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The Peano Curve of Schoenberg Is Nowhere Differentiable*

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Let $f(t)$ be defined in $[0, 1]$ by

$$\begin{aligned} f(t) &= 0 && \text{if} \quad 0 \leqslant t \leqslant \tfrac{1}{3}, \\ &= 3t - 1 && \text{if} \quad \tfrac{1}{3} \leqslant t \leqslant \tfrac{2}{3}, \\ &= 1 && \text{if} \quad \tfrac{2}{3} \leqslant t \leqslant 1. \end{aligned}$$

and extended to all real t by requiring that $f(t)$ should be an even function having period 2. The plane arc defined parametrically by the equations

$$x(t) = \sum_{n=0}^{\infty} \frac{f(3^{2n}t)}{2^{n+1}}, \qquad y(t) = \sum_{n=0}^{\infty} \frac{f(3^{2n+1}t)}{2^{n+1}} \qquad (0 \leqslant t \leqslant 1)$$

is known to be continuous, and to map the interval $I = \{0 \leqslant x \leqslant 1\}$ onto the entire square $I^2 = \{0 \leqslant x, y \leqslant 1\}$. (See I. J. Schoenberg, *Bull. Amer. Math. Soc.* **44** (1938), 519). Here it is shown that this arc is nowhere differentiable, meaning the following: There is no value of t such that both derivatives $x'(t)$ and $y'(t)$ exist and are finite.

1. INTRODUCTION

It came as quite a surprise to the mathematical world when, in 1875, Weierstrass constructed an everywhere continuous, nowhere differentiable function (see [1]). Equally startling though was the discovery by Giuseppe Peano [2] 15 years thereafter that the unit interval could be mapped continuously onto the entire unit square I^2.

Well known now are examples of area-filling curves, and of continuous functions which are nowhere differentiable. This paper brings together these two pathological properties by showing that the plane Peano curve of Schoenberg [3], defined in Section 3 below, lacks at every point a finite derivative (Theorem 3). An analogous space curve is similarly shown to fill the unit cube I^3 (Theorem 2), and to be nowhere differentiable (Theorem 4).

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2. An Identity on the Cantor Set Γ

The foundation of Schoenberg's curve is the continuous function $f(t)$, defined first in $[0, 1]$ by

$$
\begin{aligned}
f(t) &= 0 && \text{if } 0 \leqslant t \leqslant \tfrac{1}{3}, \\
&= 3t - 1 && \text{if } \tfrac{1}{3} \leqslant t \leqslant \tfrac{2}{3}, \\
&= 1 && \text{if } \tfrac{2}{3} \leqslant t \leqslant 1.
\end{aligned}
\tag{2.1}
$$

We then extend its definition to all real t such that $f(t)$ is an even function of period 2 (see Fig. 1). Thus

$$f(-t) = f(t), \quad f(t+2) = f(t) \qquad \text{for all } t.$$

The main property of this function is that it produces the following remarkable identity on Γ.

LEMMA 1. *If t is an element of Cantor's Set Γ, then*

$$t = \sum_{n=0}^{\infty} 2f(3^n t)/3^{n+1}. \tag{2.2}$$

Proof. If indeed $t \in \Gamma$, it can be expressed as

$$t = \sum_{n=0}^{\infty} a_n/3^{n+1} \qquad (a_n = 0, 2); \tag{2.3}$$

then (2.2) would follow from the relations

$$a_n = 2 \cdot f(3^n t) \qquad (n = 0, 1, 2, \ldots). \tag{2.4}$$

FIG. 1. The continuous function $f(t)$.

To prove (2.4) observe that (2.3) implies

$$3^n t = 3^n \left(\frac{a_0}{3} + \cdots + \frac{a_{n-1}}{3^n}\right) + \frac{a_n}{3} + \frac{a_{n+1}}{3^2} + \cdots,$$

whence

$$3^n t = M_n + \frac{a_n}{3} + \frac{a_{n+1}}{3^2} + \cdots \qquad (M_n \text{ is an even integer}). \tag{2.5}$$

From the graph of $f(t)$ we conclude the following:

$$\text{If } a_n = 0, \text{ then } M_n \leqslant 3^n t \leqslant M_n + \frac{2}{3^2} + \frac{2}{3^3} + \cdots = M_n + \frac{1}{3}$$

and therefore $f(3^n t) = 0$.

$$\text{If } a_n = 2, \text{ then } M_n + \frac{2}{3} \leqslant 3^n t \leqslant M_n + \frac{2}{3} + \frac{2}{3^2} + \cdots = M_n + 1$$

and so $f(3^n t) = 1$.

This establishes (2.4) and thus the relation (2.2).

3. Schoenberg's Curve

This function is defined parametrically by the equations

$$x(t) = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} f(3^{2n} t), \tag{3.1}$$

$$y(t) = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} f(3^{2n+1} t) \qquad (0 \leqslant t \leqslant 1). \tag{3.2}$$

The mapping $t \to (x(t), y(t))$ indeed defines a curve: its continuity follows from the expansions (3.1), (3.2) being not only termwise continuous, but dominated by the series of constants

$$\sum_{n=0}^{\infty} \frac{1}{2^{n+1}} = 1. \tag{3.3}$$

These conditions insure their uniform convergence, and therefore also the continuity of their sums.

Now if $t \in \Gamma$, hence

$$t = \sum_{n=0}^{\infty} \frac{a_n}{3^{n+1}} \qquad (a_n = 0, 2), \tag{3.4}$$

by (2.4) we may write (3.1) and (3.2) as

$$x(t) = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \cdot \frac{a_{2n}}{2}, \qquad y(t) = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \cdot \frac{a_{2n+1}}{2}. \tag{3.5}$$

We then invert these relationships: let $P = (x(t), y(t))$ be an arbitrarily preassigned point of the square $I^2 = \{0 \leqslant x, y \leqslant 1\}$, and regard (3.5) as the binary expansions of the coordinates of P. This defines a_{2n} and a_{2n+1}, and therefore also the full sequence $\{a_n\}$. With it we define t ($\in \Gamma$) by (3.4), and thus the expressions (3.5), being a consequence of (3.1) and (3.2), show that the point P is on our curve. This proves

THEOREM 1. *The mapping*

$$t \to (x(t), y(t))$$

from I into I^2 defined by (3.1), (3.2), *is continuous, and covers the square I^2, even if t is restricted to the Cantor Set Γ.*

This result extends naturally to higher dimensions. We discuss only the case of the space curve

$$X(t) = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} f(3^{3n}t), \tag{3.6}$$

$$Y(t) = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} f(3^{3n+1}t), \tag{3.7}$$

$$Z(t) = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} f(3^{3n+2}t) \qquad (0 \leqslant t \leqslant 1). \tag{3.8}$$

The continuity of $X(t)$, $Y(t)$, and $Z(t)$, as in the two-dimensional case, is guaranteed by the continuity of each of their terms and by the convergence of the series of constants (3.3). If we define t by (3.4), so $a_n = 0, 2$ for $n = 0, 1, 2, \ldots$, then again (2.4) shows that

$$X(t) = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \cdot \frac{a_{3n}}{2}, \qquad Y(t) = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \cdot \frac{a_{3n+1}}{2},$$
$$Z(t) = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \cdot \frac{a_{3n+2}}{2}. \tag{3.9}$$

If the right sides are the binary expansions of the coordinates of an arbitrarily chosen point of I^3, then this point of I^3 is reached by our space curve for the value of $t \in \Gamma$ defined by (3.4). Thus we have proven

THEOREM 2. *The mapping*

$$t \to (X(t), Y(t), Z(t))$$

from I *into* I^3 *defined by* (3.6), (3.7), (3.8), *is continuous, and fills the cube* I^3, *even if* t *is restricted to the Cantor Set* Γ.

Theorems 1 and 2 raise a interesting question. Just *how* does the plane curve, for example, fill the square as t varies from 0 to 1? Though by no means may this question be answered completely, we can gain some feeling for the curve's path by viewing it as the point-for-point limit of the sequence of continuous mappings

$$t \to (x_k(t), y_k(t)) \qquad (k = 0, 1, 2, \ldots), \tag{3.10}$$

where x_k and y_k are the kth partial sums of the series (3.1) and (3.2) defining x and y. The graph of this sequence for $k = 0, 1, 2$ and $0 \leqslant t \leqslant 1$ is shown in Fig. 2. (The origin is at the lower left corners, with x_k and y_k on the

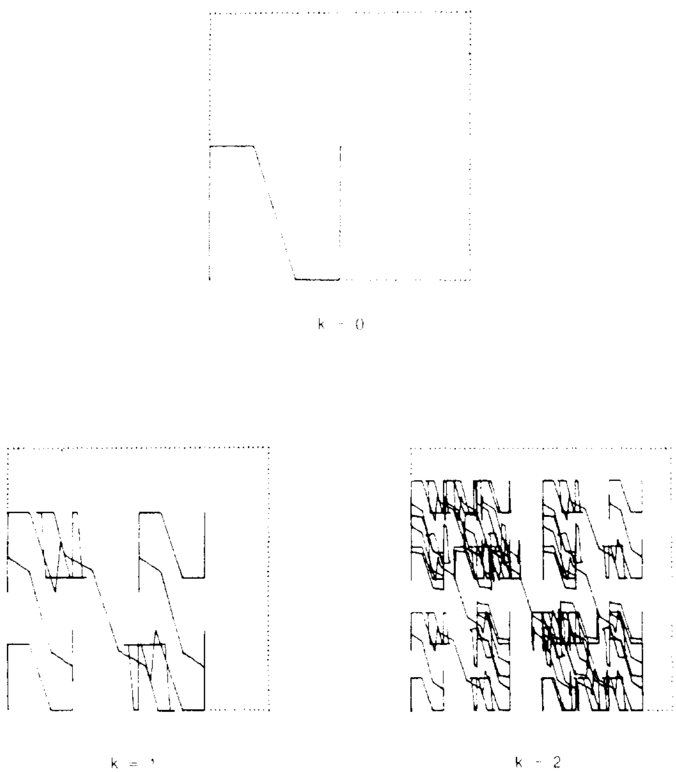

FIG. 2. The approximation curves $t \to (x_k(t), y_k(t))$ for $k = 0, 1, 2$.

horizontal and vertical axes, respectively. The dotted lines delineate the boundary of I^2.)

Note in particular in Fig. 2 that the curves lack the one-to-one property for $k = 1, 2$. This fact, together with the promise for increased complexity in these approximation curves as $k \to \infty$, suggests that the limit curve itself may be many-to-one.

The implication is indeed correct, and not only for the case at hand. If an area-filling curve were one-to-one, it would be a homeomorphism. The unit interval and I^n (for $n \geq 2$), however, are not homeomorphic, since the removal of any interior point disconnects I but not I^n.

The point $(\frac{1}{2}, \frac{1}{2})$ of I^2 nicely illustrates this many-to-one property for Schoenberg's curve (3.1), (3.2). Since the number $\frac{1}{2}$ can be expressed in binary form either as .1000... or .0111..., (3.4) and (3.5) imply that $(x(t_0), y(t_0)) = (\frac{1}{2}, \frac{1}{2})$ is the image of *four* distinct elements of the Cantor Set Γ, namely,

$$t_0 = \frac{1}{9}, \frac{11}{36}, \frac{25}{36}, \frac{8}{9}.[1]$$

In fact, the set of all (x, y) with four preimages in Γ is dense in the square. Theorem 1 asserted that Γ, a set of Lebesgue measure zero, is sufficiently large to be mapped onto I^2, a set of plane measure 1. It would now seem that Γ has more points than I^2!

In the next section, we explore yet another property of Schoenberg's curve, and prove our main result.

4. The Peano Curve of Schoenberg Is Nowhere Differentiable

We say that a plane curve $(x(t), y(t))$ is *differentiable at* t_0 if both derivatives $x'(t_0)$ and $y'(t_0)$ exist and are finite. Our goal will be to prove

THEOREM 3. *For no value of t do both functions*

$$x(t) = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} f(3^{2n}t), \tag{4.1}$$

$$y(t) = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} f(3^{2n+1}t), \tag{4.2}$$

have finite derivatives $x'(t)$, $y'(t)$.

[1] More precisely, $(\frac{1}{2}, \frac{1}{2})$ is a quintuple point of the curve, having its fifth preimage, $t_0 = \frac{1}{2}$, in $[0, 1] \setminus \Gamma$.

Since $f(t)$ is an even function of period two, then so are $x(t)$ and $y(t)$. Thus it suffices to prove Theorem 3 for $t \in I = [0, 1]$. The theorem will follow from the proofs of two lemmas.

Let t be a fixed number in $[0, 1]$, expressed in ternary form by

$$t = \frac{a_0}{3} + \frac{a_1}{3^2} + \cdots + \frac{a_n}{3^{n+1}} + \cdots \qquad (a_n = 0, 1, 2), \tag{4.3}$$

and corresponding to this t, define the following disjoint sets:

$$N_0 = \{n: a_{2n} = 0\},$$
$$N_1 = \{n: a_{2n} = 1\},$$
$$N_2 = \{n: a_{2n} = 2\}.$$

The first of our lemmas is

LEMMA 2. *$x'(t)$ does not exist finitely if $N_0 \cup N_2$ is an infinite set.*

In the proof we make use of several properties of the function $f(t)$:

$$f(t+2) = f(t) \qquad \text{for all } t. \tag{4.4}$$

If M is an integer and $t_1 \in [M, M + \frac{1}{3}]$, $t_2 \in [M + \frac{2}{3}, M + 1]$, then

$$|f(t_1) - f(t_2)| = 1. \tag{4.5}$$

$f(t)$ also satisfies the Lipschitz condition

$$|f(t_1) - f(t_2)| \leqslant 3 \cdot |t_1 - t_2| \qquad \text{for any } t_1, t_2. \tag{4.6}$$

Let us now assume that $m \in N_0 \cup N_2$, hence $a_{2m} = 0$ or $a_{2m} = 2$. For such m, we define the increment

$$\begin{aligned} \delta_m &= \tfrac{2}{3} 9^{-m} \qquad \text{if} \quad a_{2m} = 0, \\ &= -\tfrac{2}{3} 9^{-m} \qquad \text{if} \quad a_{2m} = 2, \end{aligned} \tag{4.7}$$

and seek to estimate the corresponding difference quotient

$$\frac{x(t+\delta_m) - x(t)}{\delta_m} = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \gamma_{n,m}, \tag{4.8}$$

where

$$\gamma_{n,m} = \frac{f(9^n(t+\delta_m)) - f(9^n t)}{\delta_m}. \tag{4.9}$$

We must distinguish three cases.

(i) $n > m$. By (4.7), $9^n\delta_m = \pm \frac{2}{3}9^{n-m}$, which is an even integer. Thus by (4.4), we conclude that

$$\gamma_{n,m} = 0 \qquad \text{if} \quad n > m, \tag{4.10}$$

regardless of the value of a_{2m}.

(ii) $n < m$. Here we make use of the Lipschitz inequality (4.6) to show that

$$|\gamma_{n,m}| \leqslant 3 \cdot |9^n\delta_m|/|\delta_m|,$$

whence

$$|\gamma_{n,m}| \leqslant 3 \cdot 9^n \qquad \text{for} \quad n < m. \tag{4.11}$$

(iii) $n = m$. By (4.3), we see that

$$9^m t = 3^{2m} t = M + \frac{a_{2m}}{3} + \frac{a_{2m+1}}{3^2} + \cdots \qquad (M \text{ is an integer}). \tag{4.12}$$

Here we must distinguish two subcases:

If $a_{2m} = 0$, and so, by (4.7), $9^m\delta_m = 2/3$, (4.12) implies that $M \leqslant 9^m t \leqslant M + 2/3^2 + 2/3^3 + \cdots$. Since $2/3^2 + 2/3^3 + \cdots = 1/3$, we find that $M \leqslant 9^m t \leqslant M + 1/3$, and therefore that $M + 2/3 \leqslant 9^m t + 9^m\delta_m \leqslant M + 1$.

If $a_{2m} = 2$, then, by (4.7), $9^m\delta_m = -2/3$. From (4.12), $M + 2/3 \leqslant 9^m t \leqslant M + 2/3 + 2/3^2 + \cdots = M + 1$, while $M \leqslant 9^m t + 9^m\delta_m \leqslant M + 1/3$.

In either subcase, we can apply (4.5) to conclude that

$$|\gamma_{m,m}| = 1/|\delta_m| = \tfrac{3}{2}9^m, \tag{4.13}$$

regardless of the value of a_{2m}.

The results (4.10), (4.11), and (4.13) hold under the sole assumption

$$m \in N_0 \cup N_2.$$

Applying them to the difference quotient

$$DQ_m = \frac{x(t+\delta_m) - x(t)}{\delta_m}, \tag{4.14}$$

we find by (4.8) that

$$\begin{aligned} |DQ_m| &= \left| \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \gamma_{n,m} \right| \\ &= \left| \sum_{n=0}^{m} \frac{1}{2^{n+1}} \gamma_{n,m} \right| \end{aligned}$$

$$\geqslant \frac{1}{2^{m+1}} |\gamma_{m,m}| - \sum_{n=0}^{m-1} \frac{1}{2^{n+1}} |\gamma_{n,m}|$$

$$\geqslant \frac{1}{2^{m+1}} \cdot \frac{3}{2} 9^m - \sum_{n=0}^{m-1} \frac{1}{2^{n+1}} \cdot 3 \cdot 9^n$$

$$= \frac{3}{4} \left(\frac{9}{2}\right)^m - \frac{3}{7} \left[\left(\frac{9}{2}\right)^m - 1 \right].$$

and finally

$$\left| \frac{x(t+\delta_m) - x(t)}{\delta_m} \right| \geqslant \frac{9}{28} \left(\frac{9}{2}\right)^m + \frac{3}{7} \qquad \text{if} \quad m \in N_0 \cup N_2. \tag{4.15}$$

This establishes Lemma 2 if, in (4.15), we let $m \to \infty$ through the elements of the infinite sequence $N_0 \cup N_2$.

We now turn out attention to the digits of t having odd subscripts, and define the sets

$$N'_0 = \{n: a_{2n+1} = 0\},$$

$$N'_1 = \{n: a_{2n+1} = 1\},$$

$$N'_2 = \{n: a_{2n+1} = 2\}.$$

Now if

$$t = \frac{a_0}{3} + \frac{a_1}{3^2} + \cdots + \frac{a_{2n+1}}{3^{2n+2}} + \cdots,$$

then for $\tau = 3t$ we find

$$\tau = a_0 + \frac{a_1}{3} + \cdots + \frac{a_{2n+1}}{3^{2n+1}} + \cdots.$$

As the same time

$$x(\tau) = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} f(3^{2n}\tau) = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} f(3^{2n+1}t) = y(t).$$

Applying Lemma 2 to $x(t)$ at the point $\tau = 3t$, we see that the digits a_{2n+1} are the digits of τ having *even* subscripts. We thus obtain

Corollary 1. *$y'(t)$ does not exist finitely if $N'_0 \cup N'_2$ is an infinite set.*

By Lemma 2 and Corollary 1 we can conclude that the only t for which

$x'(t)$ and $y'(t)$ might both exist and be finite, is one whose sets $N_0 \cup N_2$ and $N'_0 \cup N'_2$ are finite. This is the case if and only if the digits

$$a_n = 1 \qquad \text{for all sufficiently large } n. \tag{4.16}$$

On the other hand, to prove the *non*differentiability of the mapping $t \to (x(t), y(t))$, it suffices to show that one of the derivatives $x'(t)$, $y'(t)$ fails to exist.

LEMMA 3. *If t is such that* (4.16) *holds, then $x'(t)$ does not exist finitely.*

The simplest t satisfying (4.16) is the one for which all $a_n = 1$, or

$$t = \sum_{n=0}^{\infty} \frac{1}{3^{n+1}} = \frac{1}{2}$$

We must, however, treat the general case, where

$$t = \frac{a_0}{3} + \frac{a_1}{3^2} + \cdots + \frac{a_{2m-1}}{3^{2m}} + \frac{1}{3^{2m+1}} + \frac{1}{3^{2m+2}} + \cdots, \tag{4.17}$$

with $a_n = 0, 1, 2$ for $n = 0, 1, \ldots, 2m - 1$. To prove the lemma, we proceed as in Lemma 2 by estimating the difference quotient

$$\frac{x(t + \delta_m) - x(t)}{\delta_m} = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \gamma_{n,m}, \tag{4.18}$$

where

$$\gamma_{n,m} = \frac{f(9^n(t + \delta_m)) - f(9^n t)}{\delta_m}. \tag{4.19}$$

Here, though, we must abandon our former choice for the increment δ_m in favor of

$$\delta_m = \tfrac{2}{9} 9^{-m}. \tag{4.20}$$

We will once again examine the quantity $\gamma_{n,m}$ in terms of three cases:

(i) $n > m$. From (4.20), $9^n \delta_m = \frac{2}{9} 9^{n-m}$, which is an even integer. Thus, by property (4.4), the periodicity of $f(t)$, we see that

$$\gamma_{n,m} = 0 \qquad \text{if} \quad n > m. \tag{4.21}$$

(ii) $n < m$. In this case, we again use the Lipschitz condition (4.6) to conclude that

$$|\gamma_{n,m}| \leqslant 3 \cdot 9^n \qquad \text{if} \quad n < m. \tag{4.22}$$

(iii) $n=m$. By (4.17),

$$9^m t = 3^{2m} t = M + \frac{1}{3} + \frac{1}{3^2} + \cdots \qquad (M \text{ is an integer}),$$

whence

$$9^m t = M + \tfrac{1}{2}$$

while

$$9^m \delta_m = \tfrac{2}{9}. \tag{4.24}$$

From the graph of $f(t)$, Fig. 1,

$$f(N+\tfrac{1}{2}) = f(\tfrac{1}{2}) = \tfrac{1}{2} \qquad \text{for any integer } N, \tag{4.25}$$

and so from (4.23),

$$f(9^m t) = \tfrac{1}{2}. \tag{4.26}$$

The addition of (4.23) and (4.24) gives

$$9^m t + 9^m \delta_m = M + 13/18,$$

and since $2/3 < 13/18 < 1$, Fig. 1 shows us that

$$\begin{aligned} f(9^m t + 9^m \delta_m) &= 0 \qquad \text{if } M \text{ is odd}, \\ &= 1 \qquad \text{if } M \text{ is even}. \end{aligned} \tag{4.27}$$

Regardless of the value of M, (4.26) and (4.27) imply that

$$|f(9^m t + 9^m \delta_m) - f(9^m t)| = \tfrac{1}{2},$$

and therefore, by (4.19) and (4.20), that

$$|\gamma_{m,m}| = \frac{1/2}{|\delta_m|} = \frac{9}{4}\, 9^m. \tag{4.28}$$

Applying the results (4.21), (4.22), and (4.28) to the difference quotient (4.18),

$$\begin{aligned} |DQ_m| &= \left| \frac{x(t+\delta_m) - x(t)}{\delta_m} \right| = \left| \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \gamma_{n,m} \right| \\ &= \left| \sum_{n=0}^{m} \frac{1}{2^{n+1}} \gamma_{n,m} \right| \end{aligned}$$

$$\geqslant \frac{1}{2^{m+1}} |\gamma_{m,m}| - \sum_{n=0}^{m-1} \frac{1}{2^{n+1}} |\gamma_{n,m}|$$

$$\geqslant \frac{1}{2^{m+1}} \cdot \frac{9}{4} 9^m - \sum_{n=0}^{m-1} \frac{1}{2^{n+1}} \cdot 3 \cdot 9^n,$$

which yields

$$|DQ_m| \geqslant \frac{39}{56} \left(\frac{9}{2}\right)^m + \frac{3}{7}. \tag{4.29}$$

If, in (4.29), we let $m \to \infty$, $\delta_m \to 0$, hence x is not differentiable at t. This establishes Lemma 3, and therefore also Theorem 3.

While Lemma 3 alone is sufficient to prove the nondifferentiability of the mapping

$$t \to (x(t), y(t)) \tag{4.30}$$

for t defined by (4.17), $y'(t)$ as well may be shown not to exist for such t. This claim is easily verified by the same argument that produced Corollary 1.

5. The Generalization of Theorem 3

Analogous to Schoenberg's plane Peano curve (4.1), (4.2) is the space curve

$$X(t) = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} f(3^{3n} t), \tag{5.1}$$

$$Y(t) = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} f(3^{3n+1} t), \tag{5.2}$$

$$Z(t) = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} f(3^{3n+2} t) \qquad (0 \leqslant t \leqslant 1), \tag{5.3}$$

introduced in Section 3. By way of Theorem 2, we saw that these functions define a Peano curve filling the unit cube I^3. Here, in a similar fashion, we seek to extend Theorem 3 to higher dimensions.

THEOREM 4. *The Peano curve defined by* (5.1), (5.2), (5.3) *above is nowhere differentiable.*

The technique of proof used for Theorem 3 will apply nicely; again we shall have two lemmas and a corollary.

Indeed, with t defined by

$$t = \frac{a_0}{3} + \frac{a_1}{3^2} + \cdots + \frac{a_n}{3^{n+1}} + \cdots \qquad (a_n = 0, 1, 2),$$

we define the corresponding sets of integers

$$M_0 = \{n: a_{3n} = 0\}, \qquad M_1 = \{n: a_{3n} = 1\}, \qquad M_2 = \{n: a_{3n} = 2\},$$

and state

LEMMA 4. *The derivative $X'(t)$ does not exist finitely if $M_0 \cup M_2$ is an infinite set.*

For $m \in M_0 \cup M_2$, we define the increment

$$\begin{aligned} \delta_m &= \tfrac{2}{3} 3^{-3m} \qquad \text{if} \quad a_{3m} = 0, \\ &= -\tfrac{2}{3} 3^{-3m} \qquad \text{if} \quad a_{3m} = 2, \end{aligned}$$

and investigate the difference quotient

$$DQ_m = \frac{X(t+\delta_m) - X(t)}{\delta_m} = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \gamma_{n,m},$$

where

$$\gamma_{n,m} = \frac{f(3^{3n}(t+\delta_m)) - f(3^{3n}t)}{\delta_m}.$$

Proceeding as in the proof of Lemma 2, we find that

$$|DQ_m| \geqslant \frac{3}{4}\left(\frac{27}{2}\right)^m - \frac{3}{25}\left(\left(\frac{27}{2}\right)^m - 1\right),$$

which proves Lemma 4, if we let $m \to \infty$ through the elements of $M_0 \cup M_2$.

Using the identities $Y(t) = X(3t)$, $Z(t) = X(3^2 t)$, we obtain the following:

COROLLARY 2. (i) *If the sets $M_0' = \{n: a_{3n+1} = 0\}$, $M_2' = \{n: a_{3n+1} = 2\}$ are such that $M_0' \cup M_2'$ is an infinite set, then $Y'(t)$ does not exist finitely.*

(ii) *If the sets $M_0'' = \{n: a_{3n+2} = 0\}$, $M_2'' = \{n: a_{3n+2} = 2\}$ are such that $M_0'' \cup M_2''$ is an infinite set, then $Z'(t)$ does not exist finitely.*

The only t for which all the derivatives $X'(t)$, $Y'(t)$, $Z'(t)$ might still exist is one whose sets

$$M_0 \cup M_2, \qquad M_0' \cup M_2', \qquad M_0'' \cup M_2''$$

are all finite. This condition is true if and only if

$$a_n = 1 \qquad \text{for all sufficiently large } n. \tag{5.4}$$

We now state our final

LEMMA 5. *Suppose t satisfies* (5.4). *Then none of the derivatives $X'(t)$, $Y'(t)$, $Z'(t)$ exists and is finite.*

The proof of the claim for $X'(t)$ follows from the choice of

$$\delta_m = \tfrac{2}{9} 3^{-3m},$$

and those for $Y'(t)$ and $Z'(t)$ from arguments similar to the proof of Corollary 1 in Section 4.

6. A FINAL REMARK

With its complete lack of differentiability, Schoenberg's plane curve provides an interesting contrast to the Peano curve from which it is derived, that of Lebesgue (see [3]).

Under Lebesgue's mapping $L(t)$, each (x_0, y_0) of I^2, expressed as

$$x_0 = \frac{\alpha_0}{2} + \frac{\alpha_2}{2^2} + \frac{\alpha_4}{2^3} + \cdots$$

$$y_0 = \frac{\alpha_1}{2} + \frac{\alpha_3}{2^2} + \frac{\alpha_5}{2^3} + \cdots \qquad (\alpha_i = 0, 1),$$

is the image of a point t_0 in Cantor's Set Γ of the form

$$t_0 = \frac{2\alpha_0}{3} + \frac{2\alpha_1}{3^2} + \frac{2\alpha_2}{3^3} + \cdots.$$

This correspondence we now recognize as a restatement of the relations (3.5). As such, $L(t)$ coincides with Schoenberg's curve on Γ, and thus must lack a finite derivative there.

Lebesgue then extends the domain of $L(t)$ to all of $[0, 1]$ by means of linear interpolation over each of the open intervals which comprise the complement of Γ. Defined in this manner, $L(t)$ must indeed be differentiable on $[0, 1] \setminus \Gamma$, and hence constitutes an example of a Peano curve which, unlike Schoenberg's, is differentiable almost everywhere.

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