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Graphs whose endomorphism monoids are regular☆

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Abstract

In this paper, we give several approaches to construct new End-regular (-orthodox) graphs by means of the join and the lexicographic product of two graphs with certain conditions. In particular, the join of two connected bipartite graphs with a regular (orthodox) endomorphism monoid is explicitly described.

MSC: 05C25; 20M20

Keywords: Endomorphism monoids; Join of graphs; Lexicographic product of graphs; Regular; Orthodox

1. Introduction and preliminary concepts

Endomorphism monoids of graphs (or just monoids of graphs) are a generalizations of automorphism groups of graphs (or just groups of graphs). In recent years much attention has been paid to endomorphism monoids of graphs and many interesting results concerning graphs and their endomorphism monoids have been obtained. The aim of the research in this line is to establish the relationship between graph theory and algebraic theory of semigroup and to apply the theory of semigroups to graph theory. Just as Petrich and Reilly pointed out in [13], in the great range of special classes of semigroups, regular semigroups take a central position from the point of view of richness of their structural "regularity". So it is natural to ask for which graph G the endomorphism monoid of G is regular (such an open question was raised in [11]). However, it seems difficult to obtain a general answer to this question. So the strategy for solving this question is finding various kinds of regularity conditions for various kinds of graphs. In [14], the connected bipartite graphs whose endomorphism monoids are regular were explicitly found. The joins of two trees with regular endomorphism monoids were characterized in [9]. The split graphs with regular endomorphism monoids were studied in [10]. In this paper, we give several approaches to construct End-regular graphs by means of joins and lexicographic products of two graphs with certain conditions. In particular, we determine the End-regular (End-orthodox) joins of two connected bipartite graphs.

The graphs considered in this paper are finite undirected graphs without loops and multiple edges. Let X be a graph. The vertex set of X is denoted by $V(X)$ and the edge set of X is denoted by $E(X)$. If two vertices x_1 and x_2 are adjacent

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in graph *X*, the edge connecting x_1 and x_2 is denoted by $\{x_1, x_2\}$ and write $\{x_1, x_2\} \in E(X)$. For a vertex v of *X*, denote by $N_X(v)$ (or briefly by $N(v)$) the set $\{x \in V(X)|\{x, v\} \in E(X)\}$ and call it the *neighborhood* of v in *X*. A subgraph *H* is called an *induced subgraph* of *X* if for any $a, b \in V(H)$, $\{a, b\} \in E(H)$ if and only if $\{a, b\} \in E(X)$. A graph *X* is called *bipartite* if *X* has no odd cycle. It is known that, if a graph *X* is a bipartite graph, then its vertex set can be partitioned into two disjoint non-empty subsets, such that no edge joins two vertices in the same set. A set $S \subseteq V(X)$ is called an *independent set* of *X* if for any $a, b \in S$, $\{a, b\} \notin E(X)$. We call the length of a shortest cycle in *X* the *girth* of *X* and denote it by $\mathrm{girth}(X)$. We call the length of a shortest odd cycle in *X* the *odd girth* of *X*.

Let X_1 and X_2 be two graphs. The *join* of X_1 and X_2, denoted by $X_1 + X_2$, is a graph such that $V(X_1 + X_2) = V(X_1) \cup V(X_2)$ and $E(X_1 + X_2) = E(X_1) \cup E(X_2) \cup \{\{x_1, x_2\}|x_1 \in V(X_1), x_2 \in V(X_2)\}$. The *lexicographic product* of X_1 and X_2, denoted by $X_1[X_2]$, is a graph with vertex set $V(X_1[X_2]) = V(X_1) \times V(X_2)$, and edge set $E(X_1[X_2]) = \{\{(x, y), (x_1, y_1)\}|\{x, x_1\} \in E(X_1)$, or $x = x_1$ and $\{y, y_1\} \in E(X_2)\}$. The *generalized lexicographic product* of a graph *G* with a family of graphs $\{B_i|i \in V(G)\}$, denoted by $G(B_i)_{i\in V(G)}$, is defined as a graph whose vertex set $V(G(B_i)_{i\in V(G)}) = \{(x, y_x)|x \in V(G), y_x \in V(B_x)\}$, and $\{(x, y_x), (x', y'_{x'})\} \in E(G(B_i)_{i\in V(G)})$if and only if $\{x, x'\} \in E(G)$, or $x = x'$ and $\{y_x, y'_{x'}\} \in E(B_x)$.

Let *X* and *Y* be two graphs. A mapping *f* from $V(X)$ to $V(Y)$ is called a *homomorphism* if $\{a, b\} \in E(X)$ implies that $\{f(a), f(b)\} \in E(Y)$. A homomorphism from *X* to itself is called an endomorphism of *X*. An endomorphism *f* is called a *half-strong endomorphism* if $\{f(a), f(b)\} \in E(X)$ implies that there exist $x_1, x_2 \in V(X)$ with $f(x_1) = f(a)$ and $f(x_2) = f(b)$ such that $\{x_1, x_2\} \in E(X)$. An endomorphism *f* is called a *strong endomorphism* if $\{f(a), f(b)\} \in E(X)$ implies that any preimage of $f(a)$ is adjacent to any preimage of $f(b)$. An endomorphism *f* is called an *automorphism* if *f* is bijective and f^{-1} is an endomorphism. By $\mathrm{End}(X)$, $h\mathrm{End}(X)$, $s\mathrm{End}(X)$ and $\mathrm{Aut}(X)$ we, respectively, denote the set of endomorphisms, half-strong endomorphisms, strong endomorphisms and automorphisms of graph *X*. It is known that $\mathrm{End}(X)$ and $s\mathrm{End}(X)$ form monoid and $\mathrm{Aut}(X)$ forms a group.

A subgraph of *X* is called the *endomorphic image* of *X* under *f*, denoted by I_f, if $V(I_f) = f(V(X))$ and $\{f(a), f(b)\} \in E(I_f)$ if and only if there exist $c \in f^{-1}(f(a))$ and $d \in f^{-1}(f(b))$ such that $\{c, d\} \in E(X)$. Let *f* be an endomorphism of *X* and H_1, H_2 be two induced subgraph of *X*. If $f(H_1) \subseteq H_2$, we will denote by $f|_{H_1}$ the mapping from $V(H_1)$ to $V(H_2)$ such that $f|_{H_1}(x) = f(x)$ for any $x \in V(H_1)$. Clearly, $f|_{H_1}$ is a homomorphism from H_1 to H_2. Let *f* be an endomorphism of *X*. By ρ_f we denote the equivalence relation on $V(X)$ induced by *f*, i.e. for any $a, b \in V(X)$, $(a, b) \in \rho_f$ if and only if $f(a) = f(b)$. Denote by $[a]_{\rho_f}$ the equivalence class of $a \in V(X)$ under ρ_f.

An element *a* of a semigroup *S* is called *regular* if there exists $x \in S$ such that $axa = a$. A semigroup *S* is called regular if all its elements are regular. A semigroup *S* is called *orthodox* if *S* is regular and the set of all idempotents forms a subsemigroup, that is, a regular semigroup is orthodox if the product of any two of its idempotents is still an idempotent. A graph *X* is said to be End-regular (End-orthodox) if its endomorphism monoid $\mathrm{End}(X)$ is regular (orthodox). Clearly, End-orthodox graphs are End-regular.

For undefined notations and terminology in this paper the reader refer to [3,5].

We list some known results which will be used in the sequel.

Lemma 1.1 (Li [9]). *Let X be a graph and let* $f \in \mathrm{End}(X)$. *Then*

(1) $f \in h\mathrm{End}(X)$ *if and only if* I_f *is an induced subgraph of X.*
(2) *If f is regular*, *then* $f \in h\mathrm{End}(X)$.

Lemma 1.2 (Li [8]). *Let X be a graph and let* $f \in \mathrm{End}(X)$. *Then f is regular if and only if there exist* $g, h \in \mathrm{Idpt}(X)$ *such that* $\rho_g = \rho_f$ *and* $I_h = I_f$.

Lemma 1.3 (Li [9]). *Let X and Y be two graphs. If* $X + Y$ *is End-regular*, *then both X and Y are End-regular*.

Lemma 1.4 (Li [9]). *Let X be a graph. Then X is End-regular if and only if* $X + K_n$ *is End-regular for any* $n \geqslant 1$.

Lemma 1.5 (Hou et al. [4]). *Let X and Y be two graphs. If* $X + Y$ *is End-orthodox*, *then both of X and Y are End-orthodox.*

Lemma 1.6 (*Wilkeit [14]*). *Let X be a connected bipartite graph. Then X is End-regular if and only if X is one of the following graphs*:

(1) *Complete bipartite graph.*
(2) *Tree T with* $d(T) = 3$.
(3) *Cycle* C_6 *and* C_8.
(4) *Path with five vertices, i.e.* P_5.

Lemma 1.7 (*Li [9]*). *Let* T_1 *and* T_2 *be two trees. Then* $T_1 + T_2$ *is End-regular if and only if either* (1) *one of them is End-regular and the other is* K_1 *or* K_2, *or* (2) $d(T_1) = d(T_2) = 2$.

Lemma 1.8 (*Fan [2]*). *Let X be a bipartite graph. Then X is End-orthodox if and only if X is one of the following graphs*: $K_1, K_2, P_3, P_4, C_4, 2K_1, K_1 \cup K_2$.

2. Endomorphism regular graphs

In this section, we shall give various End-regular graphs by means of the join and the lexicographic product of two graphs with certain conditions.

Recall that a graph X is said to be *E–S*-unretractive if $\mathrm{End}(X) = s\mathrm{End}(X)$. It is known that if X is finite, then $s\mathrm{End}(X)$ is regular. Thus *E–S*-unretractive graphs give a family of End-regular graphs. The following lemma gives some results about *E–S*-unretractive graph, which will be used in the sequel.

Lemma 2.1 (*Knauer [6]*). *Let X and Y be two E–S-unretractive graph. Then*:

(1) $\mathrm{End}(X + Y) = s\mathrm{End}(X + Y)$.
(2) $\mathrm{End}(C_{2m+1}[X]) = s\mathrm{End}(C_{2m+1}[X])$.
(3) $\mathrm{End}(X(Y_x)_{x\in X}) = s\mathrm{End}(X(Y_x)_{x\in X})$, *where* $(Y_x)_{x\in X}$ *be graphs with* $E(Y_x) = \phi$ *for all* $x \in X$.

A *proper coloring* of a graph X is a map from $V(X)$ into some finite set of colors such that no two adjacent vertices are assigned the same colors. If X can be properly colored with a set of k colors, then we say that X can be properly k-colored. The least value of k for which X can be properly k-colored is the *chromatic number* of X, and is denoted by $\chi(X)$. We know that if there is a homomorphism from X to Y, then $\chi(X) \leqslant \chi(Y)$. A graph X is *unretractive*, if $\mathrm{End}(X) = \mathrm{Aut}(X)$. A subgraph Y of X is a *core* of X if Y is a unretractive and there is a homomorphism from X to Y. Let X and Y be two graphs. We say X and Y are *homomorphically equivalent* if there is a homomorphism from X to Y, and there is a homomorphism from Y to X. It is known that two graphs X and Y are homomorphically equivalent if and only if their cores are isomorphic.

Lemma 2.2. *Let X and Y be two* K_3*-free graphs. If both of them are non-bipartite, then for any endomorphism f of* $X + Y$, *either* $f(X) \subseteq X$ *and* $f(Y) \subseteq Y$, *or* $f(X) \subseteq Y$ *and* $f(Y) \subseteq X$.

Proof. Let f be an endomorphism of $X + Y$. We show that either $f(X) \subseteq X$, or $f(X) \subseteq Y$. Otherwise, there exist two vertices $x_1, x_2 \in V(X)$ such that $f(x_1) \in X$ and $f(x_2) \in Y$.

Since Y is not bipartite, then Y has an odd cycle as its subgraph, by the definition of endomorphism, $f(Y)$ also has an odd cycle as its subgraph, thus $f(Y)$ either has an edge in X, or has an edge in Y. Without loss of generality, suppose $\{f(y_1), f(y_2)\} \in E(Y)$ for some $y_1, y_2 \in V(Y)$. Note that $\{f(y_1), f(x_2)\} \in E(Y)$ and $\{f(y_2), f(x_2)\} \in E(Y)$, then $f(x_2), f(y_1), f(y_2)$ form a triangle in Y. It is a contradiction to Y being K_3-free.

A similar argument will show that for any endomorphism $f \in \mathrm{End}(X + Y)$, either $f(Y) \subseteq X$, or $f(Y) \subseteq Y$. Now we claim that if $f(X) \subseteq X$, then $f(Y) \nsubseteq X$. Otherwise, there exists a homomorphism from $X + Y$ to X, so $\chi(X + Y) = \chi(X) + \chi(Y) \leqslant \chi(X)$. A contradiction. Similarly, if $f(X) \subseteq Y$, then $f(Y) \nsubseteq Y$. Now the assertion follows immediately. $\square$

Lemma 2.3. *Let X and Y be two End-regular graphs. If for any* $f \in \mathrm{End}(X + Y)$, $f(X) \subseteq X$ *and* $f(Y) \subseteq Y$, *then* $X + Y$ *is End-regular*.

Proof. Let $f \in \mathrm{End}(X+Y)$. To show that f is regular, we only need to show that there exist two idempotents g and h in $\mathrm{End}(X+Y)$ such that $\rho_g = \rho_f$ and $I_h = I_f$.

Since $f(X) \subseteq X$, $f|_X$ is an endomorphism of X. As X is End-regular, there exist two idempotents g_1 and h_1 in $\mathrm{End}(X)$ such that $\rho_{g_1} = \rho_{f|X}$ and $I_{h_1} = I_{f|X}$. Similarly, there exist two idempotents g_2 and h_2 in $\mathrm{End}(Y)$ such that $\rho_{g_2} = \rho_{f|Y}$ and $I_{h_2} = I_{f|Y}$.

Let g be a mapping from $V(X+Y)$ to itself defined by

$$g(x) = \begin{cases} g_1(x) & \text{if } x \in V(X), \\ g_2(x) & \text{if } x \in V(Y). \end{cases}$$

Then $g \in \mathrm{End}(X+Y)$, $g^2 = g$ and $\rho_g = \rho_f$.

Let h be a mapping from $V(X+Y)$ to itself defined by

$$h(x) = \begin{cases} h_1(x) & \text{if } x \in V(X), \\ h_2(x) & \text{if } x \in V(Y). \end{cases}$$

Then $h \in \mathrm{End}(X+Y)$, $h^2 = h$ and $I_f = I_h$, as required. $\square$

Theorem 2.4. *Let X and Y be two K_3-free End-regular non-bipartite graphs. If the cores of X and Y are not isomorphic, then $X+Y$ is End-regular.*

Proof. Let f be an endomorphism of $X+Y$. By Lemma 2.2, either $f(X) \subseteq X$ and $f(Y) \subseteq Y$, or $f(Y) \subseteq X$ and $f(X) \subseteq Y$. In the second case, $f|_X$ is a homomorphism from X to Y and $f|_Y$ is a homomorphism from Y to X. Thus X and Y are homomorphically equivalent and so the cores of them are isomorphic. A contradiction. Hence $f(X) \subseteq X$ and $f(Y) \subseteq Y$. By Lemma 2.3, $\mathrm{End}(X+Y)$ is regular. $\square$

Let X and Y be two graphs. Recall that if $\chi(X) \neq \chi(Y)$, or X and Y have different odd girth, then the cores of X and Y are not isomorphic. As a direct consequence of Theorem 2.4, we have

Corollary 2.5. *Let X and Y be two K_3-free End-regular non-bipartite graph. If $\chi(X) \neq \chi(Y)$, or X and Y have different odd girth, then $X+Y$ is End-regular.*

Example 2.6. Let X and Y be two K_3-free non-bipartite graph as shown in Fig. 1. Theorem 2.4 shows that if the cores of X and Y are not isomorphic, then $X+Y$ is End-regular if and only if both of X and Y are End-regular. In the following, we give an example to show that if the cores of X and Y are isomorphic, then both of X and Y are End-regular may not imply that $X+Y$ is End-regular. Let X and Y be two graphs as shown in the following:

Then both of X and Y are End-regular. Now let

$$f = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 & y_8 \\ y_1 & y_2 & y_3 & y_2 & y_1 & y_8 & y_7 & y_8 & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 \end{pmatrix}.$$

It is a routine matter to check f is an endomorphism of $X+Y$. Now $\{x_4, x_5\} \in E(X+Y)$, $f^{-1}(x_4) = y_4$ and $f^{-1}(x_5) = y_5$. Since $\{y_4, y_5\} \notin E(X+Y)$, I_f is not an induced subgraph of $X+Y$. By Lemma 1.1 f is not regular.

Theorem 2.7. *Let X and Y be two K_3-free End-regular non-bipartite graph. If X or Y is unretractive, then $X+Y$ is End-regular.*

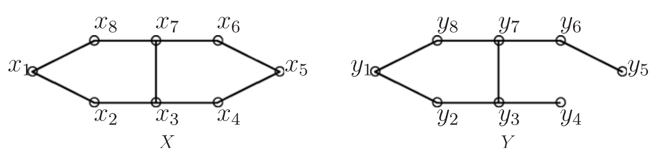

Fig. 1. Illustration for Example 2.6.

Proof. Without loss of generality, suppose X is unretractive. If the core of Y is not isomorphic to X, then $X + Y$ is End-regular by Theorem 2.4.

If the core of Y is isomorphic to X, then for any $f \in \mathrm{End}(X + Y)$, either $f(X) \subseteq X$ and $f(Y) \subseteq Y$, or $f(X) \subseteq Y$ and $f(Y) \subseteq X$. In the first case, by the proof of Lemma 2.3, f is regular. In the second case, $f(X)$ is a core of Y and $f(Y) = X$. Thus $f(X + Y)$ is isomorphic to $X + X$. So it is an induced subgraph of $X + Y$. Define a mapping from $X + Y$ to itself by

$$g(x) = \begin{cases} x & \text{if } x \in V(X), \\ f^{-1}(f(x)) \cap f(X) & \text{if } x \in V(Y). \end{cases}$$

Then $g \in \mathrm{End}(X + Y)$, $g^2 = g$, $\rho_g = \rho_f$ and $I_g = I_f$. By Lemma 1.2, f is regular. Hence $X + Y$ is End-regular. □

Theorem 2.8. *Let X be a bipartite graph and Y be a K_3-free non-bipartite graph. Then $X + Y$ is End-regular if and only if both of X and Y are End-regular.*

Proof. The direct part follows directly from Lemma 1.3.

Conversely, let f be an endomorphism of $X + Y$. We prove that $f(X) \subseteq X$ and $f(Y) \subseteq Y$. There are two cases.

Case 1: $E(X) = \phi$. First we show that $f(X) \subseteq X$. Otherwise, there exist a vertex $x_1 \in V(X)$ such that $f(x_1) \in Y$. Since Y is non-bipartite, Y contains an odd cycle. So $f(Y)$ also has an odd cycle. As $E(X) = \phi$, then $f(Y)$ has an edge in Y, say $\{f(y_1), f(y_2)\}$. Then $f(y_1)$, $f(y_2)$, $f(x_1)$ form a triangle in Y. A contradiction.

Next we prove $f(Y) \subseteq Y$. Otherwise, there exists a vertex $y_1 \in V(Y)$ such that $f(y_1) \in X$. Since $\{y_1, x\} \in E(X+Y)$ for any $x \in V(X)$, then $\{f(y_1), f(x)\} \in E(X + Y)$. Note that $f(y_1), f(x) \in V(X)$, so $\{f(y_1), f(x)\} \in E(X)$. This contradicts to $E(X) = \phi$.

Case 2: $E(X) \neq \phi$. First we show that $f(X) \subseteq X$. Assume that $f(X) \nsubseteq X$. Then either $f(X) \subseteq Y$, or there exist two vertices x_1 and x_2 in $V(X)$ such that $f(x_1) \in X$ and $f(x_2) \in Y$. In the first case, since X contains at least one edge, $f(X)$ contains at least one edge, say $\{a, b\}$. Now we claim $f(Y) \subseteq X$. Otherwise, there exists a vertex $y_0 \in V(Y)$ such that $f(y_0) \in V(Y)$, then $a, b, f(y_0)$ form a triangle. A contradiction. Hence $f|_Y$ is a homomorphism from Y to X, and we have $\chi(Y) \leqslant \chi(X)$. Since X is bipartite and Y is non-bipartite, so $\chi(X) = 2$ and $\chi(Y) \geqslant 3$. This is a contradiction. In the second case, since Y contains an odd cycle, $f(Y)$ also contains an odd cycle. Thus $f(Y)$ either has an edge in X or has an edge in Y. With a similar proof as that of Lemma 2.2, we have either there exists a triangle in X or there exists a triangle in Y. A contradiction in both cases. Hence $f(X) \subseteq X$.

Next we prove $f(Y) \subseteq Y$. Otherwise, there exists a vertex $y_1 \in V(Y)$ such that $f(y_1) \in V(X)$ and $f(y_1)$ is adjacent to every vertex of $f(X)$. Since $f(X)$ contains at least one edge, X contains a triangle. A contradiction. □

Now the assertion follows from Lemma 2.3.

Theorem 2.9. *Let X be a K_3-free End-regular non-bipartite graph and Y be an End-regular graph which has at least one triangle. If $\chi(Y) < \chi(X) + 1$, then $X + Y$ is End-regular.*

Proof. We show that $f(Y) \nsubseteq X$. Otherwise, there exists a homomorphism from Y to X. Since any homomorphism f maps a triangle to a triangle and Y has at least one triangle, then X also has at least one triangle. A contradiction. Hence either $f(Y) \subseteq Y$, or there exist two vertices y_1 and y_2 in Y such that $f(y_1) \in Y$ and $f(y_2) \in X$.

In the second case, if $f(X) \subseteq X$, then $f|_X$ is a homomorphism from X to itself, so $\chi(X) = \chi(I_{f|_X})$. Note that $f(y_2)$ is adjacent to every vertex of $I_{f|_X}$, then $\chi(X) \geqslant \chi(I_{f|_X}) + 1$. A contradiction. If $f(X) \subseteq Y$, then $f|_X$ is a homomorphism from X to Y and $f(y_1)$ is adjacent to every vertex of $I_{f|_X}$, thus $\chi(Y) \geqslant \chi(I_{f|_X}) + 1 \geqslant \chi(X) + 1$. A contradiction. If there exist two vertices x_1 and x_2 in X such that $f(x_1) \in X$, $f(x_2) \in Y$, then both of $f(X)$ and $f(Y)$ have no edge in X, otherwise, there exists a triangle in X. This is impossible, because X is a K_3-free graph.

Now $f(Y) \subseteq Y$. If $f(X) \nsubseteq X$, then there exists a vertex $x \in V(X)$ such that $f(x) \in Y$ and $f(x)$ is adjacent to every vertex in $V(I_{f|_Y})$. Thus we have $\chi(Y) \leqslant \chi(I_{f|_Y}) + 1 = \chi(Y) + 1$. A contradiction. Hence $f(X) \subseteq X$ and by Lemma 2.3, $X + Y$ is End-regular. □

At the end of this section, we consider endomorphism regularity of the lexicographic product of two graphs. To this aim, we need the following result which is due to Fan [1]. For the definition of the wreath product of semigroups the reader refer to [12].

Lemma 2.10 (*Fan [1]*). *Let X and Y be two K_3-free connected graphs. If* $\mathrm{girth}(X)$ *or* $\mathrm{girth}(Y)$ *is odd, then* $\mathrm{End}(X[Y]) = \mathrm{End}(X)[\mathrm{End}(Y)]$, *where* $\mathrm{End}(X)[\mathrm{End}(Y)]$ *is the wreath product of the monoids* $\mathrm{End}(X)$ *and* $\mathrm{End}(Y)$.

Let X and Y be two K_3-free connected graphs such that $\mathrm{girth}(X)$ or $\mathrm{girth}(Y)$ is odd. In [2] Fan proved that if both of X and Y are End-regular, and one of them is unretractive, then $X[Y]$ is End-regular. Here we shall prove that if X is an E–S-unretractive graph with $|N(x_1)| = |N(x_2)|$ for any two vertices $x_1, x_2 \in V(X)$, and Y is End-regular graph, then $X[Y]$ is End-regular.

Recall that every graph is a generalized lexicographic product of an S-unretractive graph with sets (see [7]). Thus if X is an E-S-unretractive graph with $|N(x_1)| = |N(x_2)|$ for any two vertices $x_1, x_2 \in V(X)$, then X is a lexicographic product of an unretractive graph with a set. In this case, $X = U[T]$ for some unretractive graph U and some set T. We need the following result which is due to Knauer (for details, the reader refer to [7]).

Lemma 2.11. *Let* $X = U(Y_u)_{u\in U}$ *be a graph with* $U = X|_v$ *and* $U < \propto$. *Let H be a small category with* $ObH = \{Y_u | u \in U\}$ *and* $\mathrm{Mor}(Y_u, Y_v) = \mathrm{Map}(Y_u, Y_v)$ $(u, v \in U)$. *Then* $s\mathrm{End}(X) = \mathrm{Aut}U[H]$.

Let X be an E–S-unretractive graph with $|N(x_1)| = |N(x_2)|$ for any two vertices $x_1, x_2 \in V(X)$. Then $X = U[T]$ for some unretractive graph U and some set T. Hence $X[Y] = U[T][Y]$ for some graph Y. In this case, every vertex of $X[Y]$ can be written to form (u, t, y), where $u \in U$, $t \in T$ and $y \in Y$. For each $u \in U$, let $T_u = \{(u, t, y) | t \in T, y \in Y\}$.

Theorem 2.12. *Let X and Y be two K_3-free connected graphs with* $\mathrm{girth}(X)$ *odd or* $\mathrm{girth}(Y)$ *odd, and let*

(1) *X is E–S-unretractive with* $|N(x_1)| = |N(x_2)|$ *for any two vertices* $x_1, x_2 \in V(X)$,
(2) *Y is End-regular.*

Then $X[Y]$ *is End-regular.*

Proof. Let X and Y be two graphs satisfying the assumptions of the theorem. To show that $X[Y]$ is End-regular, we shall prove that for any $\delta \in \mathrm{End}(X[Y])$, there exists an endomorphism $\tau \in \mathrm{End}(X[Y])$ such that $\delta\tau\delta = \delta$.

Since X is E–S-unretractive with $|N(x_1)| = |N(x_2)|$ for any two vertices $x_1, x_2 \in V(X)$, $X = U[T]$ for some unretractive graph U and some set T, then by Lemmas 2.10 and 2.11, $\mathrm{End}(X[Y]) = (\mathrm{Aut}(U)[\mathrm{End}(T)])[\mathrm{End}(Y)]$, where $\mathrm{End}(T)$ is the set of all mapping from $V(T)$ to itself. Hence $\delta = (s, f, g)$ for some $s \in \mathrm{Aut}(U)$, $f \in \mathrm{End}(T)^U$ and $g \in \mathrm{End}(Y)^X$.

Now we define an endomorphism $\tau = (r, h, k) \in \mathrm{End}(X[Y])$. Let u be an arbitrary vertex of U. Since s is an automorphism of U, there exists a vertex $u_1 \in V(U)$ such that $su_1 = u$. First let $r = s^{-1}$, where s^{-1} is the inverse of s in $\mathrm{Aut}(U)$. Next let $h \in \mathrm{End}(T)^U$ and $h_u = h(u)$ be an automorphism of T such that for any $t \in f_{u_1}(T_{u_1})$, $h_u(t) \in f_{u_1}^{-1}(t)$. Since $\mathrm{End}(T)$ contains all the permutations of $V(T)$, clearly h_u exists. Without loss of generality, suppose $h_u(t_i) = t_i'$ for any $t_i \in V(T)$. Now $g_{(u_1,t_i')} \in \mathrm{End}(Y)$. Since $\mathrm{End}(Y)$ is regular, there exists $g'_{(u_1,t_i')} \in \mathrm{End}(Y)$ such that $g_{(u_1,t_i')} g'_{(u_1,t_i')} g_{(u_1,t_i')} = g_{(u_1,t_i')}$. Now let $k \in \mathrm{End}(Y)^X$ such that $k_{(u,t_i)} = k(u, t_i) = g'_{(u_1,t_i')}$. It remains to show that $\delta\tau\delta = \delta$. Let (u_1, t_i', y) be an arbitrary vertex of $X[Y]$. Then

$$\begin{aligned}
\delta\tau\delta(u_1, t_i', y) &= \delta\tau(su_1, f_{u_1}(t_i'), g_{(u_1,t_i')}(y)) \\
&= \delta(u_1, h_u f_{u_1}(t_i'), k_{(u,t_i)} g_{(u_1,t_i')}(y)) \\
&= (su_1, f_{u_1} h_u f_{u_1}(t_i'), g_{(u_1,t_i')} k_{(u,t_i)} g_{(u_1,t_i')}(y)) \\
&= (su_1, f_{u_1}(t_i'), g_{(u_1,t_i')}(y)) \\
&= \delta(u_1, t_i', y).
\end{aligned}$$

Since (u_1, t_i', y) is an arbitrary vertex of $X[Y]$, we have $\delta\tau\delta = \delta$, as required. $\square$

3. End-regular (orthodox) joins of two bipartite graphs

Recall that End-regular bipartite graphs are characterized in Lemma 1.6, and End-regular joins of two trees are determined in [9]. In this section, we shall characterize the End-regular (orthodox) joins of two connected bipartite graphs.

Theorem 3.1. *Let B_1 and B_2 be two connected bipartite graphs. Then $B_1 + B_2$ is End-regular if and only if*

(1) *One of them is End-regular and the other is K_1 or K_2, or*
(2) $B_1 + B_2 = T_1 + T_2$, *where T_1 and T_2 are trees with diameter* 2, *or*
(3) $B_1 + B_2 = K_{m_1,n_1} + K_{m_2,n_2}$, *where K_{m_i,n_i} $(i = 1, 2)$ denotes complete bipartite graphs*, *or*
(4) $B_1 + B_2 = T + K_{m,n}$, *where $K_{m,n}$ denotes a complete bipartite graph and T denotes a tree with diameter* 2.

Proof. *Sufficiency*: In case (1), using Lemma 1.4, we have immediately $B_1 + B_2$ is End-regular. Case (2) is given in Lemma 1.7. In case (3), since K_{m_i,n_i} is *E*–*S*-unretractive graphs, by Lemma 2.1, $K_{m_1,n_1} + K_{m_2,n_2}$ is End-regular. In case (4), $T + K_{m,n} = (K_1 + \overline{K_m}) + K_{m,n} = K_1 + (\overline{K_m} + K_{m,n})$. Now since both of $\overline{K_m}$ and $K_{m,n}$ are *E*–*S*-unretractive. By Lemma 2.1, $\overline{K_m} + K_{m,n}$ is End-regular. Using Lemma 1.4, $K_1 + (\overline{K_m} + K_{m,n})$ is End-regular.

Necessity: For the join of two trees were considered in [9], we only need to show that $B_1 + B_2$ is not End-regular for the following nine cases (see Fig. 2 for graphs appeared in all cases). The main idea of the proof is that, for each cases, we will find an endomorphism $f \in \mathrm{End}(B_1 + B_2)$ such that $f \notin h\mathrm{End}(B_1 + B_2)$

Case 1: $K_{m,n} + T(s, t)$

$$f = \begin{pmatrix} c_1 & \cdots & c_s & d & e & f_1 & \cdots & f_t & a_1 & \cdots & a_{n-1} & a_n & b_1 & \cdots & b_m \\ c_1 & \cdots & c_1 & a_1 & e & a_n & \cdots & a_n & b_1 & \cdots & b_1 & b_1 & d & \cdots & d \end{pmatrix}.$$

Case 2: $K_{m,n} + C_6$

$$f = \begin{pmatrix} a_1 & a_2 & \cdots & a_n & b_1 & \cdots & b_m & 1 & 2 & 3 & 4 & 5 & 6 \\ b_1 & b_1 & \cdots & b_1 & 2 & \cdots & 2 & 1 & a_1 & 3 & a_n & 3 & a_1 \end{pmatrix}.$$

Case 3: $K_{m,n} + P_5$

$$f = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & a_1 & \cdots & a_{n-1} & a_n & b_1 & \cdots & b_m \\ x_1 & a_1 & x_3 & a_n & x_3 & b_1 & \cdots & b_1 & b_1 & x_2 & \cdots & x_2 \end{pmatrix}.$$

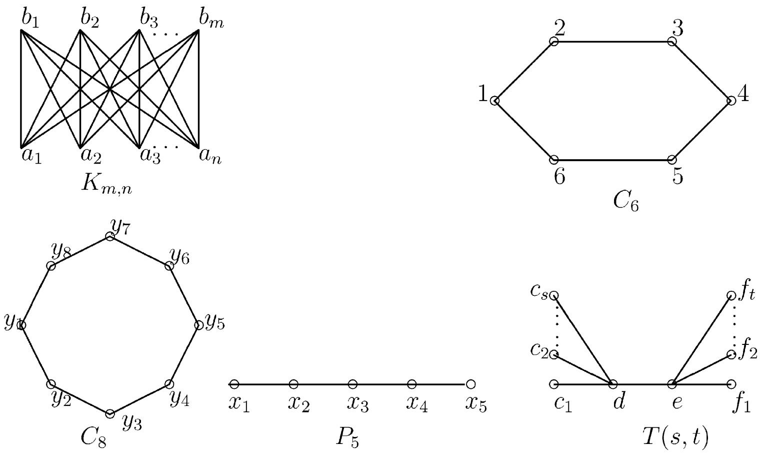

Fig. 2. Graphs $K_{m,n}$, C_6, C_8, P_5 *and* $T_{(s,t)}$.

Case 4: $T(s,t)+C_6$

$$f=\begin{pmatrix} c_1 & \cdots & c_s & d & e & f_1 & \cdots & f_t & 1 & 2 & 3 & 4 & 5 & 6 \\ c_1 & \cdots & c_1 & 1 & e & 3 & \cdots & 3 & 2 & d & 2 & d & 2 & d \end{pmatrix}.$$

Case 5: C_6+P_5

$$f=\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & x_1 & x_2 & x_3 & x_4 & x_5 \\ 2 & x_2 & 2 & x_2 & 2 & x_2 & x_1 & 1 & x_3 & 3 & x_3 \end{pmatrix}.$$

Case 6: $K_{m,n}+C_8$

$$f=\begin{pmatrix} a_1 & \cdots & a_n & b_1 & \cdots & b_m & y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 & y_8 \\ b_1 & \cdots & b_1 & y_2 & \cdots & y_2 & y_1 & a_1 & y_3 & a_n & y_3 & a_n & y_3 & a_1 \end{pmatrix}.$$

Case 7: $T(s,t)+C_8$

$$f=\begin{pmatrix} c_1 & \cdots & c_s & d & e & f_1 & \cdots & f_t & y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 & y_8 \\ d & \cdots & d & y_2 & d & y_2 & \cdots & y_2 & y_1 & c_1 & y_3 & e & y_3 & e & y_3 & c_1 \end{pmatrix}.$$

Case 8: C_6+C_8

$$f=\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 & y_8 \\ 2 & y_2 & 2 & y_2 & 2 & y_2 & y_1 & 1 & y_3 & 3 & y_3 & 3 & y_3 & 1 \end{pmatrix}.$$

Case 9: C_8+P_5

$$f=\begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 & y_8 \\ x_1 & y_1 & x_3 & y_3 & x_3 & y_2 & x_2 & y_2 & x_2 & y_2 & x_2 & y_2 & x_2 \end{pmatrix}.$$

The proof is completed. □

Proposition 3.2. *Let G_1 and G_2 be two graphs. Then G_1+G_2 is End-orthodox if and only if*

(1) *G_1+G_2 is End-regular*, *and*
(2) *Both of G_1 and G_2 are End-orthodox.*

Proof. The direct part follows immediately from Lemma 1.5.

Conversely, since G_1+G_2 is End-regular, to show G_1+G_2 is End-orthodox, we only need to prove that the composition of any two idempotent endomorphisms of G_1+G_2 is also an idempotent.

Let f be an idempotent of $\mathrm{End}(G_1+G_2)$. Then $f(G_1)\subseteq G_1$. Otherwise, there exists a vertex $x\in V(G_1)$ such that $f(x)\in V(G_2)$. Since $f^2=f$, then $f(f(x))=f^2(x)=f(x)$. Note that $\{x,f(x)\}\in E(G_1+G_2)$, then $\{f(x),f(x)\}$ is a loop of G_1+G_2. A contradiction. A similar argument will show that $f(G_2)\subseteq G_2$.

If f_1 and f_2 are two idempotents of $\mathrm{End}(G_1+G_2)$, let $g_1=f_1|_{G_1}$, $g_2=f_1|_{G_2}$ and $h_1=f_2|_{G_1}$, $h_2=f_2|_{G_2}$. Then $g_1,h_1\in\mathrm{Idpt}(G_1)$ and $g_2,h_2\in\mathrm{Idpt}(G_2)$. Since both of G_1 and G_2 are End-orthodox, then $g_1h_1\in\mathrm{Idpt}(G_1)$ and $g_2h_2\in\mathrm{Idpt}(G_2)$. Now $f_1f_2|_{G_1}=g_1h_1$ and $f_1f_2|_{G_2}=g_2h_2$ imply that f_1f_2 is an idempotent of $\mathrm{End}(G_1+G_2)$. Consequently, G_1+G_2 is End-orthodox. □

Theorem 3.3. *Let B_1 and B_2 be two connected bipartite graphs. Then B_1+B_2 is End-orthodox if and only if*

(1) *One of them is End-orthodox and the other is K_1 or K_2, or*
(2) *$B_1+B_2=P_3+C_4$.*

Proof. *Sufficiency*: It follows directly from Lemma 1.8, Theorem 3.1 and Proposition 3.2.

Necessity: By Theorem 3.1, P_3+P_4 and P_4+C_4 are not End-regular, so they cannot be End-orthodox.

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