Discrete Applied Mathematics 157 (2009) 3519–3530

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

On minimally b-imperfect graphs

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ARTICLE INFO

Article history:
Received 2 December 2007
Received in revised form 30 January 2009
Accepted 24 February 2009
Available online 9 April 2009

Keywords:
Coloration
b-coloring
a-chromatic number
b-chromatic number

ABSTRACT

A b-coloring is a coloring of the vertices of a graph such that each color class contains a vertex that has a neighbour in all other color classes. The b-chromatic number of a graph G is the largest integer k such that G admits a b-coloring with k colors. A graph is b-perfect if the b-chromatic number is equal to the chromatic number for every induced subgraph H of G. A graph is minimally b-imperfect if it is not b-perfect and every proper induced subgraph is b-perfect. We give a list $\mathcal{F}$ of minimally b-imperfect graphs, conjecture that a graph is b-perfect if and only if it does not contain a graph from this list as an induced subgraph, and prove this conjecture for diamond-free graphs, and graphs with chromatic number at most three.

1. Introduction

A proper coloring of a graph G is a mapping c from the vertex-set $V(G)$ of G to the set of positive integers (colors) such that any two adjacent vertices are mapped to different colors. Each set of vertices colored with one color is a stable set of vertices of G, so a coloring is a partition of V into stable sets. The smallest number k for which G admits a coloring with k colors is the chromatic number $\chi(G)$ of G.

Many graph invariants related to colorings have been defined. Most of them try to minimize the number of colors used to color the vertices under some constraints. For some other invariants, it is meaningful to try to maximize this number. The b-chromatic number is such an example. When we try to color the vertices of a graph, we can start from a given coloring and try to decrease the number of colors by eliminating color classes. One possible such procedure consists in trying to reduce the number of colors by transferring every vertex from a fixed color class to a color class in which it has no neighbour, if any such class exists. A *b-coloring* is a proper coloring in which this is not possible, that is, every color class i contains at least one vertex that has a neighbour in all the other color classes. Any such vertex will be called a *b-vertex* of color i. The *b-chromatic number* $b(G)$ is the largest integer k such that G admits a b-coloring with k colors.

The behavior of the b-chromatic number can be surprising. For example, the values of k for which a graph admits a b-coloring with k colors do not necessarily form an interval of the set of integers; in fact any finite subset of $\{2, \ldots\}$ can be the set of these values for some graph [5]. Irving and Manlove [7,12] proved that deciding whether a graph G admits a b-coloring with a given number of colors is an NP-complete problem, even when it is restricted to the class of bipartite graphs [11]. On the other hand, they gave a polynomial-time algorithm that solves this problem for trees. The NP-completeness results have incited researchers to establish bounds on the b-chromatic number in general or to find its exact values for subclasses of graphs (see [3,9,10,2,4,8]).

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doi:10.1016/j.dam.2009.02.023

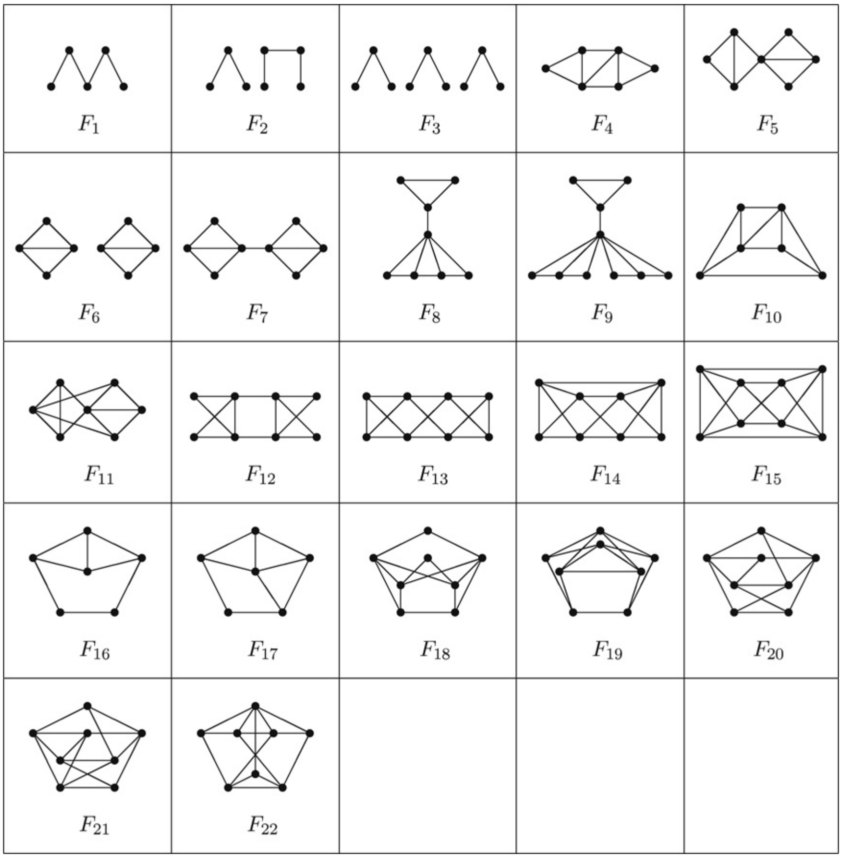

Fig. 1. Class $\mathcal{F} = \{F_1, \ldots, F_{22}\}$.

Clearly every $\chi(G)$-coloring of a graph G is a b-coloring, and so every graph G satisfies $\chi(G) \le b(G)$. As usual with such an inequality, it may be interesting to look at the graphs that satisfy it with equality. However, graphs such that $\chi(G) = b(G)$ do not have a specific structure; to see this, we can take any arbitrary graph G and add a component that consists of a clique of size $b(G)$; we obtain a graph G' that satisfies $\chi(G') = b(G') = b(G)$. This led Hoàng and Kouider [6] to introduce the class of *b-perfect* graphs: a graph G is called b-perfect if every induced subgraph H of G satisfies $\chi(H) = b(H)$. Hoàng and Kouider [6] proved the b-perfectness of some classes of graphs, and asked whether b-perfectness can be characterized in some way. Here we propose a precise conjecture in this direction and some evidence for its validity. For a fixed graph F, we say that a graph G is *F-free* if it does not contain an induced subgraph that is isomorphic to F. For a set $\mathcal{F}$ of graphs, we say that a graph G is *$\mathcal{F}$-free* if it does not have an induced subgraph that is isomorphic to a member of $\mathcal{F}$. Let us say that a graph is *minimally b-imperfect* if it is not b-perfect and each of its proper induced subgraphs is b-perfect. Let $\omega(G)$ denote the number of vertices in a largest clique of G.

Let $\mathcal{F} = \{F_1, \ldots, F_{22}\}$ be the set of graphs depicted in Fig. 1.

Conjecture 1. *A graph is b-perfect if and only if it is $\mathcal{F}$-free.*

One direction of this conjecture is easy to establish, namely that the graphs in class $\mathcal{F}$ are b-imperfect. More precisely, for $i \in \{1, 2, 3\}$, we have $\chi(F_i) = 2$ and $b(F_i) = 3$; a b-coloring of F_i with three colors is obtained by giving colors 1, 2, 3 to the three vertices of degree two and coloring the remaining vertices in such a way that the first three are b-vertices. For $i \in \{4, \ldots, 22\}$, we have $\chi(F_i) = 3$ and $b(F_i) = 4$; a b-coloring of F_i with four colors is obtained by giving colors 1, 2, 3, 4 to four carefully chosen vertices of degree at least three and coloring the remaining vertices in such a way that the chosen vertices are b-vertices: for $i \in \{4, 5, 6, 7, 8, 9, 12, 16\}$ there is only one choice of four such vertices; for $i \in \{13, 14, 15\}$, choose the two leftmost and the two rightmost vertices; for other values of i we omit the details. Moreover, it is a routine matter to check (and we omit the details) that every proper induced subgraph of every member of $\mathcal{F}$ is b-perfect; so every graph in class $\mathcal{F}$ is minimally b-imperfect.

Conjecture 2. *A minimally b-imperfect graph G that is not triangle-free has $b(G) = 4$ and $\omega(G) = 3$.*

The *diamond* is the graph with four vertices and five edges. The purpose of this paper is to prove the following two theorems.

Theorem 1.1. *A diamond-free graph is b-perfect if and only if it is $\{F_1, F_2, F_3, F_{18}, F_{20}\}$-free.*

Theorem 1.2. *Let G be a graph with chromatic number at most 3. Then G is b-perfect if and only if it does not contain F_i as induced subgraph for $i = 1, 2, \ldots, 22$.*

The following results were proved by Hoàng and Kouider [6].

Theorem 1.3 (*Hoàng and Kouider [6]*).

- *A bipartite graph is b-perfect if and only if it is $\{F_1, F_2, F_3\}$-free.*
- *A P_4-free graph is b-perfect if and only if it is $\{F_3, F_6\}$-free.*

Thus we can view Theorems 1.1–1.3 as evidence for Conjecture 1.

The rest of this paper is organized as follows. We give preliminary results in Section 2, the proof of Theorem 1.1 in Section 3, and the proof of Theorem 1.2 in Section 4. In the remainder of this section, we introduce the definitions needed in the proofs.

Let G be a graph, and x be a vertex of G. The *neighbourhood* of v is the set $N(v) = \{u \in V(G) \mid uv \in E\}$ and the *degree* of v is $\deg(v) = |N(v)|$. If a vertex v is adjacent to a vertex x, then we say that v *sees* x, otherwise we say v *misses* x. Vertices v and x are *comparable* if $N(v) - \{x\} \subseteq N(x)$, or $N(x) - \{v\} \subseteq N(v)$. v and x are *twins* if $N(v) = N(x)$, in particular, twins miss each other. A set X of vertices is *homogeneous* if $2 \leq |X| < |V(G)|$ and every vertex in $G - X$ either sees all, or misses all vertices of X. A *homogeneous clique* is a clique that is a homogeneous set. For integer $k \geq 1$, we denote by P_k the chordless path with k vertices. For integer $k \geq 3$, we denote by C_k the chordless cycle with k vertices.

2. Preliminary results

Lemma 2.1 (*Hoàng and Kouider [6]*). *Let G be a minimal b-imperfect graph. Then no component of G is a clique.* □

Lemma 2.2. *Let G be a minimal b-imperfect graph and x be any simplicial vertex of G. Then x is not a b-vertex for any b-coloring of G with $b(G)$ colors.*

Proof. Suppose that x is a b-vertex for some b-coloring c of G with $b(G)$ colors. Then all $b(G)$ colors of c appear in the clique formed by x and its neighbours. Thus $b(G) \leq \omega(G) \leq \chi(G) < b(G)$, a contradiction. □

Lemma 2.3. *Let G be a minimal b-imperfect graph, let u, v be two non-adjacent vertices of G such that $N(u) \subseteq N(v)$, and let c be any b-coloring with $b(G)$ colors. Then $c(u) \neq c(v)$, and u is not a b-vertex. In particular, if $N(u) = N(v)$, then none of u, v is a b-vertex.*

Proof. Suppose that $c(u) = c(v) = 1$. Consider the restriction of c to $G \setminus u$. Every b-vertex z of color $i \geq 2$ in G is still a b-vertex in $G \setminus u$, because it cannot be that u is the only neighbour of z of color 1. Moreover, it cannot be that u is the only b-vertex of G of color 1, because if it is a b-vertex then v is also a b-vertex. But then $b(G\setminus u) \geq b(G) > \chi(G) \geq \chi(G\setminus u)$, so $G\setminus u$ is b-imperfect, a contradiction. Thus $c(u) \neq c(v)$. This implies that u cannot be a b-vertex, for it has no neighbour of color $c(v)$. In particular, if $N(u) = N(v)$, then the preceding argument works both ways, which leads to the desired conclusion. □

Lemma 2.4. *Let G be a minimal b-imperfect $\mathcal{F}$-free graph. Then G is connected.*

Proof. Suppose that G has several components $G_1, \ldots, G_p$, $p \geq 2$. By Lemma 2.1, each G_i has a subset S_i of three vertices that induce a chordless path. Then G is P_4-free, for otherwise, since a P_4 is in one component of G, G contains an F_2. But then Theorem 1.3 is contradicted. Thus the lemma holds. □

In the remainder this section, we assume that G is an $\mathcal{F}$-free graph that contains an induced C_5, and we try in the following claims to describe the structure of G as precisely as possible.

Claim 2.1. *Let $C = \{c_1, \ldots, c_5\}$ be the vertex-set of an induced C_5 in G, with edges c_ic_{i+1}, $i = 1, \ldots, 5$ and with the subscripts taken modulo 5. Let v be any vertex of $V(G) \setminus C$ that has a neighbour in C. Then either:*

- $N(v) \cap C = C$, *or*
- $N(v) \cap C = \{c_i, c_{i+2}, c_{i+3}\}$ *for some* $i \in \{1, \ldots, 5\}$, *or*
- $N(v) \cap C = \{c_i, c_{i+2}\}$ *for some* $i \in \{1, \ldots, 5\}$.

Proof. Suppose that the claim does not hold. Then $N(v) \cap C$ is equal to either $\{c_i\}$ or $\{c_i, c_{i+1}\}$ or $\{c_i, c_{i+1}, c_{i+2}\}$ or $\{c_i, c_{i+1}, c_{i+2}, c_{i+3}\}$ for some $i \in \{1, \ldots, 5\}$. In the first two cases, $\{v, c_i, c_{i-1}, c_{i-2}, c_{i-3}\}$ induces an F_1. In the third case, $C \cup \{v\}$ induces an F_{16}. In the last case, $C \cup \{v\}$ induces an F_{17}. In either case we have a contradiction to G being $\mathcal{F}$-free. So the claim holds. □

Since G has an induced C_5, $V(G)$ contains five disjoint and non-empty subsets $A_1, \ldots, A_5$ such that (with subscripts modulo 5) every vertex of A_i sees every vertex of $A_{i-1} \cup A_{i+1}$ and misses every vertex of $A_{i-2} \cup A_{i+2}$. We choose these five sets such that their union $A_1 \cup \cdots \cup A_5$ is as large as possible. Now we define additional subsets of vertices as follows, for $i = 1, \ldots, 5$:

- Let T be the set of vertices of $V(G) \setminus (A_1 \cup \cdots \cup A_5)$ that see all of $A_1 \cup \cdots \cup A_5$;
- Let Z be the set of vertices of $V(G) \setminus (A_1 \cup \cdots \cup A_5)$ that see none of $A_1 \cup \cdots \cup A_5$;
- Let D_i be the set of vertices of $V(G) \setminus (A_1 \cup \cdots \cup A_5)$ that see all of $A_i \cup A_{i-2} \cup A_{i+2}$ and miss all of $A_{i-1} \cup A_{i+1}$;
- Let B_i^- be the set of vertices of $V(G) \setminus (A_1 \cup \cdots \cup A_5)$ that see all of $A_{i-1} \cup A_{i+1}$, miss all of $A_i \cup A_{i+2}$ and see at least one but not all vertices of A_{i-2};
- Let B_i^+ be the set of vertices of $V(G) \setminus (A_1 \cup \cdots \cup A_5)$ that see all of $A_{i-1} \cup A_{i+1}$, miss all of $A_i \cup A_{i-2}$ and see at least one but not all vertices of A_{i+2}.

Claim 2.2. *The sets A_i, B_i^-, B_i^+, D_i ($i = 1, \ldots, 5$) and T, Z form a partition of $V(G)$.*

Proof. It is easy to check that the sets are pairwise disjoint, from their definition. So we need only prove that every vertex v of $V(G) \setminus (A_1 \cup \cdots \cup A_5)$ lies in one of the remaining sets. For this purpose, pick a vertex a_i in A_i for each $i = 1, \ldots, 5$. Put $C = \{a_1, \ldots, a_5\}$ and $s = |N(v) \cap C|$. We choose C so that s is as large as possible. By Claim 2.1, we have $s \in \{0, 2, 3, 5\}$. First suppose that $s = 5$. Then, for each $i = 1, \ldots, 5$, vertex v sees all of A_i, for otherwise $(C \setminus a_i) \cup \{v, x_i\}$ induces an F_{17} for any $x_i \in A_i \setminus N(v)$. So v lies in T. Now suppose that $s = 3$. By Claim 2.1 and up to symmetry, we may assume that $N(v) \cap C = \{a_1, a_3, a_4\}$. Then v has no neighbour in $A_2 \cup A_5$, for otherwise the choice of C that maximizes s is contradicted. Then v sees all of A_1, for otherwise $\{v, a_4, a_5, x_1, a_2\}$ induces an F_1 for any $x_1 \in A_1 \setminus N(v)$. Then v either sees all of A_3 or all of A_4, for otherwise $\{v, a_1, a_2, x_3, x_4\}$ induces an F_1 for any $x_3 \in A_3 \setminus N(v)$ and $x_4 \in A_4 \setminus N(v)$. So v lies in D_1 or in B_2^+ or B_5^-. Now suppose that $s = 2$. By Claim 2.1 and up to symmetry, we may assume that $N(v) \cap C = \{a_1, a_3\}$. Then v has no neighbour in $A_2 \cup A_4 \cup A_5$, for otherwise the choice of C that maximizes s is contradicted. Then v sees all of A_1, for otherwise $\{a_5, x_1, a_2, a_3, v\}$ induces an F_1 for any $x_1 \in A_1 \setminus N(v)$; and similarly, v sees all of A_3. But then we could add v to A_2 and contradict the maximality of $A_1 \cup \cdots \cup A_5$. Finally, if $s = 0$ then v lies in Z. Thus the claim holds. □

In the following claims, for each $i = 1, \ldots, 5$, we let a_i denote a fixed (arbitrary) vertex of A_i.

Claim 2.3. *For $i = 1, \ldots, 5$, the set A_i is a stable set.*

Proof. For if u, v are two adjacent vertices in, say, A_1, then $\{u, v, a_2, a_3, a_4, a_5\}$ induces an F_{16}, a contradiction. □

Claim 2.4. *For $i = 1, \ldots, 5$, every vertex of $B_i^- \cup D_{i+1}$ sees all of $B_{i+3}^+ \cup D_{i+2}$ and misses all of D_{i-1}. Every vertex of B_i^- misses every vertex of D_{i+1}.*

Proof. Put $i = 1$. Pick any $x \in B_1^- \cup D_2$. So x sees a_2, a_5, misses a_1, a_3, and has a neighbour $u_4 \in A_4$. If x misses a vertex $y \in B_4^+ \cup D_3$, then there is a vertex $u_1 \in A_1$ that sees y, and $\{x, y, u_1, a_3, u_4\}$ induces an F_1, a contradiction. If x sees a vertex $d_5 \in D_5$, then $\{x, d_5, a_2, a_3, u_4, a_5\}$ induces an F_{10}. This proves the first part of the claim. For the second part, let x be in B_1^-; so x misses a vertex $v_4 \in A_4$. If x sees a vertex $d_2 \in D_2$, then $\{x, d_2, a_2, a_3, v_4, a_5\}$ induces an F_{17}, a contradiction. Thus the claim holds. □

Claim 2.5. *At least three of the D_i's are empty.*

Proof. For suppose the contrary, that is, at least three of the D_i's are not empty. For $i = 1, \ldots, 5$, pick any d_i in any non-empty D_i. Up to symmetry there are two cases. In the first case, D_1, D_2, D_3 are not empty. By the preceding claim we have edges d_1d_2, d_2d_3 and no edge d_1d_3, and then $\{a_3, a_4, a_5, d_1, d_2, d_3\}$ induces an F_{10}. In the second case, D_1, D_2, D_4 are not empty. By the preceding claim we have an edge d_1d_2 and no edge d_1d_4 nor d_2d_4, and then $\{a_1, \ldots, a_5, d_1, d_2, d_4\}$ induces an F_{22}. In either case we have a contradiction. Thus the claim holds. □

Claim 2.6. *Suppose that B_i^- is not empty. Then, all the $B_j^{\pm}$'s are empty, except possibly B_{i+3}^+, and $D_i = \emptyset$ and $D_{i+3} = \emptyset$.*

Proof. Suppose the contrary, that is, there exists a vertex x in one of the sets we claim are empty. Put $i = 1$. Pick any $b \in B_1^-$. So b sees all of $A_2 \cup A_5$, misses all of $A_1 \cup A_3$, and there are vertices $u_4, v_4 \in A_4$ such that b sees u_4 and misses v_4. By Claim 2.3, u_4 and v_4 are not adjacent.

First suppose that x lies in D_1 or B_5^-. So x sees a_1, u_4, v_4, misses a_2, a_5, and has a neighbour $u_3 \in A_3$. If x misses b, then $\{a_5, b, a_2, u_3, x\}$ induces an F_1, while if x sees b, then $\{b, x, u_3, u_4, v_4, a_5\}$ induces an F_{10}, a contradiction. Thus, we have $D_1 = \emptyset, B_5^- = \emptyset$. This argument shows that if $B_i^- \neq \emptyset$ then $B_{i-1}^- = \emptyset$ for all i. Thus, if $B_2^- \neq \emptyset$ then $B_1^- = \emptyset$, a contradiction to our choice of b. Therefore, we also have $B_2^- = \emptyset$.

Now suppose that x lies in B_3^- or D_4. So x sees a_2, u_4, v_4, misses a_3, a_5, and has a neighbour $u_1 \in A_1$. Note that b misses u_1. If x misses b, then $\{b, x, u_1, a_2, a_3, u_4, v_4, a_5\}$ induces an F_{20}, while if x sees b, then $\{u_1, a_2, b, x, u_4, a_5\}$ induces an F_{10}, a contradiction. Thus, we have $B_3^- = \emptyset$ and $D_4 = \emptyset$. This argument shows that if $B_i^- \neq \emptyset$ then $B_{i+2}^- = \emptyset$ for all i. Thus, if $B_4^- \neq \emptyset$ then $B_1^- = \emptyset$, a contradiction to our choice of b. Therefore, we also have $B_4^- = \emptyset$.

Now suppose that $x \in B_1^+$. So x sees a_2, a_5, misses a_1, u_4, v_4, and there are vertices $u_3, v_3 \in A_3$ such that x sees u_3 and misses v_3. By Claim 2.3, u_3 and v_3 are not adjacent. Note that b misses u_3, v_3. If x misses b, then $\{b, x, a_2, u_3, v_3, u_4, v_4, a_5\}$ induces an F_{20}, while if x sees b, then $\{b, x, a_2, u_3, u_4, a_5\}$ induces an F_{10}, a contradiction. Thus $B_1^+ = \emptyset$.

Now suppose that $x \in B_2^+$. So x sees a_1, a_3 and misses a_2, a_5. If x misses b, then $\{b, a_5, a_1, x, a_3\}$ induces an F_1. So x sees b. Suppose that x sees u_4. If x misses v_4, then $\{b, x, a_3, u_4, v_4, a_5\}$ induces an F_{17}, while if x sees v_4, then the same set induces an F_{10}. So x misses u_4. If x sees v_4, then $\{b, x, a_1, a_2, a_3, u_4, v_4, a_5\}$ induces an F_{20}. So x misses v_4. This argument actually implies that x cannot have any neighbour in A_4, and this contradicts the definition of B_2^+. Thus $B_2^+ = \emptyset$.

Now suppose that $x \in B_3^+$. So x sees a_2, u_4, v_4, misses a_1, a_3, and there are vertices $u_5, v_5 \in A_5$ such that x sees u_5 and misses v_5. By Claim 2.3, u_5 and v_5 are not adjacent. Note that b sees u_5, v_5. If x misses b, then $\{b, x, u_4, v_4, u_5, v_5\}$ induces an F_{10}, while if x sees b then $\{b, x, a_1, a_2, u_4, v_5\}$ induces an F_{17}, a contradiction. Thus $B_3^+ = \emptyset$.

Finally suppose that x lies in B_5^+. So x sees a_1, u_4, v_4, misses a_3, a_5, and has a neighbour $u_2 \in A_2$. If x misses b, then $\{b, x, a_1, u_2, a_3, u_4, v_4, a_5\}$ induces an F_{20}, while if x sees b then $\{b, x, a_1, u_2, u_4, a_5\}$ induces an F_{10}, a contradiction. Thus $B_5^+ = \emptyset$. This completes the proof of the claim. □

Claim 2.7. *For $i = 1, \ldots, 5$, the set $A_i \cup B_i^- \cup B_i^+$ is a stable set.*

Proof. Suppose on the contrary, and up to symmetry, that there exist two adjacent vertices $b, c \in A_1 \cup B_1^- \cup B_1^+$. By Claim 2.3 and by the definition of $B_1^{\pm}$, vertices b, c are not in A_1. By Claim 2.6, and up to symmetry, we may assume that they are both in B_1^-. So b, c both see a_2, a_5 and miss a_1, a_3. By the definition of B_1^-, vertex b has a non-neighbour $v_4 \in A_4$. If v_4 also misses c, then $\{b, c, a_2, a_3, v_4, a_5\}$ induces an F_{16}. So v_4 sees c. Vertex c has a non-neighbour $w_4 \in A_4$, and by the same argument, w_4 sees b. By Claim 2.3, v_4 and w_4 are not adjacent. But then $\{b, c, a_3, v_4, w_4, a_5\}$ induces an F_{17}, a contradiction. Thus the claim holds. □

Claim 2.8. *There is no edge between Z and $B_i^{\pm}$.*

Proof. Suppose on the contrary, and up to symmetry, that there is an edge zb with $b \in B_1^-$. By the definition of B_1^-, vertex b has a non-neighbour $v_4 \in A_4$. Then $\{z, b, a_2, a_3, v_4\}$ induces an F_1, a contradiction. Thus the claim holds. □

Claim 2.9. *Every vertex of T is adjacent to every vertex of D_i and $B_i^{\pm}$ ($i = 1, \ldots, 5$).*

Proof. Suppose on the contrary, and up to symmetry, that some $t \in T$ is not adjacent to a vertex x in $B_1^- \cup D_2$. Vertex x sees a_2, a_5, misses a_1, a_3, and has a neighbour $u_4 \in A_4$. Then $\{d, t, a_1, a_2, u_4, a_5\}$ induces an F_{10}, a contradiction. □

By Claim 2.6, and up to symmetry, we may assume that all the $B_j^{\pm}$'s are empty, except possibly B_1^- and B_4^+, and also D_1 and D_4 are empty.

Claim 2.10. *Any two non-adjacent vertices of $X_1 = A_1 \cup B_1^- \cup D_2$ have inclusionwise comparable neighbourhoods in $V(G) \setminus X_1$.*

Proof. Suppose the contrary, that is, there are non-adjacent vertices $x, y \in X_1$ and $x', y' \in V(G) \setminus X_1$ with edges xx', yy' and none of the edges $xy', x'y$. By the definition of these sets and by previous claims, x' and y' are in $A_4 \cup B_4^+ \cup D_3 \cup Z$. So they both miss a_2. Then $x'y'$ is an edge, for otherwise $\{a_2, x, y, x', y'\}$ induces an F_1. If x', y' both miss a_3, then $\{a_3, a_2, x, x', y'\}$ induces an F_1. Now we may assume that x' sees a_3, and so it is in $A_4 \cup B_4^+ \cup D_3$. If y' also sees a_3, then $\{x, a_2, a_3, y, a_5, x'\}$ induces an F_{17}. So y' misses a_3 and therefore is in Z. Then x' is in D_3, so $x' \neq a_4$ and x' misses a_4. Also y is in D_2, so y sees a_4. Then x sees a_4, else $\{a_2, x, y, x', a_4\}$ induces an F_1. But now $\{x, y, x', y', a_4, a_5\}$ induces an F_{17}. Thus the claim holds. □

Claim 2.11. *Any two non-adjacent vertices in D_5 have inclusionwise comparable neighbourhoods in $V(G) \setminus D_5$.*

For suppose on the contrary that there are non-adjacent vertices $x, y \in D_5$ and vertices x', y' with edges xx', yy' and non-edges $xy', x'y$. By the definition of the sets and previous claims, x' and y' are in Z. If x', y' are not adjacent, then $\{x', x, a_5, y, y'\}$ induces an F_1. If x', y' are adjacent, then $\{y', x', x, a_2, a_1\}$ induces an F_1. Thus the claim holds. □

Claim 2.12. *Every component of Z is a clique.*

Proof. For in the opposite case, Z has three vertices that induce a chordless path x–y–z, and then $\{a_1, a_2, a_3, a_4, x, y, z\}$ induces an F_2. □

Claim 2.13. *If $D_5 \neq \emptyset$, then there is no edge between Z and $D_2 \cup D_3$.*

Proof. For suppose that there is a vertex $d_5 \in D_5$ and an edge zx with $z \in Z$ and (up to symmetry) $x \in D_2$. No vertex in D_5 sees a vertex of $D_2 \cup D_3$ by Claim 2.4. If z misses d_5, then $\{z, x, a_5, d_5, a_3\}$ induces an F_1. If z sees d_5, then $\{a_1, a_2, a_3, a_4, a_5, x, d_5, z\}$ induces an F_{18}. □

Claim 2.14. *If a vertex of D_i has a neighbour in a component of Z, then it sees all of that component.*

Proof. Suppose on the contrary that some vertex $d \in D_1$ has a neighbour u and a non-neighbour v in a component of Z. We may assume that u, v are adjacent. Then $\{u, v, d, a_1, a_2\}$ induces an F_1, a contradiction. Thus the claim holds. □

Let us say that a set A of vertices is *complete* (respectively, *anti-complete*) to a set B if every vertex of A sees (respectively, misses) every vertex of B.

We can summarize the preceding claims as follows.

Lemma 2.5. *Let G be an $\mathcal{F}$-free graph that contains a C_5. Then $V(G)$ can be partitioned into sets $X_1, \dots, X_6, T, Z$ such that:*

1. *Each $X_1, \dots, X_5$ is non-empty.*
2. *For every j modulo 5, X_j is complete to X_{j+1}.*
3. *For every j modulo 5 and $j \neq 4$, X_j is anti-complete to X_{j+2}, and some vertex of X_1 misses a vertex X_4.*
4. *X_6 is complete to $X_2 \cup X_3 \cup X_5$ and anti-complete to $X_1 \cup X_4$.*
5. *X_2, X_3, X_5 are stable sets.*
6. *The sets $X_1' = \{x \in X_1 \mid x \text{ has a non-neighbour in } X_4\}$ and $X_4' = \{x \in X_4 \mid x \text{ has a non-neighbour in } X_1\}$ are stable sets, and there is no edge between X_1' and $X_1 \setminus X_1'$ and no edge between X_4' and $X_4 \setminus X_4'$.*
7. *One of $X_1 \setminus X_1', X_4 \setminus X_4', X_6$ is empty.*
8. *Any two non-adjacent vertices of X_1 have inclusionwise comparable neighbourhoods in $V(G) \setminus X_1$, and the same holds for X_4 and X_6.*
9. *T is complete to $X_1 \cup \dots \cup X_6$.*
10. *Z is anti-complete to $X_1' \cup X_2 \cup X_3 \cup X_4' \cup X_5$; and if $X_6 \neq \emptyset$, then Z is anti-complete to $X_1 \cup X_2 \cup X_3 \cup X_4 \cup X_5$.*
11. *Every component of Z is a clique and is a homogeneous set in $G \setminus T$.*

Proof. Consider the sets defined before this lemma, and set $X_1 = A_1 \cup B_1^- \cup D_2$, $X_2 = A_2$, $X_3 = A_3$, $X_4 = A_4 \cup B_4^+ \cup D_3$, $X_5 = A_5$, $X_6 = D_5$. Then the lemma is a reformulation of Claims 2.2–2.14. □

Theorem 2.6. *Let G be an $\mathcal{F}$-free graph. Suppose that G contains a C_5 and that, with the notation of Lemma 2.5, X_1, X_4, X_6 are stable sets and $T = \emptyset$. If there exists a b-coloring c with $b(G)$ colors such that there is a b-vertex of color i in $X_1 \cup \dots \cup X_6$ for every color $i = 1, \dots, b(G)$, then G is not minimally b-imperfect.*

Proof. As usual, for $j = 2, 3, 5$ let a_j be an arbitrary vertex in X_j, and for $j = 1, 4$ let a_1, a_4 be non-adjacent vertices of X_1 and X_4 respectively. We start with some observations:

For $j = 2, 3, 5$, any two vertices in X_j are twins. (1)

This follows directly from Lemma 2.5 (properties 2, 3, 4, 5, 10).

For $j = 1, 4, 6$, any two vertices in X_j have inclusionwise comparable neighbourhoods. (2)

This follows from Lemma 2.5 (properties 2, 3, 4, 8, 10) and the hypothesis that X_1, X_4 and X_6 are stable sets.

Let u_i be a b-vertex of color i for each $i = 1, \dots, b(G)$. It follows from (1), (2) and Lemma 2.3 that each of $X_1, \dots, X_6$ contains at most one u_i.

Suppose that G is minimally b-imperfect, so $b(G) > \chi(G)$, and let c be a b-coloring with $b(G)$ colors. For $i = 1, \dots, b(G)$, by the hypothesis, there is a b-vertex u_i of color i in $X_1 \cup \dots \cup X_6$. By (1), (2) and Lemma 2.3, in each of the sets $X_1, \dots, X_6$, all vertices have different colors, and each of these sets contains at most one b-vertex of c. Thus $b(G) \leq 6$. Moreover, for $j = 2, 3, 5$, if X_j contains a b-vertex, then $|X_j| = 1$. Also, if X_1 contains a b-vertex, then either $|X_1| = 1$ or this vertex has a neighbour in $X_4 \cup Z$, and therefore in X_4; and similarly for X_4; and if X_6 contains a b-vertex, then either $|X_6| = 1$ or this vertex has a neighbour in Z (by condition 8 of Lemma 2.5).

Note that $\chi(G) \geq 3$ since G contains a C_5, and so $b(G) \geq 4$. Thus $b(G) \in \{4, 5, 6\}$.

At least one of X_1, X_4 and X_6 does not contain any of $u_1, \dots, u_{b(G)}$. (3)

For suppose on the contrary that there are vertices $u_i \in X_1, u_j \in X_4, u_k \in X_6$ for three different integers $i, j, k \in \{1, \dots, b(G)\}$. Since $X_6 \neq \emptyset$, by property 10 of Lemma 2.5, u_i and u_j have no neighbour in Z. Vertex u_i must have a neighbour v_k of color k, and since $N(u_i) \setminus X_4 \subseteq N(u_k)$, we must have $v_k \in X_4$. Likewise, u_j has a neighbour w_k or color k, and we must have $w_k \in X_1$. Now if u_i, u_j are not adjacent, then $\{v_k, u_i, a_2, w_k, u_j\}$ induces an F_1; and if u_i, u_j are adjacent, then $\{u_i, a_2, a_3, u_j, a_5, u_k, v_k, w_k\}$ induces an F_{22}, a contradiction. Thus (3) holds.

At least one of X_2, X_3 and X_5 does not contain any of $u_1, \dots, u_{b(G)}$. (4)

For suppose on the contrary that there are vertices $u_i \in X_2, u_j \in X_3, u_k \in X_5$ for three different integers $i, j, k \in \{1, \dots, b(G)\}$. As observed above, we have $|X_j| = 1$ for $j = 2, 3, 5$. Vertex u_i must have a neighbour w_k of color k, and since $N(u_i) \setminus X_3 \subset N(u_k)$, it must be that w_k is in X_3; but this is impossible since the unique vertex of X_3 has color j. Thus (4) holds.

Now it follows from (3) and (4) that $b(G) = 4$.

At least one of X_1 and X_4 does not contain any of $u_1, \dots, u_4$. (5)

For suppose on the contrary that $u_1 \in X_1$ and $u_4 \in X_4$. Then X_6 contains no b-vertex by (3). Also $|X_5| \le 2$, because all vertices of X_5 have different colors and they cannot have color 1 or 4. So either X_5 has two vertices, of color 2 and 3, and no b-vertex by (1) and Lemma 2.3, or X_5 has only one vertex, which (up to symmetry) has color 2; and in either case we may assume that $u_3 \in X_3$.

We are going to prove that $|X_5| = 2$. Vertex u_1 must have a neighbour v_3 of color 3. Vertex u_3 must have a neighbour w_1 of color 1, and necessarily we have $w_1 \in X_4 \cup X_6$. If w_1 is in X_6, then by property 10 of Lemma 2.5 u_1, u_4 have no neighbour in Z. If $w_1 \in X_4$, then u_1 has a non-neighbour w_1 in X_4, so $u_1 \in X_1'$, so u_1 again has no neighbour in Z. In either case, it follows that $N(u_1) \setminus X_5 \subset N(u_3)$, and so $v_3 \in X_5$; so $|X_5| = 2$, as announced, which restores the symmetry between colors 2 and 3, and we may assume that $u_2 \in X_2$, and u_4 has no neighbour in Z.

Vertex u_1 must have a neighbour v_4 of color 4, and necessarily v_4 is in X_4. Since vertices in X_4 must have different colors, we have $v_4 = u_4$. So u_1, u_4 are adjacent. Vertex u_2 must have a neighbour w_4 of color 4, and necessarily we have $w_4 \in X_1 \cup X_6$. If both w_1, w_4 are in X_6, then $u_1, u_2, u_3, u_4, w_1, w_4$ and the two vertices of X_5 induce an F_{15}. If only one of w_1, w_4 is in X_6, then the same eight vertices induce an F_{21}. Thus we must have $w_1 \in X_4$ and $w_4 \in X_1$. Note that $|X_1| = 2$ since the vertices of X_1 have colors different from 2, 3; and similarly $|X_4| = 2$. Then w_4 misses w_1, for otherwise $\{u_1, u_2, u_4, w_1, w_4\}$ induces an F_1. But then the six vertices $u_1, \dots, u_4, w_1, w_4$ plus the two vertices of X_5 induce an F_{19}. Thus (5) holds.

By (3)–(5) and up to symmetry, we may assume that $u_1 \in X_6$, $u_4 \in X_4$ and X_1 does not contain u_2, u_3. Since $X_6 \neq \emptyset$, vertices in $X_1 \cup X_4$ have no neighbour in Z. Vertex u_4 must have a neighbour v_1 of color 1, and necessarily $v_1 \in X_1$. Vertex u_1 must have a neighbour v_4 or color 4, and necessarily $v_4 \in X_2 \cup Z$. If v_4 is in Z, then $\{v_4, u_1, a_3, u_4, v_1\}$ induces an F_1. So we have $v_4 \in X_2$. By Lemma 2.3, if $|X_2| \ge 2$, then it contains no b-vertex. Since X_2 already contains v_4, it cannot contain a b-vertex of color 2 or 3, so we may assume that $u_3 \in X_3$ and $u_2 \in X_5$. Vertex u_2 must have a neighbour v_3 of color 3, and necessarily $v_3 \in X_1$. Vertex u_3 must have a neighbour v_2 of color 2, and necessarily $v_2 \in X_2$. Now $\{u_1, \dots, u_4, v_1, \dots, v_4\}$ induces an F_{21} (if u_4, v_3 are not adjacent) or an F_{15} (if u_4, v_3 are adjacent), a contradiction. This completes the proof of Theorem 2.6. □

3. Proof of Theorem 1.1

In this section we assume that G is a diamond-free $\mathcal{F}$-free graph, and we prove that G is b-perfect. For this purpose, we may assume on the contrary that G is minimally b-imperfect. We have $b(G) > \chi(G)$. Let c be a b-coloring of G with $b(G)$ colors. By Theorem 1.3, we may assume that G is not bipartite, so $\chi(G) \ge 3$ and $b(G) \ge 4$.

(I) First assume that G contains an induced C_5. We use the notation of Lemma 2.5. For $j = 2, 3, 5$, let a_j be a vertex of X_j, and let $a_1 \in X_1$ and $a_4 \in X_4$ be non-adjacent vertices.

$$T = \emptyset. \tag{6}$$

For if t is any vertex in T, then $\{t, a_1, a_2, a_3\}$ induces a diamond.

$$X_1, X_4 \text{ are stable sets.} \tag{7}$$

For suppose, without loss of generality, that there are adjacent vertices $x, y \in X_1$. Then $\{x, y, a_2, a_5\}$ induces a diamond. Thus (7) holds.

$$|X_6| \le 1. \tag{8}$$

For suppose that there are two vertices $x, y \in X_6$. If x, y are adjacent, then $\{x, y, a_2, a_5\}$ induces a diamond. If they are not adjacent, then $\{x, y, a_2, a_3\}$ induces a diamond. Thus (8) holds.

$$Z \text{ contains no b-vertex for } c. \tag{9}$$

For suppose that some vertex $z \in Z$ is a b-vertex. By Lemma 2.2, z has two neighbours u, v that are not adjacent. Let Y be the component of Z that contains z. By property 11 of Lemma 2.5 and since $T = \emptyset$, Y is a homogeneous clique, so u, v are in $(X_1 \setminus X_1') \cup (X_4 \setminus X_4') \cup X_6$. Then $Y = \{z\}$, for otherwise two vertices of Y and u, v would induce a diamond. But now we have $N(z) \subset N(a_5)$, and so z cannot be a b-vertex, a contradiction. Thus (9) holds.

It follows from the preceding facts that G satisfies the hypotheses of Theorem 2.6, so it is not minimally b-imperfect, a contradiction.

(II) Now we may assume that G contains no induced C_5. By Lemma 2.4, G is connected. A theorem due to Bacsó and Tuza [1] states that every connected, P_5-free and C_5-free graph has a *dominating* clique, that is, a clique Q such that every vertex of $G \setminus Q$ has a neighbour in Q. We choose a dominating clique Q of size as large as possible. Clearly, $|Q| \ge 2$.

Suppose that $|Q| = 2$, and let $Q = \{x_1, x_2\}$. For $i = 1, 2$, let $A_i = N(x_i) \setminus \{x_{3-i}\}$. Note that no vertex z of G sees both x_1, x_2, for otherwise $\{x_1, x_2, z\}$ would be a dominating clique of size 3, contradicting the choice of Q. So $A_1 \cup \{x_1\}$ and $A_2 \cup \{x_2\}$ form a partition of $V(G)$, and there is no edge between A_i and x_{3-i} for $i = 1, 2$. Note that, for $i = 1, 2$, the subgraph of G induced by A_i contains no P_3 (for otherwise, adding x_i, we would obtain a diamond), and so each component of $G[A_i]$ is a clique. We may assume that x_i has color $c(x_i) = i$ for $i = 1, 2$. Let y_3 be a b-vertex with color $c(y_3) = 3$. Without loss of generality, we have $y_3 \in A_2$. Let Y be the (clique) component of $G[A_2]$ that contains y_3. Since y_3 is a b-vertex, it has a neighbour y_1 with color $c(y_1) = 1$, and since $y_1 \notin A_1$, we have $y_1 \in Y$. Since $Y \cup \{x_2\}$ is a clique, we have $|Y \cup \{x_2\}| \le \chi(G) < b(G)$, and so there is a color used by c, say color 4, that does not appear in $Y \cup \{x_2\}$. Vertex y_3 must have a neighbour z_4 with color $c(z_4) = 4$, and so $z_4 \in A_1$. Let Z be the (clique) component of A_1 that contains z_4. Note that z_4 misses every vertex $y \in Y \setminus y_3$, for otherwise $\{z_4, y, y_3, x_2\}$ induces a diamond. Then y_3 sees every vertex $u \in A_1 \setminus Z$, for otherwise $\{u, x_1, z_4, y_3, y_1\}$ induces an F_1 or C_5. Since $Y \cup \{x_2\}$ is a clique of size at least 3, it is not dominating, so there exists a vertex z' that has no neighbour in that clique, and we must have $z' \in Z \setminus z_4$. Then z_4 sees every vertex $v \in A_2 \setminus Y$, for otherwise $\{v, x_2, y_3, z_4, z'\}$ induces an F_1 or C_5. In fact we have $A_2 \setminus Y = \emptyset$, for if u was any vertex in that set, then $\{z', z_4, u, x_2, y_1\}$ would induce an F_1 (z' misses u, for otherwise we have a diamond with vertices u, z', z_4, x_1). Likewise, we have $A_1 \setminus Z = \emptyset$, for if v was any vertex in that set, then $\{z', x_1, v, y_3, y_1\}$ would induce an F_1. Now we have $V(G) = \{x_1, x_2\} \cup Y \cup Z$, and z_4 is the only vertex of G with color 4. So all the b-vertices of any color different from 4 must be neighbours of z_4. Since $N(z_4) = (Z \setminus z_4) \cup \{x_1, y_3\}$, it follows that x_1 is the only b-vertex of color 1. Since $N(x_1) = Z \cup \{x_2\}$ and $c(x_2) = 2$, it follows that each of the colors $3, \ldots, b(G)$ must appear in Z, and so $b(G) - 2 \le |Z| \le \omega(G) - 1$ (because $Z \cup \{x_1\}$ is a clique) $\le \chi(G) - 1 \le b(G) - 2$. Thus we must have equality throughout, which implies that Z contains no vertex of color 2, and then z_4 cannot be a b-vertex, a contradiction.

Now suppose that $|Q| \ge 3$. Put $q = |Q|$ and $Q = \{x_1, \ldots, x_q\}$. Every vertex z of $G \setminus Q$ sees at least one vertex of Q, because Q is dominating, and it sees at most one, for otherwise either $Q \cup \{z\}$ would be a larger dominating clique or $\{z, x_i, x_j, x_k\}$ would induce a diamond for any $x_i, x_j \in N(z), x_k \notin N(z)$. For $i = 1, \ldots, q$, let $A_i = N(x_i) \setminus Q$. So $Q, A_1, \ldots, A_q$ form a partition of $V(G)$, and for $i = 1, \ldots, q$, any vertex of A_i misses every vertex of $Q \setminus x_i$. We may assume that $c(x_i) = i$ for each $i = 1, \ldots, q$. We have $3 \le q \le \omega(G) \le \chi(G) < b(G)$, so c uses at least $q + 1 \ge 4$ colors. Let z be a b-vertex with the largest color $b(G) \ge q + 1$. We may assume that $z \in A_1$. Since z is a b-vertex, it has neighbours $y_2, \ldots, y_{b(G)-1}$ with colors $2, \ldots, b(G) - 1$ respectively, and they are not in Q. Put $Y = \{y_2, \ldots, y_{b(G)-1}\}$. We claim that

Y is either a stable set or a clique. (10)

For in the opposite case, Y contains three vertices y, y', y'' that induce a subgraph with either one edge or two edges. If it induces two edges, then $\{z, y, y', y''\}$ induces a diamond. So suppose it induces one edge $y'y''$. If $y' \in A_1$, then $y'' \in A_1$, for otherwise $\{x_1, z, y', y''\}$ induces a diamond; then $y \notin A_1$, for otherwise $\{x_1, y, z, y'\}$ induces a diamond; then, up to symmetry, $y \in A_2$, and $\{y', z, y, x_2, x_3\}$ induces an F_1, a contradiction. Thus $y' \notin A_1$, and similarly $y'' \notin A_1$. So, up to symmetry, $y' \in A_2$. Then $y'' \notin A_2$, for otherwise $\{x_2, y', y'', z\}$ induces a diamond. So, up to symmetry, $y'' \in A_3$. Then, up to symmetry we have $y \notin A_3$, and then $\{x_2, x_3, y'', z, y\}$ induces an F_1 or C_5, a contradiction. Thus (10) is established.

Suppose that Y is a stable set. Since $b(G) \ge 4$, we have $|Y| \ge 2$. Consider vertices $y, y' \in Y$. We cannot have both $y, y' \in A_1$ for otherwise G contains a diamond with vertices x_1, z, y, y'. Thus, we may assume $y \in A_2$. We cannot have $y' \in A_j$ with $j \notin \{1, 2\}$, for otherwise $\{z, y, y', x_2, x_j\}$ induces a C_5. It follows that $Y \cap A_j = \emptyset$ for $j > 3$. If $y' \in A_1$, then $\{x_3, x_2, y, z, y'\}$ induces an F_1. It follows that $Y \subseteq A_2$. But this implies vertices y_2 and x_2 are adjacent and have the same color, a contradiction. So Y is not a stable set.

Therefore Y induces a clique. Put $Z = Y \cup \{z\}$. Suppose that some $x_i \in Q$ has two neighbours in Z. Then it sees all of Z, for otherwise $\{x_i, y, y', y''\}$ induces a diamond for any $y, y' \in Z \cap N(x_i), y'' \in Z \setminus N(x_i)$. Then $i = 1$, for otherwise z sees both x_1 and x_i, a contradiction. But $Z \cup \{x_1\}$ is a clique of size $b(G)$ implying $\chi(G) \ge b(G)$, a contradiction to our assumption on G. So no vertex of Q sees two vertices of Z. Since every vertex of Z has exactly one neighbour in Q, we have $|Z| = |Q|$, so $q = b(G) - 1$. The vertices of Z can be renamed $z_1, \ldots, z_q$ such that $z_i x_i$ is an edge for each $i = 1, \ldots, q$ and there is not other edge between Z and Q. Consider any vertex $u \in V(G) \setminus (Q \cup Z)$. We have $u \in A_i$ for some i. If u has two neighbours in Z, then it sees all of Z, for otherwise $\{u, y, y', y''\}$ induces a diamond for any $y, y' \in Z \cap N(u), y'' \in Z \setminus N(u)$. But then $\{u, x_i, z_i, z_j\}$ induces a diamond for any $j \ne i$. So u has at most one neighbour in Z. If it sees z_i or no vertex of Z, then $\{u, x_i, x_j, z_j, z_k\}$ induces an F_1 for any $j, k \ne i$. If it sees z_j for some $j \ne i$, then $\{u, x_i, z_j, x_k, z_k\}$ induces a C_5 for any $k \ne i, j$. Thus such a vertex u cannot exist, that is, $V(G) = Q \cup Z$. Now x_2, y_2 are the only vertices of color 2 in G. However, x_2 is not a b-vertex because it has no neighbour of color $q + 1$, and y_2 is not a b-vertex because it has no neighbour of color 1, a contradiction. This completes the proof of Theorem 1.1. □

4. Proof of Theorem 1.2

Suppose that the theorem is false. Let G be a counterexample to the theorem with the smallest number of vertices, and let c be a b-coloring of G with $b(G) > \chi(G)$ colors. If G is diamond-free, then the result follows from Theorem 1.1. So we may assume that G contains a diamond. Thus $\chi(G) = 3$. If $b(G) > 4$, then the subgraph of G induced by the vertices of colors $1, \ldots, 4$ is also a counterexample to the theorem, which contradicts the minimality of G. So $b(G) = 4$. For any integer $k \ge 4$, the *k-wheel* is the (complete) join of a vertex and a chordless cycle of length k. Note that G contains no 5-wheel, since a 5-wheel cannot be colored with 3 colors. Likewise, G contains no K_4.

If $\{u, v, x, y\}$ induces a diamond, where u, v are not adjacent, then $N(u) \subseteq N(v)$ or $N(v) \subseteq N(u)$ and (consequently), u, v have different colors. (11)

For suppose that none of the two inclusions holds. So there is a vertex u' that sees u and misses v, and there is a vertex v' that sees v and misses u. If x misses both u' and v', then either $\{u', u, x, v, v'\}$ induces an F_1, or $\{u', u, x, v, v', y\}$ induces an F_{16}, F_{17} or a 5-wheel. So, up to symmetry, x sees u'. Then u' misses y, for otherwise $\{u, u', x, y\}$ induces a K_4. By symmetry, y sees v', and v' misses x. But then $\{u, u', v, v', x, y\}$ induces an F_4 or F_{10}. Thus one of the inclusions $N(u) \subseteq N(v)$ or $N(v) \subseteq N(u)$ holds; and it follows from Lemma 2.3 that u, v have different colors. Thus (11) holds.

G does not contain a C_5. (12)

For suppose that G contains a C_5. Then it admits a partition into sets $X_1, \dots, X_6, T, Z$ as in Lemma 2.5. For $j = 2, 3, 5$ let a_j be an arbitrary vertex in X_j, and let $a_1 \in X_1$ and $a_4 \in X_4$ be non-adjacent vertices. We claim that G satisfies the hypotheses of Theorem 2.6. We have $T = \emptyset$ because G contains no 5-wheel. The set X_1 is a stable set, for if it contained two adjacent vertices x, y, then $\{x, y, a_4, a_5\}$ would induce a K_4. Likewise X_4 is a stable set. Also X_6 is a stable set, for if it contained two adjacent vertices x, y then $\{x, y, a_2, a_3\}$ would induce a K_4. Finally, suppose that some vertex $z \in Z$ is a b-vertex, say of color 1. Let Y be the component of Z that contains z. By property of Lemma 2.5 and since $T = \emptyset$, Y is a homogeneous clique. By Lemma 2.2, z has two neighbours u, v that are not adjacent, and so they are both in $V(G) \setminus Z$. We have $|Y| \leq 2$, for otherwise $Y \cup \{u\}$ induces a clique of size at least 4. If z has a neighbour in X_6 then it cannot have a neighbour in $X_1 \cup X_4$ by property 10 of Lemma 2.5. So either z has no neighbour in $(X_1 \setminus X_1') \cup (X_4 \setminus X_4')$ or z has no neighbour in X_6. If $|Y| = 1$, then we have $N(z) \subset N(a_5)$, with strict inclusion, which contradicts Lemma 2.3. So Y has two elements z, y. By (11), we may assume that y has color 2 and u, v have color respectively 3, 4. If u, v are in $(X_1 \setminus X_1') \cup (X_4 \setminus X_4')$, then, since they are not adjacent, and by the definition of X_1' and X_4', and up to symmetry, they are both in $(X_1 \setminus X_1')$. Then $\{a_4, a_5, u, v\}$ induces a diamond, and so one of a_4, a_5 is a b-vertex of color 1. If u, v are in X_6, then, $\{a_2, a_3, u, v\}$ induces a diamond, and so one of a_2, a_3 is a b-vertex of color 1. Thus, there are b-vertices of all four colors in $X_1 \cup \dots \cup X_6$. So G satisfies the hypotheses of Theorem 2.6, so G is not minimally b-imperfect, a contradiction. Thus (12) holds.

G does not contain a 4-wheel. (13)

For if G contains a 4-wheel, then, by (11), all the vertices of the 4-wheel must have different colors, which is impossible since c is a 4-coloring. Thus (13) holds.

Call *3-diamond* a graph that consists of five vertices u, v, w, x, y and seven edges $xy, ux, uy, vx, vy, wx, wy$.

G does not contain a 3-diamond. (14)

For if G contains a 3-diamond, with the above notation, then, by (11), vertices u, v, w have three different colors that are also different from the two colors of x, y, which is impossible since c is a 4-coloring. Thus (14) holds.

Call *gem* any graph that consists of five vertices u, v, w, x, y and seven edges $uv, vw, wx, uy, vy, wy, xy$.

G does not contain a gem. (15)

For suppose that G contains a gem, with vertices u, v, w, x, y and edges $uv, vw, wx, uy, vy, wy, xy$. By (11) and up to symmetry, we may assume that $c(u) = c(x) = 1$, $c(v) = 2$, $c(w) = 3$, $c(y) = 4$. Thus v, w, y are b-vertices of colors 2, 3, 4. By (11) again we have $N(u) \subset N(w)$ and $N(x) \subset N(v)$, and by Lemma 2.3, vertices u and x are not b-vertices. Let z be a b-vertex of color 1; so $z \neq u, x$. If z sees v, then in the graph $G \setminus \{u\}$ (with the same colors) vertices z, v, w, y are b-vertices of colors $1, \dots, 4$, which contradicts the minimality of G. Therefore z misses v and similarly w. In summary, z misses all of u, v, w, x.

Suppose that z sees y. Let z_2, z_3 be two neighbours of z of color 2 and 3 respectively. So $z_2 \neq v$, z_2 misses v and (since $N(x) \subset N(v)$) misses x too. Likewise $z_3 \neq w$ and z_3 misses both u, w. Suppose that z_2 and z_3 are not adjacent. Since $\{u, v, w, x, z, z_2, z_3\}$ cannot induce an F_2, it must be that one of z_2, z_3 has a neighbour in $\{u, v, w, x\}$, and we may assume, up to symmetry, that z_2 sees one of u, w. Then z_2 must see both u and w, for otherwise $\{z, z_2, u, v, w\}$ induces an F_1. Then z_3 sees x, for otherwise $\{z_3, z, z_2, w, x\}$ induces an F_1. But then $\{u, z_2, z, z_3, x\}$ induces an F_1. Thus z_2 and z_3 are adjacent. Since $\{u, v, w, x, y, z, z_2, z_3\}$ cannot induce an F_8, it must be that one of z_2, z_3 has a neighbour in $\{u, v, w, x\}$, and we may assume, up to symmetry, that z_2 sees one of u, w. Then z_2 must see both u, w, for otherwise $\{z, z_2, u, v, w\}$ induces an F_1. Then z_2 misses y, for otherwise $\{u, v, w, y, z_2\}$ induces a 4-wheel. If z_3 sees x, then z_3 sees v (since $N(x) \subset N(v)$) and misses y (for otherwise $\{v, w, x, y, z_3\}$ induces a 4-wheel); but then $\{u, v, w, x, y, z, z_2, z_3\}$ induces an F_{22}. So z_3 misses x. Then z_3 misses v, for otherwise $\{z, z_3, v, w, x\}$ induces an F_1; and z_3 sees y, for otherwise $\{z_3, z_2, u, y, x\}$ induces an F_1. But then $\{u, v, w, y, z, z_2, z_3\}$ induces an F_{11}. Therefore z misses y.

Since G is connected, there is a path z-p_1-$\cdots$-p_h such that p_h has a neighbour in $X = \{u, v, w, x, y\}$ and the path is as short as possible. So the path is chordless and its vertices other than p_h have no neighbour in X. We have $h \leq 3$ since G contains no F_1. If $h = 3$, then there is still an F_1, induced by z, p_1, p_2, p_3 and a neighbour of p_3 in X. If $h = 2$, then p_2 must see all of X, for otherwise there is still an F_1 induced by z, p_1, p_2 and some two adjacent vertices of X; but then $X \cup \{p_2\}$ contains a K_4. So $h = 1$. Let us now write p instead of p_1. We claim that p sees y. For suppose not. If p sees v, then it sees x (for otherwise $\{z, p, v, y, x\}$ induces an F_1) and u (for otherwise $\{z, p, x, y, u\}$ induces an F_1); p sees w, for otherwise $\{z, p, u, y, w\}$ induces an F_1; thus p must have color 4, and so the diamond induced by $\{p, y, w, x\}$ contradicts (11). So p misses v and similarly w, and so it must see one of u, x, say u; but then $\{z, p, u, v, w\}$ induces an F_1. So p sees y as claimed. We may assume up to symmetry that p has color 2. So p misses v, it also misses x because $N(x) \subset N(v)$. Vertex p also misses w, for otherwise $\{v, w, y, p\}$ induces a diamond that contradicts (11). Then p misses u, for otherwise $\{p, u, v, w, x\}$ induces an F_1. Let z_4 be a neighbour of z of color 4. So z_4 and y are different and not adjacent. If z_4 misses p, then it sees u, for otherwise $\{z_4, z, p, y, u\}$ induces an F_1; and similarly z_4 sees v; but then $\{u, v, y, z_4\}$ induces a diamond that contradicts (11). So z_4 sees p. Since $\{u, v, w, x, y, p, z, z_4\}$ cannot induce an F_8, it must be that z_4 has a neighbour in the path $P = u$-v-w-x. If z_4 has only one neighbour in P, then some three consecutive vertices of P plus z and z_4 induce an F_1. On the other hand, if z_4 has two consecutive neighbours in P, then these two neighbours plus y and z_4 induce a diamond that contradicts (11). So z_4 has exactly two neighbours in P, and they are not adjacent. If these two neighbours are u and x, then $\{u, v, w, x, y, z_4\}$ induces an F_{17}. So the two neighbours of z_4 in P

are either u and w or v and x. In either case, $\{u, y, x, z, z_4\}$ (not necessarily in this order) induces an F_1. Thus (15) holds.

If $D = \{u, v, x, y\}$ is a diamond in G, where u, v are not adjacent, then any vertex in $G \setminus D$ sees at most two vertices of D, and if it sees two, then these two are u and v. (16)

This is an immediate consequence of the preceding claims.

G does not contain two vertex-disjoint diamonds. (17)

For suppose that G has two vertex-disjoint diamonds $D = \{u, v, x, y\}$ and $D' = \{u', v', x', y'\}$ where u, v are not adjacent and u', v' are not adjacent. By (16), there are at most two edges between $\{x, y\}$ and $\{x', y'\}$, and if there are two, then they form a matching.

Suppose that there is no edge between $\{x, y\}$ and $\{u', v'\}$ and no edge between $\{x', y'\}$ and $\{u, v\}$. If there is no edge between $\{u, v\}$ and $\{u', v'\}$, then $D \cup D'$ induces an F_6, F_7 or F_{12}. So let u see u'. Then v sees u', for otherwise $\{v, x, u, u', y'\}$ induces an F_1; and similarly, v' sees u, and v' sees v; but then $D \cup D'$ induces an F_{13}, F_{14} or F_{15}. So we may assume, up to symmetry, that there is an edge between $\{x, y\}$ and $\{u', v'\}$, say the edge xu'.

By (16), u' misses u, y, v and x misses x', y'. By (16), y misses a vertex z' among x', y'. If x misses v', then $\{y, x, u', z', v'\}$ induces an F_1 or C_5. So x sees v', and u', v' have no neighbour in $D \setminus \{x\}$.

Suppose that one of x', y', say x', sees one of u, v. Then, by similar arguments, we obtain that x' see both u, v and there is no other edge between D and D' except possibly yy'. Consider any vertex w not in $D \cup D'$. If w sees u, then it misses x and y by (16), and it sees u', for otherwise $\{w, u, x, u', y'\}$ induces an F_1 or C_5. But then w misses x' by (16), and $\{y, u, w, u', y'\}$ induces an F_1 or C_5. Therefore w misses u, and, by symmetry, it misses v, u' and v'. If w sees y, then it misses x by (16), and $\{w, y, x, u', x'\}$ induces an F_1 or C_5. So w misses y, and similarly y'.

Moreover, w does not see both x, x', for otherwise $\{w, x, x', y, y'\}$ induces an F_1 or C_5. Define $X = \{z \notin D \cup D' \mid N(z) \cap (D \cup D') = \{x\}\}$, $X' = \{z \notin D \cup D' \mid N(z) \cap (D \cup D') = \{x'\}\}$, and $Z = \{z \notin D \cup D' \mid N(z) \cap (D \cup D') = \emptyset\}$. We have established that $V(G) = D \cup D' \cup X \cup X' \cup Z$. If there are vertices $w \in X$ and $w' \in X'$, then $\{w, x, u, x', w'\}$ induces an F_1 or C_5. So we may assume that $X' = \emptyset$. If $Z \neq \emptyset$, then, since G is connected, there is an edge zw with $z \in Z$ and $w \in X$. But then $\{z, w, x, u, x'\}$ induces an F_1. So $Z = \emptyset$. Thus $V(G) = D \cup D' \cup X$. By Lemma 2.3, u, v, u', v' are not b-vertices. By (11), we may assume that u, v, x, y have colors respectively 1, 2, 3, 4. Consequently, $c(x') \in \{3, 4\}$ and one of the colors 1, 2, say color 1, does not have a b-vertex in $D \cup D'$. So there must be a b-vertex w of color 1 in X. By Lemma 2.2, w has two neighbours w', w'' that are not adjacent, and necessarily $w', w'' \in X$. But then $\{w, w', w'', y, u, x', u'\}$ induces an F_2.

Now we may assume that x' and y' do not see any of u, v. Consider any vertex w not in $D \cup D'$. If w sees u, then it misses x and y by (16), and it sees u', for otherwise $\{w, u, x, u', y'\}$ induces an F_1 or C_5. But then w misses x' by (16) and $\{y, u, w, u', y'\}$ induces an F_1 or C_5. Therefore w misses u, and, by symmetry, it misses v. If w sees y, then it misses x by (16), and it sees u', for otherwise $\{w, y, x, u', x'\}$ induces an F_1 or C_5; but then $\{u, y, w, u', x'\}$ induces an F_1. So w misses y. If w sees one of u', v', then it sees both, for otherwise $\{w, u', x', v', u, v, y\}$ induces an F_2. In this case w is in the set $U' = \{z \notin D \cup D' \mid N(z) \cap (D \cup D') = \{u', v'\}$ or $\{u', v', x\}\}$. Next, suppose that $w \notin D \cup D' \cup U'$. Then w misses x', for otherwise either $\{w, x', u', x, u\}$ induces an F_1 or $\{u, x, w, x', y'\}$ induces an F_1. Similarly w misses y'. So in this case w is either in the set $X = \{z \notin D \cup D' \mid N(z) \cap (D \cup D') = \{x\}\}$ or in the set $Z = \{z \notin D \cup D' \mid N(z) \cap (D \cup D') = \emptyset\}$.

If $Z \neq \emptyset$, then since G is connected there is an edge zw with $z \in Z$ and $w \in U' \cup X$. But then either $\{z, w, x, u', x'\}$ induces an F_1 (if $w \in X$) or $\{z, w, u', x', u, y, v\}$ induces an F_2 (if $w \in U'$). So $Z = \emptyset$. Thus $V(G) = D \cup D' \cup U' \cup X$. By Lemma 2.3, u, v, u', v' are not b-vertices. By (11), we may assume that $c(x') = 1$, $c(y') = 2$, $c(u') = 3$, $c(v') = 4$. Consequently, x has color 1 or 2, so one of the colors 3, 4, say color 4, has no b-vertex in $D \cup D'$. So there is a b-vertex w of color 4 in $U' \cup X$. In fact $w \in U'$ is not possible since w, v' have color 4; so $w \in X$. Vertex w has a neighbour w_3 of color 3, and necessarily $w_3 \in X$. Also w has a neighbour w_2 of color 2. If $w_2 \in U'$, then w_2 sees w_3, for otherwise $\{w_3, w, w_2, u', x'\}$ induces an F_1; also, w_2 misses x, for otherwise $\{w_2, w, w_3, x\}$ induces a K_4; but then $\{u, v, x, y, w, w_2, w_3\}$ induces an F_5. So $w_2 \in X$, and w_2, w_3 are not adjacent, for otherwise $\{x, w, w_2, w_3\}$ induces a K_4. But then $\{w, w_2, w_3, u, y, v, u', x', v'\}$ induces an F_3. Thus (17) holds.

Let u, v, x, y be four vertices of G that induce a diamond, where u and v are not adjacent. Put $D = \{u, v, x, y\}$. By (16), no vertex of $G \setminus D$ can see two adjacent vertices of D. By (11), we may assume that $N(v) \subseteq N(u)$. Thus if we set $U = N(u) \setminus D$, $X = N(x) \setminus D$, $Y = N(y) \setminus D$, and $Z = \{z \in V(G) \mid N(z) \cap D = \emptyset\}$, then D, U, X, Y, Z form a partition of $V(G)$. We may assume that u, v, x, y have color respectively 1, 2, 3, 4. By Lemma 2.3 and the assumption $N(v) \subseteq N(u)$, v is not a b-vertex. Let z be a b-vertex of color 2 (the color of v). First, we claim that:

z is not in U. (18)

Suppose that z is in U. Let z_3, z_4 be neighbours of z of color respectively 3 and 4 (such vertices exist since z is a b-vertex). If z_3 misses u, then $\{z_3, z, u, x, v\}$ induces an F_1 or a C_5. So z_3 sees u. Likewise, z_4 sees u. By (16), z_3 and z_4 both miss x and y. Then z_3 misses z_4, for otherwise $\{u, z, z_3, z_4\}$ induces a K_4. Now if v sees both z_3, z_4, then the seven vertices $\{u, v, x, y, z, z_3, z_4\}$ induces an F_{11}; if it misses both, then the seven vertices induce an F_5; and if it sees exactly one of them, say z_3, then $\{x, v, z_3, z, z_4\}$ induces an F_1. So (18) holds.

Now, we claim that:

z is not in Z. (19)

Suppose that z is in Z. Since G is connected, there is a path z-p_1-$\cdots$-p_h such that p_h has a neighbour in D and the path is as short as possible. So the path is chordless and its vertices other than p_h have no neighbour in D, and $p_h \in U \cup X \cup Y$. We have $h \leq 3$ since G contains no F_1. If $h = 3$, then there is still an F_1, induced by z, p_1, p_2, p_3 and a neighbour of p_3 in D. If $h = 2$, then there is still an F_1, induced by z, p_1, p_2 and some two adjacent vertices of D. So $h = 1$. Let us now write p instead of p_1. Suppose that $p \notin X \cup Y$. So p sees at least one of u, v, and it actually sees both, for otherwise $\{z, p, u, x, v\}$ induces an F_1. Up to symmetry we may assume that p has color 3. Let z_1, z_4 be neighbours of z of color respectively 1, 4. So $z_1, z_4 \notin D$. Vertex z_1 misses u (which has color 1) and consequently misses v too. Then z_1 sees p, for otherwise $\{z_1, z, p, u, x\}$ induces an F_1 or C_5. Then z_4 misses both p, z_1, for otherwise $\{p, z, z_1, z_4\}$ induces either a K_4 or a diamond disjoint from D, which contradicts (17). If z_4 sees u, then z_1, z, z_4, u, y induces an F_1 or a C_5. So z_4 misses u, and consequently it misses v. But then $\{z_4, z, p, u, y\}$ induces an F_1. Therefore, we have $p \in X \cup Y$, say, up to symmetry, $p \in X$. So p sees x and, by (16), it misses u, v, y. Let z_3 be a neighbour of z of color 3. So $z_3 \notin D$. If z_3 misses p, then z_3, z, p, x and one of u, v, y induce an F_1. So z_3 sees p. Let z' be a neighbour of z whose color is not 2, 3 or the color of p (z' exists since z is a b-vertex). Then z' misses both p, z_3, for otherwise $\{p, z, z_3, z'\}$ induces either a K_4 or a diamond disjoint from D. Then z' sees x, for otherwise z', z, p, x and one of u, v, y induce an F_1. By (16), z' misses u, v, y. But then z_3, z, z', x and one of u, v, y induce an F_1 or C_5. Thus (19) holds. By (18) and (19) we have:

z is in $X \cup Y$. (20)

Without loss of generality, we may assume that z is in X. So z sees x and, by (16), misses u, v, y. Let z_1, z_4 be neighbours of z of color respectively 1, 4 (such vertices exist since z is a b-vertex). Note that z_1 misses u, because they both have color 1, and v, because $N(v) \subseteq N(u)$. Clearly z_4 misses y. Now we claim that:

$N(u) = N(v)$. (21)

We already have $N(v) \subseteq N(u)$. Suppose that there exists a vertex t that sees u and not v. By (16), t misses x and y. Then t misses z, for otherwise $\{z, t, u, y, v\}$ induces an F_1. So $t \neq z_1, z_4$. Then z_1 sees x, for otherwise $\{z_1, z, x, u, t\}$ induces an F_1 or C_5. By (16), z_1 misses y. Then z_1 misses t, for otherwise $\{z_1, t, u, y, v\}$ induces an F_1. If z_1 misses z_4, then either $\{z_1, z, z_4, t, u, y, v\}$ induces an F_2 or z_1, z, z_4 and some two adjacent vertices of t, u, y, v induce an F_1. So z_1 sees z_4. Then z_4 misses x, for otherwise $\{z, z_1, z_4, x\}$ induces a K_4; and it sees u, for otherwise $\{z_4, z_1, x, u, t\}$ induces an F_1 or C_5. But then either $\{z_1, z_4, u, y, v\}$ induces an F_1 (if z_4 misses v), or $\{u, v, x, y, z, z_1, z_4\}$ induces an F_{11} (if z_4 sees v). Therefore no such t exists, so (21) holds.

Now u and v play symmetric roles, and u is not a b-vertex. Let w be a b-vertex of color 1 (the color of u). By symmetry, (20) holds with w replacing z, that is, $w \in X \cup Y$. We claim that:

w is in X. (22)

For suppose that $w \in Y$. So w sees y and, by (16) and (21), it misses u, v, x. Let w_2, w_3 be neighbours of w of color respectively 2, 3. Clearly w_3 misses x. By (21), w_2 misses u and v. If w, z are adjacent, then z_4 sees w, for otherwise $\{z_4, z, w, y, u\}$ induces an F_1 or C_5; and by symmetry w_3 sees z; and w_3 misses z_4, for otherwise $\{w, z, w_3, z_4\}$ induces a K_4; but then $\{w, z, w_3, z_4\}$ induces a diamond disjoint from D, a contradiction to (17). So w and z are not adjacent, and consequently $w \neq z_1$ and $z \neq w_2$. Then w_3 sees y, for otherwise $\{w_3, w, y, x, z\}$ induces an F_1 or a C_5. And by (16), w_3 misses u and v. Similarly, z_4 sees x and misses u and v. If both xz_1, yw_2 are edges, then either the two sets $\{x, z, z_1, z_4\}$ and $\{y, w, w_2, w_3\}$ induce two disjoint diamonds or one of them induces a K_4, a contradiction. So, up to symmetry, x misses z_1. Then z_1 sees y, for otherwise $\{z_1, z, x, y, w\}$ induces an F_1. Then z_4 sees z_1, for otherwise $\{z_4, z, z_1, y, u\}$ induces an F_1 or a C_5. Thus $D_z = \{z, z_1, z_4, x\}$ induces a diamond. Then w_2 misses both y, w_3, for otherwise $\{y, w, w_2, w_3\}$ induces either a K_4 or a diamond disjoint from D_z, which contradicts (17). Then w_2 sees x, for otherwise $\{w_2, w, y, x, z\}$ induces an F_1. But now $\{u, x, w_2, w, w_3\}$ induces an F_1. Thus (22) holds.

Therefore w sees x and, by (16) and (21), it misses u, v, y. Let w_2, w_4 be neighbours of w of color respectively 2, 4. Clearly w_4 misses y. By (21), w_2 misses both u and v. We claim that:

w misses z. (23)

For suppose that w sees z. If w, z have a common neighbour t of color 4, then t misses x, for otherwise $\{x, t, w, z\}$ induces a K_4, but then $\{u, v, x, y, w, z, t\}$ induces an F_5 (if t misses both u, v) or an F_{11} (if t sees both u, v). So w and z do not have a common neighbour of color 4. Thus $w_4 \neq z_4$ and $\{w_4, w, z, z_4\}$ induces a P_4. Then u misses both w_4, z_4, for otherwise $\{u, w_4, w, z, z_4\}$ induces an F_1 or C_5, and similarly v misses both w_4, z_4. But then $\{w_4, w, z, z_4, u, v, y\}$ induces an F_2. Thus (23) holds.

We claim that:

Either z_1 sees z_4 or w_2 sees w_4. (24)

For suppose the contrary. So both $\{z, z_1, z_4\}$ and $\{w, w_2, w_4\}$ induce a P_3. If $w_4 = z_4$, then $\{z, w, z_1, w_2, z_4\}$ induces an F_1 or C_5. So, $w_4 \neq z_4$. Write $P = \{u, y, v\}$, and $Q = \{z, z_1, z_4, w, w_2, w_4\}$. If z_4 sees a vertex in P, then by (21) it sees both u and v; but then $\{y, u, z_4, z, z_1\}$ induces an F_1 or C_5. So, z_4 misses all of P. Similarly, w_4 misses all of P. If z_1 sees y, then $\{z_4, z, z_1, y, u\}$ induces an F_1. Thus z_1, and similarly, w_2 have no neighbour in P, i.e., there is no edge between P and Q. We may assume that Q does not contain a P_4, for otherwise this P_4 and P induce an F_2. It follows that:

z_1 misses w_2, for otherwise $\{z, z_1, w_2, w\}$ induces a P_4;

z_1 misses w_4, for otherwise $\{z_1, w_4, w, w_2\}$ induces a P_4;

z misses w_4, for otherwise $\{z, w_4, w, w_2\}$ induces a P_4;
z_4 misses every $w' \in \{w, w_2\}$, for otherwise $\{w', z_4, z, z_1\}$ induces a P_4.
But now $P \cup Q$ induce an F_3, a contradiction. Thus (24) holds.

By (24) and by symmetry, we may assume that $\{z, z_1, z_4\}$ induces a K_3. If x sees z_1, then it misses z_4, for otherwise, $\{x, z, z_1, z_4\}$ induces a K_4, and z_1 misses y by (16); but then $\{u, v, x, y, z, z_1, z_4\}$ induces either an F_5 or F_{11}. So x misses z_1. If x sees z_4, then z_4 misses u and v by (16), and z_1 sees y, for otherwise again $\{u, v, x, y, z, z_1, z_4\}$ induces an F_5. But now $\{x, z, z_1, z_4\}$ induces a diamond in which the two non-adjacent vertices do not have the same neighbourhood; by the mapping $z \to x, z_4 \to y, z_1 \to u, x \to v$, we have a contradiction to (21). So x misses z_4. If x misses a vertex $w' \in \{w_2, w_4\}$, then w' sees z, for otherwise $\{z_1, z, x, w, w'\}$ induces an F_1 or C_5. So $w' = w_4$. Then w_4 misses z_1, for otherwise $\{z_1, w_4, w, x, y\}$ induces an F_1 or C_5. And so $w_4 \neq z_4$. Then z_1 misses y and w_4 misses u, for otherwise $\{w_4, z, z_1, y, u\}$ induces an F_1 or C_5; and by (21) w_4 misses v. But then $\{z_1, z, w_4, w, u, y, v\}$ induces an F_2. Therefore x sees both w_2, w_4. So w_2, w_4 are not adjacent, for otherwise $\{x, w, w_2, w_4\}$ induces a K_4; and, by (16), they both miss u, v and y. We have the following implications:
z_1 misses y, for otherwise $\{z_4, z_1, y, x, w\}$ induces an F_1 or C_5;
z_4 misses w, for otherwise $\{z_1, z_4, w, x, y\}$ induces an F_1;
z_4 misses u, for otherwise $\{z_1, z_4, u, x, w\}$ induces an F_1;
z_4 misses v by (21);
z_4 misses w_2, for otherwise $\{z_4, w_2, w, w_4, u, y, v\}$ induces an F_2;
z_1 misses each $w' \in \{w_2, w_4\}$, for otherwise $\{z_4, z_1, w', x, y\}$ induces an F_1.
But now $\{u, v, x, y, z, z_1, z_4, w, w_2, w_4\}$ induces an F_9. This completes the proof of Theorem 1.2. □

Acknowledgements

The first author's research is supported by NSERC. The second author's research is supported by CAPES/COFECUB project number 359/01. The third author's research is supported by ADONET (Marie-Curie Research Training Network of the European Community).

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