ELSEVIER

Available online at www.sciencedirect.com

SCIENCE DIRECT®

J. Differential Equations 226 (2006) 250–268

Journal of Differential Equations

www.elsevier.com/locate/jde

On the quasi-ordinary cuspidal foliations in $(\mathbb{C}^3, 0)$ ☆

Percy Fernández-Sánchez [a,*], Jorge Mozo-Fernández [b]

[a] *Instituto de Matemática y Ciencias Afines, UNI – PUCP, Casa de las Trece Monedas, Jr. Ancash 536, Lima 1, Peru*

[b] *Departamento de Matemática Aplicada, Fac. de Ciencias, Universidad de Valladolid, Prado de la Magdalena, s/n, 47005 Valladolid, Spain*

Received 21 February 2005; revised 13 September 2005

Available online 26 October 2005

Abstract

In this paper, we study a class of singularities of codimension 1 holomorphic germs of foliations in $(\mathbb{C}^3, 0)$, namely those ones having only one separatrix, that is a quasi-ordinary surface, and whose reduction of singularities agrees with the combinatorial desingularization of the separatrix. We show that the analytic classification of these germs can be read in the holonomy of a certain component of the exceptional divisor of the desingularization.

Keywords: Holomorphic foliations; Analytic classification; Reduction of singularities

1. Introduction and motivation

We would like to study the reduction of the singularities and the analytic classification, in some cases that we shall describe, of germs of singular holomorphic foliations in $(\mathbb{C}^3, 0)$, with non-zero linear part. Consider, more generally, a germ ω in $(\mathbb{C}^n, 0)$ of an integrable 1-form, and let

$$\omega = \omega_1 + \omega_2 + \cdots$$

☆ Both authors partially supported by CONCYTEC (Peru) under the research project 551-OAJ (2004), and by Junta de Castilla y León (Spain) under project VA123/04.

* Corresponding author. Fax: +34 983423013.
E-mail addresses: pefernan@pucp.edu.pe (P. Fernández-Sánchez), jmozo@maf.uva.es (J. Mozo-Fernández).

0022-0396/$ – see front matter
doi:10.1016/j.jde.2005.09.006

be its decomposition in homogeneous forms ($\omega_i = \sum_{j=1}^n A_{ij}\,dx_i$, A_{ij} is a homogeneous polynomial of degree i). Suppose, moreover, that $\omega_1 \not\equiv 0$. In general, we can write

$$\omega_1 = \sum_{1\leqslant i,j\leqslant n}^{n} c_{ij}x_j\,dx_i, \quad c_{ij} \in \mathbb{C}.$$

The integrability condition $\omega \wedge d\omega = 0$ implies that $\omega_1 \wedge d\omega_1 = 0$. Let C be the matrix $(c_{ij})_{i,j} \in M_{n\times n}(\mathbb{C})$. Writing down explicitly the integrability condition, the coefficient of $dx_i \wedge dx_j \wedge dx_k$ $(i < j < k)$ in $\omega_1 \wedge d\omega_1$ is

$$c_i(c_{kj} - c_{jk}) - c_j(c_{ki} - c_{ik}) + c_k(c_{ji} - c_{ij}),$$

where $c_i = \sum_{j=1}^n c_{ij}x_j$. Two cases appear:

(1) C is a symmetric matrix.
(2) C is not symmetric. So, it exists (j,k) with $c_{kj} \neq c_{jk}$.

In the last case, the polynomials c_i, c_j, c_k are linearly dependent for every i, j, k, and so $\mathrm{rk}(C) \leqslant 2$. Moreover, $d\omega_1(0) = d\omega(0) \neq 0$, so we are in presence of a Kupka-type phenomenon and, in fact, there exists a biholomorphism f such that $f^*\omega \wedge \eta = 0$, where η is a form in 2-variables. For bidimensional phenomena, lots of works have been done.

We then focus on the symmetric case. A linear change of coordinates changes C in P^tCP, P invertible, so we can suppose C diagonal and moreover,

$$\omega_1 = \sum_{i=1}^r x_i\,dx_i, \quad r \leqslant n.$$

If $r = n$, G. Reeb, in his thesis [19] shows that there always exists a holomorphic first integral. The behaviour of the foliation is, then, the behaviour of a function. Using Malgrange's singular Frobenius theorem [13], we recover this result.

If $r < n$, some work was done by R. Moussu [17] under additional hypothesis. The fundamental paper of Mattei and Moussu [14] completes the mentioned results. Let us recall in this case, briefly, the 2-dimensional situation. The foliations studied are defined by 1-forms $y\,dy + \cdots$. Following Takens [20] such a foliation has a formal normal form

$$\omega_N = d\left(y^2 + x^m\right) + x^p U(x)\,dy,$$

where $m \geqslant 3$, $p \geqslant 2$, $U(x) \in \mathbb{C}[\![x]\!]$, $U(0) \neq 0$.

The generic case ($m = 3$) was studied by Moussu [18] and a generalization ($m \geqslant 3$, $2p > m$) by Cerveau and Moussu [5]. In both cases, the reduction of the singularities of ω (and ω_N) agrees with the reduction of the curve $y^2 + x^m = 0$. Projective holonomy classifies and, generically, there is a rigidity phenomenon formal/analytic. If m is even and $2p = m$, it has been studied by Meziani [15] under some restrictions on the values of $U(0)$. If $2p < m$ the study was done (not in full generality) by Berthier, Meziani and Sad [1]. We shall call "cuspidal" these foliations.

The goal of the present work is to generalize this situation to dimension three. We want to study foliations whose linear part is given by $d(x^2 + y^2)$ or by $d(z^2)$. In this paper we shall focus in the case $d(z^2)$. A surface that controls the resolutions of the singularities, with an equation $z^2 + \cdots = 0$ will appear in the considered cases.

Let us recall some results about reduction of the singularities of a complex surface, following Hironaka [9]. A surface X in $\mathbb{C}^3$ has an equation

$$f = f_\nu + f_{\nu+1} + \cdots = 0,$$

where f_i is homogeneous of degree i. For such a surface, define, at the origin:

(1) The tangent cone, C_X, as the cone $f_\nu = 0$.
(2) The Zariski tangent cone T_X, as $\operatorname{Spec}(\mathcal{M}/\mathcal{M}^2)$, $\mathcal{M}$ being the maximal ideal corresponding to the origin of $\mathbb{C}[\![x, y, z]\!]/(f)$. This is the smallest linear space containing C_X.
(3) The strict tangent cone S_X, as the largest linear subspace T of T_X such that $C_X = C_X + T$. The codimension of S_X is the minimum number of variables required to write down the equations of C_X.

The resolution of singularities of an analytic surface X is a problem that may be stated as follows: to find a non-singular surface $\widetilde{X}$ and a birational morphism $\widetilde{X} \to X$ composed of quadratic (point blow-ups) and monoidal (curve blow-ups) transformations. These must be done in a precise order. The main case to consider is when the three tangent spaces defined above coincide, and the most difficult case is when, moreover, $\dim S_X = 2$. In this case, the tangent cone can be written as z^ν. The resolution may be controlled by Hironaka's characteristic polyhedra of the singularities [9]. The precise sequence of blow-ups needed can be read in the polyhedra.

A kind of surface singularities whose resolution is particularly simple, and combinatorial, are quasi-ordinary singularities. To define them, consider a finite projection $X \xrightarrow{\pi} \mathbb{C}^2$ and let Δ be discriminant locus of π (i.e., the projection of the apparent contour). If Δ has normal crossings the singularities of X are called quasi-ordinary.

Quasi-ordinary singularities are studied not only because they are relatively simple, but because they arise in the Jungian approach to desingularization. First of all desingularize the discriminant locus in order to obtain quasi-ordinary singularities. Then, the problem (simpler) is to reduce the singularities of a quasi-ordinary surface. Some good references of this are the articles of Giraud [8] and Cossart [7].

Quasi-ordinary singularities can be parametrized by fractional power series, as branches of curves:

$$\begin{cases} x = x, \\ y = y, \\ z = \sum_{i,j} c_{ij} x^{i/n} y^{j/n}. \end{cases}$$

By the condition of the discriminant, it can be seen that the set of points $\{(i, j) \in \mathbb{R}^2\colon c_{ij} \neq 0\}$ is contained in a quadrant $(a, b) + \mathbb{R}^2_+$, where $c_{ab} \neq 0$. Characteristic pairs may be defined for this parametrization, as is the case of curves, and they still determine the local topology of the singularity, while the converse is not known [11].

Coming back to foliations, this is related with the case we shall study. More precisely, we search a class of foliations in $(\mathbb{C}^3, 0)$ whose reduction process can be read in a quasi-ordinary surface. For the case considered $\omega_1 = d(z^2)$, by Weierstrass preparation theorem and Tschirnhausen transformations we find that, in appropriate coordinates, the surface is $z^2 + \varphi(x, y) = 0$, that is not necessarily a separatrix. The natural generalization of cuspidal foliations will be those with an equation

$$\omega = d\big(z^2 + \varphi(x, y)\big) + A(x, y)\,dz.$$

In fact, in a recent work, Frank Loray [12] finds an analytic normal form as

$$\omega = dF + z\,dG + z\,dz,$$

where $F, G \in \mathbb{C}\{x, y\}$, for integrable holomorphic foliations with linear part not tangent to the radial vector field. Note that a coordinate change $z \to z - G(x, y)$ in Loray's form gives an equation like our expression for the foliations. This is integrable if and only if $d\varphi \wedge dA = 0$, i.e., if φ, A are analytically dependent. As are shall restrict to the quasi-ordinary case, we have that $\varphi(x, y) = x^p y^q U(x, y)$, with U a unit. A convenient change of variable in x, y, allows us to suppose that $\varphi(x, y) = x^p y^q$. Let $d = \gcd(p, q)$ $p = dp'$ $q = dq'$. The integrability condition $d\varphi \wedge dA = 0$ is then that $A(x, y) = L(x^{p'} y^{q'})$ where $L(u) \in \mathbb{C}\{u\}$.

The plan of this paper is as follows. In Section 2 we shall review the notion of simple singularity of a foliation, in the sense defined by Cano and Cerveau [3], and its analytic classification according to Cerveau and Mozo [6]. Section 3 is devoted to describe the resolution of singularities of the quasi-ordinary foliations we are going to study, and the topology of the exceptional divisor. In Section 4, we construct a Hopf fibration associated to the quasi-ordinary foliations, making a reduction of the separatrix to a canonical form. Finally, Section 5 is devoted to present the main result of the paper: In the considered cases, the holonomy of a certain component of the exceptional divisor classifies analytically the foliation. The cases we study, as we shall see, are essentially the sames that are studied in dimension two.

Some notations used throughout the paper are presented here. $\mathrm{Diff}(\mathbb{C}, 0)$ will denote the group (under composition) of germs of analytic diffeomorphisms of $(\mathbb{C}, 0)$. If Ω denotes a holomorphic integrable 1-form, defining a foliation, and D is a component of the divisor obtained after reduction of singularities, $\mathcal{H}_{\Omega,D} : \pi_1(D \setminus \mathcal{S}) \to \mathrm{Diff}(\mathbb{C}, 0)$ is the holonomy representation, defined over a transversal to D (omitted from the notation), where $\mathcal{S}$ is the singular set of the reduced foliation.

2. Simple singularities of foliations and analytic classification

The process of reduction of singularities for a holomorphic foliation is well known in dimension two. After a finite number of point blow-ups performed in any order, a germ of analytic space and a foliation are obtained, and around the singular points, the foliation is generated by a one-form

$$\omega = (\lambda y + \text{h.o.t.})\, dx + (\mu x + \text{h.o.t.})\, dy,$$

with $\mu \neq 0$, $\lambda/\mu \notin \mathbb{Q}_{<0}$.

The analytic classification is well studied in a wide variety of cases:

(1) If $\lambda/\mu \notin \mathbb{R}_{\geqslant 0}$, ω is analytically linearizable, i.e., there exists an analytic diffeomorphism $\phi : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ such that

$$\phi^*\omega \wedge (\lambda y\, dx + \mu x\, dy) = 0.$$

(2) If $\lambda/\mu \in \mathbb{R}_{>0} \setminus \mathbb{Q}$, but it is not "well-approached" by rational numbers, it is also linearizable. If it is well-approached, we face a problem of small divisors, and the situation becomes more complicated.

(3) If $\lambda = 0$ or $\lambda/\mu \in \mathbb{Q}_+$, Martinet and Ramis find a large moduli space formal/analytic. In this case the classification of the foliation agrees with the classification of the holonomy of a strong separatrix (i.e., a separatrix in the direction of a non-zero eigenvalue). Moreover, in the resonant case ($\frac{\lambda}{\mu} \in \mathbb{Q}_+$) or in the saddle-node case ($\lambda = 0$) with analytic center manifold, the conjugation of the foliation is fibered (see [1] for the saddle-node case). This means the following: choose coordinates x, y such that the axis are the separatrices, $y = 0$ being a strong one (this means that it is tangent to the eigenspace that corresponds to a non-zero eigenvalue); the foliations are defined by 1-forms

$$\omega_i = yA_i(x, y)\, dx + \mu/\lambda x\bigl(1 + B_i(x, y)\bigr)\, dy,$$

with $i = 1, 2$. Let $h^{(i)}(x)$ be the holonomies of $y = 0$, supposed conjugated. Then the foliations are conjugated by a diffeomorphism $\phi(x, y) = (x, yg(x, y))$.

The singularities obtained after this reduction process are called simple or reduced. The class of simple singularities is stable under blow-ups. Let us observe that the notion of simple singularity is not only analytic, but formal: if ω_1, ω_2 are analytic 1-forms, and $\hat{\phi}$ is a local diffeomorphism such that $\hat{\phi}^*\omega_1 \wedge \omega_2 = 0$, then ω_1 has a simple singularity if and only if ω_2 has.

If the dimension of the ambient space is greater or equal than three, the notion of simple singularity has been developed in [2,3], and its analytic classification studied in [6]. The reduction of singularities is only achieved when the dimension of the ambient space is at most three, and in this case, simple singularities are the final ones obtained after the reduction process. Let us summarize here, for convenience of the reader, the main results in dimension three.

First of all, let us recall the notion of "dimensional type." A foliation has dimensional type r if there exist analytic (resp. formal) coordinates such that the foliation is defined by an integrable 1-form ω that can be written in coordinates $x_1, \ldots, x_r$ ($r \leqslant n$), but not less.

So, a three-dimensional singularity of foliation has dimensional type 2 and 3. For instance, if we are in presence of a Kupka phenomenon, the dimensional type is 2. The notions of formal dimensional type or analytic dimensional type are equivalent, as seen in [6]. So, we have simple singularities of dimensional types 2 and 3. If the dimensional type is 2, simple singularities are defined by a simple 2-dimensional 1-form. They have 2 separatrices, of which at most one is formal.

If the dimensional type is three, simple singularities are the ones that admit one of the following formal normal forms:

$$\omega = xyz\left(\alpha \frac{dx}{x} + \beta \frac{dy}{y} + \gamma \frac{dz}{z}\right), \tag{1}$$

with $\alpha/\gamma, \beta/\gamma, \alpha/\beta \notin \mathbb{Q}_-$ (and $\alpha\beta\gamma \neq 0$, as the dimensional type is 3). This is the linearizable case. If, for instance, some of the quotients is not real, the linearization is analytic [4].

$$\omega_N = xyz\left(x^p y^q z^r\right)^s \left[\alpha \frac{dx}{x} + \beta \frac{dy}{y} + \left(\lambda + \frac{1}{(x^p y^q z^r)^s}\right)\left(p\frac{dx}{x} + q\frac{dy}{y} + r\frac{dz}{z}\right)\right], \tag{2}$$

where $p, q, r \in \mathbb{N}$, $qr \neq 0$, $s \in \mathbb{N}^*$, α, β constants, not both zero. This is the *resonant case*. Several things can be said about foliations that are formally equivalent to this normal form:

(1) $\mathcal{F}$ has three separatrices, of which at most one is formal (which, in the preceding coordinates, would be $x = 0$). This is a confluence of simple two-dimensional singularities defined along the axis. Saddle-nodes only appear if $p = 0$, and only in this case the existence of a formal, nonconvergent separatrix is possible.
(2) The holonomy group of $z = 0$ (strong separatrix) classifies analytically the foliation. Moreover, the conjugations is fibered if the three separatrices are convergent.
(3) If $\alpha/\beta \notin \mathbb{Q}$, there is a rigidity phenomenon: every such foliation is analytically equivalent to ω_N.

A typical case in which we are in presence of a simple singularity and that will appear in the sequel, is when the foliation is defined by a 1-form

$$\omega = xyz\left[\left(p + A(x,y,z)\right)\frac{dx}{x} + \left(q + B(x,y,z)\right)\frac{dy}{y} + \left(r + C(x,y,z)\right)\frac{dz}{z}\right], \tag{3}$$

with $p, q, r \in \mathbb{N}^*$, $\nu(A), \nu(B), \nu(C) > 0$.

More can be said: the transformation ϕ that converts ω in its formal normal form ω_N, even if it is not analytic, it is transversally formal and fibered. This means in particular that such a ϕ can be found in the form

$$\phi(x,y,z) = \left(x, y, \varphi(x,y,z)\right).$$

The existence of local holomorphic first integrals, according to Mattei and Moussu [14], is equivalent to the periodicity of the holonomy group. Moreover, an integrable 1-form ω, that generates a reduced foliation of dimensional type three, has a holomorphic first integral if and only if there exists analytic coordinates (x, y, z) such that

$$\omega \wedge (pyz\,dx + qxz\,dy + rxy\,dz) = 0,$$

where $p, q, r \in \mathbb{N}^*$.

3. Reduction of singularities and topology of the divisor

In this paper, we shall study the analytic classification of quasi-ordinary cuspidal foliations in dimension three, i.e., foliations such that, in appropriate coordinates, can be defined by an integrable 1-form

$$\omega = d\big(z^2 + x^p y^q\big) + A(x, y)\,dz.$$

The integrability condition here is equivalent to $d(x^p y^q) \wedge dA = 0$. So, let $d = \gcd(p, q)$, $p = dp'$, $q = dq'$. Such a 1-form can be written as

$$\omega = d\big(z^2 + x^p y^q\big) + \big(x^{p'} y^{q'}\big)^k h\big(x^{p'} y^{q'}\big)\,dz,$$

where $h(u) \in \mathbb{C}\{u\}$, $h(0) \neq 0$. Fixing p, q, we shall call Σ_{pq} the set of holomorphic foliations that are analytically equivalent to the foliation defined by one of these 1-forms.

As it will become clear from the development of the paper, the separatrices of this foliation have the equation

$$z^2 + x^p y^q + \text{h.o.t.} = 0,$$

and Weierstrass preparation theorem and Tschirnhausen transformation show that this separatrix is analytically equivalent to $z^2 + x^p y^q = 0$.

The reduction of singularities for these foliations is quite simple, similar to plane curves, and the main objective of this section is their detailed analysis. For convenience, we divide the problem in three cases:

Case 1. p, q even.
Case 2. p even, q odd.
Case 3. p, q odd.

Case 1. Suppose p, q are even and $d = 2d'$. If $k > d'$, the reduction of the singularities is obtained after $\frac{p+q}{2}$ blow-ups:

(a) First of all, blow up $\frac{p}{2}$ times the y-axis. We obtain a sequence of divisors $D_1, \dots, D_{p/2}$, topologically germs $(\mathbb{P}^1_{\mathbb{C}} \times \mathbb{C}, \mathbb{P}^1_{\mathbb{C}})$. The intersection of two consecutive components is a germ of a line $(\mathbb{C}, 0)$, $L_i = D_i \cap D_{i+1}$, $1 \leqslant i < \frac{p}{2}$. In the appropriate chart, these blow-ups have the equations

$$\begin{cases} x = x, \\ y = y, \\ t_{i-1} = x \cdot t_i, \end{cases}$$

where $t_0 = z$, $1 \leqslant i \leqslant \frac{p}{2}$.

(b) Then blow-up $\frac{q}{2}$ times the x-axis, obtaining again a sequence of divisors $D_{p/2+1}, \dots, D_{(p+q)/2}$, topologically equal to $(\mathbb{P}^1_{\mathbb{C}} \times \mathbb{C}, \mathbb{P}^1_{\mathbb{C}})$. Again, the intersection between two consecutive components is a line $L_i = D_i \cap D_{i+1}$, $\frac{p}{2} + 1 \leqslant i < \frac{p+q}{2}$. Now, the coordinates of the blow-ups are

$$\begin{cases} x = x, \\ y = y, \\ t_{i-1} = y \cdot t_i, \end{cases}$$

with $\frac{p}{2} < i < \frac{p+q}{2}$.

The result of the composition of all the blow-ups in the preceding charts is the map $\pi(x, y, t_{(p+q)/2}) = (x, y, x^{\frac{p}{2}} \cdot y^{\frac{q}{2}} \cdot t_{(p+q)/2})$. The pull-back of the foliation is given by

$$\pi^*\omega = x^{p-1}y^{q-1} \cdot \biggl[2xyt\,dt + (t^2+1)xy\left(p\frac{dx}{x} + q\frac{dy}{y}\right) + \left(x^{p'}y^{q'}\right)^{k-d'} h\left(x^{p'}y^{q'}\right)xyt \cdot \left(\frac{p}{2}\frac{dx}{x} + \frac{q}{2}\frac{dy}{y} + \frac{dt}{t}\right)\biggr]$$

(here $t = t_{(p+q)/2}$).

The foliation, now, is reduced. Let $\mathcal{S}$ be the singular locus of this reduced foliation. $\mathcal{S}$ is an analytic, normal crossing space of dimension one, composed by (see Fig. 1):

(i) The lines L_i of intersection of the divisors. These are resonant singular points of dimensional type two.

(ii) The lines L, L' in $D_{(p+q)/2}$ of equations $(y = 0, t = i)$, $(y = 0, t = -i)$, and also the lines M', M'' in $D_{p/2}$ of equations $(x = 0, t = i)$, $(x = 0, t = -i)$ (in the last chart). These lines are the intersections of the two separatrices S', S'' with the divisors.

(iii) The intersection $P_i := D_{p/2} \cap D_i$, $\frac{p}{2} < i \leqslant \frac{p+q}{2}$ is a projective line composed of points of dimensional type two, except at the corners:

(A) $m_i = P_i \cap L_i = D_{p/2} \cap D_i \cap D_{i+1}$, $\frac{p}{2} < i < \frac{p+q}{2}$. These are the resonant singular points of dimensional type three, having $D_{p/2}$, D_i, D_{i+1} as separatrices.

(B)
$$m' := D_{p/2} \cap D_{(p+q)/2} \cap S' = L' \cap M' \cap P_{(p+q)/2}, \quad \text{and}$$
$$m'' := D_{q/2} \cap D_{(p+q)/2} \cap S'' = L'' \cap M'' \cap P_{(p+q)/2}.$$

These are the resonant singular points of dimensional type three corresponding to the separatrices of the foliations.

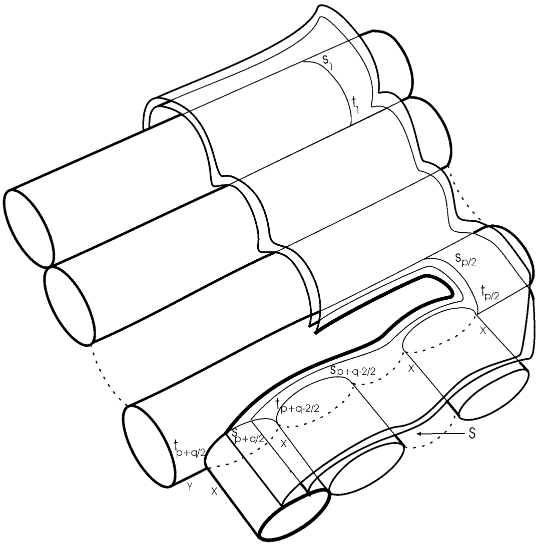

Fig. 1. Final form in Case 1.

According to the preceding description of the resolution of the singularities, we have all the information about the topology of $D_i \setminus \mathcal{S}$, and more precisely about the fundamental group of these components. We have:

- $D_1 \setminus \mathcal{S}$ is topologically $\mathbb{C} \times \mathbb{C}$, so simply connected.
- $D_i \setminus \mathcal{S}$ $(1 < i < \frac{p}{2})$ is topologically $\mathbb{C}^* \times \mathbb{C}$. The generator of the fundamental group is a loop γ_i that turns around L_i (or γ_i^{-1} around L_{i-1}).
- $D_{p/2+1} \setminus \mathcal{S} \cong \mathbb{C}^* \times \mathbb{C}$. The fundamental group is generated by a loop α_i around $P_{p/2+1}$.
- $D_i \setminus \mathcal{S} \cong \mathbb{C}^* \times \mathbb{C}^*$ $(\frac{p}{2}+1 < i < \frac{p+q}{2})$. The fundamental group has generators γ_i around L_i and α_i around P_i, that commute.
- $D_{(p+q)/2} \setminus \mathcal{S} \cong (\mathbb{C} \setminus \{m', m''\}) \times \mathbb{C}^*$. We have one loop $\alpha_{(p+q)/2}$ around $P_{(p+q)/2}$ and loops γ', γ'' around the separatrices (i.e., around m', m'').
- $D_{p/2} \setminus \mathcal{S} \cong \mathbb{C}^2 \setminus \mathcal{C}$, where $\mathcal{C}$ is the curve with coordinates $t_{p/2}^2 + y^q = 0$, composed of two smooth branches that meet tangentially at the origin. In this case (see [10]), $\pi_1(\mathbb{C}^2 \setminus \mathcal{C})$ is the group, written in terms of generators and relations as

$$\pi_1\left(\mathbb{C}^2 \setminus \mathcal{C}\right) = \left\langle \alpha, \beta; \alpha^{\frac{q}{2}} \beta = \beta \alpha^{\frac{q}{2}} \right\rangle.$$

These loops go as follows. Consider the curve $t_{p/2}^2 + y^q = 0$ on $\mathbb{C}^2$, and cut by $y = 1$. You obtain $\mathbb{C} \setminus \{m', m''\}$; then α is a loop in $y = 1$ that turns around these two points m', m'', and β is a loop in $t_{p/2} = 0$ that turns around the origin. At the end of the reduction process, α is going to be a loop in $D_{p/2}$ around the two separatrices, and β a loop around $P_{(p+q)/2}$ “between S' and S''.”

The case $k = d'$ ($2k = d$) is almost identical, except for some values of the coefficient $h(0)$. More precisely, after $\frac{p+q}{2}$ blow-ups, in order to obtain the complete reduction of singularities (i.e., simple singular points) it is necessary and sufficient that

$$h(0)^2 \neq \frac{(16+r)^2}{16+2r}, \quad \forall r \in \mathbb{Q}_{>0}.$$

Moreover, if in the preceding expression we put $r = 0$, we have then $h(0) = \pm 4$. In this case, only one separatrix is obtained, but it is a three-dimensional saddle-node, the divisor being the weak separatrix (then convergent). We shall assume that this is not the case, i.e., if $k = d'$ we shall assume that

$$h(0)^2 \neq \frac{(16+r)^2}{16+2r}, \quad \forall r \in \mathbb{Q}_{\geqslant 0}.$$

The reader may verify that this condition is equivalent to $\mathcal{P}_2$ property in [15,16] (i.e., $h(0) \neq \pm 2(\sqrt{r} + \frac{1}{\sqrt{r}})$, $\forall r \in (0,1] \cap \mathbb{Q}$).

Suppose now that $k < d'$. In this case, the reduction of singularities is achieved blowing-up kp' times the y-axis and kq' times the x-axis. After these, in the last chart we obtain as singularities the sets $L' = (x = t = 0)$, $M' = (y = t = 0)$, $L'' = (x = 0, t = 1)$, $M'' = (y = 0, t = 1)$. These are also two singular points of dimensional type three, namely $m' = L' \cap M' \cap P_{k(p'+q')}$, $m'' = L'' \cap M'' \cap P_{k(p'+q')}$ (with analogous notations as before), corresponding respectively to the points $(0,0,0)$ and $(0,0,1)$. But now m'' is a saddle-node, so the separatrix S'' is maybe formal. In this paper, we shall assume that always S'' is convergent, i.e., there is a center manifold.

Case 2. Suppose p even, q odd. If $k > d$, the reduction of singularities is obtained after the following sequence of blow-ups.

(a) First, blow-up $\frac{p}{2}$ times the y-axis, obtaining divisors $D_1, \ldots, D_{p/2}$ linked by lines $L_1, \ldots, L_{p/2-1}$. The equations of these blow-ups are

$$\begin{cases} x = x, \\ y = y, \\ t_i = x \cdot t_{i+1}, \end{cases}$$

where $t_0 := z$, $i < \frac{p}{2}$.

(b) Blow-up $\frac{q-1}{2}$ times the x-axis, obtaining $D_{p/2+1}, \ldots, D_{(p+q-1)/2}$ joined by lines $L_i = D_i \cap D_{i+1}$, and D_i joined to $D_{p/2}$ by a projective P_i. The equations are

$$\begin{cases} x = x, \\ y = y, \\ t_i = y \cdot t_{i+1}, \end{cases}$$

$\frac{p}{2} \leqslant i < \frac{p+q-1}{2}$.

(c) It appears a tangency in the singular locus. In order to break it, blow-up again the x-axis and take a chart centered in the point corresponding to $t_{(p+q-1)/2}$. The equations are now

$$\begin{cases} x = x, \\ y = s \cdot t_{(p+q-1)/2}, \\ t_{(p+q-1)/2} = t_{(p+q-1)/2}, \end{cases}$$

and we obtain a new component D' such that $D' \cap D_{(p+q-1)/2} = L_{(p+q-1)/2}$, $D' \cap D_{p/2} = P'$.

(d) Finally, blow-up again the x-axis, in order to obtain normal crossings. We obtain a final component D'' and the only separatrix S of the foliation cuts D' transversely in a line L (and $D_{p/2}$ in a line M). We have $L' = D' \cap D''$ and $P'' = D'' \cap D_{p/2}$.

The singular points of dimensional type three are

$$\begin{aligned} &m_i := D_{p/2} \cap D_i \cap D_{i+1} \quad \frac{p}{2} < i < \frac{p+q-1}{2}, \\ &m_{(p+q-1)/2} := D_{p/2} \cap D_{(p+q-1)/2} \cap D', \\ &m' = D_{p/2} \cap D' \cap D'' \quad \text{and} \quad m = D_{p/2} \cap D' \cap S. \end{aligned}$$

The topology of the components is as in Case 1 (see Fig. 2). If $\mathcal{S}$ is the singular locus, $D_1 \setminus \mathcal{S} \cong \mathbb{C}^2$ is simply connected, $D_i \setminus \mathcal{S} \cong \mathbb{C}^* \times \mathbb{C}$ if $1 < i < \frac{p}{2}$, $D_{p/2+1} \setminus \mathcal{S} \cong \mathbb{C}^* \times \mathbb{C}$, $D_i \setminus \mathcal{S} \cong \mathbb{C}^* \times \mathbb{C}^*$ if $\frac{p}{2} + 1 < i < \frac{p+q-1}{2}$, $D' \setminus \mathcal{S} \cong (\mathbb{C} \setminus \{m, m''\}) \times \mathbb{C}^*$, $D'' \setminus \mathcal{S} \cong \mathbb{C} \times \mathbb{C}^*$. Finally, $D_{p/2} \setminus \mathcal{S} \cong \mathbb{C}^2 \setminus \mathcal{C}$, where $\mathcal{C}$ is the curve with coordinates $t_{p/2}^2 + y^q = 0$. As before,

$$\pi_1(\mathbb{C}^2 \setminus \mathcal{C}) = \langle \alpha, \beta; \alpha^q = \beta^2 \rangle.$$

When $k < d$, as in dimension two, the situation is as in Case 1, with $k < d'$.

Case 3. p, q odd. Now, the resolution is something different than before. First, blow-up $\frac{p-1}{2}$ times the y-axis and $\frac{q-1}{2}$ times the x-axis obtaining $D_1, \ldots, D_{(p-1)/2}, D_{(p+1)/2}, \ldots, D_{(p+q)/2-1}$. In the new coordinates $(x, y, t := t_{(p+q)/2-1})$ the singular locus is given by the three coordinate axis, that corresponds to the intersection of the divisors and the intersection of the cone $t^2 + xy = 0$ with the divisors.

Now, blow-up the origin, obtaining P, a projective $\mathbb{P}^2_{\mathbb{C}}$. The three coordinate axis, now transverse to P, continue being singular. Over P, the singular locus is composed by two projective lines and a conic tangent to both lines. In order to finish, blow-up twice each of the axis x and y transverse to P, obtaining $D'_{(1)}$, $D''_{(1)}$, $D'_{(2)}$, $D''_{(2)}$ (see Fig. 3).

With respect to the topology of the divisors, the only interesting case (i.e., not similar to the preceding ones) to comment is $P \setminus \mathcal{S}$. As we said before, $P \cap \mathcal{S}$ is composed by two lines and a regular conic, so

$$\pi_1(P \setminus \mathcal{S}) \cong \langle \alpha, \beta; \alpha^2\beta = \beta\alpha^2 \rangle.$$

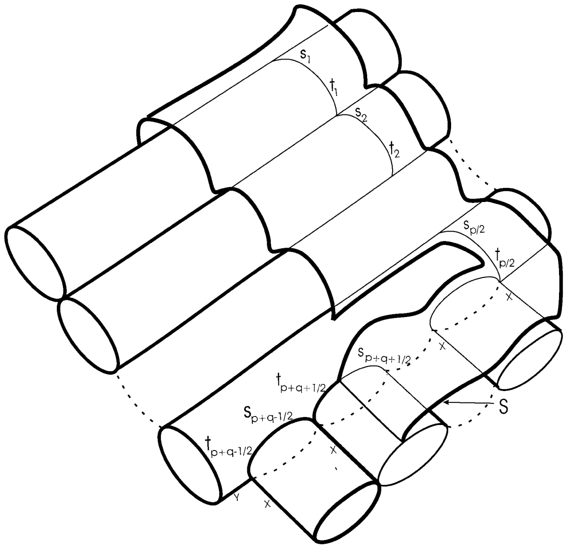

Fig. 2. Final form in Case 2.

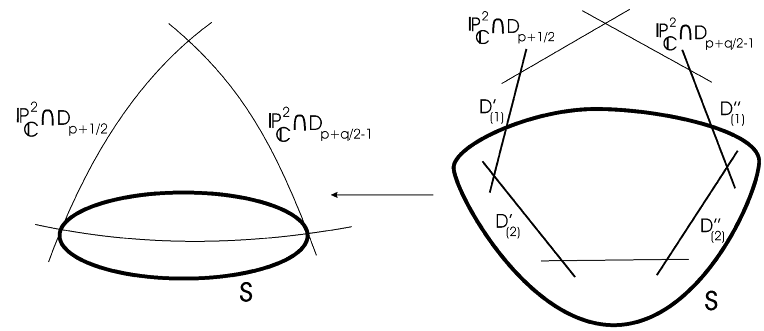

Fig. 3. Final form in Case 3.

4. Reduction of the separatrix to a canonical form

Let $\mathcal{F}$ be a germ of a singular foliation defined on $(\mathbb{C}^3, 0)$, and let $\pi : (M, D) \to (\mathbb{C}^3, 0)$ be the minimal reduction of the singularities of $\mathcal{F}$ in Cano–Cerveau sense, as described above [3]. Let $\tilde{\mathcal{F}}$ be the strict transform of the foliation $\mathcal{F}$ by π and let D_i be a component of the exceptional divisor D.

We recall, that a *Hopf fibration* $\mathcal{H}_{\mathcal{F}_\Omega}$ adapted to $\mathcal{F}_\Omega \in \sum_{pq}$ is a holomorphic transversal fibration $f : M \to D_i$ to the foliation $\mathcal{F}_\Omega$, i.e.:

(1) f is a retraction, more precisely, f is a submersion and $f|_{D_i} = \mathrm{Id}_{D_i}$.
(2) The fibers $f^{-1}(p)$ of $\mathcal{H}_{\mathcal{F}_\Omega}$ are contained in the separatrices of $\mathcal{F}_\Omega$, for all $p \in D_i \cap \mathrm{Sing}(\tilde{\mathcal{F}}_\Omega)$.

(3) The fibers $f^{-1}(p)$ of $\mathcal{H}_{\mathcal{F}_\Omega}$ are transversal to the foliation $\mathcal{F}_\Omega$, for all $p \in D_i \setminus \operatorname{Sing}(\tilde{\mathcal{F}}_\Omega)$.

We shall be interested in finding a Hopf fibration adapted to the foliation, relative to a particular component of the exceptional divisor. For, if p is even, call $\tilde{D} := D_{p/2}$, i.e., the last component obtained after the first sequence of line blow-ups. If p and q are odd, $\tilde{D} := P$, i.e., the projective obtained after the (only) point blow-up.

The task of finding a Hopf fibration associated to the foliation $\mathcal{F}_\Omega$ is not easy in the actual coordinates (x, y, z). As it is done in the two-dimensional case, to overcome this obstacle, we analyze the desingularization of $\mathcal{F}_\Omega$ in order to obtain a simple equation for the separatrices.

From Section 3 we know that the foliation $\mathcal{F}_\Omega \in \Sigma_{pq}$ defined by the one-form

$$\Omega = d\big(z^2 + \big(x^{p'} y^{q'}\big)^d\big) + \big(x^{p'} y^{q'}\big)^k h\big(x^{p'} y^{q'}\big)\, dz,$$

has a separatrix analytically equivalent to $S : z^2 + (x^{p'} y^{q'})^r = 0$ for some $r \in \mathbb{N}$. In order to find a Hopf fibration $\mathcal{H}_\mathcal{F}$ of the foliation $\mathcal{F}$, we need to normalize the one-form Ω such that the foliation defined by this normal form has exactly $S : z^2 + (x^{p'} y^{q'})^r = 0$ as separatrix, for certain r. So, the strict transformed of S by the desingularization is an hyperplane in these coordinates and invariant by Hopf fibration.

Proposition 1. *The foliation $\mathcal{F}_\Omega$ is analytically equivalent to a foliation defined by the one-form*

$$d\big(z^2 + \big(x^{p'} y^{q'}\big)^r\big) + g\big(x^{p'} y^{q'}, z\big).x^{p'} y^{q'} z\left(2\frac{dz}{z} - p'\frac{dx}{x} - q'\frac{dy}{y}\right),$$

where $r = d$ if $2k \geqslant d$ and $r = 2k$ if $2k < d$. In particular, the separatrix of the foliation $\mathcal{F}_\Omega$ is analytically equivalent to S: $z^2 + (x^{p'} y^{q'})^r = 0$.

Proof. The foliation $\mathcal{F}_\Omega$ is defined by the 1-form

$$\Omega = d\big(z^2 + \big(x^{p'} y^{q'}\big)^d\big) + \big(x^{p'} y^{q'}\big)^k h\big(x^{p'} y^{q'}\big)\, dz,$$

where $(p, q) = d$, $p = p'd$, $q = q'd$. That is, Ω is the pull-back of the 1-form $\Omega_0 = d(z^2 + u^d) + u^k h(u)\, dz$ by the ramified fibration

$$\rho : \big(\mathbb{C}^3, 0\big) \to \big(\mathbb{C}^2, 0\big), \quad (x, y, z) \to \big(x^{p'} y^{q'}, z\big) = (u, z).$$

The equation of the separatrices of Ω is of the form $z^2 + u^d + h.o.t. = 0$ (if d is even, this is a joint equation, i.e., the product of the two separatrices).

Using Weierstrass' preparation theorem, we can assume that the local equation of the separatrix is a polynomial in z: $z^2 + a(u)z + b(u) = 0$, with $a(0) = b(0) = 0$. If $\Phi_1(u, z) = (u, z - a(u)/2)$ is the Tschirnhausen transformation, then the pull-back $\Phi_1^*\Omega_0$ has $z^2 + c(u) = 0$ as separatrix, with $c(u) = b(u) - a(u)^2/4 = u^r f(u)$, $f(0) \neq 0$. If $d > 2$ (cuspidal case), we have that $\nu(a) > 1$, $\nu(b) > 2$, and then $r > 2$. In fact, $r = d$ when $2k > d$ or $r = 2k$ when $2k \leqslant d$ (see [1,2,5]). Similar computations are valid when $d = 1$ or $d = 2$ (in these cases, $2k \geqslant d$).

Let us write this reduced equation of the separatrices as

$$\frac{z^2}{f(u)} + u^r = 0,$$

and let $f(u)^{1/2}$ be a square root of the unit $f(u)$. If $\Phi_2(u,z) = (u, z \cdot f(u)^{1/2})$, and $\Phi := \Phi_1 \circ \Phi_2$, then $\Phi^*\Omega_0$ has $z^2 + u^r = 0$ as separatrix. This map has the form

$$\Phi(u,z) = \left(u, z \cdot f(u)^{1/2} - \frac{a(u)}{2}\right).$$

Consider the diagram

$$\begin{array}{ccc} \mathbb{C}^3 & \xrightarrow{\rho} & \mathbb{C}^2 \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle \Phi} \\ \mathbb{C}^3 & \xrightarrow{\rho} & \mathbb{C}^2. \end{array}$$

We want to find a diffeomorphism $F = (F_1, F_2, F_3)$ that makes commutative the diagram, i.e., such that

$$\left(F_1^{p'} F_2^{q'}, F_3\right) = \left(x^{p'} y^{q'}, zf\left(x^{p'} y^{q'}\right)^{\frac{1}{2}} - \frac{a(x^{p'} y^{q'})}{2}\right).$$

For, we may choose $F_1 = x$, $F_2 = y$, $F_3 = z \cdot f(x^{p'} y^{q'})^{1/2} - a(x^{p'} y^{q'})/2$. The form $\Phi^*\Omega_0$, having $z^2 + u^r = 0$ as a separatrix is, up to a unit, $d(z^2 + u^r) + g(u,z)(2u\,dz - dz\,du)$, so $F^*\Omega_0$ defines the same foliation that

$$d\left(z^2 + \left(x^{p'} y^{q'}\right)^r\right) + g\left(x^{p'} y^{q'}, z\right).x^{p'} y^{q'} z\left(2\frac{dz}{z} - p'\frac{dx}{x} - q'\frac{dy}{y}\right).$$

We reproduce part of the proof presented in [5] in order to find the transformation F fibered. □

As a consequence of this normal form for $\mathcal{F}_\Omega$, there exists coordinates (x,y,z), such that the separatrix S of the normal form is given by the equation: $z^2 + (x^{p'} y^{q'})^r = 0$, where r is as in the Proposition 1, and not only "analytically equivalent to." Now, we can find a Hopf fibration, from a holomorphic vector field X_1 for which S is an invariant set, that is,

$$X_1 = \begin{cases} x\frac{\partial}{\partial x} + \frac{p}{2} z \frac{\partial}{\partial z}, & p \text{ is even}, \\ x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \left(\frac{p+q}{2}\right) z \frac{\partial}{\partial z}, & p \text{ and } q \text{ are odd}. \end{cases}$$

So, we have that the Hopf fibration $\mathcal{H}_{\mathcal{F}_\Omega}$ ($f : M \to \tilde{D}$), adapted to the foliation defined by the one-form $\Omega \in \sum_{pq}$ will be determined (not uniquely) by a linearizable singularity of a holomorphic vector field $X = X_1 + X_2 + \cdots$.

Having defined a Hopf fibration adapted to $\mathcal{F}_\Omega$, we can define the holonomy of the leaf $\tilde{D} \setminus \mathrm{Sing}(\tilde{\mathcal{F}}_\Omega)$ respect to this fibration. In order to determine it, we fix a point $p_0 \in \tilde{D} \setminus \mathrm{Sing}(\tilde{\mathcal{F}}_\Omega)$. Over this point we have a transversal $f^{-1}(p_0)$ and by path lifting construction, a representation of the fundamental group of $\tilde{D} \setminus \mathrm{Sing}(\tilde{\mathcal{F}}_\Omega)$ in $\mathrm{Diff}(\mathbb{C}, 0)$ is determined, denoted by $\mathcal{H}_{\Omega,\tilde{D}}$

$$\mathcal{H}_{\Omega,\tilde{D}} : \pi_1\big(\tilde{D} \setminus \mathrm{Sing}(\tilde{\mathcal{F}}_\Omega), p_0\big) \to \mathrm{Diff}(\mathbb{C}, 0).$$

This representation is independent of p_0 modulo conjugacy and its image will be called the *exceptional holonomy* and denoted $H_{\Omega,\tilde{D}}$.

5. Classification of the singularities

From Section 3 we know that the homotopy group $\pi_1(\tilde{D} \setminus \mathrm{Sing}(\tilde{\mathcal{F}}_\Omega); t_0)$ can be generated by two elements α and β in all the cases considered, with different relations in each case:

(1) If p and q are even: $\alpha^{\frac{q}{2}}\beta = \beta\alpha^{\frac{q}{2}}$,
(2) If p is even and q is odd: $\alpha^q = \beta^2$,
(3) If p and q are odd: $\alpha^2\beta = \beta\alpha^2$.

If γ is an element of the homotopy group, let us denote h_γ its image by the map $\mathcal{H}_{\Omega,\tilde{D}}$ in the exceptional holonomy. This holonomy can be generated by h_α, h_β, which at least satisfy the same relations than α, β. But in some cases, these relations may be improved. The following proposition collects some of these improvements:

Proposition 2.

(1) *If p is even,* $h_\alpha^{\frac{p}{2}} = \mathrm{id}$.
(2) *If p is even and q is odd,* $h_\alpha^{\frac{p}{2}} = h_\beta^{p'} = \mathrm{id}$.

Proof. Consider p even. After $\frac{p}{2}$ blow-ups, the strict transform of the separatrix $\mathcal{S}$ is given by a surface analytically equivalent to $t_{p/2}^2 + y^q = 0$. This singular surface is a cylinder over a curve, that is either a cuspidal curve of characteristic pair $(2, q)$ or a couple of regular curves tangent at the origin at order $\frac{q}{2}$. Applying Picard–Lefschetz techniques, it can be seen that the loop α is a simple curve contained in the plane $y = \varepsilon$, with $|\varepsilon|$ small enough, that turns around the points $(t, y) = (\pm i \cdot \varepsilon^{\frac{q}{2}}, \varepsilon)$. Thus, the holonomy h_α is completely determined by the holonomy of a loop that turns around the line $D_{p/2-1} \cap D_{p/2}$. Along this line, the foliation is a reduced foliation of dimensional type 2 (in fact, we are in presence of a Kupka phenomenon), and its analytic type is determined by a two-dimensional section transversal to the y-axis. This foliation has a linearizable, periodic holonomy, and $h'_\alpha(0) = e^{-2\pi i \cdot \frac{p-2}{p}}$.

If, moreover, q is odd, the periodicity of h_α implies the periodicity of h_β, and so, h_β is linearizable, $h'_\beta(0) = e^{2\pi i q'/p'}$. Nevertheless, it does not mean necessary that the holonomy group $H_{\Omega,\tilde{D}}$ is linearizable, since in particular we don't know if it is abelian or not. □

The following theorem contains the main result of the paper. In the proof, several techniques from [1,5,6,15] are frequently used, and we shall not enter in details about them.

Theorem 1. *Let Ω_1, Ω_2 be elements of $\sum_{pq}$. Consider the foliations $\mathcal{F}_{\Omega_1}$ and $\mathcal{F}_{\Omega_2}$, and their exceptional holonomies $H_{\Omega_i,\tilde{D}} = \langle h^i_\alpha, h^i_\beta \rangle$, $i = 1, 2$, defined as before. Then, the foliations are analytically conjugated if and only if the couples (h^i_α, h^i_β) are also analytically conjugated, i.e., if and only if there exists $\Psi \in \mathrm{Diff}(\mathbb{C}, 0)$ such that $\Psi^* h^1_\gamma = h^2_\gamma$, where $\gamma = \alpha, \beta$.*

Proof. If the foliations are conjugated then clearly their exceptional holonomies are also conjugated. Conversely, suppose that the exceptional holonomies are conjugated via Ψ, and let $\tilde{\mathcal{F}}_{\Omega_1}$, $\tilde{\mathcal{F}}_{\Omega_2}$ be the desingularized, reduced foliations. Because of the existence of the Hopf fibration relative to $\tilde{D}$, Ψ can be extended to a neighbourhood of $\tilde{D}$, away from the singular points. These singular points are the intersections of $\tilde{D}$ with the other components of the divisor, and with the separatrix (the separatrices in the even–even case). All these points are singular points of dimensional types two or three, and for all of them, $\tilde{D}$ is a strong separatrix. In this situation, the conjugation of the holonomies of $\tilde{D}$ implies conjugation of the reduced foliations in a neighbourhood of the singular points [6].

So, we have that $\tilde{\mathcal{F}}_{\Omega_1}$, $\tilde{\mathcal{F}}_{\Omega_2}$ are conjugated in a neighbourhood of $\tilde{D}$. Suppose now that p is even. We need to conjugate the foliations also in a neighbourhood of $D_1, \ldots, D_{p/2-1}$. As D_1 is simply connected, its holonomy is trivial. So, the holonomy of D_2, generated by one loop around $L_1 = D_1 \cap D_2$ is periodic (the argument is the same as in [14]). The same argument shows that D_i has a periodic holonomy, $1 \leqslant i < \frac{p}{2}$, and so, the foliations have first integrals in a neighbourhood of each L_i, $1 \leqslant i < \frac{p}{2}$. These are points of dimensional type two. By analogous reasons as in the two-dimensional case, Ψ can be extended to a neighbourhood of the exceptional divisor, so, $\mathcal{F}_{\Omega_1}$, $\mathcal{F}_{\Omega_2}$ are conjugated outside the singular locus, which has codimension two. We conclude using Hartogs' theorem to extend the conjugation to a neighbourhood of the origin.

Suppose now that p, q are odd. $\tilde{\mathcal{F}}_{\Omega_1}$ and $\tilde{\mathcal{F}}_{\Omega_2}$ are conjugated in a neighbourhood of $\tilde{D}$ (that is a projective $\mathbb{P}^2_{\mathbb{C}}$ in this case). The fundamental group of $D_{(p-1)/2}$ is generated by only one loop, that, after the resolution, can be seen as a loop around $D_{(p-1)/2} \cap D''_{(1)}$. This is one of the loops that generates the holonomy of $D_{(p-1)/2}$ locally at the reduced singular points, and following similar arguments as in the preceding cases, and as the two-dimensional case, the foliation is linearizable around these points. Let us detail, in this case, how the use of first integrals allows the extension of the conjugation.

Consider, for instance, the singular point $D_{(p-1)/2} \cap D_{(p+q)/2-2} \cap D_{(p+q)/2-1}$, with coordinates (x', s', t') as in Fig. 4.

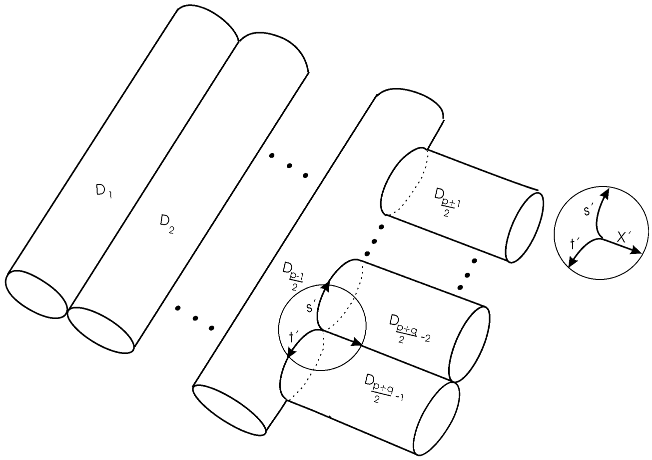

Fig. 4. Coordinates (x', s', t') the singular point $D_{(p-1)/2} \cap D_{(p+q)/2-2} \cap D_{(p+q)/2-1}$.

We have a conjugation Ψ between the foliations $\mathcal{F}_{\Omega_1}$ and $\mathcal{F}_{\Omega_2}$ defined over an annulus

$$\{|x'| < \varepsilon\} \times \{|s'| < \varepsilon\} \times \{c_1 < |t'| < c_2\}$$

that respects the fibration. In these coordinates, the foliation is given by $x't' = \text{cst.}$, $s't'^2 = \text{cst.}$, and the first integral of $\mathcal{F}_{\Omega_j}$ is $x'^{p-1}s'^{q-1}t'^{q-3} \cdot U_j(x', s', t')$, $U_j(0) = 1$. This first integral may be extended to

$$\{|x'| < \varepsilon\} \times \{|s'| < \varepsilon\} \times \{|t'| < c\},$$

where $c_1 < c$, eventually making ε small enough. We look first for a diffeomorphism Ψ_j that transforms this first integral into $x'^{p-1}s'^{q-1}t'^{q-3}$, respecting the fibration. This diffeomorphism is

$$\Psi_j(x', s', t') = (x' \cdot V_{1j}, s' \cdot V_{2j}, t' \cdot V_{3j}),$$

and the conditions mean that

$$V_1^{p-1} \cdot V_2^{q-1} \cdot V_3^{q-3} = U_j, \qquad V_1 \cdot V_3 = 1, \qquad V_2 \cdot V_3^2 = 1.$$

So, $V_{1j} = U_j^{-(p+q)}$; $V_{2j} = U_j^{-2(p+q)}$; $V_{3j} = U_j^{p+q}$.

Consider now the diffeomorphism $\tilde{\Psi} := \Psi_1 \circ \Psi \circ \Psi_2^{-1}$. It respects both the fibration and the first integral $x'^{p-1}s'^{q-1}t'^{q-3}$. Write $\tilde{\Psi} = (\theta_1, \theta_2, \theta_3)$. The conditions above mean that

$$\theta_1 \cdot \theta_3 = x't', \qquad \theta_2 \cdot \theta_3^2 = s't'^2,$$
$$\theta_1^{p-1} \cdot \theta_2^{q-1} \cdot \theta_3^{q-3} = x'^{p-1}s'^{q-1}t'^{q-3} \cdot g\left(x'^{p-1}s'^{q-1}t'^{q-3}\right),$$

with $g(0) \neq 0$. As before, we have that $\theta_1 = x' \cdot g^{-(p+q)}$; $\theta_2 = s' \cdot g^{-2(p+q)}$; $\theta_3 = t' \cdot g^{p+q}$.

This is a map defined, in the considered chart, over a set of the type $\{|x'^{p-1}s'^{q-1}t'^{q-3}| < \varepsilon\}$, and this set intersects the domain of definition of Ψ. So, $\Psi = \Psi_1^{-1} \circ \tilde{\Psi} \circ \Psi_2$ may be extended to a neighbourhood of $L_{(p+q)/2-1} = D_{(p-1)/2} \cap D_{(p+q)/2-1}$.

Repeating the argument, we extend the conjugation to a neighbourhood of $D_{(p-1)/2} \cap (D_{(p+1)/2} \cup \cdots \cup D_{(p+q)/2})$. Now, similar arguments as in the preceding situations, and as in the two-dimensional case, allow us to extend Ψ to a neighbourhood of the exceptional divisor, and again Hartogs' theorem completes the result. □

Acknowledgments

This paper was begun during the stay of the first author at the University of Valladolid and was finished at the Instituto de Matemática y Ciencias Afines (IMCA), of Lima, and the Pontificia Universidad Católica de Perú. We want to thank these institutions. We also want to thank the referee for his suggestions.

References

[1] M. Berthier, R. Meziani, P. Sad, On the classification of nilpotent singularities, Bull. Sci. Math. 123 (1999) 351–370.
[2] F. Cano, Reduction of the singularities of foliations and applications, in: Collection: Singularities Symposium – Lojasiewicz 70, Krakow, 1996; Warsaw, 1996, in: Banach Center Publ., vol. 44, Polish Acad. Sci., Warsaw, 1998, pp. 51–71.
[3] F. Cano, D. Cerveau, Desingularization of non-dicritical holomorphic foliations and existence of separatrices, Acta Math. 169 (1992) 1–103.
[4] D. Cerveau, A. Lins-Neto, Formes tangentes à des actions commutatives, Ann. Fac. Sci. Toulouse VI (1984) 51–85.
[5] D. Cerveau, R. Moussu, Groupes d'automorphismes de $(\mathbb{C}, 0)$ et équations différentielles $y\,dy + \cdots = 0$, Bull. Soc. Math. France 116 (1988) 459–488.
[6] D. Cerveau, J. Mozo Fernández, Classification analytique des feuilletages singuliers réduits de codimension 1 en dimension $n \geqslant 3$, Ergodic Theory Dynam. Systems 22 (2002) 1041–1060.
[7] V. Cossart, Desingularization in dimension two, in: Resolution of Surface Singularities, in: Lecture Notes in Math., vol. 1101, Springer, 1984, pp. 79–98.
[8] J. Giraud, Desingularization in low dimension, in: Resolution of Surface Singularities, in: Lecture Notes in Math., vol. 1101, Springer, 1984, pp. 51–78.
[9] H. Hironaka, Desingularization of excellent schemes, in: Resolution of Surface Singularities, in: Lecture Notes in Math., vol. 1101, Springer, 1984, pp. 99–132.
[10] F. Lazzeri, Analytic singularities, in: Singularities of Analytic Spaces, CIME, Bressanone, 1974, Edizioni Cremonese, Roma, 1975.
[11] J. Lipman, Quasi-ordinary singularities of surfaces in $\mathbb{C}^3$, Proc. Sympos. Pure Math. 40 (2) (1983) 161–172.
[12] F. Loray, A preparation theorem for codimension one foliations, preprint, ArXiv: math.DG/0402411, February 25, 2004.
[13] B. Malgrange, Frobenius avec singularités. Codimension un, Inst. Hautes Études Sci. Publ. Math. 46 (1976) 163–173.
[14] J.F. Mattei, R. Moussu, Holonomie et intégrales premiéres, Ann. Sci. École Norm. Sup. 13 (1980) 469–523.
[15] R. Meziani, Classification analytique d'équations différentielles $y\,dy + \cdots = 0$ et espaces de modules, Bol. Soc. Brasil. Mat. 27 (1996) 23–53.
[16] R. Meziani, Problème de modules pour des équations différentielles dégénérées de $(\mathbb{C}^2, 0)$, PhD thesis, Univ. Rennes I, 1992.

[17] R. Moussu, Sur l'existence d'intégrales premières pour un germe de forme de Pfaff, Ann. Inst. Fourier 26 (2) (1976) 171–220.
[18] R. Moussu, Holonomie evanescente des équations différentielles dégénérées transverses, in: Singularities and Dynamical Systems, North-Holland, 1985, pp. 151–173.
[19] G. Reeb, Sur certaines propriétés topologiques des variétés feuilletées, thesis, Herman, 1952.
[20] F. Takens, Singularities of vectors fields, Inst. Hautes Études Sci. Publ. Math. 43 (1974) 47–100.