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Isomorphism between Systems of Equivariant Singularities*

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In this article isomorphisms between systems of singularities equivariant under different Lie group actions are investigated and a sufficient condition for two systems to be isomorphic is given. With this sufficiency theorem we show that the system of $\mathbf{O}(n)$-equivariant singularities in its irreducible representation on $\mathbb{R}^n$ is isomorphic to that of one-dimensional $\mathbb{Z}_2$-equivariant singularities and the system of $\frac{1}{2}n(n+1)$-dimensional $\mathbf{O}(n)$-equivariant singularities is isomorphic to that of n-dimensional $\mathbf{S}_n$-equivariant singularities.

1. INTRODUCTION

Equivariant singularity theory is an important tool for the study of local bifurcations with symmetry and its main topics include normal forms (the simplest possible forms), recognition problems (finding conditions for

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determining a singularity to be equivalent to a normal form), and universal unfoldings (a comprehensive way to understand the degeneracy and the complexity of a bifurcation problem). The general theory of equivariant singularities was developed in the two volume books by M. Golubitsky and D. G. Schaeffer [3], and Golubitsky, Stewart, and Schaeffer [4].

From the results of [2, 7, 8, 9] we found that normal forms and universal unfoldings of $\mathbf{O}(n)$-equivariant singularities in its irreducible representation on $\mathbb{R}^n$ have the same expressions as those of one-dimensional $\mathbb{Z}_2$-equivariant singularities. M. Golubitsky suggested that we also consider a similar problem for the irreducible $\mathbf{O}(3)$-action on a five-dimensional space and the $\mathbf{S}_3$-action on the plane stated in [4]. Both examples motivate us to consider the correspondence between systems of equivariant singularities (see Section 2 for exact definitions of related concepts).

The remainder of the article is organized as follows. In Section 2 we introduce the concept of systems of equivariant singularities and we give the definition of isomorphisms between two such systems. We conclude that if two systems of equivariant singularities are isomorphic then normal forms and universal unfoldings of one system correspond, respectively, to those of the other via the isomorphism (Theorem 2.5). We also give a sufficient condition which guarantees two systems of equivariant singularities to be isomorphic (Theorem 2.6). In Sections 3 and 4 we construct two concrete isomorphisms. The first example is between $\mathbf{O}(n)$-equivariant singularities in its irreducible representation on $\mathbb{R}^n$ and one-dimensional $\mathbb{Z}_2$-equivariant singularities, and is well known. The second example is between the equivariant singularities based on the $\frac{1}{2}n(n+1)$-dimensional action of $\mathbf{O}(n)$ on $n \times n$ symmetric matrices and the system of n-dimensional $\mathbf{S}_n$-equivariant singularities. The latter one with additional restrictions on the spaces gives an isomorphism between $\mathbf{O}(n)$-equivariant singularities based on the action of $\mathbf{O}(n)$ on $n \times n$ symmetric matrices with trace zero (which is of dimension $\frac{1}{2}(n-1)(n+2)$) and the system of $(n-1)$-dimensional $\mathbf{S}_n$-equivariant singularities. Particularly, our result for the case $n = 3$ recovers the result in [4] mentioned in the preceding text (see Remark 4.5 for details). By these examples we show that the problem of finding normal forms and universal unfoldings for some systems of equivariant singularities may be reduced to that of a system of lower dimensional singularities equivariant under a simpler Lie group action.

2. ISOMORPHISMS BETWEEN SYSTEMS OF EQUIVARIANT SINGULARITIES

In this section we discuss isomorphisms between systems of equivariant singularities.

First of all, we recall some important notations and concepts in the theory of equivariant singularities. Let V be a finitely dimensional vector space and let $\mathscr{L}(V)$ be the set of all linear transformations on V. Let ρ: $\Gamma \to \mathscr{GL}(V)$ be a representation of a Lie group Γ on V. Here $\mathscr{GL}(V)$ is the group of all invertible transformations in $\mathscr{L}(V)$. We write

$$\gamma \cdot x := \rho(\gamma) x, \qquad \forall \gamma \in \Gamma,\ x \in V.$$

Then Γ acts on $\mathscr{L}(V)$ by similarity as follows,

$$\gamma \cdot L := \rho(\gamma) \circ L \circ \rho(\gamma^{-1}), \qquad \forall \gamma \in \Gamma,\ L \in \mathscr{L}(V).$$

Denote by $\vec{\mathscr{E}}_{V,k}(\Gamma)$ the set of all smooth (C^∞) Γ-equivariant germs g: $(V \times \mathbb{R}^k, 0) \to V$,

$$g(\gamma \cdot x, \mu) = \gamma \cdot g(x, \mu), \qquad \forall x \in V,\ \mu \in \mathbb{R}^k, \text{ and } \gamma \in \Gamma;$$

by $\overleftrightarrow{\mathscr{E}}_{V,k}(\Gamma)$ the set of all smooth Γ-equivariant germs S: $(V \times \mathbb{R}^k, 0) \to \mathscr{L}(V)$,

$$S(\gamma \cdot x, \mu) = \gamma \cdot S(x, \mu), \qquad \forall x \in V,\ \mu \in \mathbb{R}^k, \text{ and } \gamma \in \Gamma,$$

by $\mathscr{E}_{1+k}$ the set of all smooth germs $(\mathbb{R} \times \mathbb{R}^k, 0) \to \mathbb{R}$ and by $\overleftrightarrow{\mathscr{E}}_k$ the set of all smooth germs $(\mathbb{R}^k, 0) \to \mathbb{R}^k$.

Define

$$\mathscr{D}_V(\Gamma) := \Big\{(S, X, \Lambda) \in \overleftrightarrow{\mathscr{E}}_{V,1}(\Gamma) \times \vec{\mathscr{E}}_{V,1}(\Gamma) \times \mathscr{E}_1: \\ S(0,0), (dX)_{0,0} \in \mathscr{L}_\Gamma(V)^\circ, X(0,0) = 0, \Lambda(0) = 0, \Lambda'(0) > 0\Big\},$$

where $\mathscr{L}_\Gamma(V)$ is the set of all Γ-equivariant invertible linear transformations on V and $\mathscr{L}_\Gamma(V)^\circ$ is the connected component of $\mathscr{L}_\Gamma(V)$ containing the identity. With a suitably defined binary operation (see [5] for exact formulation) $\mathscr{D}_V(\Gamma)$ becomes a group and acts on $\vec{\mathscr{E}}_{V,1}(\Gamma)$ as

$$((S, X, \Lambda) g)(x, \lambda) := S(x, \lambda) g(X(x, \lambda), \Lambda(\lambda)),$$

$$(S, X, \Lambda) \in \mathscr{D}_V(\Gamma), \qquad g \in \vec{\mathscr{E}}_{V,1}(\Gamma).$$

Then two Γ-equivariant germs g and h in $\vec{\mathscr{E}}_{V,1}(\Gamma)$ are said to be Γ-*equivalent* if they are in the same $\mathscr{D}_V(\Gamma)$-orbit.

Let $(x, \lambda, \alpha) \in V \times \mathbb{R} \times \mathbb{R}^k$ and $g \in \vec{\mathscr{E}}_{V,1}(\Gamma)$. A Γ-unfolding of g with unfolding parameters $\alpha = (\alpha_1, \dots, \alpha_k)$ is a germ $G \in \vec{\mathscr{E}}_{V,1+k}(\Gamma)$ such that $G(x, \lambda, 0) = g(x, \lambda)$ for all $x \in V$ and $\lambda \in \mathbb{R}$. Define

$$\mathscr{D}_{V,k}(\Gamma) := \Big\{(S, X, \Lambda, A) \in \overleftrightarrow{\mathscr{E}}_{V,1+k}(\Gamma) \times \vec{\mathscr{E}}_{V,1+k}(\Gamma) \times \mathscr{E}_{1+k} \times \vec{\mathscr{E}}_k: \\ S(x, \lambda, 0) \equiv \mathrm{id}, X(x, \lambda, 0) \equiv x, \Lambda(\lambda, 0) \equiv \lambda, A(0) = 0, \det DA(0) \neq 0\Big\}.$$

With a binary operation similarly defined as that for $\mathscr{D}_V(\Gamma)$, $\mathscr{D}_{V,k}(\Gamma)$ is a group and acts on $\vec{\mathscr{E}}_{V,1+k}(\Gamma)$ as,

$$((S, X, \Lambda, A)G)(x, \lambda, \alpha) \coloneqq S(x, \lambda, \alpha)G(X(x, \lambda, \alpha), \Lambda(\lambda, \alpha), A(\alpha)).$$

Two germs G and H in $\vec{\mathscr{E}}_{V,1+k}(\Gamma)$ are said to be *equivalent* if they are in the same $\mathscr{D}_{V,k}(\Gamma)$-orbit. It is easy to see that equivalent germs in $\vec{\mathscr{E}}_{V,1+k}(\Gamma)$ are Γ-unfoldings of the same unperturbed germ. A Γ-unfolding $G \in \vec{\mathscr{E}}_{V,1+k}(\Gamma)$ of $g \in \vec{\mathscr{E}}_{V,1}(\Gamma)$ is said to be *versal* if for any other Γ-unfolding, say $H \in \vec{\mathscr{E}}_{V,1+l}(\Gamma)$, there is a reparameterization $B\colon (\mathbb{R}^l, 0) \to (\mathbb{R}^k, 0)$ such that H is equivalent to $B^*G \in \vec{\mathscr{E}}_{V,1+l}(\Gamma)$, where B^*G is a Γ-unfolding induced from G via B,

$$(B^*G)(x, \lambda, \beta) \coloneqq G(x, \lambda, B(\beta)). \tag{2.1}$$

Universal Γ-*unfoldings* of $g \in \vec{\mathscr{E}}_{V,1}(\Gamma)$ are defined as the versal ones with the least number of unfolding parameters and this number, denoted by $\operatorname{codim}_\Gamma g$, is called the Γ-*codimension* of g. In addition, g is said to be of *infinite* Γ-*codimension* if it has no versal Γ-unfolding with finite number of unfolding parameters. A universal unfolding exhibits all the complex nature and the codimension characterizes the degeneracy of an unperturbed germ.

DEFINITION 2.1. For each Lie group representation $\rho\colon \Gamma \to \mathscr{GL}(V)$ the set,

$$\mathscr{S}_\rho \coloneqq \prod_{k=0}^{\infty} \vec{\mathscr{E}}_{V,1+k}(\Gamma)$$

is called *a system of equivariant singularities* (*with respect to* ρ). Let $\rho\colon \Gamma \to \mathscr{GL}(V)$ and $\rho'\colon \Sigma \to \mathscr{GL}(W)$ be two representations of Lie groups. A linear map $\Phi\colon \mathscr{S}_\rho \to \mathscr{S}_{\rho'}$ is called a *homomorphism* of these two systems of equivariant singularities if Φ is defined by a sequence of linear maps $\{\Phi_k\colon \vec{\mathscr{E}}_{V,1+k}(\Gamma) \to \vec{\mathscr{E}}_{W,1+k}(\Sigma)\}_{k=0}^{\infty}$ satisfy the following conditions:

(a) for any $g \in \vec{\mathscr{E}}_{V,1}(\Gamma)$ and for any $(S, X, \Lambda) \in \mathscr{D}_V(\Gamma)$ there is a triple $(\tilde{S}, \tilde{X}, \tilde{\Lambda}) \in \mathscr{D}_W(\Sigma)$ such that

$$\Phi_0((S, X, \Lambda)g) = (\tilde{S}, \tilde{X}, \tilde{\Lambda})(\Phi_0 g);$$

(b) for any positive integer k, for any $G \in \vec{\mathscr{E}}_{V,1+k}(\Gamma)$, and for any $(S, X, \Lambda, A) \in \mathscr{D}_{V,k}(\Gamma)$ there is a quadruple $(\tilde{S}, \tilde{X}, \tilde{\Lambda}, \tilde{A}) \in \mathscr{D}_{W,k}(\Sigma)$ such that

$$\Phi_k((S, X, \Lambda, A)G) = (\tilde{S}, \tilde{X}, \tilde{\Lambda}, \tilde{A})(\Phi_k G);$$

(c) for any positive integers k and l, for any $G \in \vec{\mathscr{E}}_{V,1+k}(\Gamma)$, and for any reparameterization B: $(\mathbb{R}^l, 0) \to (\mathbb{R}^k, 0)$, there is a reparameterization $\tilde{B}$: $(\mathbb{R}^l, 0) \to (\mathbb{R}^k, 0)$ such that

$$\Phi_l(B^* G) = \tilde{B}^*(\Phi_k G),$$

where B^* and $\tilde{B}^*$ are defined as in Eq. (2.1);

(d) for any positive integer k and for any $G \in \vec{\mathscr{E}}_{V,1+k}(\Gamma)$ the following equality holds

$$(\Phi_k G)(y, \lambda, 0) = (\Phi_0 g)(y, \lambda), \qquad \forall y \in (W, 0),\ \lambda \in (\mathbb{R}, 0),$$

where $g(x, \lambda) = G(x, \lambda, 0)$.

We denote by $\mathrm{Hom}(\mathscr{S}_\rho, \mathscr{S}_{\rho'})$ the set of all homomorphisms from $\mathscr{S}_\rho$ to $\mathscr{S}_{\rho'}$.

The following proposition can be easily proved.

PROPOSITION 2.2. (a) *Let* ρ_i: $\Gamma_i \to \mathscr{GL}(V_i)$, $i = 1, 2, 3$, *be representations of Lie groups. Then for any* $\Phi \in \mathrm{Hom}(\mathscr{S}_{\rho_1}, \mathscr{S}_{\rho_2})$ *and for any* $\Psi \in \mathrm{Hom}(\mathscr{S}_{\rho_2}, \mathscr{S}_{\rho_3})$, *the composition of* Φ *and* Ψ, *which is given by*

$$\Psi \circ \Phi := \{\Psi_k \circ \Phi_k\}_{k=0}^{\infty}, \tag{2.2}$$

is a homomorphism in $\mathrm{Hom}(\mathscr{S}_{\rho_1}, \mathscr{S}_{\rho_3})$.

(b) *The composition of homomorphisms is associative.*

(c) *For any representation* ρ: $\Gamma \to \mathscr{GL}(V)$, $\mathrm{Id} := \{\mathrm{Id}_k\}_{k=0}^{\infty}$ *is the unit in* $\mathrm{Hom}(\mathscr{S}_\rho, \mathscr{S}_\rho)$, *where* Id_k *is the identity on* $\vec{\mathscr{E}}_{V,1+k}(\Gamma)$. ∎

DEFINITION 2.3. A homomorphism Φ in $\mathrm{Hom}(\mathscr{S}_{\rho_1}, \mathscr{S}_{\rho_2})$ is called an *isomorphism* if there is some Ψ, which is called an *inverse* of Φ, in $\mathrm{Hom}(\mathscr{S}_{\rho_2}, \mathscr{S}_{\rho_1})$ such that $\Psi \circ \Phi$ and $\Phi \circ \Psi$ are, respectively, the units in $\mathrm{Hom}(\mathscr{S}_{\rho_1}, \mathscr{S}_{\rho_1})$ and $\mathrm{Hom}(\mathscr{S}_{\rho_2}, \mathscr{S}_{\rho_2})$. Two systems of equivariant singularities $\mathscr{S}_{\rho_1}$ and $\mathscr{S}_{\rho_2}$ are said to be *isomorphic* if there exists an isomorphism in $\mathrm{Hom}(\mathscr{S}_{\rho_1}, \mathscr{S}_{\rho_2})$.

By Proposition 2.2 we know that the inverse of an isomorphism Φ is unique and we denote it by Φ^{-1}. Moreover, it is easy to see that $\Phi = \{\Phi_k\}_{k=0}^{\infty}$ is invertible if and only if each Φ_k is invertible. When Φ is invertible we have $\Phi^{-1} = \{\Phi_k^{-1}\}_{k=0}^{\infty}$.

EXAMPLE 2.4. Let ρ: $\Gamma \to \mathscr{GL}(\mathrm{V})$ be a representation of Γ. Then every $(S, X, \Lambda) \in \mathscr{D}_V(\Gamma)$ determines an isomorphism Φ in $\mathrm{Hom}(\mathscr{S}_\rho, \mathscr{S}_\rho)$,

$$(\Phi_0 g)(x, \lambda) = S(x, \lambda) g(X(x, \lambda), \Lambda(\lambda)), \qquad \forall g \in \vec{\mathscr{E}}_{V,1}(\Gamma),$$

$$(\Phi_k G)(x, \lambda, \alpha) = S(x, \lambda) G(X(x, \lambda), \Lambda(\lambda), \alpha), \qquad \forall G \in \vec{\mathscr{E}}_{V,1+k}(\Gamma).$$

THEOREM 2.5. *Let* $\rho\colon \Gamma \to \mathscr{GL}(V)$ *and* $\rho'\colon \Sigma \to \mathscr{GL}(W)$ *be representations of Lie groups. Suppose* $\Phi \in \mathrm{Hom}(\mathscr{S}_\rho, \mathscr{S}_{\rho'})$ *is an isomorphism. Then*

(a) g *and* h *in* $\vec{\mathscr{E}}_{V,1}(\Gamma)$ *are* Γ*-equivalent if and only if* $\Phi_0 g$ *and* $\Phi_0 h$ *are* Σ*-equivalent.*

(b) $G \in \vec{\mathscr{E}}_{V,1+k}(\Gamma)$ *is a versal (universal)* Γ*-unfolding of* $g \in \vec{\mathscr{E}}_{V,1}(\Gamma)$ *if and only if* $\Phi_k G$ *is a versal (universal)* Σ*-unfolding of* $\Phi_0 g$.

(c) $\mathrm{codim}_\Gamma\, g = \mathrm{codim}_\Sigma(\Phi_0 g)$.

Proof. Conclusion (a) is a direct consequence of Definition 2.1(a). Conclusion (b) can be proved by an argument combining (b), (c), and (d) in Definition 2.1. Conclusion (c) follows from conclusion (b) immediately. ∎

THEOREM 2.6. *Let* $\rho\colon \Gamma \to \mathscr{GL}(V)$ *and* $\rho'\colon \Sigma \to \mathscr{GL}(W)$ *be representations of Lie groups. Suppose* $h\colon \Sigma \to \Gamma$ *is a group homomorphism and* $i\colon W \to V$ *and* $\pi\colon V \to W$ *are linear maps such that* $\pi \circ i = \mathrm{id}_W$ *and* $i(\sigma \cdot w) = h(\sigma) \cdot i(w)$ *for all* $w \in W$ *and for all* $\sigma \in \Sigma$. *Define*

$$\Phi_k\colon \vec{\mathscr{E}}_{V,1+k}(\Gamma) \to \vec{\mathscr{E}}_{W,1+k}(\Sigma), \qquad \Phi_k(G)(w,\mu) = \pi G(i(w),\mu), \qquad k = 0,1,2,\ldots . \tag{2.3}$$

If

(a) $i(W) = \mathrm{Fix}(\Delta) := \{v \in V \mid \delta \cdot v = v,\ \forall \delta \in \Delta\}$ *for some subgroup* $\Delta \subset \Gamma$, *then* $\{\Phi_k\}_{k=0}^{\infty} \in \mathrm{Hom}(\mathscr{S}_\rho, \mathscr{S}_{\rho'})$.

If, in addition,

(b) *the map* $p\colon \Gamma \times W \to V$, $(\gamma, w) \mapsto \gamma \cdot i(w)$, *is surjective;*

(c) *for each* $g \in \vec{\mathscr{E}}_{W,1+k}(\Sigma)$ *there exists a unique germ* $G \in \vec{\mathscr{E}}_{V,1+k}(\Gamma)$ *such that the following diagram commutes*

$$\begin{array}{ccc} \Gamma \times W \times \mathbb{R}^{1+k} & \xrightarrow{\mathrm{id}_\Gamma \times g} & \Gamma \times W, \\ {\scriptstyle p \times \mathrm{id}_{\mathbb{R}^{1+k}}}\downarrow & & \downarrow{\scriptstyle p} \\ V \times \mathbb{R}^{1+k} & \xrightarrow{G} & V \end{array} \tag{2.4}$$

and

(d) *for each* $T \in \overleftrightarrow{\mathscr{E}}_{W,1+k}(\Sigma)$ *there exists a germ* $S \in \overleftrightarrow{\mathscr{E}}_{V,1+k}(\Gamma)$ *such that*

$$S(p(\gamma, w_1), \mu) p(\gamma, w_2) = \gamma \cdot i(T(w_1,\mu) w_2), \qquad \forall \gamma \in \Gamma, \quad \text{and} \quad w_1, w_2 \in W, \tag{2.5}$$

then $\{\Phi_k\}_{k=0}^{\infty}$ *is an isomorphism in* $\mathrm{Hom}(\mathscr{S}_\rho, \mathscr{S}_{\rho'})$.

Proof. It is easy to see that $i\colon W \to i(W)$ and $\pi|i(W)\colon i(W) \to W$ are linear isomorphisms and such that

$$i \circ \pi|i(W) = \mathrm{id}_{i(W)};$$

$$\pi(h(\sigma)\cdot v) = \sigma \cdot \pi(v), \qquad \forall \sigma \in \Sigma, v \in i(W).$$

Let $G \in \vec{\mathscr{E}}_{V,1}(\Gamma)$ and $(S, X, \Lambda) \in \mathscr{D}_V(\Gamma)$. Assumption (a) implies that $G(i(w), \lambda) \in i(W)$ and $S(i(w), \lambda)i(w') \in i(W)$ for all $w, w' \in W$ and $\lambda \in \mathbb{R}$. Then for all $(w, \lambda) \in (W \times \mathbb{R}, 0)$ we have

$$\begin{aligned}\Phi_0((S,X,\Lambda)G)(w,\lambda) &= \pi\big[S(i(w),\lambda)G(X(i(w),\lambda),\Lambda(\lambda))\big]\\ &= \pi S(i(w),\lambda)i\pi G(i\pi X(i(w),\lambda),\Lambda(\lambda))\\ &= T(w,\lambda)(\Phi_0 G)((\Phi_0 X)(w,\lambda),\Lambda(\lambda)), \quad (2.6)\end{aligned}$$

where $T\colon (W \times \mathbb{R}, 0) \to \mathscr{L}(W)$ is given by $T(w, \lambda) = \pi S(i(w), \lambda)i$ and is Σ-equivariant. Because $S(0,0) \in \mathscr{L}_\Gamma(V)^\circ$ there is a continuous map $[0,1] \ni t \mapsto \tilde{S}^t \in \mathscr{L}_\Gamma(V)$ such that $\tilde{S}^0 = \mathrm{id}_V$ and $\tilde{S}^1 = S(0,0)$. Write $\tilde{T}^t = \pi\tilde{S}^t i$. Then for any $w \in \mathrm{Ker}(\tilde{T}^t)$ we have $\tilde{S}^t(i(w)) \in i(W) \cap \mathrm{Ker}(\pi) = \{0\}$ and hence $w = 0$. This shows $\tilde{T}^t \in \mathscr{L}_\Sigma(W)$. Because the map $[0,1] \ni t \mapsto \tilde{T}^t \in \mathscr{L}_\Sigma(W)$ is continuous and $\tilde{T}^0 = \mathrm{id}_W$ we have $T(0,0) = \tilde{T}^1 \in \mathscr{L}_\Sigma(W)^\circ$. Similarly we can prove $(d\Phi_0 X)_{0,0} \in \mathscr{L}_\Sigma(W)^\circ$. Therefore $(T, \Phi_0 X, \Lambda) \in \mathscr{D}_W(\Sigma)$ and Eq. (2.6) shows that condition (a) of Definition 2.1 holds. A similar argument shows that condition (b) of Definition 2.1 holds. Conditions (c) and (d) can be easily checked. Consequently, $\{\Phi_k\}_{k=0}^\infty \in \mathrm{Hom}(\mathscr{S}_\rho, \mathscr{S}_{\rho'})$.

To prove the second part of the theorem note that assumption (c) implies that each Φ_k is a linear isomorphism which satisfies that, for every $g \in \vec{\mathscr{E}}_{W,1}(\Sigma)$, $\Phi_k^{-1}g$ is the unique germ such that diagram (2.4) commutes.

According to (c), for each $g \in \vec{\mathscr{E}}_{W,1+k}(\Sigma)$, there exists a unique germ $G \in \vec{\mathscr{E}}_{V,1+k}(\Gamma)$ such that

$$G(p(\gamma,w),\mu) = p(\gamma, g(w,\mu)), \qquad \forall w \in W, \quad \text{and} \quad \mu \in \mathbb{R}^{1+k}.$$

Then

$$\Phi_k G(w,\mu) = \pi G(i(w),\mu) = g(w,\mu).$$

Therefore, Φ_k is a linear isomorphism and $\Phi_k^{-1}g = G$.

Let $(T, Y, \Lambda) \in \mathscr{D}_W(\Sigma)$. By (d) there is an $S \in \overleftrightarrow{\mathscr{E}}_{V,1}(\Gamma)$ such that

$$S(p(\gamma,w_1),\lambda)p(\gamma,w_2) = \gamma\cdot i(T(w_1,\lambda)w_2),$$
$$\forall \gamma \in \Gamma, w_1, w_2 \in W, \lambda \in \mathbb{R}. \quad (2.7)$$

Hence, for any $g \in \vec{\mathscr{E}}_{W,1}(\Sigma)$ and for any $v(=p(\gamma, w)) \in V$ we have

$$\begin{aligned}\left(\Phi_0^{-1}(T, Y, \Lambda) g\right)(v, \lambda) &= \gamma \cdot i\left[T(w, \lambda) g(Y(w, \lambda), \Lambda(\lambda))\right] \\ &= S(v, \lambda)\left(\Phi_0^{-1} g\right)\left(\left(\Phi_0^{-1} Y\right)(v, \lambda), \Lambda(\lambda)\right). \quad (2.8)\end{aligned}$$

In order to prove $(S, \Phi_0^{-1}Y, \Lambda) \in \mathscr{D}_V(\Gamma)$ it is sufficient to show that if $B \in \mathscr{L}_\Sigma(W)^\circ$ and A: $V \to V$ is a Γ-equivariant linear map such that $Ap(\gamma, w) = p(\gamma, Bw)$, then we must have $A \in \mathscr{L}_\Gamma(V)^\circ$.

Suppose $B \in \mathscr{L}_\Sigma(W)^\circ$. Then there is a continuous map f: $[0,1] \to \mathscr{L}_\Sigma(W)$ such that $f(0) = \mathrm{id}_W$ and $f(1) = B$. For any fixed $t \in [0,1]$ the map $W \times \mathbb{R} \ni (w, \lambda) \mapsto f(t)w \in W$ is a C^∞ Σ-equivariant germ. By (d) there is an $\tilde{F}_t \in \vec{\mathscr{E}}_{V,1}(\Gamma)$ such that

$$\tilde{F}_t\left(p(\gamma, w_1), \lambda\right) p(\gamma, w_2) = p\left(\gamma, f(t) w_2\right), \qquad \forall \gamma \in \Gamma, w_1, w_2 \in W, \lambda \in \mathbb{R}.$$

Write $F(t) = \tilde{F}_t(0,0)$. Then

$$F(t) p(\gamma, w) = p\left(\gamma, f(t) w\right), \qquad \forall \gamma \in \Gamma, w \in W.$$

It is easy to see that $F(0) = \mathrm{id}_V$ and $F(1) = A$. Hence for any $\gamma \in \Gamma$ and for any $w \in W$ we have

$$|w| = \left|\pi\left(\gamma^{-1} p(\gamma, w)\right)\right| \le |\gamma^{-1}|\,|\pi|\,|p(\gamma, w)|,$$

and therefore,

$$\begin{aligned}&\left|F(t_1) p(\gamma, w) - F(t_2) p(\gamma, w)\right| \\ &\quad = \left|\gamma \cdot i\left(f(t_1) w - f(t_2) w\right)\right| \\ &\quad \le |\gamma|\,|i|\,\left|f(t_1) - f(t_2)\right|\,|w| \\ &\quad \le |\gamma|\,|\gamma^{-1}|\,|i|\,\left|f(t_1) - f(t_2)\right|\,|\pi|\,|p(\gamma, w)|.\end{aligned}$$

Because p is surjective we have

$$\left|F(t_1) - F(t_2)\right| \le \sup_{\gamma \in \Gamma} \left\{|\gamma|\,|\gamma^{-1}|\,|i|\,|\pi|\,\left|f(t_1) - f(t_2)\right|\right\}.$$

So $[0,1] \ni t \mapsto F(t) \in \mathscr{L}(V)$ is continuous. Note that $f(t)$ is invertible and hence so is $F(t)$. Therefore $A = F(1) \in \mathscr{L}_\Gamma(V)^\circ$.

The preceding argument shows that condition (a) of Definition 2.1 holds for Φ_0^{-1}. Other conditions can be proved in a similar way and consequently $\Phi \in \mathrm{Hom}(\mathscr{S}_\rho, \mathscr{S}_{\rho'})$ is an isomorphism. ∎

Remark 2.7. In fact we have shown that Φ and Φ^{-1} defined previously satisfy stronger conditions than those required in Definition 2.1 because $(\tilde{S}, \tilde{X}, \Lambda)$, $(\tilde{S}, \tilde{X}, \Lambda, A)$ and B do not depend on the concrete germs, namely, the following diagrams commute

$$\begin{tikzcd} \vec{\mathscr{E}}_{V,1}(\Gamma) \arrow[r, "\Phi_0"] \arrow[d, "{(S, X, \Lambda)}"'] & \vec{\mathscr{E}}_{W,1}(\Sigma) \arrow[d, "{(\tilde{S}, \tilde{X}, \Lambda)}"] \\ \vec{\mathscr{E}}_{V,1}(\Gamma) \arrow[r, "\Phi_0"] & \vec{\mathscr{E}}_{W,1}(\Sigma), \end{tikzcd} \tag{2.9}$$

$$\begin{tikzcd} \vec{\mathscr{E}}_{V,1+k}(\Gamma) \arrow[r, "\Phi_k"] \arrow[d, "{(S, X, \Lambda, A)}"'] & \vec{\mathscr{E}}_{W,1+k}(\Sigma) \arrow[d, "{(\tilde{S}, \tilde{X}, \Lambda, A)}"] \\ \vec{\mathscr{E}}_{V,1+k}(\Gamma) \arrow[r, "\Phi_k"] & \vec{\mathscr{E}}_{W,1+k}(\Sigma), \end{tikzcd} \tag{2.10}$$

$$\begin{tikzcd} \vec{\mathscr{E}}_{V,1+k}(\Gamma) \arrow[r, "\Phi_k"] \arrow[d, "B^*"'] & \vec{\mathscr{E}}_{W,1+k}(\Sigma) \arrow[d, "B^*"] \\ \vec{\mathscr{E}}_{V,1+k}(\Gamma) \arrow[r, "\Phi_l"] & \vec{\mathscr{E}}_{W,1+k}(\Sigma). \end{tikzcd} \tag{2.11}$$

COROLLARY 2.8. *Let $\rho\colon \Gamma \to \mathscr{L}(V)$ and $\rho'\colon \Sigma \to \mathscr{L}(W)$ be two isomorphic representations of Lie groups. Then the corresponding systems of singularities $\mathscr{S}_\rho$ and $\mathscr{S}_{\rho'}$ are isomorphic.*

Proof. According to the definition, two representations of groups are isomorphic if there is an isomorphism $h\colon \Sigma \to \Gamma$ of the corresponding groups and a linear isomorphism $A\colon W \to V$ such that

$$A(\sigma \cdot w) = h(\sigma) \cdot A(V).$$

We take $i = A$ and $\pi = A^{-1}$. For each $g \in \vec{\mathscr{E}}_{W,1+k}(\Sigma)$ and for each $T \in \overleftrightarrow{\mathscr{E}}_{W,1+k}(\Sigma)$ we take

$$G(v, \mu) = Ag\big(A^{-1}v, \mu\big),$$
$$S(v_1, \mu)v_2 = AT\big(A^{-1}v_1, \mu\big)A^{-1}v_2.$$

Then all the conditions of Theorem 2.6 are satisfied and hence $\mathscr{S}_\rho$ and $\mathscr{S}_{\rho'}$ are isomorphic. ∎

EXAMPLE 2.9. Let ρ be the action of the dihedral group $\mathbf{D}_3$ on the real plane $\mathbb{R}^2 \cong \mathbb{C}$,

$$\theta \cdot z := e^{i\theta}z, \qquad \theta = 0, \frac{\pi}{3}, \frac{2\pi}{3};$$
$$\kappa \cdot z := \bar{z},$$

and let ρ' be the action of the symmetric group $\mathbf{S}_3$ on the two-dimensional linear space of 3×3 real diagonal matrices with trace zero,

$$\sigma \cdot A := M_\sigma^{-1} A M_\sigma ,$$

where M_σ is the permutation matrix corresponding to $\sigma \in \mathbf{S}_3$. By [4, Lemma XV.6.3] these two actions are isomorphic and hence by Corollary 2.8 the corresponding systems of equivariant singularities are isomorphic.

3. ISOMORPHISM BETWEEN SYSTEMS OF SINGULARITIES EQUIVARIANT UNDER ORTHOGONAL GROUP ACTIONS

In this section we consider $\mathbf{O}(n)$-equivariant singularities in its irreducible representation on $\mathbb{R}^n$,

$$\rho_n : \mathbf{O}(n) \hookrightarrow \mathscr{GL}(\mathbb{R}^n), \qquad \gamma \cdot x = \gamma x, \qquad \forall \gamma \in \mathbf{O}(n),\, x \in \mathbb{R}^n.$$

In the case of $n = 1$ the representation is

$$\rho_1 : \mathbb{Z}_2 = \mathbf{O}(1) \to \{\pm 1\}, \qquad (-1) \cdot x = -x, \qquad \forall x \in \mathbb{R}.$$

We prove that $\mathscr{S}_{\rho_n}$ is isomorphic to $\mathscr{S}_{\rho_1}$ for any $n \in \mathbb{N}$.

Remark 3.1. In this section we take $V = \mathbb{R}^n$ and $W = \mathbb{R}$ and replace all the spaces V, W appearing in the subscripts of the notations introduced in the previous section with their dimensions.

Let $h : \mathbb{Z}_2 \to \mathbf{O}(n)$ be the group homomorphism,

$$h(\sigma) = \begin{pmatrix} \sigma & 0 \\ 0 & I_{n-1} \end{pmatrix}, \qquad \sigma = \pm 1.$$

Let $i : \mathbb{R} \to \mathbb{R}^n$ and $\pi : \mathbb{R}^n \to \mathbb{R}$ be, respectively, the embedding and the projection,

$$i(y) = (y, 0, \ldots, 0), \qquad \forall y \in \mathbb{R}, \tag{3.1}$$

$$\pi(x) = x_1, \qquad \forall x = (x_1, \ldots, x_n) \in \mathbb{R}^n. \tag{3.2}$$

Define $\Phi_k : \vec{\mathscr{E}}_{n,1+k}(\mathbf{O}(n)) \to \vec{\mathscr{E}}_{1,1+k}(\mathbb{Z}_2)$ as

$$\Phi_k(G)(y, \mu) = \pi G(i(y), \mu), \qquad k = 0, 1, 2, \ldots . \tag{3.3}$$

Because $i(\mathbb{R}) = \mathrm{Fix}(\Delta)$, where

$$\Delta = \left\{ I_n, \begin{bmatrix} 1 & 0 \\ 0 & -I_{n-1} \end{bmatrix} \right\},$$

by Theorem 2.6 $\{\Phi_k\}_{k=0}^{\infty} \in \mathrm{Hom}(\mathscr{S}_{\rho_n}, \mathscr{S}_{\rho_1})$.

Let ρ: $\mathbf{O}(n) \times \mathbb{R} \to \mathbb{R}^n$ be the map defined by

$$p(\gamma, y) = \gamma i(y), \qquad \forall \gamma \in \mathbf{O}(n), \quad \text{and} \quad y \in \mathbb{R}. \tag{3.4}$$

We have

LEMMA 3.2. (a) *The map p is surjective.*

(b) *Let k be a nonnegative integer. Then every $g \in \vec{\mathscr{E}}_{n,1+k}(\mathbf{O}(n))$ can be written in the form,*

$$g(x, \mu) = r\left(|x|^2, \mu\right)x, \tag{3.5}$$

where $r \in \mathscr{E}_{1,1+k}$.

(c) *For each germ $g \in \vec{\mathscr{E}}_{1,1+k}(\mathbb{Z}_2)$ there exists a unique germ $G \in \vec{\mathscr{E}}_{n,1+k}(\mathbf{O}(n))$ such that the following diagram commutes*

$$\begin{tikzcd} \mathbf{O}(n) \times \mathbb{R} \times \mathbb{R}^{1+k} \arrow[r, "\mathrm{id}_{\mathbf{O}(n)} \times g"] \arrow[d, "p \times \mathrm{id}_{\mathbb{R}^{1+k}}"'] & \mathbf{O}(n) \times \mathbb{R} \arrow[d, "p"] \\ \mathbb{R}^n \times \mathbb{R}^{1+k} \arrow[r, "G"] & \mathbb{R}^n. \end{tikzcd} \tag{3.6}$$

(d) *For each $T \in \overleftrightarrow{\mathscr{E}}_{1,1+k}(\mathbb{Z}_2)$ there exists a germ $S \in \overleftrightarrow{\mathscr{E}}_{n,1+k}(\mathbf{O}(n))$ such that*

$$S(p(\gamma, y), \mu)p(\gamma, y') = \gamma i(T(y, \mu)y'),$$
$$\forall y, y' \in \mathbb{R}, \quad \textit{and} \quad \gamma \in \mathbf{O}(n).$$

Proof. (a) Because $\mathbf{O}(n)$ acts transitively on any sphere centered at the origin it follows that for each $x \in \mathbb{R}^n$ there exists an orthogonal transformation $\gamma \in \mathbf{O}(n)$ such that $\gamma x = i(|x|)$ and hence,

$$x = p\left(\gamma^{-1}, |x|\right).$$

This shows that p is surjective.

(b) The conclusion is well known for the special case $n = 1$ ([3]). We suppose $n > 1$. By (a), $i(\mathbb{R})$ is invariant under g and it is easy to see that the restriction of g on $i(\mathbb{R})$ commutes the action of the group,

$$\mathbb{Z}_2 = \left\{ \begin{bmatrix} \pm 1 & 0 \\ 0 & I_{n-1} \end{bmatrix} \right\}.$$

Hence there is a germ $r \in \mathscr{E}_{1,1+k}$ such that $g(x, \mu) = r(|x|^2, \mu)x$ for all $x \in i(\mathbb{R})$ and $\mu \in \mathbb{R}^{1+k}$. By (a) for each $x \in \mathbb{R}^n$ there is a $\gamma \in \mathbf{O}(n)$ such that $x = p(\gamma, |x|)$. Then we have

$$\begin{aligned} g(x, \mu) &= g(p(\gamma, |x|), \mu) \\ &= \gamma g(i(|x|), \mu) \\ &= \gamma r\left(|x|^2, \mu\right) i(|x|) \\ &= r\left(|x|^2, \mu\right) x. \end{aligned}$$

(c) Let g be any germ in $\vec{\mathscr{E}}_{1,1+k}(\mathbb{Z}_2)$. Then by (b) there is a germ $r \in \mathscr{E}_{1,1+k}$ such that

$$g(y, \mu) = r\left(y^2, \mu\right) y.$$

Define G: $(\mathbb{R}^n \times \mathbb{R}^{1+k}, 0) \to \mathbb{R}^n$,

$$G(x, \mu) = r\left(|x|^2, \mu\right) x.$$

Then $G \in \vec{\mathscr{E}}_{n,1+k}(\mathbf{O}(n))$ and

$$G(p(\gamma, y), \mu) = \gamma r\left(y^2, \mu\right) i(y) = p(\gamma, g(y, \mu)),$$

i.e., diagram (2.4) commutes. The uniqueness can be derived from the following claim.

Claim. For any $y_1, y_2 \in \mathbb{R}$ and for any $\gamma_1, \gamma_2 \in \mathbf{O}(n)$,

$$p\left(\gamma_1, h(\gamma_1, \mu)\right) = p\left(\gamma_2, h(y_2, \mu)\right), \qquad \forall \mu \in \mathbb{R}^{1+k}, \tag{3.7}$$

provided

$$p(\gamma_1, y_1) = p(\gamma_2, y_2). \tag{3.8}$$

Now we prove the claim. By Eq. (3.8) we have $|y_1| = |y_2|$. For the case when $y_1 = y_2 = 0$ Eq. (3.7) holds because the $\mathbb{Z}_2$-equivariance of h ensures that $h(0, \mu) = 0$, and for the case when y_1 and y_2 are nonzero we have

$$\frac{1}{y_1} h(y_1, \mu) = \frac{1}{y_2} h(y_2, \mu),$$

for all $\mu \in \mathbb{R}^{1+k}$ and hence,

$$\begin{aligned} p(\gamma_1, h(y_1, \mu)) &= \gamma_1 i(h(y_1, \mu)) \\ &= \frac{1}{y_2} h(y_2, \mu)\gamma_1 i(y_1) \\ &= \frac{1}{y_2} h(y_2, \mu)\gamma_2 i(y_2) \\ &= p(\gamma_2, h(y_2, \mu)). \end{aligned}$$

(d) Let T be any germ in $\overleftrightarrow{\mathscr{E}}_{1,1+k}(\mathbb{Z}_2)$. It is known that

$$T(y, \mu) = r(y^2, \mu)\mathrm{id}_{\mathbb{R}}$$

for some germ $r \in \mathscr{E}_{1,1+k}$. Define S: $(\mathbb{R}^n \times \mathbb{R}^{1+k}, 0) \to \mathscr{L}(\mathbb{R}^n)$,

$$S(x, \mu) = r(|x|^2, \mu)I_n.$$

It is easy to see that $S \in \overleftrightarrow{\mathscr{E}}_{n,1+k}(\mathbf{O}(n))$ and

$$S(p(\gamma, y), \mu)p(\gamma, y') = \gamma i(T(y, \mu)y').$$

∎

The following theorem is a corollary of Lemma 3.2 and Theorem 2.6.

THEOREM 3.3. *Let* Φ_k *be defined by Eq.* (3.3) *for any* $k \in \{0\} \cup \mathbb{N}$. *Then* $\Phi = \{\Phi_k\}_{k=0}^{\infty}$ *is an isomorphism in* $\mathrm{Hom}(\mathscr{S}_{\rho_n}, \mathscr{S}_{\rho_1})$. ∎

It is easy to see that if $g \in \overrightarrow{\mathscr{E}}_{n,1+k}(\mathbf{O}(n))$ and $g(x, \mu) = r(|x|^2, \mu)x$, $\forall (x, \mu) \in (\mathbb{R}^n \times \mathbb{R}^{1+k}, 0)$, then $(\Phi_k g)(y, \mu) = r(y^2, \mu)y$, $\forall (y, \mu) \in (\mathbb{R} \times \mathbb{R}^{1+k}, 0)$. Conversely, if $h \in \overrightarrow{\mathscr{E}}_{1,1+k}(\mathbb{Z}_2)$ and $h(y, \mu) = r(y^2, \mu)y$, $\forall (y, \mu) \in (\mathbb{R} \times \mathbb{R}^{1+k}, 0)$, then $(\Phi_k^{-1} h)(x, \mu) = r(|x|^2, \mu)x$, $\forall (x, \mu) \in (\mathbb{R}^n \times \mathbb{R}^{1+k}, 0)$. Then by Theorems 2.5, 3.3, and properties of the linear isomorphisms defined in Eq. (3.3) we have

THEOREM 3.4. (a) *Let* $f, g \in \overrightarrow{\mathscr{E}}_{n,1}(\mathbf{O}(n))$. *Then* f *and* g *are* $\mathbf{O}(n)$*-equivalent if and only if* $\Phi_0 f$ *and* $\Phi_0 g$ *are* $\mathbb{Z}_2$*-equivalent*; *The codimension of any* $f \in \overrightarrow{\mathscr{E}}_{n,1}(\mathbf{O}(n))$ *is the same as that of* $\Phi_0 f \in \overrightarrow{\mathscr{E}}_{1,1}(\mathbb{Z}_2)$.

(b) $r(|x|^2, \lambda)x$ *is a normal form in* $\overrightarrow{\mathscr{E}}_{n,1}(\mathbf{O}(n))$ *if and only if* $r(y^2, \lambda)y$ *is a normal form in* $\overrightarrow{\mathscr{E}}_{1,1}(\mathbb{Z}_2)$, *where* $r \in \mathscr{E}_{1,1}$.

(c) $r(|x|^2, \lambda, \alpha)x$ *is a universal* $\mathbf{O}(n)$*-unfolding of the* $\mathbf{O}(n)$*-equivariant germ* $r(|x|^2, \lambda, 0)x$ *in* $\overrightarrow{\mathscr{E}}_{n,1}(\mathbf{O}(n))$ *if and only if* $r(y^2, \lambda, \alpha)y$ *is a universal* $\mathbb{Z}_2$*-unfolding of the corresponding* $\mathbb{Z}_2$*-equivariant germ* $r(y^2, \lambda, 0)y \in \overrightarrow{\mathscr{E}}_{1,1}(\mathbb{Z}_2)$, *where* $r \in \mathscr{E}_{1,1+k}$. ∎

In [8, 9] the normal forms and universal unfoldings for $\mathbf{O}(n)$-equivariant singularities in its irreducible representation on $\mathbb{R}^n$ with $\mathbf{O}(n)$-codimension less than 5 were obtained by direct calculations and they share the same expressions, as those for one-dimensional $\mathbb{Z}_2$-equivariant singularities previously obtained in [2, 7]. These observations are formulated more exactly by Theorem 3.4.

4. ISOMORPHISM BETWEEN $\frac{1}{2}n(n+1)$-DIMENSIONAL $\mathbf{O}(n)$-EQUIVARIANT SINGULARITIES AND n-DIMENSIONAL $\mathbf{S}_n$-EQUIVARIANT SINGULARITIES

Let $n \geq 2$ and $m := \frac{1}{2}n(n+1)$. Let V be the m-dimensional space of $n \times n$ real symmetric matrices and ρ: $\mathbf{O}(n) \to \mathscr{GL}(V)$ be the representation by the similarity,

$$\gamma \cdot A := \rho(\gamma)A := \gamma^{-1}A\gamma, \qquad \forall \gamma \in \mathbf{O}(n), \quad \text{and} \quad \forall A \in V. \quad (4.1)$$

Let $\mathbf{S}_n$ be the permutation group of $\{1, 2, \dots, n\}$ and let ρ': $\mathbf{S}_n \to \mathscr{GL}(\mathbb{R}^n)$ be the representaiton given by

$$\sigma \cdot x := \rho'(\sigma)x := (x_{\sigma(1)}, \dots, x_{\sigma(n)}),$$
$$\forall x = (x_1, \dots, x_n) \in \mathbb{R}^n, \quad \text{and} \quad \forall \sigma \in \mathbf{S}_n. \quad (4.2)$$

In this section we prove for these two representations the corresponding systems of equivariant singularities $\mathscr{S}_\rho$ and $\mathscr{S}_{\rho'}$ are isomorphic.

For $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n) \in \mathbb{R}^n$ and $k \in \mathbb{N}$ we write

$$xy := (x_1 y_1, \dots, x_n y_n),$$
$$x^k := (x_1^k, \dots, x_n^k),$$
$$x^0 := 1_n := (1, \dots, 1).$$

LEMMA 4.1. *Write $s_j(x) = x_1^j + \cdots + x_n^j$ for $x = (x_1, \dots, x_n)$ and $1 \leq j \leq n$. Then*

(a) *for each $\mathbf{S}_n$-invariant C^∞ germ f: $(\mathbb{R}^n, 0) \to \mathbb{R}$ there exists a C^∞ germ q: $(\mathbb{R}^n, 0) \to \mathbb{R}$ such that*

$$f(x) = q(s_1(x), \dots, s_n(x)); \quad (4.3)$$

(b) *for each $\mathbf{S}_n$-equivariant C^∞ germ g: $(\mathbb{R}^n, 0) \to \mathbb{R}^n$ there exist C^∞ germs q_j: $(\mathbb{R}^n, 0) \to \mathbb{R}$ for $j = 0, \dots, n-1$, such that*

$$g(x) = \sum_{j=0}^{n-1} q_j(s_1(x), \dots, s_n(x))x^j; \quad (4.4)$$

(c) *for each* $\mathbf{S}_n$*-equivariant* C^∞ *(matrix-valued) germ* S: $(\mathbb{R}^n, 0) \to \mathscr{L}(\mathbb{R}^n)$ *there exist* C^∞ *germs* $q_0, \ldots, q_{n-1}, q_{0,0}, \ldots, q_{n-1,n-2}$: $(\mathbb{R}^n, 0) \to \mathbb{R}$ *such that*

$$S(x)y = \sum_{k=0}^{n-1} q_k(s_1(x), \ldots, s_n(x))x^k y + \sum_{k=0}^{n-1} \sum_{l=0}^{n-2} \sum_{j=1}^{n} q_{k,l}(s_1(x), \ldots, s_n(x))x_j^l y_j x^k. \quad (4.5)$$

Proof. By Schwarz's Theorem ([4, Theorem XII.4.3]) and Poénaru's Theorem ([4, Theorem XII.5.3]) it is sufficient to prove that these conclusions are valid in the special case that all the smooth germs mentioned in this lemma are polynomials.

(a) The conclusion follows from the fact that all $\mathbf{S}_n$-invariant polynomials can be written as polynomials in $s_1, \ldots, s_n$ (see [6, Theorem 4.3.7]).

(b) Let $g = (g_1, \ldots, g_n)$: $\mathbb{R}^n \to \mathbb{R}^n$ be an $\mathbf{S}_n$-equivariant polynomial. By the $\mathbf{S}_n$-equivariance of g there exists a polynomial r such that

$$g_j(x) = r\big(x_j, s_1(x) - x_j, s_2(x) - x_j^2, \ldots, s_{n-1}(x) - x_j^{n-1}\big), \quad j = 1, \ldots, n. \quad (4.6)$$

Hence Eq. (4.4) follows from the following claim.

Claim. For each polynomial h in n indeterminates there exist n polynomials $q_0, \ldots, q_{n-1}$ such that

$$r(x_1, \tilde{s}_1(x), \tilde{s}_2(x), \ldots, \tilde{s}_{n-1}(x)) = \sum_{j=0}^{n-1} q_j(s_1(x), \ldots, s_n(x))x_1^j, \quad (4.7)$$

where $\tilde{s}_j(x) = s_j(x) - x_1^j$ $(j = 1, \ldots, n-1)$.

We prove the claim by induction on the degree of r. If $\deg(r) = 0$ we take $q_0 = r$ and $q_1 = \cdots = q_{n-1} = 0$ and Eq. (4.7) holds. Suppose that our claim is valid for all polynomials of degree less than m and let r be a polynomial of degree m. We may write r as

$$r(y_1, \ldots, y_n) = c + y_1 r_1(y_1, \ldots, y_n) + \cdots + y_n r_n(y_1, \ldots, y_n),$$

where c is a constant and $r_1, \ldots, r_n$ are polynomials of degree less than m. Hence,

$$\begin{aligned} r(x_1, \tilde{s}_1(x), \ldots, \tilde{s}_{n-1}(x)) &= c + x_1 r_1(x_1, \tilde{s}_1(x), \ldots, \tilde{s}_{n-1}(x)) \\ &\quad + \sum_{j=1}^{n-1} \big(s_j(x) - x_1^j\big) r_{j+1}(x_1, \tilde{s}_1(x), \ldots, \tilde{s}_{n-1}(x)). \end{aligned} \quad (4.8)$$

However, as we supposed, the claim is valid for $r_1, \ldots, r_n$. Note that

$$x_1^n - a_1(x)x_1^{n-1} + \cdots + (-1)^{n-1}a_{n-1}(x)x_1 + (-1)^n a_n(x) = 0,$$

where $a_1, \ldots, a_n$ are the primitive symmetric polynomials in n indeterminates, i.e.,

$$a_j(x) = \sum_{1 \le i_1 < \cdots < i_j \le n} x_{i_1} \cdots x_{i_j}, \qquad j = 1, 2, \ldots, n.$$

By (a) $a_1, \ldots, a_n$ can be expressed as polynomials in $s_1, \ldots, s_n$. It is not difficult to see that each term in the right-hand side of Eq. (4.8) can be written in the form of a linear sum of $1, x_1, \ldots, x_1^{n-1}$ with n polynomial in $s_1, \ldots, s_n$ as the coefficients. By the principle of mathematical induction the claim is true for all polynomials in n indeterminates.

(c) Let $S\colon \mathbb{R}^n \to \mathscr{L}(\mathbb{R}^n)$ be a matrix-valued $\mathbf{S}_n$-equivariant polynomial map. Then there are polynomials a and b such that

$$[S(x)]_{i,i} = a(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}),$$
$$[S(x)]_{i,j} = b(x_{\tau(1)}, x_{\tau(2)}, \ldots, x_{\tau(n)}),$$

where $1 \le i, j \le n$, $i \ne j$, σ is the transposition $(1, i)$, and τ is the product of transpositions $(1, i)$ and $(2, j)$. Hence $a(x_1, x_2, \ldots, x_n)$ is invariant under permutations of $\{2, 3, \ldots, n\}$ and $b(x_1, x_2, \ldots, x_n)$ is invariant under permutations of $\{3, \ldots, n\}$. Therefore by the claim which we state in the proof of (b) there are polynomials $\tilde{q}_0, \ldots, \tilde{q}_{n-1}, q_{0,0}, \ldots, q_{n-1,n-2}$, and $r_{l,k}, 0 \le k < n$, $l = 1, 2, \ldots,$ such that

$$a(x_1, \ldots, x_n) = \sum_{k=0}^{n-1} \tilde{q}_k(s_1(x), \ldots, s_n(x))x_1^k,$$

$$b(x_1, \ldots, x_n) = \sum_{k=0}^{n-1} \sum_{l=0}^{n-2} q_{k,l}(s_1(x), \ldots, s_n(x))x_1^k x_2^l,$$

$$x_i^l = \sum_{k=0}^{n-1} r_{l,k}(s_1(x), \ldots, s_n(x))x_i^k.$$

So we have

$$S(x)y = \sum_{k=0}^{n-1} q_k(s_1(x), \ldots, s_n(x))x^k y$$
$$+ \sum_{k=0}^{n-1} \sum_{l=0}^{n-2} \sum_{j=1}^{n} q_{k,l}(s_1(x), \ldots, s_n(x))x_j^l y_j x^k,$$

where $q_k = \tilde{q}_k - \sum_{j=0}^{n-1} \sum_{l=0}^{n-2} \tilde{q}_{j,l} r_{j+l,k}$. ∎

Let $h\colon \mathbf{S}_n \to \mathbf{O}(n)$ be the group homomorphism which maps each permutation σ to the $n \times n$ matrix M_σ whose entry at the cross of ith row and jth column is $\delta_{\sigma(i), j}$. Define $i\colon \mathbb{R}^n \to V$, $\pi\colon V \to \mathbb{R}^n$, and $p\colon \mathbf{O}(n) \times \mathbb{R}^n \to V$,

$$i(x_1, \dots, x_n) = \operatorname{diag}(x_1, \dots, x_n); \tag{4.9}$$

$$\pi\big((a_{i,j})_{n\times n}\big) = (a_{1,1}, \dots, a_{n,n}); \tag{4.10}$$

$$p(\gamma, x) = \gamma \cdot i(x). \tag{4.11}$$

For any nonnegative integer k define $\Phi_k\colon \vec{\mathscr{E}}_{m,1+k}(\mathbf{O}(n)) \to \vec{\mathscr{E}}_{n,1+k}(\mathbf{S}_n)$ as

$$\Phi_k(G)(x, \mu) = \pi(G(i(x), \mu)),$$

$$\forall x \in \mathbb{R}^n,\ \mu \in \mathbb{R}^{1+k}, \quad \text{and} \quad G \in \vec{\mathscr{E}}_{m,1+k}(\mathbf{O}(n)). \tag{4.12}$$

LEMMA 4.2. (a) $i(\mathbb{R}^n) = \operatorname{Fix}(T)$, *where* $T \subset \mathbf{O}(n)$ *is the subgroup of all real* $n \times n$ *diagonal matrices whose entries on the diagonal are* $+1$ *or* -1.

(b) $i\colon \mathbb{R}^n \to V$ *and* $\pi\colon V \to \mathbb{R}^n$ *satisfy* $\pi \circ i = \mathrm{id}_{\mathbb{R}^n}$ *and* $i(\sigma \cdot x) = h(\sigma) \cdot i(x)$ *for all* $\sigma \in \mathbf{S}_n$ *and* $x \in \mathbb{R}^n$. ∎

By this lemma and Theorem 2.6 we have $\{\Phi_k\}_{k=0}^{\infty} \in \operatorname{Hom}(\mathscr{S}_\rho, \mathscr{S}_{\rho'})$.

PROPOSITION 4.3. *Let k be a nonnegative integer. Then*

(a) $p\colon \mathbf{O}(n) \times \mathbb{R}^n \to V$ *is surjective.*

(b) *Every* $G \in \vec{\mathscr{E}}_{m,1+k}(\mathbf{O}(n))$ *can be written in the form,*

$$G(A, \mu) = q_0(\operatorname{tr}(A), \dots, \operatorname{tr}(A^n), \mu) I_n + \sum_{j=1}^{n-1} q_j(\operatorname{tr}(A), \dots, \operatorname{tr}(A^n), \mu) A^j, \tag{4.13}$$

where $\operatorname{tr}(A)$ *is the trace of A and* $q_i \in \mathscr{E}_{n,1+k}$, $0 \le i < n$.

(c) *Let g be any germ in* $\vec{\mathscr{E}}_{n,1+k}(\mathbf{S}_n)$. *Then there is a unique germ* $G \in \vec{\mathscr{E}}_{m,1+k}(\mathbf{O}(n))$ *such that the following diagram commutes*

$$\begin{array}{ccc} \mathbf{O}(n) \times \mathbb{R}^n \times \mathbb{R}^{1+k} & \xrightarrow{\mathrm{id}_{\mathbf{O}(n)} \times g} & \mathbf{O}(n) \times \mathbb{R}^n \\ \Big\downarrow{\scriptstyle p \times \mathrm{id}_{\mathbb{R}^{1+k}}} & & \Big\downarrow{\scriptstyle p} \\ V \times \mathbb{R}^{1+k} & \xrightarrow{\quad G \quad} & V. \end{array} \tag{4.14}$$

(d) *For any* $T \in \overleftrightarrow{\mathscr{E}}_{n,1+k}(\mathbf{S}_n)$ *there is an* $S \in \overleftrightarrow{\mathscr{E}}_{m,1+k}(\mathbf{O}(n))$ *such that*

$$S(p(\gamma, x), \mu) p(\gamma, y) = y \cdot i(T(x, \mu) y),$$

for all $\gamma \in \mathbf{O}(n)$ *and all* $x, y \in \mathbb{R}^n$.

Proof. (a) The conclusion is known as the fact that every real symmetric matrix can be diagonalized under orthogonal similar transformation.

(b) Let $G \in \vec{\mathscr{E}}_{m,1+k}(\mathbf{O}(n))$. Then $\Phi_k G \in \vec{\mathscr{E}}_{n,1+k}(\mathbf{S}_n)$. By Lemma 4.1(b) there are germs $q_0, \ldots, q_{n-1} \in \mathscr{E}_{n,1+k}$ such that

$$\Phi_k G(x, \mu) = q_0(s_1(x), \ldots, s_n(x), \mu) 1_n + \sum_{j=1}^{n-1} q_j(s_1(x), \ldots, s_n(x), \mu) x^j.$$

By (a) for each $A \in V$ there is a pair $(\gamma, x) \in \mathbf{O}(n) \times \mathbb{R}^n$ such that $A = \gamma i(x) \gamma^{-1}$. Hence we have

$$\begin{aligned} G(A, \mu) &= \gamma \cdot i \circ \pi G(i(x), \mu) \\ &= \gamma \Bigg[q_0(s_1(x), \ldots, s_n(x), \mu) I_n \\ &\qquad + \sum_{j=1}^{n-1} q_j(s_1(x), \ldots, s_n(x), \mu) i(x)^j \Bigg] \gamma^{-1} \\ &= q_0(\operatorname{tr}(A), \ldots, \operatorname{tr}(A^n), \mu) I_n \\ &\qquad + \sum_{j=1}^{n-1} q_j(\operatorname{tr}(A), \ldots, \operatorname{tr}(A^n), \mu) A^j. \end{aligned}$$

(c) Let $g \in \vec{\mathscr{E}}_{n,1+k}(\mathbf{S}_n)$. For any $A \in V$ there is a pair $(\gamma, x) \in \mathbf{O}(n) \times \mathbb{R}^n$ such that $A = p(\gamma, x)$. Suppose (γ, x) and (σ, y) are two such pairs, i.e., $A = p(\gamma, x) = p(\sigma, y)$. Then $s_j(x) = \operatorname{tr}(A^j) = s_j(y)$, for $j = 1, \ldots, n$. By Lemma 4.1(b) g can be written as

$$g(x, \mu) = q_0(s_1(x), \ldots, s_n(x), \mu) 1_n + \sum_{j=1}^{n-1} q_j(s_1(x), \ldots, s_n(x), \mu) x^j,$$

where $q_0, \ldots, q_{n-1} \in \mathscr{E}_{n,1+k}$. Then

$$\begin{aligned} p(\gamma, g(x, \mu)) &= \gamma \cdot \Bigg[q_0(s_1(x), \ldots, s_n(x), \mu) I_n \\ &\qquad + \sum_{j=1}^{n-1} q_j(s_1(x), \ldots, s_n(x), \mu) [i(x)]^j \Bigg] \\ &= q_0(\operatorname{tr}(A), \ldots, \operatorname{tr}(A^n), \mu) I_n \\ &\qquad + \sum_{j=1}^{n-1} q_j(\operatorname{tr}(A), \ldots, \operatorname{tr}(A^n), \mu) A^j \\ &= p(\sigma, g(y, \mu)). \end{aligned}$$

Hence there is a unique germ G: $(V \times \mathbb{R}^{1+k}, 0 \to V$ such that diagram (4.14) commutes. In fact,

$$G(A, \mu) = q_0(\mathrm{tr}(A), \dots, \mathrm{tr}(A^n), \mu) I_n + \sum_{j=1}^{n-1} q_j(\mathrm{tr}(A), \dots, \mathrm{tr}(A^n), \mu) A^j.$$

Therefore $G \in \vec{\mathscr{E}}_{m,1+k}(\mathbf{O}(n))$.

(d) For $T \in \overleftrightarrow{\mathscr{E}}_{n,1+k}(\mathbf{S}_n)$ it follows from Lemma 4.1(c) that

$$T(x, \mu)y = \sum_{k=0}^{n-1} q_k(s_1(x), \dots, s_n(x), \mu)x^k y + \sum_{k=0}^{n-1} \sum_{l=0}^{n-2} \sum_{j=1}^{n} q_{k,l}(s_1(x), \dots, s_n(x), \mu)x_j^l y_j x^k,$$

where $q_k, q_{k,l} \in \mathscr{E}_{n,1+k}$, $0 \le k < n$, $0 \le l \le n-2$. Define

$$S(A, \mu)B = \sum_{k=0}^{n-1} q_k(\mathrm{tr}(A), \dots, \mathrm{tr}(A^n), \mu) A^k B + \sum_{k=0}^{n-1} \sum_{l=0}^{n-2} q_{k,l}(\mathrm{tr}(A), \dots, \mathrm{tr}(A^n), \mu)\mathrm{tr}(A^l B) A^k.$$

It is easy to see that $S \in \overleftrightarrow{\mathscr{E}}_{m,1+k}(\mathbf{O}(n))$ and

$$S(p(\gamma, x), \mu)p(\gamma, y) = \gamma \cdot S(i(x), \mu)i(y) = \gamma \cdot i(T(x, \mu)y).$$

∎

The following theorem is a corollary of Theorem 2.6.

THEOREM 4.4. $\{\Phi_k\}_{k=0}^{\infty} \in \mathrm{Hom}(\mathscr{S}_\rho, \mathscr{S}_{\rho'})$ *is an isomorphism.* ∎

Remark 4.5. By Eqs. (4.1) and (4.2) it is easy to see that

$$V_0 = \{n \times n \text{ real symmetric matrices with trace zero}\}$$

is an $\mathbf{O}(n)$-invariant subspace of V with $\dim V_0 = \frac{1}{2}(n-1)(n+2)$ and

$$W_0 = \{x \in \mathbb{R}^n | x_1 + \cdots + x_n = 0\}$$

is an $\mathbf{S}_n$-invariant subspace of $\mathbb{R}^n$ with $\dim W_0 = n - 1$. Confined to these subspaces ρ and ρ' give representations in lower dimensions and $\{\Phi_k\}_{k=0}^{\infty}$

constructed in Eq. (4.12) also gives an isomorphism between the corresponding systems of equivariant singularities. For the case $n = 3$ this recovers a result obtained in [4].

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