Wandering $r$-tuples for unitary systems

Xunxiang Guo 1

Institute of Mathematics, Southwestern University of Finance and Economics, Chengdu, Sichuan, PR China, 611130

A R T I C L E   I N F O

Article history:
Received 12 March 2010
Available online 27 August 2010
Submitted by P.G. Lemarie-Rieusset

Keywords:
Wandering $r$-tuples
Unitary system
Multiwavelet
Local commutant
Dilation
Translation

A B S T R A C T

The properties of the set $W^r(\mathcal{U})$ of all complete wandering $r$-tuples for a system $\mathcal{U}$ of unitary operators acting on a Hilbert space $\mathcal{H}$ are investigated by parameterizing $W^r(\mathcal{U})$ in terms of a fixed wandering $r$-tuple $\Psi$ and the set of all unitary operators which locally commute with $\mathcal{U}$ at $\Psi$. The special case of greatest interest is the system $\langle D, T \rangle$ of dilation (by $2$) and translation (by $1$) unitary operators acting on $L^2(\mathbb{R})$, for which the complete wandering $r$-tuples are precisely the orthogonal multiwavelets with multiplicity $r$. We also give some examples for its application.

© 2010 Elsevier Inc. All rights reserved.

1. Introduction

A unitary system is a set of unitary operators $\mathcal{U}$ acting on a Hilbert space $\mathcal{H}$ which contains the identity operator $I$ of $B(\mathcal{H})$. A wandering $r$-tuple for $\mathcal{U}$ is a tuple of $r$ unit vectors $S = (x_1, x_2, \ldots, x_r)$ in $\mathcal{H}$ with the property that

$$\mathcal{U}S := \{ Ux_1, Ux_2, \ldots, Ux_r : U \in \mathcal{U} \}$$

is an orthonormal set; it is called complete if $\mathcal{U}S$ is an orthonormal basis for $\mathcal{H}$. A wandering $r$-tuple system means a unitary system which has a complete wandering $r$-tuple.

Let $W^r(\mathcal{U})$ denote the set of complete wandering $r$-tuples for a unitary system $\mathcal{U}$. Some properties are forced on $\mathcal{U}$ by the presence of a wandering $r$-tuple. Similar to [1], it must be countable if it acts separably, and it must be discrete in the strong operator topology because if $U \neq V \in \mathcal{U}$ and if $S = (x_1, x_2, \ldots, x_r)$ is a wandering $r$-tuple for $\mathcal{U}$, then for any $x_i, i = 1, 2, \ldots, r$, $\|Ux_i\| = \|Vx_i\| = 1$ and $Ux_i \perp Vx_i$, so

$$\|U - V\| \geq \|Ux_i - Vx_i\| = \sqrt{2}.$$

A more immediate purpose is to study the properties of $W^r(\mathcal{U})$ for special systems $\mathcal{U}$ which are relevant to orthogonal multiwavelet theory which generalizes the results in [1], where it focuses on scalar orthogonal wavelet.

In operator theory, wandering vectors and wandering subspaces have been studied for unitary systems and isometry systems (see [2,3]). In the past decades, wavelet theory has undergone a vast development, and many different aspects of the theory have been studied extensively in the literature. For more information on wavelets, we refer readers to [4–7]. The following definition is from Chui’s book [4]:

An orthogonal wavelet is a unit vector $\psi(t)$ in $L^2(\mathbb{R}, \mu)$, with $\mu$ Lebesgue measure, such that $\{2^n \psi(2^n t - l) : n, l \in \mathbb{Z}\}$ constitutes an orthogonal basis for $L^2(\mathbb{R}, \mu)$.
From the above definition, it is natural to view orthogonal wavelets as wandering vectors for dilation–translation unitary systems, which is one of the main ideas in [1], and the knowledge about the more general setting leads to many deep results on orthogonal wavelets.

With the development of wavelets, it is found that the scalar wavelet cannot possess the features as compact support, orthogonality, symmetry and vanishing moments at the same time which is known to be important in signal processing. That’s why multiwavelets were created, which can have all of these properties simultaneously. This suggests that multiwavelets could perform better in various applications. There are also many reference papers in this subject. We refer readers to [8,9]. The following is the well-known definition of an orthogonal multiwavelet with multiplicity r:

An orthogonal multiwavelet with multiplicity r is an r-tuple of unit vectors \( \Psi = (\psi_1(t), \psi_2(t), \ldots, \psi_r(t)) \) with each \( \psi_i(t) \) in \( L^2(\mathbb{R}, \mu) \), and \( \mu \) Lebesgue measure, such that \( 2^t \psi_i(2^t l - i) : n, l \in \mathbb{Z}, \ i = 1, 2, \ldots, r \) constitutes an orthogonal basis for \( L^2(\mathbb{R}, \mu) \).

The similar ideas of viewing orthogonal multiwavelets with multiplicity \( r \) as wandering \( r \)-tuples for dilation–translation unitary systems have been applied by others, see [10–12]. In this paper, we will investigate the properties of the set \( \mathcal{W}^r(\mathcal{U}) \) of all complete wandering \( r \)-tuples for a system \( \mathcal{U} \) of unitary operators acting on a Hilbert space.

2. Preliminaries

Let \( T \) and \( D \) be the operators on \( \mathcal{H} = L^2(\mathbb{R}) \) defined by

\[
(Tf)(t) = f(t-1) \quad \text{and} \quad (Df)(t) = \sqrt{2}f(2t), \quad f \in L^2(\mathbb{R}), \ t \in \mathbb{R}.
\]

These are unitary operators and are in fact bilateral shifts of infinite multiplicity, with wandering subspaces \( L^2([0, 1]) \) and \( L^2([-2, -1] \cup [1, 2]) \), respectively. They are not commutative, in fact, we have \( TD = DT^2 \). Hence \( \mathcal{U}_{D,T} := \{T^nD^l : n, l \in \mathbb{Z}\} \) is an example of countable unitary systems which consists of non-commuting unitary operators and it doesn’t form a group of unitary operators. It is well known (see [1]) that the group generated by \( \{D, T\} \) is

\[
\text{Group}(D, T) = \{D^nT^\beta : n \in \mathbb{Z}, \ \beta \in \mathcal{D}\},
\]

where \( \mathcal{D} \) denotes the set of dyadic rational numbers, and for each \( \beta, T^\beta \) denotes the translation unitary operator

\[
(T^\beta f)(t) = f(t - \beta),
\]

and the \( \text{Group}(D, T) \) fails to have wandering vectors. Following the similar argument, we can show that it fails to have wandering tuples with any length. However, if we consider the subset \( \mathcal{U}_{D,T} \), then the complete wandering \( r \)-tuples for \( \mathcal{U}_{D,T} \) are precisely the orthogonal multiwavelets with multiplicity \( r \). So \( \mathcal{W}^r(\mathcal{U}_{D,T}) \) is far from empty, by the existence of orthogonal multiwavelets in [13], although the reversed set \( \mathcal{U}_{T,D} = \{T^nD^l : n, l \in \mathbb{Z}\} \) fails to have wandering vectors and hence fails to have wandering tuples with any length by [1].

The following notations and terminologies will be needed in this paper. We use \([\cdot]\) to denote closed linear span. If \( S \) is a set of operators, we use \( U(S) \) to denote the set of unitary operators in \( S \). If \( S \) is a linear space of operators, an \( r \)-tuple of vectors \( X = (x_1, x_2, \ldots, x_r) \), where each \( x_i \in \mathcal{H} \), is called a cyclic \( r \)-tuple for \( S \) if \( [Sx_1, Sx_2, \ldots, Sx_r] = \mathcal{H} \), and \( X \) is a separating \( r \)-tuple for \( S \) if the map \( A \rightarrow AX = (Ax_1, Ax_2, \ldots, Ax_r) : S \rightarrow \mathcal{H}^r \), is injective. We use \( S' \) to denote the commutant of \( S \). In this paper Hilbert spaces will be separable and \( C \) denote the complex filed.

3. Main results and proofs

Let \( S \subseteq B(H) \) be a set of operators, and let \( X \in \mathcal{H}^r \) be a non-zero \( r \)-tuple of vectors. Let us define

\[
C_X(S) := \{ A \in B(H) : (AS - SA)X = 0, \ S \in S\}
\]

\[
= \{ A \in B(H) : (AS - SA)x_i = 0, \ S \in S, \ i = 1, 2, \ldots, r\}.
\]

We call this the local commutant of \( S \) at \( X \). It is obvious that \( S' \subseteq C_X(S) \), so local commutant can be viewed as a generalization of the commutant. It can be a useful concept, especially when \( X \) is a cyclic \( r \)-tuple for the linear span of \( S \) and \( S \) is not a semigroup. (If \( S \) is a semigroup and \( X \) is a cyclic \( r \)-tuple for the linear span of \( S \), it reduces to be the commutant by item (ii) below.) \( C_X(S) \) is clearly a linear subspace of \( B(H) \), and it is easy to see that it is closed in the strong operator topology and weak operator topology. In fact, assume that \( A_n \in C_X(S) \) and \( A_n \) converges to \( A \) strongly, then \( (A_nS - SA_n)x_i = 0 \) for all \( S \in S \) and \( i = 1, 2, \ldots, r \), and \( A_nSx_i \) converges to \( ASx_i \), \( SA_nx_i \) converges to \( SAx_i \). Hence \( (A_nS - SA_n)x_i \) converges to \( (AS - SA)x_i \), so \( (AS - SA)x_i = 0 \) for any \( S \in S \) and \( i = 1, 2, \ldots, r \), i.e., \( A \in C_X(S) \). If we assume that \( A_n \) converges to \( A \) weakly, then \( (A_nx, y) \) converges to \( (Ax, y) \) for any \( x, y \in \mathcal{H} \), so \( 0 = (A_nS - SA_n)x_i, y) = (A_nSx_i, y) - (SA_nx_i, y) \) converges to \( (ASx_i, y) - (SAx_i, y) = ((AS - SA)x_i, y) \), hence \( (AS - SA)x_i = 0 \) for any \( S \in S \) and \( i = 1, 2, \ldots, r \), i.e., \( A \in C_X(S) \).

**Lemma 3.1.** If \( S \subseteq B(H) \) is a set of operators and if \( X \in \mathcal{H}^r \) is an \( r \)-tuple of vectors which is a cyclic \( r \)-tuple for \( S \), i.e., \( [SX] = \mathcal{H} \), then:

(i) The \( r \)-tuple of vectors \( X \) is a separating \( r \)-tuple for \( C_X(S) \).

(ii) If \( S \) is a semigroup, then \( C_X(S) = S' \).
(iii) If \( A \) is an element of \( C_X(S) \) with dense range, then \( AX = (Ax_1, Ax_2, \ldots, Ax_i) \in \mathcal{H}^r \) is also a cyclic \( r \)-tuple for \( S \).

(iv) Suppose \( X \) is also a separating \( r \)-tuple for \( S \). Then if \( S, T \in S \) with \( ST \in S \) and \( TS \in S \), and \( ST \neq TS \), then neither \( S \) nor \( T \) is in \( C_X(S) \).

(v) Suppose \( S = S_1S_2 \) where \( S_1 \) is a semigroup. Then \( C_X(S) \subseteq S' \).

(vi) Suppose \( S = S_1S_2 \) and \( S \subseteq S_1 \cap S_2 \). Then \( C_X(S) \subseteq C_X(S_1) \cap C_X(S_2) \).

(vii) If \( V \in C_X(S) \) is invertible, then \( C_X(S) = C_X(S)V^{-1} \).

(viii) For any \( A \in S' \) and \( S \in C_X(S) \) we have \( AS \in C_X(S) \), which means that \( C_X(S) \) is a left module of \( S' \).

Proof. (i) If \( A \in C_X(S) \) and \( AX = (Ax_1, Ax_2, \ldots, Ax_i) = 0 \), then for any \( S \in S \) and \( i = 1, 2, \ldots, r \), we have \( ASx_i = SAx_i = 0 \) by the definition of \( C_X(S) \). So \( A[SX] = A[H] = 0 \), i.e., for any \( x \in \mathcal{H} \), we have \( Ax = 0 \), hence \( A = 0 \). So \( X \) is separating for \( C_X(S) \).

(ii) The inclusion \( C_X(S) \supseteq S' \) is trivial. It is sufficient to show that \( C_X(S) \subseteq S' \). Suppose that \( A \in C_X(S) \). Then for each \( S, T \in S \), since \( S \) is a semigroup, we have \( ST \in S \), so \( AS(TX) = A(ST)x = (ST)Ax = S(TAx) = S(ATx) = SA(Tx) \) for each \( x_i \), \( i = 1, 2, \ldots, r \). So \( AS(TX) = SA(TX) \). Since \( T \in S \) is arbitrary and \( [SX] = H \), it follows that \( AS = SA \), i.e., \( A \in S' \). So \( C_X(S) \subseteq S' \).

(iii) Since \( A \in C_X(S) \), for any \( S \in S \) we have \( SAX = ASX \). But it is assumed that \( [SAX] = H \) and \( A \) has dense range, so \( [SAX] = [AH] = H \), hence \( AX \) is a separating \( r \)-tuple for \( S \).

(iv) If \( S \in C_X(S) \), then since \( S \in S \), we have \( (ST - TS)x_i = 0, i = 1, 2, \ldots, r \) so \( STX = TSX \). By item (i), \( X \) is a separating \( r \)-tuple for \( C_X(S) \), we have \( ST = TS \), which contradicts the assumption that \( ST \neq TS \), so \( S \notin C_X(S) \). Similarly, \( T \notin C_X(S) \).

(v) Since \( S = S_1S_2 \) and \( S_1 \) is a semigroup, so \( S_1S \subseteq S \). Let \( A \in C_X(S) \) and \( R \in S_1 \). Then \( ASx_1 = SAx_1 \) for \( i = 1, 2, \ldots, r \). Since \( R \in S \), we have \( A(RS)x_i = (RAS)x_i = R(SAx_i) = R(AS)x_i \) for \( i = 1, 2, \ldots, r \). Hence \( AS(RS)x_i = (RAS)x_i \) for \( i = 1, 2, \ldots, r \) and \( S \in S \), and \( [AR](Sx) = [RA](Sx) \). Since \( [Sx] = H \), this implies that \( AR = RA \), i.e., \( A \in S' \), so \( C_X(S) \subseteq S' \).

(vi) If \( I \in S_1 \cap S_2 \), then \( S_1 = S_1 \cdot I \subseteq S_1 \subseteq S_1S_2 = S \) and \( S_2 = I \cdot S_2 \subseteq S_1 \subseteq S_1S_2 = S \), so \( C_X(S_1) \subseteq C_X(S) \) and \( C_X(S_2) \subseteq C_X(S) \). Hence \( C_X(S) \subseteq C_X(S_1) \cap C_X(S_2) \).

(vii) By definition we have

\[
C_X(S) = \{ A \in B(H) : (AS - SA)VX = 0, \ S \in S \} = \{ A \in B(H) : (AS - SA)Vx_i = 0, \ S \in S, \ i = 1, 2, \ldots, r \} = \{ A \in B(H) : (AVS - SAV)x_i = 0, \ S \in S, \ i = 1, 2, \ldots, r \} = \{ A \in B(H) : (AVS - SAV)x = 0, \ S \in S \} = \{ A \in B(H) : AV \in C_X(S) \} = C_X(S) \cdot V^{-1}.
\]

The third equality is based on \( V \in C_X(S) \).

(viii) For any \( A \in S' \), \( S \in C_X(S) \) and \( B \in S \), we have \( (AS)Bx_i = (ABS)x_i = (AB)Sx_i = (BA)Sx_i = B(AS)x_i \) for \( i = 1, 2, \ldots, r \). Hence \( (AS)B = B(AS) \), i.e., \( A \in C_X(S) \), since \( B \) is arbitrary.

Corollary 3.2. If \( S \) contains a semigroup \( S_0 \) with \( S' = S_0' \) and \( X = (x_1, x_2, \ldots, x_i) \in \mathcal{H}^r \) is a cyclic \( r \)-tuple for \( S_0 \) in the sense that \( [S_0X] = H \), then \( C_X(S) \subseteq C_X(S_0) \).

Proof. Since \( S_0 \) is a semigroup and \( [S_0X] = H \), by item (ii) in Lemma 3.1, we have that \( C_X(S_0) = S_0' \). By the condition that \( S' = S_0' \), we have \( C_X(S_0) = S' \). But it is obvious that \( C_X(S) \supseteq C_X(S_0) \supseteq S' \), so \( C_X(S) = C_X(S_0) = S' \).

Remark 3.3. Most of the results in Lemma 3.1 are generalizations of the corresponding results in Lemma 1.1 in [1]. It is interesting to know that although \( C_X(S) = \bigcap_{i=1}^{r} C_X(S_i) \) and \( [SX] = \bigcup_{i=1}^{r} [Sx_i] \), the similar results are still true for wandering \( r \)-tuple systems.

Theorem 3.4. Let \( S \subseteq B(H) \) and \( X \in \mathcal{H}^r \) be arbitrary. Then

\[
(C_X(S))_\perp = \overline{\text{span}} \{ \{S, x_i \otimes y \} : \ S \in S, \ y \in H, \ i = 1, 2, \ldots, r \},
\]

where for \( x, y \in \mathcal{H} \), \( x \otimes y \) denotes the rank-1 operator defined by \( (x \otimes y)w = (w, y)x \), \( w \in H \), and \( [S, x \otimes y] = S(x \otimes y) - (x \otimes y)S = Sx \otimes y - x \otimes y \). And \( C_X(S) \) denotes the preannihilator of \( C_X(S) \) in \( C_1(H) \), the subspace of trace class operators.
For arbitrary \(A \in B(H)\), we have the trace equation

\[
\text{Tr}(A[S, x_i \otimes y]) = \text{Tr}(A(Sx_i \otimes y - x_i \otimes S^*y))
= \text{Tr}(ASx_i \otimes y) - \text{Tr}(Ax_i \otimes S^*y)
= (ASx_i, y) - (Ax_i, S^*y)
= (ASx_i, y) - (S Ax_i, y)
= (AS - SA)x_i, y.
\]

From this it follows that \(A \in C_X(S)\) iff \((AS - SA)x_i = 0\) for \(i = 1, 2, \ldots, r\) and any \(S \in S\) iff \(A\) is annihilated by all trace class operators of the form \([S, x_i \otimes y]\) for \(S \in S\) and \(i = 1, 2, \ldots, r\); \(y \in H\). So \((C_X(S))_\perp = \text{span}((S, x_i \otimes y) \mid S \in S, \ y \in H, \ i = 1, 2, \ldots, r)\). \(\square\)

**Theorem 3.5.** Let \(U\) be a unitary system in \(B(H)\). Suppose \(\Psi = (\psi_1, \psi_2, \ldots, \psi_r) \in \mathcal{W}^\Psi(U)\). Then

\[
\mathcal{W}^\Psi(U) = \{V\Psi = (V\psi_1, V\psi_2, \ldots, V\psi_r) : V \in \text{U}(C_\Psi(U))\}.
\]

Moreover, the correspondence

\[
V \rightarrow V\Psi, \quad \text{U}(C_\Psi(U)) \rightarrow \mathcal{W}^\Psi(U),
\]

is one-to-one.

**Proof.** Let \(V \in \text{U}(C_\Psi(U))\). Let \(\Phi = (\phi_1, \phi_2, \ldots, \phi_r) \equiv V\Psi = (V\psi_1, V\psi_2, \ldots, V\psi_r)\). For \(U \in U\) and \(i = 1, 2, \ldots, r\) we have

\[
U\phi_i = VU\psi_i = VU\psi_i,
\]

since \(U\) commutes with \(V\) at \(\Psi = (\psi_1, \psi_2, \ldots, \psi_r)\), hence \(U\) commutes with \(V\) at each \(\psi_i\) for \(i = 1, 2, \ldots, r\). Thus \(U\Phi = VU\Psi\), and so \(U\Phi\) is an orthonormal basis for \(H\) since \(V\) is unitary and \(U\Psi\Phi\) is an orthonormal basis. So \(\Phi \in \mathcal{W}^\Psi(U)\).

Conversely, let \(\Phi = (\phi_1, \phi_2, \ldots, \phi_r) \in \mathcal{W}^\Psi(U)\) be arbitrary. Since \(U\Phi = \{U\psi_1 : U \in U, \ i = 1, 2, \ldots, r\}\) and \(U\Psi = \{U\psi_i : U \in U, \ i = 1, 2, \ldots, r\}\) are orthonormal bases for \(H\), there is a unique unitary operator \(V\) such that

\[
VU\psi_i = U\phi_i, \quad i = 1, 2, \ldots, r, \ U \in U.
\]

Then \(V\psi_i = \phi_i\) for \(i = 1, 2, \ldots, r\), since \(U \in U\). So \(VU\psi_i = U\phi_i = UV\psi_i\) for \(U \in U\) and \(i = 1, 2, \ldots, r\). Thus \(V \in C_\Phi(U)\).

By item (i) in Lemma 3.1, \(\Psi\) separates points of \(C_\Phi(U)\). Thus the map \(V \rightarrow V\Psi\) is one-to-one. \(\square\)

Theorem 3.5 shows that if \(U\) is a unitary system with \(\mathcal{W}^\Psi(U) \neq \emptyset\), then given any \(\Psi = (\psi_1, \psi_2, \ldots, \psi_r) \in \mathcal{W}^\Psi(U)\), the entire set \(\mathcal{W}^\Psi(U)\) can be parameterized in a natural way by the set of unitary operators in the local commutant of \(U\) at \(\Psi = (\psi_1, \psi_2, \ldots, \psi_r)\). This result is simple but beautiful and is key to our approach which generalizes the similar results in [1].

It is easy to see that if \(G \subset B(H)\) is a unitary group, and if \(X = (x_1, x_2, \ldots, x_r)\) and \(Y = (y_1, y_2, \ldots, y_r)\) are cyclic \(r\)-tuples for \(G\), then for all \(i, j = 1, 2, \ldots, r\), the functionals defined on \(G\) by \(W_{x_i}(g) = \langle gx_i, x_j \rangle\) and \(W_{y_i}(g) = \langle gy_i, y_j \rangle\) agree on \(G\) iff there exists \(V \in U(G)\) with \(VX = Y\). In fact, if there exists \(V \in U(G)\) with \(VX = Y\), then for \(i = 1, 2, \ldots, r\), we have \(Vx_i = y_i\). So \(W_{x_i}(g) = \langle gx_i, y_j \rangle = \langle gxVx_i, Vx_j \rangle = \langle Vgx_i, Vx_j \rangle = \langle gx_i, x_j \rangle = W_{y_i}(g)\).

Conversely, if for \(g \in G\) and \(i, j = 1, 2, \ldots, r\) we have \(W_{x_i}(g) = W_{y_i}(g)\). Then for \(i, j = 1, 2, \ldots, r\) and \(g \in G\), we have \(\langle gx_i, x_j \rangle = \langle gy_i, y_j \rangle\). For \(g_1, g_2, \ldots, g_r \in G\) and \(\lambda_1, \lambda_2, \ldots, \lambda_r \in \mathbb{C}\), we define

\[
V : \text{span}(G\mathcal{X}) \rightarrow \text{span}(G\mathcal{Y}), \quad V\left(\sum \lambda_1g_1x_i\right) = \sum \lambda_1g_1y_i.
\]

Since

\[
\left\langle \sum \lambda_1g_1x_i, \sum \lambda_1g_1x_i \right\rangle = \sum \lambda_1^2\lambda_2^2(\langle g_1^*g_2x_i, x_j \rangle)
= \sum \lambda_1^2\lambda_2^2(\langle g_1^*g_2y_i, y_j \rangle) = \left\langle \sum \lambda_1g_1y_i, \sum \lambda_1g_1y_i \right\rangle.
\]

\(V\) is isometric and onto. Since \(X\) and \(Y\) are cyclic \(r\)-tuples for \(G\), it can be extended to be a unitary operator in \(B(H)\) and we use the same notation to denote it. Since \(i \in G\), so \(Vx_i = y_i\), \(i = 1, 2, \ldots, r\), hence we have \(VX = Y\). Moreover, for \(i = 1, 2, \ldots, r\), we have \(V(gx_i) = gy_i = gVx_i\), it follows that \(Vg = gV\) at point \(X\), but by item (ii) in Lemma 3.1, we have \(V \in G\). Theorem 3.5 can be thought of as a special case of the following generalization of this to unitary systems.
**Theorem 3.6.** Let \( \mathcal{U} \) be a unitary system in \( \mathcal{B}(H) \). Suppose \( X = (x_1, x_2, \ldots, x_r) \), \( Y = (y_1, y_2, \ldots, y_r) \) \( \in \mathcal{H}^r \) with \( \|Ux\| = \|UY\| = \mathcal{H} \).

Then

\[
\langle U_1x_i, U_2x_j \rangle = \langle U_1y_i, U_2y_j \rangle
\]

for all \( U_1, U_2 \in \mathcal{U} \) and \( i, j = 1, 2, \ldots, r \) if and only if there is a unitary \( V \in \mathcal{C}_X(\mathcal{U}) \) with \( VX = Y \).

**Proof.** Fix \( U_1, U_2, \ldots, U_r \in \mathcal{U} \) and \( \lambda_1, \lambda_2, \ldots, \lambda_r \in \mathbb{C} \). If \( \langle U_1x_i, U_2x_j \rangle = \langle U_1y_i, U_2y_j \rangle \) for \( U_1, U_2 \in \mathcal{U} \) and \( i, j = 1, 2, \ldots, r \), then

\[
\left( \sum_{i,j} \lambda_i U_i x_i \right) = \sum_{i,j} \lambda_i U_i y_i.
\]

This shows that the map

\[
V : \text{span}(\mathcal{U}X) \to \text{span}(\mathcal{U}Y)
\]

defined by \( V(\sum \lambda_i U_i x_i) = \sum \lambda_i U_i y_i \), preserves the inner product and is onto, thus it is isometric and onto. Since \( \|\mathcal{U}X\| = \|\mathcal{U}Y\| = \mathcal{H} \), it can be extended to be a unitary operator and we use the same notation to denote it. Since \( I \in \mathcal{U} \), so \( Vx_i = y_i \) for \( i = 1, 2, \ldots, r \), it follows that \( VX = Y \) and for \( U \in \mathcal{U} \), \( UVx_i = Uy_i = UVx_i \) for \( i = 1, 2, \ldots, r \), hence \( VX = Y \), i.e., \( V \in \mathcal{C}_X(\mathcal{U}) \).

If there is a unitary \( V \in \mathcal{C}_X(\mathcal{U}) \) with \( VX = Y \), then for \( i = 1, 2, \ldots, r \), we have \( Vx_i = y_i \). Then for all \( U_1, U_2 \in \mathcal{U} \), \( (U_1y_i, U_2y_j) = (U_1Vx_i, U_2Vx_j) = (VUx_i, Vx_j) = (U_1x_i, U_2x_j) \) for \( i, j = 1, 2, \ldots, r \). \( \square \)

**Remark 3.7.** Given a unitary system \( \mathcal{U} \), we can define an equivalence relation among all its cyclic \( r \)-tuples by:

\[
X, Y \in \mathcal{H}^r \text{ are two cyclic } r \text{-tuples for } \mathcal{U}, \text{ if and only if } \text{ there is a unitary } V \in \mathcal{C}_X(\mathcal{U}) \text{ with } VX = Y.
\]

By Theorem 3.6, the above definition makes sense, and by Theorem 3.2, all wandering \( r \)-tuples \( \mathcal{V}^r(\mathcal{U}) \) constitute one equivalence class.

In [1], we know that in certain cases new wandering vectors can be obtained by “interpolating” between a known pair. This result gives us a method to construct new wavelets from linear combination of the known ones. The following result will show that for wandering \( r \)-tuples we also have the similar result and we prove it more directly.

**Theorem 3.8.** Let \( \mathcal{U} \) be a unitary system, let \( \Psi = (\psi_1, \psi_2, \ldots, \psi_r) \), \( \Phi = (\phi_1, \phi_2, \ldots, \phi_r) \) \( \in \mathcal{V}^r(\mathcal{U}) \), and let \( V \) be the unique unitary operator in \( \mathcal{C}_\Psi(\mathcal{U}) \) with \( V\Psi = \Phi \). Suppose \( V^2 = I \). Then for all \( m, n \in \mathbb{C} \) with \( |m|^2 + |n|^2 = 1 \) and \( \bar{m} \cdot n \in \mathbb{R} \), we have

\[
m \cdot \Psi + in \cdot \Phi \in \mathcal{V}^r(\mathcal{U}) \text{ and in particular, } \cos \alpha \cdot \Psi + i \sin \alpha \cdot \Phi \in \mathcal{V}^r(\mathcal{U}) \text{ for all } 0 \leq \alpha \leq 2\pi.
\]

**Proof.** Since \( V\Psi = \Phi \), we have \( m \cdot \Psi + in \cdot \Phi = (m \cdot I + in \cdot V)\Psi \). Since \( V \in \mathcal{C}_\Psi(\mathcal{U}) \), so \( m \cdot I + in \cdot V \in \mathcal{C}_\Psi(\mathcal{U}) \) as well. By Theorem 3.5, it is sufficient to show that \( m \cdot I + in \cdot V \) is a unitary operator. Since \( V^2 = I \) and \( V \) is a unitary operator, so \( V = V^* \).

\[
(m \cdot I + in \cdot V)^* (m \cdot I + in \cdot V) = (m \cdot I + in \cdot V)(m \cdot I + in \cdot V)^*
\]

\[
= (m \cdot I + in \cdot V)(\bar{m} \cdot I - in \cdot V^*)
\]

\[
= (|m|^2 + |n|^2) \cdot I + imm \cdot V - imm \cdot V^* = I.
\]

So \( m \cdot I + in \cdot V \) is a unitary operator. \( \square \)

**Remark 3.9.** Fix \( \Psi \in \mathcal{V}^r(\mathcal{U}) \), since \( \mathcal{U} \), the commutant of \( \mathcal{U} \), is a von Neumann algebra and \( \mathcal{U} \subseteq \mathcal{V}^r(\mathcal{U}) \), so the unitary group contained in \( \mathcal{U} \) parameterizes a norm-path-connected subset of \( \mathcal{V}^r(\mathcal{U}) \) that contains \( \Psi \) via the correspondence \( U \to U\Psi \). The above gives another norm-path-connected-path which connects \( \Psi \) and \( \Phi \) in \( \mathcal{V}^r(\mathcal{U}) \).

**Theorem 3.10.** Let \( S \) be a unital semigroup of unitaries in \( \mathcal{B}(H) \). Suppose \( \mathcal{V}^r(S) \neq \emptyset \). Then \( S \) is a group.
Proof. Let $\Psi = (\psi_1, \psi_2, \ldots, \psi_r) \in \mathcal{W}^r(S)$. If $S$ is not a group, there exists $U \in S$ such that $U^{-1} \notin S$. Then for $V \in S$ and $i, j = 1, 2, \ldots, r$, we have
\[
\langle U^{-1} \psi_i, V \psi_j \rangle = \langle \psi_i, UV \psi_j \rangle = 0,
\]
since $S$ is semigroup, so $UV \in S$ and $UV \neq I$. Hence $U^{-1} \psi_i$ is a non-zero vector which is orthogonal to $[S \Psi] = \mathcal{H}$, a contradiction. \qed

Theorem 3.11. Let $\mathcal{U}$ be a unitary system, and suppose $[\mathcal{W}^r(\mathcal{U})] = \mathcal{H}^r$. Then:

(i) If $C_{\Psi}(\mathcal{U})$ is an algebra for some $\Psi = (\psi_1, \psi_2, \ldots, \psi_r) \in \mathcal{W}^r(\mathcal{U})$, then $C_{\Phi}(\mathcal{U}) = \mathcal{U}$ for every $\Phi = (\phi_1, \phi_2, \ldots, \phi_r) \in \mathcal{W}^r(\mathcal{U})$. In particular, $C_{\Psi}(\mathcal{U})$ is an algebra for all $\Phi \in \mathcal{W}^r(\mathcal{U})$.

(ii) If $\mathcal{U}(C_{\Psi}(\mathcal{U}))$ is a semigroup for some $\Psi \in \mathcal{W}^r(\mathcal{U})$, then $\mathcal{U}(C_{\Phi}(\mathcal{U})) = \mathcal{U}$ for every $\Phi \in \mathcal{W}^r(\mathcal{U})$. In particular, $\mathcal{U}(C_{\Psi}(\mathcal{U}))$ is a group for all $\Phi \in \mathcal{W}^r(\mathcal{U})$.

Proof. Let $\Phi = (\phi_1, \phi_2, \ldots, \phi_r) \in \mathcal{W}^r(\mathcal{U})$ be arbitrary. By Theorem 3.5, there is a unique $V \in U(C_{\Psi}(\mathcal{U}))$ with $\Psi = V \Phi$. Then $C_{\Psi}(\mathcal{U}) = C_{\Phi}(\mathcal{U})V^*$ by Lemma 3.1(ii). Hence $V^* \in C_{\Psi}(\mathcal{U})$, since $I \in C_{\Psi}(\mathcal{U})$.

For item (i), if $C_{\Psi}(\mathcal{U})$ is closed under multiplication, then $C_{\Psi}(\mathcal{U})V^* \subseteq C_{\Psi}(\mathcal{U})$. It follows that
\[
C_{\Psi}(\mathcal{U}) = C_{\Psi}(\mathcal{U}) \cdot V^* \subseteq C_{\Psi}(\mathcal{U}) \cdot V = C_{\Psi}(\mathcal{U}).
\]
So if $A \in C_{\Psi}(\mathcal{U})$, then $A \in C_{\Phi}(\mathcal{U})$ for all $\Phi \in \mathcal{W}^r(\mathcal{U})$. Thus $(AU - UA)\Phi = 0$ for all $U \in \mathcal{U}$ and for all $\Phi \in \mathcal{W}^r(\mathcal{U})$. Since $[\mathcal{W}^r(\mathcal{U})] = \mathcal{H}^r$, this implies that $AU = UA$ for $U \in \mathcal{U}$, i.e., $A \in \mathcal{U}'$. So $C_{\Psi}(\mathcal{U}) \subseteq \mathcal{U}'$, but the opposite inclusion is obvious, so $C_{\Psi}(\mathcal{U}) = \mathcal{U}'$. Again let $\Phi \in \mathcal{W}^r(\mathcal{U})$ be arbitrary, and $W$ be the unique unitary in $C_{\Psi}(\mathcal{U}) = \mathcal{U}'$ with $\Phi = W \Psi$. Then $W^* \in C_{\Psi}(\mathcal{U}) = \mathcal{U}'$ since $W$ is a unitary system. So
\[
C_{\Psi}(\mathcal{U}) = C_{\Psi}(\mathcal{U})W^* = \mathcal{U}'.
\]
For item (ii), if $U(C_{\Psi}(\mathcal{U}))$ is a semigroup then $U(C_{\Phi}(\mathcal{U}))V^* \subseteq U(C_{\Psi}(\mathcal{U}))$. Since $C_{\Psi}(\mathcal{U}) = C_{\Phi}(\mathcal{U})V^*$ and $V^*$ is a unitary, we also have $U(C_{\Psi}(\mathcal{U})) = U(C_{\Phi}(\mathcal{U}))V^*$, this implies
\[
U(C_{\Psi}(\mathcal{U})) = U(C_{\Phi}(\mathcal{U}))V^* \subseteq U(C_{\Phi}(\mathcal{U})) \cdot V = U(C_{\Phi}(\mathcal{U})).
\]
If $S \in U(C_{\Psi}(\mathcal{U}))$, then $S \in U(C_{\Phi}(\mathcal{U}))$ for arbitrary $\Phi \in \mathcal{W}^r(\mathcal{U})$. So
\[
(SU - US)\Phi = 0, \quad U \in \mathcal{U}, \quad \Phi \in \mathcal{W}^r(\mathcal{U}).
\]
Since $[\mathcal{W}^r(\mathcal{U})] = \mathcal{H}^r$, we have $SU = US$ for all $U \in \mathcal{U}$, i.e., $S \in \mathcal{U}'$. The rest is identical to (i). \qed

Theorem 3.12. Let $\mathcal{U}$ be a unitary system, and suppose $C_{\Psi}(\mathcal{U})$ is abelian for some $\Psi \in \mathcal{W}^r(\mathcal{U})$. Then $C_{\Psi}(\mathcal{U})$ is abelian for all $\Phi \in \mathcal{W}^r(\mathcal{U})$.

Proof. Suppose $C_{\Psi}(\mathcal{U})$ is abelian and let $\Phi \in \mathcal{W}^r(\mathcal{U})$ be arbitrary. Let $V \in U(C_{\Psi}(\mathcal{U}))$ with $\Psi = V \Phi$. Then $C_{\Phi}(\mathcal{U}) = C_{\Phi}(\mathcal{U})V^*$. So $V^* \in C_{\Psi}(\mathcal{U})$. Since $V^* \in (C_{\Phi}(\mathcal{U}))'$ and $V$ is normal, $V \in (C_{\Phi}(\mathcal{U}))'$. So $C_{\Phi}(\mathcal{U}) = C_{\Phi}(\mathcal{U})V^*$ is abelian. \qed

4. Examples

Example 4.1. Let $\{e_n\}_{n=1}^\infty$ be an orthonormal basis for a separable Hilbert space $\mathcal{H}$, and let $S_{e_n} = e_{n+1}$ be the bilateral shift of multiplicity one. Let $\mathcal{U} = \{S^{2n} : n \in \mathbb{Z}\}$ be the group generated by $S^2$. Then $\Psi = (e_0, e_1)$ is in $\mathcal{W}^2(\mathcal{U})$. By Lemma 3.1(ii) and Theorem 3.5,
\[
\mathcal{W}^2(\mathcal{U}) = \{V \Psi : V \in U(\{S^2\}^2)\}.
\]

Example 4.2. Let $\mathcal{U} = \{D^nT^l : n, l \in \mathbb{Z}\}$, then $\mathcal{U}$ is a unitary system in $L^2(\mathbb{R})$, whose complete wandering $r$-tuples set $\mathcal{W}^r(\mathcal{U})$ consists of precisely all orthogonal multiwavelets with multiplicity $r$. One of the most important subsets of $\mathcal{W}^r(\mathcal{U})$ is the set consisting of all MRA multiwavelets with multiplicity $r$. Let $\Phi = (\phi_1, \phi_2, \ldots, \phi_r)$ be a scaling $r$-tuple associated with an MRA multiwavelet with multiplicity $r$. Let $U_0 = \{D^nT^l : n \geq 0, l \in \mathbb{Z}\}$, then $U_0 \subseteq U$. It is easy to verify that $U_0$ is a semigroup and $U' = U_0$. And by the properties of MRA, we have $[U_0 \Phi] = L^2(\mathbb{R})$. So by Corollary 3.2, we have $C_{\Phi}(U_0) = C_{\Phi}(U) = U' = U_0$. 

Example 4.3. Let $\Psi = (\psi_1, \psi_2, \ldots, \psi_r)$ be any fixed multiwavelet with multiplicity $r$ for $U_{D,T}$. Then by Lemma 3.1, we have:

(i) $W^r(U_{D,T}) = U(C_\Psi(U_{D,T}))\Psi$. The mapping

$$U(C_\Psi(U_{D,T}))\Psi \rightarrow W^r(U_{D,T}) : \Psi \rightarrow U\Psi$$

is one-to-one and onto.

(ii) $T, D \notin C_\Psi(U_{D,T})$.

(iii) $C_\Psi(U_{D,T}) \subseteq \{D\}'$.

(iv) If $\Phi \in C_\Psi(U_{D,T})$, let $V \in U(C_\Psi(U_{D,T}))$ with $V\Psi = \Phi$. Then $C_\Phi(U_{D,T}) = C_\Psi(U_{D,T})V^*$.

References