \lambda_i$ is zero or invertible gives the assertion. Hence we get the following proposition.

Proposition 4.6. *The endofinite spectrum of a finitely generated localisation $k[T, f^{-1}]$ of the polynomial ring $k[T]$ over an algebraically closed field k is identified via restriction with a full subspectrum of the endofinite spectrum of $k[T]$. More precisely, if $f = (T-\lambda_1)\cdots(T-\lambda_m)$ with $\lambda_i \in k$ is a factorisation of f in linear factors, then $\operatorname{Spec}_{\text{end}}(k[T, f^{-1}])$ is identified with the full subspectrum of $\operatorname{Spec}_{\text{end}}(k[T])$ consisting of all points except $M_{\lambda_1,n}, \ldots, M_{\lambda_m,n}$ with $n \in \mathbb{N}$.*

Hence the diagram of $\operatorname{Spec}_{\text{end}}(k[T, f^{-1}])$ is obtained from $\operatorname{Spec}_{\text{end}}(k[T])$ by removing finitely many branches.

5. Tame algebras

Let Λ be a finite dimensional algebra over an algebraically closed field k. Recall that Λ is of *tame representation type* if for all $d \in \mathbb{N}$ there are a finite number of $(\Lambda, k[T])$-

bimodules $M_1, \ldots, M_n$ which are free of rank d as $k[T]^{\mathrm{op}}$-modules, and such that every indecomposable Λ-module of dimension d is isomorphic to $M_i \otimes_{k[T]} k[T]/(T-\lambda)$ for some $1 \leqslant i \leqslant n$ and $\lambda \in k$.

We will use the following theorem from Crawley-Boevey [3] to study the endofinite spectrum of a tame algebra over an algebraically closed field k.

Theorem 5.1 [3, Introduction]. *If Λ is a finite dimensional algebra of tame representation type over an algebraically closed field k, then for every generic Λ-module G we can choose a (Λ, R_G)-bimodule M_G such that*

(1) *R_G is a finitely generated localisation of $k[T]$. As an R_G^{op}-module M_G is free of rank equal to the endolength of G. If K_G is the quotient-field of R_G, then*

$$M_G \otimes_{R_G} K_G \cong G.$$

(2) *The functor $M_G \otimes_{R_G} - : \operatorname{Mod} R_G \to \operatorname{Mod} \Lambda$ preserves isomorphism classes, indecomposability and Auslander–Reiten sequences.*
(3) *For each natural number d almost all indecomposable Λ-modules of dimension d arise as $M_G \otimes_{R_G} R_G/(r)$ for some generic module G and some non-zero $r \in R_G$.*

We want to show that the functors from this theorem preserve the endofinite spectrum by applying Theorem 1.2. Recall that by the radical of a ring we shall mean the Jacobson radical.

Lemma 5.2. *Let $f : R \to S$ be a ring homomorphism that reflects units, and let the ring S be local. Then f maps the radical of R to the radical of S and induces a monomorphism $\bar{f} : R/\operatorname{rad} \to S/\operatorname{rad}$.*

Proof. For a local ring the radical consists of all the non-invertible elements, and from the condition that f reflects units we have that f maps non-invertible elements to non-invertible elements. Hence we get the induced map $\bar{f} : R/\operatorname{rad} \to S/\operatorname{rad}$. To see that this is a monomorphism, let $f(r)$ be in $\operatorname{rad} S$. Then for all $x \in R$ we have that $f(1 + xr) = 1 + f(x)f(r)$ is a unit; hence $1 + xr$ is a unit, so $r \in \operatorname{rad} R$. □

Lemma 5.3 [3, 4.2 Lemma]. *Let $f : R \to S$ be a surjective ring homomorphism that reflects units. Then f maps the radical of R to the radical of S and induces an isomorphism $\bar{f} : R/\operatorname{rad} \to S/\operatorname{rad}$.*

Since we will deal with endomorphism rings, we have to show that certain functors reflect isomorphisms.

Lemma 5.4. *Let $f : R \to S$ be a ring epimorphism. Consider the restriction of scalars $f_* : \operatorname{Mod} S \to \operatorname{Mod} R$. Then f_* reflects isomorphisms.*

Proof. By Lemma 4.1(2), $f^* \circ f_*$ is naturally equivalent to $\mathrm{id}_{\operatorname{Mod} S}$, which of course reflects isomorphisms. Thus f_* also reflects isomorphisms. □

Lemma 5.5. *Let $f : R \to S$ be a ring epimorphism, Γ a ring and M a (Γ, R)-bimodule. Consider the functors $M \otimes_R - : \operatorname{Mod} R \to \operatorname{Mod} \Gamma$ and $(M \otimes_R S) \otimes_S - : \operatorname{Mod} S \to \operatorname{Mod} \Gamma$. If $M \otimes_R -$ reflect isomorphisms, then $(M \otimes_R S) \otimes_S -$ reflects isomorphisms too.*

Proof. The assertion follows from Lemma 5.4, since $(M \otimes_R S) \otimes_S -$ is naturally equivalent to $M \otimes_R - \circ f_*$. $\square$

Remark 5.6. Let R be a ring and M a *realisation* in the sense of [3, 5.1] such that $M \otimes_R -$ reflects isomorphisms. Let M' be a *refinement* of M, that is, a realisation $M \otimes_R R'$ over a finitely generated localisation R' of R (see again [3, 5.1]). Then from the previous lemma we immediately have that $M' \otimes_{R'} -$ reflects isomorphisms too.

The following two propositions provide additional properties of Crawley-Boevey's functors, which might be of independent interest. Note that the first proposition is not used in the sequel. For the definitions of *minimal bocses* $\mathcal{B}_i = (B_i, W_i)$, the category of *proper* $(\mathcal{B}_i, K)$-bimodules $\operatorname{Mod}^p(\mathcal{B}_i, K)$, and the map $(1, \varepsilon)^\star$ with its left inverse $(1, \omega)^\star$, we refer the reader to [3] and [2].

Proposition 5.7. *The functors $M_G \otimes_{R_G} - : \operatorname{Mod} R_G \to \operatorname{Mod} \Lambda$ from Crawley-Boevey's theorem reflect isomorphisms.*

Proof. From [3, 3.5 Theorem] one gets functors $T_i \otimes_{B_i} - : \operatorname{Mod} B_i \to \operatorname{Mod} \Lambda$ with the property that if K/k is a field extension, then there exist functors $F_i^K : \operatorname{Mod}^p(\mathcal{B}_i, K) \to \operatorname{Mod}(\Lambda, K)$ which reflect isomorphisms and whose composition

$$\operatorname{Mod}(B_i, K) \xrightarrow{(1,\varepsilon)^\star} \operatorname{Mod}^p(\mathcal{B}_i, K) \xrightarrow{F_i^K} \operatorname{Mod}(\Lambda, K)$$

with $(1, \varepsilon)^\star$ is naturally isomorphic to $T_i \otimes_{B_i} -$. Let $K = k$, then, since B_i and Λ are k-algebras, we obtain the following commutative diagram:

$$\begin{tikzcd}
\operatorname{Mod} B_i \arrow[r, "\mathrm{id}", "\cong"'] & \operatorname{Mod}(B_i, k) \arrow[r, "(1,\varepsilon)^\star"] \arrow[dr, "\mathrm{id}"'] & \operatorname{Mod}^p(\mathcal{B}_i, k) \arrow[r, "F_i^k"] \arrow[d, "(1,\omega)^\star"] & \operatorname{Mod}(\Lambda, k) \arrow[r, "\mathrm{id}", "\cong"'] & \operatorname{Mod} \Lambda \\
 & & \operatorname{Mod}(B_i, k). & &
\end{tikzcd}$$

Since $(1, \varepsilon)^\star$ has a left inverse $(1, \omega)^\star$, $(1, \varepsilon)^\star$ reflects isomorphisms. It follows that $T_i \otimes_{B_i} -$ reflects isomorphisms too. One gets $M_G \otimes_{R_G} -$ from a common refinement M_G over R_G of certain T_i (see the proof of [3, 5.4 Theorem]). From Remark 5.6 we know that $M_G \otimes_{R_G} -$ also reflects isomorphisms. $\square$

Proposition 5.8. *The functors $M_G \otimes_{R_G} - : \operatorname{Mod} R_G \to \operatorname{Mod} \Lambda$ from Crawley-Boevey's theorem induce isomorphisms $\operatorname{End}_{R_G}(N)/\operatorname{rad} \to \operatorname{End}_\Lambda(M_G \otimes_R N)/\operatorname{rad}$ for every R_G-module N with $\operatorname{End}_\Lambda(M_G \otimes_{R_G} N)$ local.*

Proof. Again we have to look at the construction of the functors and to use the left inverse $(1,\omega)^\star$ of $(1,\varepsilon)^\star$. As above, $(1,\varepsilon)^\star$ reflects isomorphisms, and $(1,\omega)^\star$ reflects isomorphisms since ω is a *reflector* by definition (see [2]). We consider again the commutative diagram in the proof of Proposition 5.7 and get for a B_i-module X an induced diagram which is commutative up to natural isomorphisms:

$$\begin{tikzcd}
\mathrm{End}_{B_i}(X) \arrow[r, "(1,\varepsilon)^\star"] \arrow[dr, "\mathrm{id}"'] \arrow[rr, bend left, "T_i\otimes_{B_i}-"] & \mathrm{End}_{\mathcal{P}(\mathcal{B}_i,k)}\big((1,\varepsilon)^\star(X)\big) \arrow[r, "F_i^k"] \arrow[d, "(1,\omega)^\star"] & \mathrm{End}_\Lambda(T_i\otimes_{B_i} X) \\
 & \mathrm{End}_{B_i}(X). &
\end{tikzcd}$$

Let N be an R_G-module such that $\mathrm{End}_\Lambda(M_G\otimes_{R_G} N)$ is local. Since M_G is a refinement of T_i, we have that $M_G\otimes_{R_G}-$ is naturally equivalent to $T_i\otimes_{B_i}-\circ f_*$ where $f: B_i\to R_G$ is a finitely generated localisation. Hence, if we set $X=f_*N$, then $\mathrm{End}_\Lambda(T_i\otimes_{B_i} X)$ is local. We know that F_i^k and $(1,\omega)^\star$ are full and reflect isomorphisms; hence from Lemma 5.3 we get induced isomorphisms modulo the radical. By Lemma 5.2, $F_i^k\circ(1,\varepsilon)^\star$ maps radical to radical and hence $(1,\varepsilon)^\star$ also does. So we get the following diagram, again commutative up to natural isomorphisms:

$$\begin{tikzcd}
\mathrm{End}_{B_i}(X)/\operatorname{rad} \arrow[r, "(1,\varepsilon)^\star"] \arrow[dr, "\mathrm{id}"'] \arrow[rr, bend left, "T_i\otimes_{B_i}-"] & \mathrm{End}_{\mathcal{P}(\mathcal{B}_i,k)}\big((1,\varepsilon)^\star(X)\big)/\operatorname{rad} \arrow[r, "F_i^k", "\cong"'] \arrow[d, "(1,\omega)^\star", "\cong"'] & \mathrm{End}_\Lambda(T_i\otimes_{B_i} X)/\operatorname{rad} \\
 & \mathrm{End}_{B_i}(X)/\operatorname{rad}. &
\end{tikzcd}$$

This gives the desired isomorphism, since for the localisation $f: B_i\to R_G$ we have the isomorphism $\mathrm{End}_{R_G}(N)\xrightarrow[\cong]{f_*}\mathrm{End}_{B_i}(f_*N)$. $\square$

Lemma 5.9. *Let Λ, R be rings and M a (Λ,R)-bimodule that is free as an R^{op} module. Then the functor $M\otimes_R-:\operatorname{Mod}R\to\operatorname{Mod}\Lambda$ preserves endofiniteness. Moreover, for X an endofinite R-module we have*

$$\operatorname{endol}(M\otimes_R X)\leqslant \operatorname{rk}(M_R)\cdot\operatorname{endol}(X). \qquad (*)$$

Proof. Since the functor $M\otimes_R-$ induces a ring homomorphism $\mathrm{End}_R(X)\to\mathrm{End}_\Lambda(M\otimes_R X)$ and we have an isomorphism $M\otimes_R X\cong X^{\operatorname{rk}(M_R)}$ as $\mathrm{End}_R(X)$-modules, we get $\operatorname{endol}(M\otimes_R X)\leqslant \ell_{\mathrm{End}_R(X)}(M\otimes_R X)=\operatorname{rk}(M_R)\cdot\ell_{\mathrm{End}_R(X)}(X)$. $\square$

Remark 5.10. In the special situation that we have in Crawley-Boevey's theorem, one can show that equality holds in $(*)$ for X an indecomposable endofinite R-module.

Proposition 5.11. *The functors $M_G \otimes_{R_G} - : \mathrm{Mod}\, R_G \to \mathrm{Mod}\, \Lambda$ from Crawley-Boevey's theorem preserve the endofinite spectrum.*

Proof. The functor $M_G \otimes_{R_G} -$ preserves isomorphism classes and indecomposability by Theorem 5.1. By Lemma 5.9, $M_G \otimes_{R_G} -$ preserves endofiniteness. So for every indecomposable R_G-module N of finite endolength, $\mathrm{End}_\Lambda(M_G \otimes_{R_G} N)$ is local, and we get an induced algebra isomorphism $\mathrm{End}_{R_G}(N)/\operatorname{rad} \to \mathrm{End}_\Lambda(M_G \otimes_{R_G} N)/\operatorname{rad}$ by Proposition 5.8. The functor $\mathrm{Hom}_R(M_G, R) \otimes_\Lambda - : \mathrm{Mod}\, \Lambda \to \mathrm{Mod}\, R$ is a left adjoint of $M_G \otimes_{R_G} -$, and moreover $M_G \otimes_{R_G} -$ preserves direct limits. From Theorem 1.2 we get that the functor $M_G \otimes_{R_G} -$ preserves the endofinite spectrum. $\square$

Hence we get Corollary 1.3 from the introduction.

Example 5.12. The Kronecker algebra $\Lambda = k(1 \cdot \underset{\beta}{\overset{\alpha}{\rightrightarrows}} \cdot 2)$. We have the unique generic Λ-module $G := k(T) \underset{T\cdot}{\overset{\mathrm{id}}{\rightrightarrows}} k(T)$ and thus the functor

$$G \otimes_{k[T]} - : \mathrm{Mod}\, k[T] \to \mathrm{Mod}\, \Lambda.$$

We consider which Λ-modules are assigned to the endofinite $k[T]$-modules:

$$G \otimes_{k[T]} k(T) \cong \left(k(T) \underset{T\cdot}{\overset{\mathrm{id}}{\rightrightarrows}} k(T)\right),$$
$$G \otimes_{k[T]} M_{\lambda,n} \cong \left(M_{\lambda,n} \underset{\overline{T}\cdot}{\overset{\mathrm{id}}{\rightrightarrows}} M_{\lambda,n}\right) \cong \left(k^n \underset{J_{\lambda,n}}{\overset{I}{\rightrightarrows}} k^n\right).$$

Thus the generic Λ-module and all regular Λ-modules except $k^n \underset{I}{\overset{J_{0,n}}{\rightrightarrows}} k^n$ with $n \in \mathbb{N}$ are images of endofinite $k[T]$-modules, and the diagrams in Example 2.3 show that the order relation is actually being preserved. In this example, we also have that $FM \leqslant FN$ implies $M \leqslant N$ for M and N indecomposable endofinite Λ-modules.

6. Minimal points and epimorphisms to simple artinian rings

In [7] Ringel studies the Cohn spectrum, a generalisation of the prime spectrum and introduced by Cohn in [1]. Ringel gives a one-to-one correspondence between the points of this spectrum—the equivalence classes of epimorphisms to simple artinian rings—and the isomorphism classes of indecomposable endofinite modules with a skew field as endomorphism ring. We show that the minimal points of the endofinite spectrum are of this form (Corollary 6.4). Again, by the radical of a ring we mean the Jacobson radical. The following lemma is well-known.

Lemma 6.1. *Let R be a ring such that R/rad is semisimple and M an R-module. Then M is semisimple if and only if $\operatorname{rad} R \cdot M = 0$.*

Lemma 6.2. *Let M be an indecomposable endofinite module. Then M is semisimple over its endomorphism ring if and only if the endomorphism ring of M is a skew field.*

Proof. Let $\operatorname{End}_R(M) =: D$ be a skew field. Then as a D-module, $M \cong D^d$ for some $d \in \mathbb{N}$; hence M is semisimple over its endomorphism ring. Now let $E := \operatorname{End}_R(M)$ and M be semisimple over E. We have that $\operatorname{rad} E \cdot M = 0$ and hence $\operatorname{rad} E \subseteq \operatorname{ann}_E(M)$. But since $\operatorname{ann}_E(M) = 0$, $\operatorname{rad} E$ is also zero. Hence E is a skew field, since it is local by Lemma 3.3. □

Proposition 6.3. *Let M be an indecomposable endofinite R-module. Then there exists an indecomposable endofinite R-module N with a skew field as endomorphism ring such that $N \leqslant M$.*

Proof. We get the R-module N inductively by reducing the endolength. Set $E_0 := \operatorname{End}_R(M)$. We know that E_0/rad is a skew field, so semisimple. If M is not semisimple as an E_0-module, then set $N_1 := \operatorname{rad} E_0 \cdot M$. We have $0 \subsetneqq N_1 \subsetneqq M$, the first inequality by Lemma 6.1 and the second inequality by Nakayama's lemma. We remark that N_1 is a submodule of M also as an R-module and that we have a ring homomorphism $E_0 = \operatorname{End}_R(M) \to \operatorname{End}_R(\operatorname{rad}(\operatorname{End}_R(M)) \cdot M) = \operatorname{End}_R(N_1) =: E_1$, since $\operatorname{rad}(\operatorname{End}_R(M))$ is an ideal of $\operatorname{End}_R(M)$. Therefore we have $\ell_{E_1} \operatorname{Hom}_R(X, N_1) \leqslant \ell_{E_0} \operatorname{Hom}_R(X, N_1) \leqslant \ell_{E_0} \operatorname{Hom}_R(X, M)$ for all finitely presented R-modules X, and in particular, for $X = R$, we have $\operatorname{endol}(N_1) < \operatorname{endol}(M)$. Hence $\chi_{N_1} < \chi_M$. Crawley-Boevey has shown in [4] that every character is a sum of irreducible characters. Now choose one of the irreducible summands of χ_{N_1}, and let M_1 be the corresponding irreducible endofinite R-module. Of course, $\operatorname{endol}(M_1) < \operatorname{endol}(M)$ and $\chi_{M_1} < \chi_M$. Since M is of finite endolength, we will get after fewer than $\operatorname{endol}(M)$ steps an $i \in \mathbb{N}$ such that $N := M_i$ is semisimple as an $\operatorname{End}_R(N)$-module. Hence by Lemma 6.2, $\operatorname{End}_R(N)$ is a skew field. □

Corollary 6.4. *The minimal points of the endofinite spectrum of a ring R have a skew field as endomorphism ring.*

In the special cases of left artinian rings and Artin algebras we get more.

Lemma 6.5. *Let R be a left artinian ring. Then the endofinite simple modules are pairwise not comparable.*

Proof. Let S and S' be non-isomorphic endofinite simple R-modules. Since simple modules over left artinian rings are always finitely presented, we compute $\chi_S(S) = 1 = \chi_{S'}(S')$ and $\chi_S(S') = 0 = \chi_{S'}(S)$. Hence S and S' are not comparable. □

Proposition 6.6. *Let R be a left artinian ring. Let M be an indecomposable endofinite R-module and S an endofinite simple submodule. Then $S \leqslant M$.*

Proof. Let X be a finitely presented R-module. If X is simple, then we have

$$\chi_S(X) = \ell_{\mathrm{End}_R(S)} \mathrm{Hom}_R(X, S) = \begin{cases} 1, & \text{if } X \cong S, \\ 0, & \text{if } X \not\cong S. \end{cases}$$

So we get $\chi_S(X) \leqslant \chi_M(X)$ since $\mathrm{Hom}_R(X, S) \subseteq \mathrm{Hom}_R(X, M)$. Of course, we have $\chi_S(X) \leqslant \chi_M(X)$ also for X semisimple. Now let X be an arbitrary finitely presented R-module. Consider $\mathrm{top}\, X = X/\mathrm{rad}_R\, X$. Since R is left artinian, we have that R/rad is semisimple. By Lemma 6.1, we get from $\mathrm{rad}\, R \cdot \mathrm{top}\, X = 0$ that $\mathrm{top}\, X$ is semisimple, hence $\chi_S(\mathrm{top}\, X) \leqslant \chi_M(\mathrm{top}\, X)$. Since $\mathrm{Hom}_R(\mathrm{top}\, X, M) \subseteq \mathrm{Hom}_R(X, M)$, we have $\chi_M(\mathrm{top}\, X) \leqslant \chi_M(X)$. Every morphism from X to a simple module factors through the top of X, hence $\mathrm{Hom}_R(\mathrm{top}\, X, S) \cong \mathrm{Hom}_R(X, S)$, so $\chi_S(X) = \chi_S(\mathrm{top}\, X)$. Finally, we have $\chi_S(X) \leqslant \chi_M(X)$. $\square$

Remark. For an Artin R-algebra Λ we have that the endomorphism ring of a finite length Λ-module M is again an Artin R-algebra; hence every Λ-module of finite length is endofinite.

Corollary 6.7. *The minimal points of the endofinite spectrum of an Artin algebra Λ are precisely the isomorphism classes of simple Λ-modules.*

We now turn our attention to the one-to-one correspondence between the points of the endofinite spectrum having a skew field as endomorphism ring and the equivalence classes of epimorphisms to simple artinian rings.

Lemma 6.8. *Let R, D be rings and M, M' be (R, D)-bimodules. Then the functors $\mathrm{Hom}_R(-, \mathrm{Hom}_D(M, M'))$ and $\mathrm{Hom}_D(M, \mathrm{Hom}_R(-, M'))$ that map from $\mathrm{mod}\, R$ to the category of $(\mathrm{End}_D(M'), \mathrm{End}_D(M))$-bimodules are naturally isomorphic.*

Proof. We clearly have an isomorphism for $X = R$. Left exactness of both functors gives, from the exact sequence $R^m \to R^n \to X \to 0$, the natural isomorphism by the Five lemma. $\square$

Lemma 6.9. *Let E be a local ring and M an (R, E)-bimodule such that M is semisimple artinian as an E-module. Let $S := \mathrm{End}_E(M)$. Then we have*

$$\ell_E \mathrm{Hom}_R(X, M) = \ell_S \mathrm{Hom}_R(X, S) \quad \textit{for all } X \in \mathrm{mod}\, R.$$

Proof. Let $\varphi : E \to E/\mathrm{rad} =: D$ be the canonical projection. D is a skew field since E is local. Since M_E is semisimple, it is also a D-module, and $\ell_D(M) = \ell_E(M) < \infty$ by Lemma 3.1. Further, we have $\ell_D(Y) = \ell_E(Y)$ for all D-modules Y of finite length by Lemma 3.2. From the previous lemma we get isomorphisms of right S-modules

$$\mathrm{Hom}_R\big(X, \mathrm{End}_D(M)\big) \cong \mathrm{Hom}_D\big(M, \mathrm{Hom}_R(X, M)\big) \quad \text{for all } X \in \mathrm{mod}\, R.$$

Since M_D is a progenerator and $S = \operatorname{End}_E(M) \cong \operatorname{End}_D(M)$, we have a Morita equivalence $\operatorname{Hom}_D(M, -) : \operatorname{Mod} D^{\mathrm{op}} \to \operatorname{Mod} S^{\mathrm{op}}$. Hence this functor preserves composition length. Since $S = \operatorname{End}_E(M) \cong \operatorname{End}_D(M) \cong \operatorname{Mat}_{\ell_D(M)}(D)$, we have $\ell_S S < \infty$, hence $\ell_D \operatorname{Hom}_R(X, M) = \ell_S \operatorname{Hom}_D(M, \operatorname{Hom}_R(X, M)) = \ell_S \operatorname{Hom}_R(X, S) < \infty$ for all $X \in \operatorname{mod} R$. Finally, we have $\ell_S \operatorname{Hom}_R(X, S) = \ell_D \operatorname{Hom}_R(X, M) = \ell_E \operatorname{Hom}_R(X, M)$ for all finitely presented R-modules X by Lemma 3.2. □

Definition 6.10. Let $f : R \to S$ be a homomorphism to a simple artinian ring. Define the *character of f*

$$\chi_f : \operatorname{mod} R \to \mathbb{N}_0, \quad X \mapsto \chi_f(X) := \ell_S \operatorname{Hom}_R(X, S).$$

One verifies that this definition gives a character for $\operatorname{mod} R$.

Recall that two ring homomorphisms $f_1 : R \to S_1$ and $f_2 : R \to S_2$ to simple artinian rings are equivalent, denoted by $f_1 \sim f_2$, if there exists a ring isomorphism $\varphi : S_1 \to S_2$ such that $\varphi \circ f_1 = f_2$.

Remark. We have $\chi_f(R) = \ell_S S$; furthermore, $f_1 \sim f_2$ implies $\chi_{f_1} = \chi_{f_2}$.

Following Ringel's assignment from [7, 2.1 Proposition], we may reformulate his result involving the characters of the epimorphisms to simple artinian rings as follows.

Proposition 6.11. *Let M be an endofinite R-module. Then the following are equivalent*:

(1) *The endomorphism ring of M is a skew field.*
(2) *There exists a ring epimorphism $f : R \to S$ to a simple artinian ring such that $\chi_f = \chi_M$.*

Proof. (1) ⇒ (2). Let $D := \operatorname{End}_R(M)$ and $d := \operatorname{endol} M$. We have that $S := \operatorname{End}_D(M) \cong \operatorname{Mat}_d(D)$ is simple artinian. Now let $f : R \to S$ be the ring homomorphism that maps an $r \in R$ to the left-multiplication map $m \mapsto r \cdot m$. Of course, M is free, and so semisimple as a D-module. From Lemma 6.9 with $E = D$ we have $\chi_f(X) = \ell_S \operatorname{Hom}_R(X, S) = \ell_E \operatorname{Hom}_R(X, M) = \chi_M(X)$ for all finitely presented R-modules X.

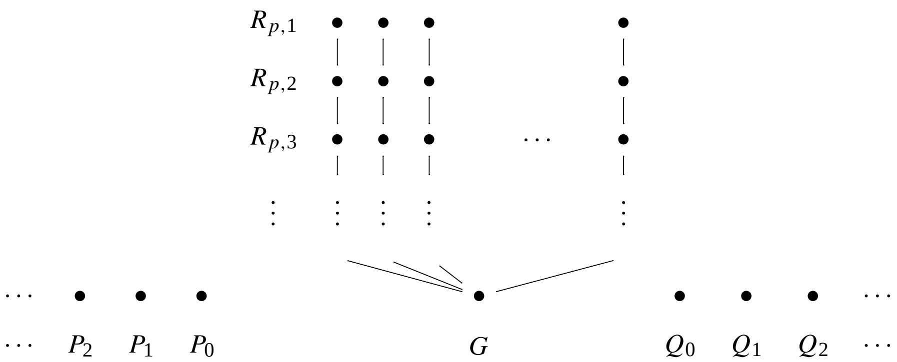
(2) ⇒ (1). Since Ringel's assignment (that is, choosing the unique simple S-module and restricting it by the ring epimorphism to an R-module) gives an endofinite R-module with a skew field as endomorphism ring, it must be isomorphic to M. □

Since the endofinite R-modules with a skew field as endomorphism ring are indecomposable, the points of the Cohn spectrum also appear in the endofinite spectrum. One may ask whether the partial order by specialisation in the Cohn spectrum and by characters in the endofinite spectrum imply each other or not. The answer is that in general there is no implication. A pair of indecomposable endofinite R-modules with skew fields as endomorphism rings and ordered by characters does in general not imply a specialisation. This can be seen by the endofinite spectrum of the Kronecker algebra that we computed in Example 2.3(2), since an implication would contradict the theory of the Cohn spectrum of finite dimensional hereditary algebras such developed by Ringel in [7]. To see that a specialisation does not in general imply an order by characters, we first recall that a simple artinian ring S is of the form $\mathrm{Mat}_d(D)$ for some $d \in \mathbb{N}$ and some skew field D, hence $\chi_f(R) = \ell_S S = d$. Now in [7, 1.6 Examples] Ringel discusses two examples of specialisations from an epimorphism $\delta : R \to \mathrm{Mat}_d(D)$ to an epimorphism $\varepsilon : R \to \mathrm{Mat}_e(E)$, in one case with $d = 1$ and $e = 2$ and in the other case with $d = 2$ and $e = 1$. This shows that in general there is no implication to either $\chi_\varepsilon \leqslant \chi_\delta$ or $\chi_\varepsilon \geqslant \chi_\delta$.

7. The normalised endofinite spectrum

In this section we consider *normalised characters*, i.e., we normalise our characters by dividing them by their degree. Here we recover Schofield's Sylvester module rank functions (see [8]) on which Crawley-Boevey's characters are based. One verifies that the analogous partial order on the normalised characters induces a partial order on the isomorphism classes of indecomposable endofinite R-modules. In this way we obtain the *normalised endofinite spectrum*.

In the normalised situation the counterexamples from the end of the previous section have been repaired. For example, for the Kronecker algebra we get the following diagram of the *normalised endofinite spectrum*:



This diagram coincides with Ringel's theory of the Cohn spectrum. We are hopeful that one can show that in the normalised case the order by characters and the order by specialisation imply each other. Another interesting aspect in the normalised situation is that we have symmetry between the ring and its opposite:

Proposition 7.1. *The normalised endofinite spectrum and its dual are isomorphic as partially ordered sets.*

Proof. For a character χ let $\overline{\chi}$ be the normalised version, i.e., $\overline{\chi}(X) := \chi(X)/\chi(R)$ for $X \in \operatorname{mod} R$. Note that $\overline{\chi}(R^n) = n$. Now let $R^m \xrightarrow{\alpha} R^n \to X \to 0$ be a free presentation of

X. We have that $\overline{D\chi}(X) = n - m + \overline{\chi}(\operatorname{Coker} \alpha^*)$. So if we have two characters χ and χ' with $\overline{\chi} \leqslant \overline{\chi'}$, then we also have $\overline{D\chi} \leqslant \overline{D\chi'}$. $\square$

The theory for the endofinite spectrum developed in Sections 3–5 can easily be verified for the normalised endofinite spectrum: in Theorem 1.2 one has to add the assumption that the functor F preserves the endolength up to a constant factor. By Lemma 4.2(1), we have that for a ring epimorphism f the restriction functor f_* has this property, so Corollary 4.3 remains unchanged. For the proof of Corollary 1.3 we have to check in addition that the functors $M_G \otimes_{R_G} - : \operatorname{Mod} R_G \to \operatorname{Mod} \Lambda$ from Crawley-Boevey's theorem also have this property. This is true, however, as noted in Remark 5.10.

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