Uniqueness results for semilinear polyharmonic boundary value problems on conformally contractible domains. II

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Abstract

We continue Part I of this paper on polyharmonic boundary value problems \((-\Delta)^m u = f(u)\) on \(\Omega \subset \mathbb{R}^n, m \in \mathbb{N}\), with Dirichlet boundary conditions. Here \(\Omega\) is a bounded or unbounded conformally contractible domain as defined in Part I. The uniqueness principle proved in Part I is applied to show the following theorems: if \(f(s) = \lambda s + |s|^{p-1}s, \lambda \leq 0\), with a supercritical \(p > (n + 2m)/(n - 2m)\) we extend the well-known non-existence result of Pucci and Serrin (Indiana Univ. Math. J. 35 (1986) 681–703) for bounded star-shaped domains to the wider class of bounded conformally contractible domains. We give two examples of domains in this class which are not star-shaped. In the case where \(1 < p < (n + 2m)/(n - 2m)\) is subcritical we give lower bounds for the \(L^\infty\)-norm of non-trivial solutions. For certain unbounded conformally contractible domains, \(1 < p < (n + 2m)/(n - 2m)\) subcritical and \(\lambda \geq 0\) we show that the only smooth solution in \(H^{2m-1}(\Omega)\) is \(u \equiv 0\). Finally, on a bounded conformally contractible domain uniqueness of non-trivial solutions for \(f(s) = \lambda(1 + |s|^{p-1}s), p > (n + 2m)/(n - 2m)\), supercritical and small \(\lambda > 0\) is proved. Solutions are critical points of a functional \(L\) on a suitable space \(X\). The theorems are proved by finding one-parameter groups of transformations on \(X\) which strictly reduce the values of \(L\). Then the uniqueness principle of Part I can be applied.

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1. Main uniqueness results

On the smooth domain $\Omega \subset \mathbb{R}^n$ we consider for $m \in \mathbb{N}$ the boundary value problem

\begin{equation}
(-\Delta)^m u = f(x,u) \quad \text{in } \Omega, \quad u = \cdots = D^{m-1}u = 0 \quad \text{on } \partial \Omega,
\end{equation}

where for a multi-index $l$ with $1 \leq |l| \leq m-1$ the expression $D^l u$ stands for the $|l|$th order derivatives of $u$. Only classical solutions $u \in C^2_m(\bar{\Omega})$ of (1) are considered. They are critical points of the following functional on a suitable function space $X$ defined later:

\[ L[u] = \int_{\Omega} \frac{1}{2} |D^m u|^2 - F(x,u) \, dx, \]

where

\[ D^m u = \begin{cases} \Delta^r u & \text{if } m = 2r, \\ \nabla \Delta^r u & \text{if } m = 2r + 1, \end{cases} \]

and $F(x,t) = \int_0^t f(x,s) \, ds$.

We describe the main uniqueness results of this paper. Let $2^* = \frac{2n}{n-2m}$ if $n > 2m$ and $2^* = \infty$ if $n \leq 2m$. Recall from Definition 1 and 2 of Part I that a vector-field $\xi = (\xi_1, \ldots, \xi_m)$ in $\mathbb{R}^n$ is called conformal if it satisfies

\begin{equation}
\partial_j \xi_i + \partial_i \xi_j = \frac{2}{n} (\text{div } \xi) \delta_{ij}, \quad i, j = 1, \ldots, n.
\end{equation}

Moreover, a domain $\Omega \subset \mathbb{R}^n$ with exterior normal $\nu(x)$ on $\partial \Omega$ is called conformally contractible if there exists a conformal vector-field $\xi$ such that $\xi(x) \cdot \nu(x) \leq 0$ on $\partial \Omega$ with strict inequality on a subset of $\partial \Omega$ of positive measure. The vector-field $\xi$ is called an associated vector-field to $\Omega$. For bounded conformally contractible domains the following Poincaré inequality holds.

**Lemma 1** (Weighted Poincaré inequality). Let $\Omega$ be a bounded conformally contractible domain with associated vector-field $\xi$ such that $\text{div } \xi \leq 0$ in $\Omega$. Then there exists a value $\tilde{\lambda}_1 > 0$ such that

\begin{equation}
\int_{\Omega} (-\text{div } \xi)|D^m u|^2 \, dx \geq \tilde{\lambda}_1 \int_{\Omega} (-\text{div } \xi) u^2 \, dx
\end{equation}

for all $u \in C^{m-1,1}_0(\Omega)$. If $\lambda_1$ denotes the first Dirichlet eigenvalue $\lambda_1$ of $(-\Delta)^m$ then the optimal value $\tilde{\lambda}_1$ in (3) satisfies $\tilde{\lambda}_1 \leq \lambda_1$ provided $n \geq 3$ or $n \geq 2$ and $m = 1$. For domains with $\text{div } \xi = \text{const} < 0$ one always has $\tilde{\lambda}_1 = \lambda_1$.

**Theorem 2.** Let $\Omega \subset \mathbb{R}^n$, $n > 2m$, be a smooth bounded conformally contractible domain with associated vector-field $\xi$ such that $\text{div } \xi \leq 0$ in $\Omega$. For $f(x,s) = \lambda s + |s|^{p-1}s$, $s \in \mathbb{R}$, problem (1) has only the zero solution if

\[ p \geq 2^*-1, \quad \lambda < \tilde{\lambda}_1 \frac{p(n-2m)-(n+2m)}{n(p-1)}. \]
The results also holds with strict inequality for \( p \) and equality permitted in \( \lambda \). Here \( \widetilde{\lambda}_1 \) is the weighted Poincaré constant from Lemma 1. For \( m = 1 \) the result holds if equality is permitted both for \( p \) and \( \lambda \).

**Remark.** In Section 3 we give an extension to a class of \( x \)-dependent non-linearities \( f(x,s) \).

For bounded star-shaped domains this result is due to Pohožaev [7] for \( m = 1 \) and Pucci and Serrin [8] for \( m \geq 1 \). For \( m = 1 \) the conformally contractible case was established by Reichel [9]. For \( m \geq 2 \) it is an open problem to include the case where both for \( p \) and \( \lambda \) equality in the above hypotheses holds, cf. [4]. For example, if \( \lambda = 0 \) and \( p = 2^* - 1 \) the only known cases are when \( \Omega = B_1(0) \) and \( u \) is positive and \( m \geq 2 \) (cf. [6]) or \( u \) is radial and \( m = 2, 3 \) (cf. [4]).

We also mention the work of Schaaf [11], where uniqueness results are given for \( m = 1 \) on a different class of domains for exponents \( p \geq p_c \), where \( p_c \) is larger than the critical Sobolev exponent. Also in the case \( m = 1 \) a completely different approach to uniqueness of positive solutions via maximum principles was found by Reichel and Zou [10].

**Theorem 3.** Let \( \Omega \subset \mathbb{R}^n \), \( n > 2m \), be a smooth bounded conformally contractible domain with associated vector-field \( \xi \) such that \( \text{div} \xi \leq 0 \) in \( \Omega \). Let \( f(x,s) = \lambda(1 + |s|^{p-1}s) \), \( s \in \mathbb{R} \), with \( p > 2^* - 1 \) and \( \lambda \geq 0 \).

(i) If \( m = 1 \) or \( m \geq 2 \) and \( \Omega = B_1(0) \) then there exists \( \widetilde{\lambda} > 0 \) such that (1) has a unique positive solution for \( \lambda \in [0, \widetilde{\lambda}] \).

(ii) Suppose \( p \geq 2 \). If \( m = 1 \) and \( \text{div} \xi < 0 \) in \( \tilde{\Omega} \) or \( m \geq 2 \) and no further restriction on \( \text{div} \xi \) then there exists \( \lambda > 0 \) such that (1) has a unique solution for \( \lambda \in [0, \lambda] \).

Under the same restrictions on \( n, m, p \) and \( \lambda \) the result holds for \( f(x,s) = 1 + \lambda |s|^{p-1}s \) and \( f(x,s) = \lambda e^s \). In particular (ii) holds for \( f(x,s) = \lambda e^s \) if \( n > 2m \).

**Remarks.** (a) Part (i) of the theorem generalizes to all those bounded domains \( \Omega \) where the positivity preserving property of \((-\Delta)^m\) holds.

(b) The problem \( \Delta^2 u = \lambda e^u \) in \( B_1(0) \) with Dirichlet boundary conditions has recently been studied by Arioli et al. [1].

Note that the uniqueness statement of part (ii) is stronger since it does not restrict to positive solutions. However, it requires \( p \) supercritical and \( p \geq 2 \) which only holds for dimensions \( n \in (2m, 6m] \). For bounded star-shaped domains and \( m = 1 \) the result of Theorem 3 was found by Schaaf [11]. A similar result for supercritical \( q \)-Laplacian boundary value problems on balls and \( q \geq 2 \) was found by Fleckinger and Reichel [3].

**Theorem 4.** Let \( \Omega \subset \mathbb{R}^n \), \( n \geq 3 \), be a smooth bounded conformally contractible domain with associated vector-field \( \xi \) such that \( \text{div} \xi \leq 0 \) in \( \Omega \). Let \( f(x,s) = \lambda s + |s|^{p-1}s \), \( s \in \mathbb{R} \), and let \( \lambda_1 \) be the weighted Poincaré constant from Lemma 1. Let \( u \) be a non-trivial solution of (1).
(i) If $1 < p < \infty$ and $\lambda < \tilde{\lambda}_1$ then
$$\|u\|_{p^{-1}\infty} \geq \tilde{\lambda}_1 - \lambda.$$  
For those domains, where $\tilde{\lambda}_1 = \lambda_1$ (the first Dirichlet eigenvalue of $(-\Delta)^m$), the estimate shows how the solution branch bifurcating at $\lambda = \lambda_1$ leaves the trivial solution.

(ii) If $1 < p < 2^* - 1$ and $\lambda < 0$ then
$$\|u\|_{p^{-1}\infty} \geq -\lambda \frac{2m(p+1)}{2n+(p+1)(2m-n)}.$$  
In the case $n > 2m$, $\lambda < 0$, the $L^\infty$-norm of any non-trivial solution blows up as $p \nearrow 2^* - 1$.

Finally, for unbounded domains and subcritical non-linearities, we have the following uniqueness result. This theorem is dual to Theorem 2 in the following sense: for bounded conformally contractible domains supercritical non-linearities create uniqueness, whereas for unbounded conformally contractible domains subcritical non-linearities create uniqueness.

**Theorem 5.** Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$, be a smooth unbounded conformally contractible domain with associated vector-field $\xi$ such that $\text{div} \xi \geq 0$. Let $f(x,s) = \lambda s + |s|^{p-1}s$, $s \in \mathbb{R}$, with $1 < p < 2^* - 1$, $\lambda \geq 0$ or $1 < p \leq 2^* - 1$, $\lambda > 0$.

(i) If $\text{div} \xi = \text{const}$ in $\Omega$ then $u \equiv 0$ is the only smooth $H^{2m-1}(\Omega) \cap L^{p+1}(\Omega)$-solution of (1).

(ii) If only $\text{div} \xi \geq 0$ is imposed then $u \equiv 0$ is the only smooth solution in the class
$$\int_\Omega (1 + |x|)|u|^{p+1} dx, \int_\Omega (1 + |x|)|D^\gamma u|^2 dx < \infty, \quad \forall \gamma \text{ with } |\gamma| \leq 2m - 1.$$  

**Remarks.** (a) For $m = 1$ and $f(x,s) = |s|^{4/(n-2)}s$ the result holds without the sign-restriction on $\text{div} \xi$.

(b) In Section 6 an extension to $x$-dependent non-linearities $f(x,s)$ is given.

(c) In the case $m = 1$ the result of Theorem 5 was established by Reichel and Zou [10] for arbitrary positive solutions on the complement of bounded star-shaped domains in $\mathbb{R}^n$ without any integrability assumption.

**Comments on the case $n = 2$.** The dimension $n = 2$ is only of interest in Theorems 4 and 5. It was shown in Part I that all simply connected domains in $n = 2$ are conformally contractible. Since it is well known that the Laplacian operator commutes with conformal maps in dimension $n = 2$ it is therefore not surprising that for $m = 1$ both theorems continue to hold, if in part (ii) of Theorem 5 the weight $(1 + |x|)$ is replaced by $(1 + \text{div} \xi)$. For $m \geq 2$ however, both theorems only hold for the restricted class of conformally contractible domains, where $\text{div} \xi$ is a linear function, i.e., $\xi$ is of the type described in part (b) of Lemma 7, Part I. No changes in the theorems are necessary.
All four theorems have one common source: the uniqueness principle of Part I of this paper. If a one-parameter transformation group \( g_\epsilon : X \to X \) acts on \( \mathcal{L} \) by strictly reducing its values then the transformation group is called a variational subsymmetry. As shown in Part I variational subsymmetries ensure uniqueness of the critical point of \( \mathcal{L} \), cf. Section 2 for more details.

Function spaces. We use the same function spaces as in Part I. Let \( C^m(\tilde{\Omega}) \) be the subspace of \( C^m(\bar{\Omega}) \) consisting of those functions satisfying Dirichlet boundary conditions of order \( m-1 \), i.e., \( u = \cdots = D^{m-1}u = 0 \) on \( \partial \Omega \). By \( H^k(\Omega), k \in \mathbb{N} \), we denote the space of \( L^2 \)-functions having \( L^2 \)-integrable derivatives of order up to \( k \). The norm in \( H^k(\Omega) \) is

\[
\|u\|_{H^k} = \sum_{|\gamma| \leq k} \left( \int_{\Omega} |D^\gamma u|^2 \, dx \right)^{1/2}.
\]

If an additional positive weight function \( \omega \) is introduced then the weighted spaces are denoted by \( H^k(\Omega ; \omega) \).

If \( \Omega \) is bounded then a suitable function space, on which the functional \( \mathcal{L} \) is still well defined, is the space \( X = C^{m-1,1}_0(\bar{\Omega}) \) of functions with continuous derivatives of order up to \( m-1 \), such that the derivatives of order \( m-1 \) are Lipschitz continuous, and the Dirichlet condition of order \( m-1 \) on \( \partial \Omega \) holds. In the cases considered in Theorem 5 where \( \Omega \) is unbounded, the functional is well defined on the space \( X = C^{m-1,1}_0(\bar{\Omega}) \cap H^m(\Omega) \cap L^{p+1}(\Omega) \).

Notation. We use standard multi-index notation \( \gamma = (\gamma_1, \ldots, \gamma_n) \), \( \gamma_i \in \mathbb{N}_0 \), with \( |\gamma| = \gamma_1 + \cdots + \gamma_n \) and

\[
D^\gamma = \frac{\partial^{|\gamma|}}{\partial \gamma_1^{\gamma_1} \cdots \partial \gamma_n^{\gamma_n}}
\]

with \( \partial_i \) a shorthand for \( \partial/\partial x_i \). Note that \( D^\gamma \) only acts on the function immediately following the symbol, i.e., \( D^\gamma u D^\delta v \) stands for \( (D^\gamma u)(D^\delta v) \). This also holds, e.g., for \( \nabla D^\gamma \theta_i u \cdot \nabla D^\delta v = (\nabla D^\gamma \theta_i u) \cdot (\nabla D^\delta v) \). For a vector function \( \xi \) we write \( D\xi \) for the Jacobian matrix \( (\partial j \xi_i)_{ij} \), and for a scalar function \( \phi \) we write \( D^2\phi \) for its Hessian matrix \( (\partial^2 \phi)_{ij} \).

Organization of the paper is as follows. In Section 2 we recall the main uniqueness principle shown in Part I of this paper. The following 4 sections are devoted to the proof of the main results of Theorems 2–5. Finally, in Section 7 we prove the Poincaré inequality of Lemma 1.

2. The uniqueness principle of Part I

In Part I we have studied one-parameter family of maps

\[
G = \left\{ g_\epsilon : C^{m-1,1}_0(\bar{\Omega}) \to C^{m-1,1}_0(\bar{\tilde{\Omega}}) \right\}_{\epsilon \in \mathbb{R}}
\]

with the group-property \( g_{\epsilon_1} \circ g_{\epsilon_2} = g_{\epsilon_1 + \epsilon_2}, g_0 = \text{Id} \). They arise from the ODE system
\[
\dot{X} = \xi(X), \quad X(0) = x, \quad (4)
\]
\[
\dot{U} = \alpha U \text{div} \xi(X), \quad U(0) = u, \quad (5)
\]

where \( \alpha \in \mathbb{R} \) and \( \xi \) is a conformal vector-field in \( \mathbb{R}^n, \ n \geq 3 \). We denote by \( g_{\epsilon}(x, u) = (\chi_{\epsilon}(x), \psi_{\epsilon}(x, u)) \) the solution of (4)–(5) at time \( \epsilon \). Due to the explicit form of the conformal vector-fields, cf. Lemma 7 in Part I, we can integrate (4) explicitly. It is then easy to show that the solutions of (4)–(5) exist for all \( \epsilon \in \mathbb{R} \), and that if \( u \in C^m_0(\bar{\Omega}) \) then \( g_{\epsilon}(\Gamma_u) \) represents the graph of a new function \( g_{\epsilon}u \in C^m_0(\bar{\Omega}) \).

This implies the following formula for the transformed function \( \tilde{u}(\tilde{x}) \):

\[
\tilde{u}(\tilde{x}) = \psi_{\epsilon}(\text{Id} \times u)\left[\chi_{\epsilon}(\text{Id})\right]^{-1}(\tilde{x}), \quad \tilde{x} \in \tilde{\Omega}, \quad (6)
\]

which defines the map

\[
g_{\epsilon} : u \mapsto \tilde{u}
\]

for functions \( u \in C^{m-1,1}_0(\tilde{\Omega}) \) and for \( \epsilon \) in a small interval containing 0. The transformed function \( \tilde{u} \) is defined on the transformed domain \( \tilde{\Omega} = \chi_{\epsilon}(\Omega) \). For the transformed function we use the notation \( g_{\epsilon}u \) as well as \( \tilde{u}(\tilde{x}) \), and for its domain of definition we write \( g_{\epsilon}\bar{\Omega} \) as well as \( \tilde{\Omega} \). The essence of Sections 2–3 in Part I is as follows.

**Proposition 6.** Suppose \( \Omega \) is a smooth conformally contractible domain and let \( u \in C^m_0(\bar{\Omega}) \) or \( u \in C^{m-1,1}_0(\tilde{\Omega}) \). For \( \epsilon > 0 \) the transformed function \( g_{\epsilon}u \) is defined on \( g_{\epsilon}\bar{\Omega} \subset \Omega \). If we extend \( g_{\epsilon}u \) by 0 outside \( g_{\epsilon}\bar{\Omega} \) then \( g_{\epsilon}u \in C^{m-1,1}_0(\tilde{\Omega}) \).

**Remark.** Note that although starting in \( C^m_0(\bar{\Omega}) \) we only end in \( C^{m-1,1}_0(\tilde{\Omega}) \) since the Dirichlet conditions of order \( m - 1 \) and the extension by 0 only take care of the \( C^{m-1} \) smoothness of the transformed function.

We recall the main uniqueness result of Theorem 4 in Part I.

**Definition 7.** Let \( \Omega \) be a conformally contractible domain with associated vector field \( \xi \) and suppose the functional \( \mathcal{L} : X \equiv C^{m-1,1}_0(\tilde{\Omega}) \cap H^m(\Omega) \cap L^{p+1}(\Omega) \to \mathbb{R} \) is well defined. Let \( G = \{g_{\epsilon}\} \) be the transformation group generated by \( \xi \). The group \( G \) is called a strict variational subsymmetry for the functional \( \mathcal{L} \) if there exists a point \( u_0 \in X \) such that

\[
\frac{d}{d\epsilon}\mathcal{L}[g_{\epsilon}u] \bigg|_{\epsilon=0} < 0 \quad \text{for all} \ u \in X \setminus \{u_0\}.
\]

**Theorem 8** (Uniqueness principle). Let \( \Omega \) be a conformally contractible domain with associated vector-field \( \xi \) and let \( \mathcal{L} : X \to \mathbb{R} \) be as in Section 1. Suppose that the transformation group \( G = \{g_{\epsilon}\} \) generated by \( \xi \) is a strict variational subsymmetry with respect to \( u_0 \). Then \( u_0 \) is the only possible critical point of \( \mathcal{L} \) within the following class of functions:

(i) \( u \) smooth in case \( \Omega \) is bounded,
(ii) $F(x, u) \div \xi$, $f(x, u)u \div \xi$, $\xi \cdot \nabla_x F(x, u) \in L^1(\Omega)$ and $u \in H^{2m-1}(\Omega; \omega)$ if $\Omega$ is unbounded, where the weight $\omega = 1$ if $\div \xi =$ const and $\omega = (1 + |x|)$ if no restriction on $\div \xi$ is imposed.

For the application of Theorem 8 to a specific functional one has to verify a strict variational subsymmetry. In all of our applications this is done through the following rate of change formula.

**Theorem 9 (Rate of change formula).** Suppose $\Omega$ is a conformally contractible domain with associated vector field $\xi$. Let $G = \{ g_\epsilon \}_{\epsilon \geq 0}$ be the transformation group generated by $\xi$. Let $u \in C^m_0(\bar{\Omega}) \cap H^m(\Omega; \omega)$, where the weight $\omega = 1$ if $\div \xi =$ const and $\omega = (1 + |x|)$ if no restriction on $\div \xi$ is imposed. Then the rate of change of the functional $L$ under the transformation group can be computed as follows:

$$
\frac{d}{d\epsilon} L[g_\epsilon u] \bigg|_{\epsilon = 0} = \int_{\Omega} \left( \alpha - \frac{m}{n} + \frac{1}{2} \right) (\div \xi) |\mathcal{D}^m u|^2 \, dx \\
- \int_{\Omega} (uf(x, u)u + F(x, u)) \div \xi + \xi \cdot \nabla_x F(x, u) \, dx,
$$

provided the last volume integral exists. This is, e.g., the case for non-linearities $|f(x, s)| \leq C(1 + |s|^p)$ provided $u \in L^{p+1}(\Omega; \omega)$.

**3. Proof of Theorem 2**

For the proof of Theorem 2 we have $f(x, s) = \lambda s + |s|^{p-1}s$. We need to verify that (4)–(5) generate a strict variational subsymmetry w.r.t. $0$. We choose $\alpha = -1/(p + 1)$ and find from the rate of change formula (7) of Theorem 9 that

$$
\frac{d}{d\epsilon} L[g_\epsilon u] \bigg|_{\epsilon = 0} = \int_{\Omega} \left( \frac{1}{p + 1} + \frac{m}{n} - \frac{1}{2} \right) (\div \xi) |\mathcal{D}^m u|^2 \, dx \\
+ \int_{\Omega} \lambda \left( \frac{1}{2} - \frac{1}{p + 1} \right) u^2 (\div \xi) \, dx.
$$

By the super-criticality assumption the coefficient of $|\mathcal{D}^m u|^2(\div \xi)$ is non-positive. Hence, with the weighted Poincaré inequality from Lemma 1 we get

$$
\frac{d}{d\epsilon} L[g_\epsilon u] \bigg|_{\epsilon = 0} \leq \int_{\Omega} \left\{ \lambda_1 \left( \frac{1}{p + 1} + \frac{m}{n} - \frac{1}{2} \right) + \lambda \left( \frac{1}{2} - \frac{1}{p + 1} \right) \right\} u^2 (\div \xi) \, dx,
$$

where $\lambda_1$ is the Poincaré constant. The assumption of Theorem 2 implies that the coefficient in $\{\ldots\}$ is strictly negative. Hence the group $G = \{ g_\epsilon \}_{\epsilon \geq 0}$ is a strict variational subsymmetry w.r.t. $u_0 = 0$. Theorem 8 shows that $u \equiv 0$ is the only solution of (1).
Remark. There is a fundamental difficulty in extending the result of Theorem 2 to the equality case $\lambda = 0$ and $p = (n + 2m)/(n - 2m)$. In this case, by Theorem 14, Part I (Pohožaev’s identity), we obtain $D^m u = 0$ on a subset of $\partial \Omega$ of positive measure. For $m = 1$ the unique continuation theorem, cf. [5], shows $u \equiv 0$. For $m > 1$ the unique continuation theorem requires the vanishing of all derivatives up to order $2m - 1$. Except for the cases mentioned in Section 1 this gap has not been closed.

We give two examples of domains in $\mathbb{R}^n$, $n \geq 3$, which are not star-shaped but conformally contractible, and the associated vector-field $\xi$ satisfies $\text{div} \xi \leq 0$.

**Example 1.** The vector-field $\xi = (-x + y, -y - x, -z)$ with $\text{div} \xi = -3$ generates a composition of a dilation and a rotation in the $(x, y)$-plane. We construct a conformally contractible domain by extending the 2d-domain $\Omega_2$ cylindrically in the $z$-direction to a 3d-domain $\Omega_3$, cf. Fig. 1. In $\Omega_2$ the trajectories of $(\dot{x}, \dot{y}) = (-x + y, -y - x)$ starting from the boundary are shown. Notice that $\Omega_2$ is positively-invariant under the flow, i.e., $\Omega_2$ is conformally contractible in the plane. By cylindrical extension this remains true for $\Omega_3$.

In $n$ dimensions we extend $\Omega_2$ to $\Omega_n = \Omega_2 \times B_1^{n-2}(0)$, where $B_1^{n-2}(0)$ is an $(n - 2)$-dimensional ball of radius 1. The associated vector-field is $\xi = (-x_1 + x_2, -x_2 - x_1, -x_3, \ldots, -x_n)$ with $\text{div} \xi = -n$.

**Example 2.** The vector-field $\xi = (-2xz, -2yz, -z^2 + x^2 + y^2)$ with $\text{div} \xi = -6z$ generates a one-parameter group of conformal maps involving inversions. We construct a conformally contractible domain by rotating a planar domain around the $z$-axis. The flow $(\dot{x}, \dot{y}, \dot{z}) = \xi(x, y, z)$ is rotation-symmetric around the $z$-axis. Figure 2 shows the 2d-cut and the trajectories starting from the boundary. The 2d-domain $\Omega_2$ is positively-invariant, and by rotation-symmetry also the 3d-domain $\Omega_3$. Hence $\Omega_3$ is conformally contractible,
and since it lies in the region $z \geq 0$ we have $\text{div} \xi \leq 0$. Figure 3 shows $\Omega_3$ from above and below.

In $n$ dimensions we rotate the boundary curve around the $x_n$-axis; the (almost flat) basis will be in the $(x_1, \ldots, x_{n-1})$-hyperplane. The associated vector-field is $\xi = (-2x_1x_n, \ldots, -2x_{n-1}x_n, -x_n^2 + x_1^2 + \cdots + x_{n-1}^2)$ with $\text{div} \xi = -2nx_n$.

Theorem 2'. Let $\Omega \subset \mathbb{R}^n$, $n > 2m$, be a smooth bounded conformally contractible domain with associated vector-field $\xi$ such that $\text{div} \xi \leq 0$ in $\Omega$. Let $(\chi_\epsilon(x), \psi_\epsilon(x,s))$ be the solution to (4)–(5). Suppose

$$\frac{F(\chi_\epsilon(x), \psi_\epsilon(x,s))}{|\psi_\epsilon(x,s)|^{2n/(n-2m)}}$$

strictly increasing in $\epsilon$ for almost all $x, s \in \bar{\Omega} \times \mathbb{R}$. (8)

Then the only smooth solution of (1) is $u \equiv 0$. Examples for (8) are

(a) $f(x, s) = |\xi(x)|^\beta |s|^{\gamma-1}s$ with $\beta \geq 0$ and $\gamma > (n + 2m + 2\beta)/(n - 2m)$,
(b) $f(s)$ such that $F(s)/|s|^{2n/(n-2m)}$ strictly increasing in $s$.

The proof follows the lines of the proof of Theorem 2. We choose $\alpha = m/n - 1/2$. Then

$$\frac{d}{d\epsilon} L\left[\begin{array}{c} g_\epsilon u \\ u \end{array}\right]_{\epsilon=0} = -\int_{\Omega} \left(\frac{2m-n}{2n} f(x, u) + F(x, u)\right) \text{div} \xi + \xi \cdot \nabla_{x} F(x, u) \, dx.$$

Differentiating (8) w.r.t. $\epsilon$ at $\epsilon = 0$ we obtain

$$\xi \cdot \nabla_{x} F(x, s) + \text{div} \xi \left(F(x, s) + \frac{2m-n}{2n} f(x, s)s\right) > 0$$
for almost all \((x, s) \in \tilde{\Omega} \times \mathbb{R}\). This shows that we have a strict variational subsymmetry with respect to 0.

4. Proof of Theorem 3

We have \(f(x, s) = \lambda(1 + s^{(p)})\), \(s \in \mathbb{R}\), where for simplicity we use the notation \(t^{(p)} = |t|^{p-1}t\) for the odd \(p\)th power. For \(\lambda = 0\) problem (1) has the unique solution \(u \equiv 0\). For small \(\lambda \in [0, \tilde{\lambda}]\) let \(u_\lambda\) be the solution of (1) obtained from the implicit function theorem.

Lemma 10. (a) If \(m = 1\) or if \(m \geq 2\) and \(\Omega = B_1(0)\) then \(u_\lambda\) is the minimal positive solution.

(b) If \(U\) is the solution of \((-\Delta)^m U = 1\) in \(\Omega\) with Dirichlet boundary conditions, then \(\|u_\lambda - \lambda U\|_{C^{2m+\alpha}(\tilde{\Omega})} = O(\lambda^{p+1})\) as \(\lambda \to 0\).

Proof. (a) Under the conditions of Theorem 3 the operator \((-\Delta)^m\) is positivity preserving, i.e., if \((-\Delta)^m w = f\) in \(\Omega\) with Dirichlet boundary conditions on \(\partial \Omega\) and \(f \geq 0\), \(f \not\equiv 0\), then \(w > 0\) in \(\Omega\), cf. [2] for \(m \geq 2\) on balls. By the implicit function theorem \(\|u_\lambda\|_{C^{2m+\alpha}(\tilde{\Omega})} \to 0\) as \(\lambda \to 0\). So for small \(\lambda > 0\) we have \((-\Delta)^m u_\lambda > 0\) in \(\Omega\), and hence \(u_\lambda > 0\) in \(\Omega\). Now fix \(\lambda > 0\). We can start the monotone iteration scheme with the subsolution \(u_0 = 0\) and define \((-\Delta)^m u_{k+1} = \lambda(1 + u_k^p)\) in \(\Omega\) with Dirichlet boundary conditions. We obtain the sequence \(0 < u_k < u_{k+1} < u_\lambda\), and by monotonicity we get the minimal positive solution \(u_\tilde{\lambda} = \lim_{k \to \infty} u_k \leq u_\lambda\). Since \(u_\tilde{\lambda} \to 0\) as \(\lambda \to 0\) the uniqueness part of the implicit function theorem implies \(u_\tilde{\lambda} = u_\lambda\) for \(\lambda > 0\) small.

(b) Since \(u_\lambda\) is increasing in \(\lambda\) we have that \(\|u_\lambda\|_{\infty}\) stays bounded as \(\lambda \to 0\). Let \(M\) be so large that \((1 + \|u_\lambda\|_{\infty}) \leq M\) for \(\lambda \in [0, \tilde{\lambda}]\). Then \(\lambda MU\) is a supersolution to (1), i.e., \(u_\lambda \leq \lambda MU\), and hence \(\|u_\lambda\|_{\infty} = O(\lambda)\) as \(\lambda \to 0\). If we define \(w_\lambda = u_\lambda / \lambda\) then \(w_\lambda\) is uniformly bounded and satisfies \((-\Delta)^m w_\lambda = 1 + \lambda^p U^p\) in \(\Omega\) with Dirichlet boundary conditions. Therefore \(w_\lambda \to U\) in \(C^{2m+\alpha}(\tilde{\Omega})\) as \(\lambda \to 0\). This finishes part (b). \(\square\)

Now we can start the proof of Theorem 3. If \(u\) is also a solution of (1) then let \(v = u - u_\lambda\). The corresponding boundary value problem for \(v\) is given by

\[ (-\Delta)^m v + \lambda(v + u_\lambda)^{(p)} - \lambda u_\lambda^{(p)} = 0 \quad \text{in} \ \Omega, \quad v = \cdots = D^{m-1}v = 0 \quad \text{on} \ \partial \Omega. \]

The corresponding functional is given by

\[ \mathcal{L}[v] = \int_\Omega \frac{1}{2}|D^m v|^2 - G(x, v) \, dx, \]

where

\[ G(x, s) = \frac{\lambda}{p+1}|s + u_\lambda(x)|^{p+1} - \lambda u_\lambda(x)^{(p)}s - \frac{\lambda}{p+1}|u_\lambda(x)|^{p+1}. \]
The last term has been added in order to have \( G(x, 0) = 0 \). Let \( \xi \) be an associated conformal vector-field such that \( \xi \cdot v \leq 0 \) on \( \partial \Omega \) and \( \text{div} \xi \leq 0 \) in \( \bar{\Omega} \). We fix a negative value \( \alpha \) such that

\[
\frac{1}{p+1} < -\alpha < \frac{n-2m}{2n}.
\]

We need to verify that \( \xi \) generates a strict variational subsymmetry w.r.t. 0. By the rate of change formula from Theorem 9 we obtain

\[
\frac{d}{d\epsilon} \mathcal{L}[g_\epsilon v] \bigg|_{\epsilon=0} = \int_{\Omega} \left( \alpha - \frac{m}{n} + \frac{1}{2} \right) (\text{div} \xi)[D^m v]^2 \, dx
\]

\[
+ \int_{\Omega} \lambda \xi \cdot \nabla u_\lambda \left( -(s + u_\lambda)^{(p)} + p|u_\lambda|^{p-1}v + u_\lambda^{(p)} \right) \, dx
\]

\[
+ \int_{\Omega} \lambda \alpha (\text{div} \xi) \left( (s + u_\lambda)^{(p)} - (u_\lambda)^{(p)} \right) v \, dx
\]

\[
+ \int_{\Omega} \frac{\lambda}{p+1} (\text{div} \xi) \left( |v + u_\lambda|^{p+1} - (p + 1)u_\lambda^{(p)} v - |u_\lambda|^{p+1} \right).
\]

Define the functions \( h_1, h_2 : [0, \bar{\lambda}] \times \Omega \times \mathbb{R} \rightarrow \mathbb{R} \) by

\[
h_1(\lambda, x, s) = \xi \cdot \nabla u_\lambda \left( -(s + u_\lambda)^{(p)} + p|u_\lambda|^{p-1}s + u_\lambda^{(p)} \right);
\]

\[
h_2(\lambda, x, s) = \alpha (\text{div} \xi) \left( (s + u_\lambda)^{(p)} - (u_\lambda)^{(p)} \right) s
\]

\[
- \frac{\text{div} \xi}{p+1} \left( |s + u_\lambda|^{p+1} - (p + 1)u_\lambda^{(p)} s - |u_\lambda|^{p+1} \right),
\]

and let \( h(\lambda, x, s) = h_1(\lambda, x, s) + h_2(\lambda, x, s) \). We discuss the behaviour of \( h_1, h_2 \) depending on the different types of hypotheses. Recall that \( \text{div} \xi \leq 0 \) is a linear function in \( \bar{\Omega} \), i.e., it has at most first order zeroes at \( \partial \Omega \).

**Case (i).** We restrict attention to positive solutions of (1). Since \( u_\lambda \) is the minimal positive solution we can assume \( v \geq 0 \), i.e., we discuss \( h_1, h_2 \) for \( s \geq 0 \). We split the domain \( \bar{\Omega} = D_1 \cup D_2 \), where \( D_1 \) is a compact subset of \( \bar{\Omega} \) and \( D_2 \) a neighbourhood of \( \partial \Omega \) such that \( \xi \cdot \nabla u_\lambda \geq 0 \). For positive \( s \) convexity implies \( -(s + u_\lambda)^{(p)} + p|u_\lambda|^{p-1}s + u_\lambda^{(p)} \leq 0 \). Hence,

\[
h_1(\lambda, x, s) \leq 0 \quad \text{for } (\lambda, x, s) \in [0, \bar{\lambda}] \times [0, \infty) \times D_2.
\]

On the set \( D_1 \) we have \( |\xi \cdot \nabla u_\lambda| \leq c\lambda (\text{div} \xi) \) for a suitable constant \( c > 0 \). Hence we have

\[
h_1(\lambda, x, s) \leq C(1 + s^p)(\text{div} \xi) \quad \text{in } D_1
\]

for all \( s \geq 0 \) uniformly for \( \lambda \in [0, \bar{\lambda}] \). Next we use the mean value theorem to get

\[
-(s + u_\lambda)^{(p)} + p|u_\lambda|^{p-1}s + u_\lambda^{(p)} = -p(p-1)|\lambda \theta|^{p-2}s^2 \quad \text{for some } \theta \in [u_\lambda(x), u_\lambda(x) + s].
\]
Since \( u_{\lambda} = \lambda U + O(\lambda^{p+1}) \) by Lemma 10 and since \( U \) is positive in \( D_1 \) we find that 
\[
|h_1(\lambda, x, s)| \leq c \text{div} \xi |\lambda|^{p-1}s^2
\]
for small \( s > 0 \), i.e.,
\[
h_1(\lambda, x, s)/(|\text{div} \xi|s^2) \text{ stays bounded as } s \to 0
\]
uniformly for \((\lambda, x) \in [0, \tilde{\lambda}] \times D_1\).

The discussion of \( h_2 \) is simpler. Since \( \alpha < -1/(p+1) \) and \((-\text{div} \xi) \geq 0\) we have that 
\[
h_2(\lambda, x, s) \leq (C - |s|^{p+1})(-\text{div} \xi)
\]
uniformly for \((\lambda, x) \in [0, \tilde{\lambda}] \times \tilde{\Omega}\). Likewise 
\[
h_2(\lambda, x, s)/(|\text{div} \xi|s^2) \text{ is bounded as } s \to 0 \text{ uniformly for } (\lambda, x) \in [0, \tilde{\lambda}] \times \tilde{\Omega}.
\] Since for large positive \( s \) the negative function \( h_2(\lambda, x, s) \) dominates \( h_1(\lambda, x, s) \) we have for their sum
\[
h(\lambda, x, s) \leq As^2(-\text{div} \xi) \tag{10}
\]
for all \((\lambda, x, s) \in [0, \tilde{\lambda}] \times [0, \infty) \times \tilde{\Omega}\) with a suitable constant \( A \).

Case (ii). Now we allow also sign-changing solution of \( (1) \), i.e., we have to consider \( h_1, h_2 \) for \( s \in \mathbb{R} \). Since \( \text{div} \xi \) has first order zeroes at most at \( \partial \Omega \), we can estimate 
\[
|\xi \cdot \nabla u_{\lambda}| \leq c(-\text{div} \xi)
\]
for a suitable constant \( c > 0 \). This is possible for \( m \geq 2 \) due to the Dirichlet boundary conditions and for \( m = 1 \) under the extra requirement \( \text{div} \xi < 0 \) in \( \tilde{\Omega} \). For \( s \to \pm \infty \) we find that 
\[
h_1(\lambda, x, s) \leq C(1 + |s|^p)(-\text{div} \xi).
\] If \( p \geq 2 \) then Taylor’s theorem shows that 
\[
h_1(\lambda, x, s)/(|\text{div} \xi|s^2) \text{ stays bounded as } s \to 0 \text{ uniformly for } (\lambda, x) \in [0, \tilde{\lambda}] \times \tilde{\Omega}.
\] Since for large positive \( s \) the negative function \( h_2(\lambda, x, s) \) dominates \( h_1(\lambda, x, s) \) we have for their sum
\[
h(\lambda, x, s) \leq As^2(-\text{div} \xi) \tag{10}
\]
for all \((\lambda, x, s) \in [0, \tilde{\lambda}] \times \mathbb{R} \times \tilde{\Omega}\). The discussion of \( h_2 \) is exactly the same as in Case (i). Thus we reach the same conclusion \((10)\) as in Case (i) for all \((\lambda, x, s) \in [0, \tilde{\lambda}] \times \mathbb{R} \times \tilde{\Omega}\).

In both cases we may estimate \((9)\) by
\[
\frac{d}{d\epsilon} L[\xi u] \bigg|_{\epsilon=0} \leq \int_{\tilde{\Omega}} (\alpha - m + \frac{1}{2}) |(\text{div} \xi)|D^m v|^2 + \lambda \tilde{A} v^2(-\text{div} \xi) dx.
\]
Since \( -\alpha < (n - 2m)/(2n) \) we continue by the weighted Poincaré inequality from Lemma 1 and get
\[
\frac{d}{d\epsilon} L[\xi u] \bigg|_{\epsilon=0} \leq \int_{\tilde{\Omega}} (\alpha - m + \frac{1}{2}) \tilde{\lambda_1} v^2 \text{div} \xi + \lambda \tilde{A} v^2(-\text{div} \xi) dx,
\]
where \( \tilde{\lambda_1} \) is the Poincaré constant. This shows that for \( \lambda > 0 \) sufficiently small we have a strict variational subsymmetry w.r.t. \( 0 \). By Theorem 8 \( v \equiv 0 \), i.e., \( u \equiv u_{\lambda} \) is the only critical point of \( L \) for sufficiently small \( \lambda \). □

5. Proof of Theorem 4

We will show that every solution \( u \) of \((1)\) with small \( L^\infty\)-norm is trivial. From the rate of change formula \((7)\) in Theorem 9 we obtain
\[
\frac{d}{d\epsilon} L[u] \bigg|_{\epsilon=0} = \int_{\tilde{\Omega}} (\alpha - m + \frac{1}{2}) |(\text{div} \xi)|D^m u|^2 dx
\] 
\[
+ \int_{\tilde{\Omega}} (-\text{div} \xi) (\alpha + \frac{1}{p+1}) |u|^{p+1} + (-\text{div} \xi) (\alpha + \frac{1}{2}) \lambda u^2 dx.
\]
We choose 
\[-\alpha \leq \min\left\{ \frac{1}{p+1} \frac{n-2m}{2n} \right\}.\]

Thus the coefficient of \((-\text{div } \xi)|D^m u|^2\) is non-positive and the coefficient of \((-\text{div } \xi) \times |u|^{p+1}\) is non-negative. Hence we can apply the weighted Poincaré inequality from Lemma 1 and get
\[
\frac{d}{d\epsilon} \mathcal{L}[g_\epsilon u] \bigg|_{\epsilon=0} \leq \int_{\Omega} \left[ \tilde{\lambda}_1 \left( -\alpha + \frac{m}{n} - \frac{1}{2} \right) + \left( \alpha + \frac{1}{2} \right) \lambda \right] (-\text{div } \xi) u^2 \, dx \\
+ \int_{\Omega} \left( \alpha + \frac{1}{p+1} \right) \|u\|_{2,\infty}^{p-1} (-\text{div } \xi) u^2 \, dx,
\]
where \(\tilde{\lambda}_1\) is the Poincaré constant. Thus we have a strict variational subsymmetry w.r.t. 0 provided
\[
\|u\|_{2,\infty}^{p-1} < \frac{\tilde{\lambda}_1 (\alpha + (n-2m)/(2n)) - \lambda (\alpha + 1/2)}{\alpha + 1/(p+1)},
\]
i.e., any solution satisfying (11) is trivial by the uniqueness principle of Theorem 8. In turn, any non-trivial solution has to satisfy the reverse inequality in (11). If we let \(\alpha \to -\infty\) then we obtain part (i) of the theorem. Part (ii) follows if we take \(\alpha = (2m-n)/(2n)\).

\[\blacksquare\]

6. Proof of Theorem 5

For Theorem 5 we have \(f(x,s) = \lambda s + |s|^{p-1}s\). Hence \(\mathcal{L}\) is well defined on \(X = C^{m-1,1}_0(\bar{\Omega}) \cap H^m(\Omega) \cap L^{p+1}(\Omega)\). To show a strict variational subsymmetry, we can follow the proof of Theorem 2. By choosing \(\alpha = -1/(p+1)\) in (7) we get the rate of change formula
\[
\frac{d}{d\epsilon} \mathcal{L}[g_\epsilon u] \bigg|_{\epsilon=0} = \int_{\Omega} \left( \frac{1}{p+1} + \frac{m}{n} - \frac{1}{2} \right) (-\text{div } \xi)|D^m u|^2 \\
+ \lambda \left( \frac{1}{2} - \frac{1}{p+1} \right) u^2 (-\text{div } \xi) \, dx.
\]
By assumption \(\text{div } \xi \geq 0\). Since \(p\) is subcritical and \(\lambda \geq 0\) (with one strict inequality) it is easy to see that we have a strict variational subsymmetry with respect to 0. The uniqueness principle of Theorem 8 completes the claim. \(\blacksquare\)

For \(x\)-dependent non-linearities we have the following generalization of Theorem 5.

**Theorem 5'.** Let \(\Omega \subset \mathbb{R}^n\) be a smooth unbounded conformally contractible domain with associated vector-field \(\xi\) such that \(\text{div } \xi \geq 0\). Let \((\chi_\epsilon(x), \psi_\epsilon(x,s))\) be the solution to (4)–(5). Suppose
\[
\frac{F(\chi_\epsilon(x), \psi_\epsilon(x,s))}{|\psi_\epsilon(x,s)|^{2n/(n-2m)}} \text{ strictly decreasing in } \epsilon \text{ for almost all } x, s \in \bar{\Omega} \times \mathbb{R}.
\]
Moreover let $\omega = 1$ if $\text{div} \xi = \text{const}$ and $\omega = (1 + |x|)$ if no restriction on $\text{div} \xi$ is imposed. Then the only smooth solution $u$ of (1) satisfying $u \in H^{2m-1}(\Omega; \omega)$ and $F(x, u) \text{div} \xi, f(x, u) u \text{div} \xi, \xi \cdot \nabla F(x, u) \in L^1(\Omega)$ is $u \equiv 0$.

Remark. The complement of the domain in Example 1 is unbounded and conformally contractible, where the associated vector-field satisfies $\text{div} \xi = 3$. Hence the above results apply. The complement of the domain in Example 2 is also unbounded and conformally contractible, but the associated vector-field $\xi$ satisfies $\text{div} \xi = 6 \xi_z$, which is sign-changing. Hence neither Theorem 5 nor Theorem $5'$ applies. However, the half-space $x_n > 0$ with $\xi = (2x_1x_n, \ldots, 2x_{n-1}x_n, x_n^2-x_1^2-\cdots-x_{n-1}^2)$ provides non-trivial examples for Theorem $5'$ if we take $f(x, s) = |\xi(x)|^\beta |s|^{\gamma-1}$ with $\beta \geq 0$ and $1 < \gamma < (n+2m+2\beta)/(n-2m)$.

7. Proof of the weighted Poincaré inequality

For $n \geq 3$ the function $-\text{div} \xi$ is linear and non-negative. Hence we may suppose that after a rotation of the coordinate system we have $-\text{div} \xi = a + bx_1 \geq 0$ in $\Omega$. To avoid trivialities assume $b < 0$ and $x_1 \leq -a/b$ for $x \in \Omega$ (a similar proof holds if $b < 0$ and $x_1 \geq -a/b$). Let $C$ denote a generic constant. First, we find

$$
\int _{\Omega} (a + bx_1)u^2 \, dx \leq C \int _{\Omega} u^2 \, dx = -\frac{C}{b} \int _{\Omega} (a + bx_1) \partial x_1 u^2 \, dx
$$

$$
\leq C \left( \int _{\Omega} (a + bx_1)u^2 \, dx \right)^{1/2} \left( \int _{\Omega} (a + bx_1)|\partial x_1 u|^2 \, dx \right)^{1/2},
$$

i.e.,

$$
\int _{\Omega} (a + bx_1)u^2 \, dx \leq C \int _{\Omega} (a + bx_1)|\partial x_1 u|^2 \, dx.
$$

Likewise, for $i = 2, \ldots, n$ we find

$$
\int _{\Omega} (a + bx_1)u^2 \, dx = -\int _{\Omega} x_i \partial x_i ((a + bx_1)u^2) \, dx \leq C \int _{\Omega} (a + bx_1)u \partial x_i u \, dx
$$

$$
\leq C \left( \int _{\Omega} (a + bx_1)u^2 \, dx \right)^{1/2} \left( \int _{\Omega} (a + bx_1)|\partial x_i u|^2 \, dx \right)^{1/2}.
$$

Hence

$$
\int _{\Omega} (a + bx_1)u^2 \, dx \leq C \int _{\Omega} (a + bx_1)|\partial x_i u|^2 \, dx \text{ for } i = 2, \ldots, n.
$$

The result in (3) of Lemma 1 now follows from iterating these inequalities. To find the relation of the best constant $\tilde{\lambda}$ in (3) and $\lambda_1$, let $\phi_1$ be the first Dirichlet eigenfunction and suppose $m$ is even.

Then, by formula (10) in the proof of Lemma 9, Part I,

$$
\int _{\Omega} (-\text{div} \xi) \Delta' \phi_1 \Delta' \phi_1 \, dx
$$

$$
= \int _{\Omega} \Delta' \left( (-\text{div} \xi) \phi_1 \right) \Delta' \phi_1 \, dx - \int _{\Omega} 2r \beta_i \Delta'^{-1} \partial_i \phi_1 \Delta' \phi_1 \, dx.
$$
The second integral vanishes, as can be seen directly or explicitly through Lemma 11, Part I. Hence integration by parts yields
\[
\int_{\Omega} (- \text{div} \xi) \Delta \phi_1 \Delta \phi_1 \, dx = \lambda_1 \int_{\Omega} (- \text{div} \xi) \phi_1^2 \, dx,
\]
which shows that the optimal constant \( \tilde{\lambda}_1 \) is smaller or equal to \( \lambda_1 \). The proof for \( m \) odd is similar. Finally, in the case \( n = 2 \) the function \( - \text{div} \xi \geq 0 \) is harmonic. Hence it also has at most simple zeroes on \( \partial \Omega \) and we can estimate it from above and below by a linear function. However, for \( n = 2 \) and \( m \geq 2 \) the relation between the optimal constant \( \tilde{\lambda}_1 \) and \( \lambda_1 \) is not clear. For \( n = 2 \) and \( m = 1 \) a similar proof as above works and shows \( \tilde{\lambda}_1 \geq \lambda_1 \). \( \square \)

References