Journal of Algebra **222,** 747–768 (1999)
Article ID jabr.1999.8073, available online at http://www.idealibrary.com on IDEAL®

On the Existence of p-Blocks of Defect 0 in p-Nilpotent Groups

Hiroshi Fukushima

Communicated by Walter Feit

Received February 19, 1998

1. INTRODUCTION

Let G be a finite group of order g. Let p be a prime and $g = p^a g'$ with $(p, g') = 1$. An irreducible ordinary character of G is called p-defect 0 if and only if its degree is divisible by p^a. By [1, Theorem 4.18], G has a character of p-defect 0 if and only if G has a p-block of defect 0.

An important question in the modular representation theory of finite groups is to find the group-theoretic conditions for the existence of characters of p-defect 0 in a finite group. If a finite group G has a character of p-defect 0, then $O_p(G) = 1$ [1, Corollary 6.9]. But the converse is not true. In this paper, we shall give sufficient conditions for a p-nilpotent group to have a character of p-defect 0.

Before describing the next examples we need to define the following notation. Let $F = GF(q^n)$ be a finite field of q^n elements. Let V be the additive group of F. Then let $T(q^n)$ (the semi-linear group) be the set of semi-linear transformations of the form $v \to av^\sigma$ with $v \in V$, $0 \neq a \in F$, and σ a field automorphism (see [8, p. 229]). Then we can consider the semi-direct product $V \rtimes T(q^n)$ (the affine semi-linear group) of V by $T(q^n)$. Now the following examples show that the converse is not true (as mentioned above).

EXAMPLE 1. Suppose p and q are two distinct primes. Let V be an elementary abelian q-group of order q^n such that p divides $q^n - 1$. Consider V the additive group of the field $GF(q^n)$ of q^n elements. Let $N = \{v \to av \mid 0 \neq a \in GF(q^n)\}$. Thus $V \rtimes N \subseteq V \rtimes T(q^n)$. Let $\langle x \rangle$ be a cyclic group of order p and let $(V \rtimes N) \wr \langle x \rangle$ be the wreath product. Set $V_0 = V \times$

0021-8693/99 $30.00

$V^x \times \cdots \times V^{x^{p-1}}$ and $N_0 = N \times N^x \times \cdots \times N^{x^{p-1}}$. Then we set $F(p, q, n) = V_0 \rtimes ((\Omega_1(O_p(N_0))[N_0, x]) \rtimes \langle x \rangle) \subseteq (V \rtimes N) \wr \langle x \rangle$, where $\Omega_1(O_p(N_0)) = \langle y \in O_p(N_0) \mid y^p = 1 \rangle$.

EXAMPLE 2. Suppose p and q are two distinct prime numbers. Let V be an elementary abelian q-group of order q^{pn} such that p divides $q^n - 1$. Consider V the additive group of the field $GF(q^{pn})$ of q^{pn} elements. Let x be an element of the Galois group $Gal(GF(q^{pn})/GF(q))$ of order p, and F_0 a subgroup of the multiplicative group $GF(q^{pn})^{\#}$ of order $(q^{pn} - 1)/(q^n - 1)$. Let $N = \{v \to av \mid a \in F_0\}$. Then p divides $|N|$. Set $E(p, q, n) = V \rtimes (N \rtimes \langle x \rangle) \subseteq V \rtimes T(q^n)$. Then $E(p, q, n)$ is determined uniquely by the three parameters p, q, and n. It is easily seen that $E(p, q, n)$ is p-nilpotent and $O_p(E(p, q, n)) = 1$.

EXAMPLE 3. Let V be an elementary abelian group of order 7^2. Then $Aut(V)$ contains a subgroup H that is isomorphic to $SL(2, 3) \times Z_3$, where Z_3 is a cyclic group of order 3. Indeed, let $L = \left\langle \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 3 & 2 \\ 2 & -3 \end{pmatrix} \right\rangle \rtimes \left\langle \begin{pmatrix} 4 & 0 \\ 1 & 2 \end{pmatrix} \right\rangle \simeq SL(2, 3)$ and let $Z = \left\langle \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \right\rangle \simeq Z_3$. Then $L \times Z \subseteq GL(2, 7)$. Thus $L \times Z$ acts naturally on V. We let J be the semi-direct product $V \rtimes H$. In Lemma 2.5, we can conclude that J is unique up to isomorphism.

In Example 1, we set $G = (V \rtimes N) \wr \langle x \rangle$. Let $H = N_0\langle x \rangle$ and $V_0 \ni v = v_1 \cdots v_p$ with $v_i \in V^{x^{i-1}}, 1 \leq i \leq p$. If $v_i = 1$ for some i, then $1 \neq O_p(N^{x^{i-1}}) \subseteq C_{O_p(N_0)}(v) \subseteq O_p(C_H(v))$. If $v_i \neq 1$ for any i, then $v = v_1 \cdots v_p$ is conjugate to $v_1 v_1^x \cdots v_1^{x^{p-1}}$ in N_0 since N acts transitively on $V^{\#}$. Since $C_H(v_1 v_1^x \cdots v_1^{x^{p-1}}) = \langle x \rangle$, $C_H(v)$ is of order p. In each case, $O_p(C_H(v)) \neq 1$.

Set $\bar{G} = G/V_0[N_0, x]$. Then $\bar{G} = \bar{N} \times \langle \bar{x} \rangle$ and $\bar{N} \simeq N$. Let y be an element of G of order p. Since N is cyclic, $\bar{y} \in \Omega_1(O_p(\bar{N})) \times \langle \bar{x} \rangle$. Let L be the inverse image of $\Omega_1(O_p(\bar{N})) \times \langle \bar{x} \rangle$. Then $L = F(p, q, n)$. Hence $1 \neq \Omega_1(O_p(C_H(v))) \subseteq L$ for $v \in V_0$ and so $1 \neq O_p(C_{H \cap L}(v))$.

Since $(|H \cap L|, |V_0|) = 1$, $O_p(I_{H \cap L}(\varphi)) \neq 1$ for any $\varphi \in Irr(V_0)$ by Lemma 2.2. Since $I_{H \cap L}(\varphi)$ has no characters of p-defect 0, L has no characters of p-defect 0 by Lemma 2.1.

In Example 2, we set $L = N \rtimes \langle x \rangle$. By [9, Prop. 1.4], L has no regular orbits on V. Hence $1 \neq C_L(v)$ for $\forall v \in V$. For $1 \neq v \in V$, $C_L(v)$ is of order p since $C_N(v) = 1$. Since $O_p(L) \neq 1$, $O_p(C_L(v)) \neq 1$ for $\forall v \in V$. By Lemmas 2.1 and 2.2, $E(p, q, n) = V \rtimes L$ has no characters of p-defect 0.

In Example 3, let Q be a subgroup of H which is isomorphic to quaternion of order 8. Then $|Q \times Z| = 24$ and $Q \times Z$ acts regularly on $V^{\#}$. Since $|V^{\#}| = 48$, $Q \times Z$ has two orbits on $V^{\#}$. Let x be an element of $SL(2, 3)$ of order 3. Then x stabilizes each $Q \times Z$-orbit. Since $O_3(H) \neq 1$, $O_3(C_H(v)) \neq 1$ for all $v \in V$. By Lemmas 2.1 and 2.2, $J = V \rtimes L$ has no characters of 3-defect 0.

Now, in this paper we shall prove the following result.

THEOREM. *Let G be a solvable p-nilpotent group for some prime p. Suppose that $O_p(G) = 1$ and G is $E(p, q, n), F(p, q, n)$-free for all possible q and n. Furthermore, if $p = 3$, assume that G is J-free. Then G has a character of p-defect 0. In particular, there exists an element $x \in O_{p'}(G)$ such that $C_G(x)$ is a p'-subgroup.*

2. PRELIMINARIES

In this section we shall prove some lemmas which will be used to prove the theorem.

Let $G \rhd V$. We let $Irr(V)$ be the set of ordinary irreducible characters of V and let $I_G(\varphi)$ be the inertia group of $\varphi \in Irr(V)$.

LEMMA 2.1. *Let $G = HV \rhd V$, where V is an abelian p'-group with $H \cap V = 1$. Let $\varphi \in Irr(V)$. Then the following are equivalent.*

(i) *There exists $\chi \in Irr(G)$ such that $\varphi \mid \chi_V$ and χ is a character of p-defect 0.*

(ii) *Let $I = I_H(\varphi) = \{h \in H | \varphi^h = \varphi\}$. Then I has a character of p-defect 0.*

Proof. Set $V_1 = \operatorname{Ker} \varphi$. Then V/V_1 is cyclic since V is abelian. Let $\overline{I_G(\varphi)} = I_G(\varphi)/V_1$. Then $\overline{I_G(\varphi)} = \bar{V} \times \bar{I}$ since $I_G(\varphi) = VI$ and there is a bijection from $Irr(IV \mid \varphi)$ onto $Irr(G \mid \varphi)$. For $\alpha \in Irr(IV)$, $|IV|_p$ divides $\alpha(1)$ if and only if $|G|_p$ divides $\alpha^G(1)$. Also φ extends to θ in $Irr(IV)$ and so $Irr(IV \mid \varphi) = \{\beta\theta \mid \beta \in Irr(IV/V)\}$. Now $(\beta\theta)^G$ has p-defect 0 if and only if β is a p-defect 0 character of $IV/V \simeq I$. ∎

LEMMA 2.2 [3, p. 231, Theorem 13.24]. *Let S act on G with S solvable and $(|G|, |S|) = 1$. Then S permutes $Irr(G)$ and S permutes the set $cl(G)$ of conjugate class of G. Then the actions of S on $Irr(G)$ and $cl(G)$ are permutation isomorphic.*

LEMMA 2.3. *Let $\langle x \rangle$ be a cyclic group of order r and V a $\langle x \rangle$-module of order q^s, where q is a prime. Suppose that every irreducible constituent of V is a faithful $\langle x \rangle$-module. Then the following hold.*

(i) *$\langle v^{x^i} \mid i = 0, \dots, r-1 \rangle$ is an irreducible $\langle x \rangle$-module for all $v \in V^{\#}$.*

(ii) *If U is a subgroup of V with $|V/U| = q$, then $V/\bigcap_{i=0}^{r-1} U^{x^i}$ is an irreducible $\langle x \rangle$-module.*

Proof. Since $(|V|, r) = 1$, V is a completely reducible $\langle x\rangle$-module. Let $V = V_1 \oplus \cdots \oplus V_n$, where V_i are faithful irreducible $\langle x\rangle$-modules, $1 \leq i \leq n$. Then we can identify V_i with the additive group of $GF(q^m)$ in such a way that $\langle x\rangle$ is contained in the set of linear transformations. Hence V_i, $1 \leq i \leq n$, are isomorphic $\langle x\rangle$-modules, and so we may assume that $v_i^x = \alpha v_i$ with fixed $\alpha \in GF(q^m)$ and $\forall v_i \in V_i$. Then every non-zero vector v is contained in an irreducible $\langle x\rangle$-module W, which must be generated as stated. Likewise every maximal subspace U of V contains an $\langle x\rangle$-invariant W such that V/W is irreducible. ∎

LEMMA 2.4. *Let P be an extra-special p-group of order p^{2r+1}, p a prime, and let $H = \{\sigma \in Aut(P) \mid \sigma$ centralizes $Z(P)\}$. We may identify $Z(P)$ with the field of p-elements. Since $P/Z(P)$ is an elementary abelian p-group, the commutator map $[x, y]$ is a non-singular, alternating bilinear form on $\bar{P} = P/Z(P)$. Any automorphism of P that centralizes $Z(P)$ must preserve this form. Then there exist hyperbolic pairs $\{u_1, v_1\} \cdots \{u_r, v_r\}$ with $(u_i, v_j) = \delta_{ij}$ and $(u_i, u_j) = (v_i, v_j) = 0$, where δ_{ij} is the Kronecker δ. Let $A = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ be the structure matrix with respect to this basis $\{u_1, \ldots, u_r, v_1, \ldots, v_r\}$ of $\bar{P}$, where I and 0 are the unit matrix and zero matrix of degree r, respectively. If $\sigma \in H$ centralizes $\langle u_1, \ldots, u_r\rangle$, then σ^p centralizes $\bar{P}$.*

Proof. Let S be the matrix of σ with respect to the basis $\{u_1, \ldots, u_r, v_1, \ldots, v_r\}$. Then $SAS^T = A$, where S^T is the transpose matrix of S. Let $S = \begin{pmatrix} I & 0 \\ K & L \end{pmatrix}$, where I and 0 are the unit matrix and zero matrix of degree r, respectively, and K, L are matrices of degree r.

Then

$$\begin{pmatrix} I & 0 \\ K & L \end{pmatrix}\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}\begin{pmatrix} I & K^T \\ 0 & L^T \end{pmatrix} = \begin{pmatrix} 0 & I \\ -L & K \end{pmatrix}\begin{pmatrix} I & K^T \\ 0 & L^T \end{pmatrix} = \begin{pmatrix} 0 & L^T \\ -L & -LK^T + KL^T \end{pmatrix} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

Hence $L = I$ and $-K^T + K = 0$. Therefore $S = \begin{pmatrix} I & 0 \\ K & I \end{pmatrix}$. Thus $S^p = \begin{pmatrix} I & 0 \\ pK & I \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$, and hence σ^p centralizes $\bar{P}$. ∎

LEMMA 2.5. *Let H_1 and H_2 be subgroups of $GL(2, 7)$ and Z_3 a cyclic group of order 3. If $H_1 \simeq H_2 \simeq SL(2, 3) \times Z_3$, then H_1 and H_2 are conjugate in $GL(2, 7)$.*

Proof. Let Q_i be a Sylow 2-subgroup of H_i $(i = 1, 2)$. Then $Q_i \simeq Q_8$, where Q_8 is a quaternion of order 8. Let S be a Sylow 2-subgroup of

$GL(2,7)$. Then S is semi-dihedral of order 32 and S has three maximal subgroups, that is, generalized quaternion, dihedral, and cyclic. Let S_0 be a generalized quaternion subgroup of S. By conjugation, we may assume that Q_1 and Q_2 are subgroups of S_0. Set $\bar{S} = S/Z(S)$. Then $\bar{S}, \bar{S}_0$ are dihedral groups of order $16, 8$, respectively. Since $\bar{Q}_1$ and $\bar{Q}_2$ are conjugate in $\bar{S}$, Q_1 and Q_2 are conjugate in S. Thus we may assume that $Q_1 = Q_2$. Therefore H_1 and H_2 are subgroups of $N_L(Q_1) \simeq GL(2,3) \times Z_3$, where $L = GL(2,7)$. Since $O^2(H_i) = H_i$ $(i = 1, 2)$ and $O^2(N_L(Q_1)) \simeq SL(2,3) \times Z_3$, $H_1 = H_2 = O^2(N_L(Q_1))$. ∎

3. PROOF OF THE THEOREM

In this section we shall prove the theorem stated in the Introduction. If G has a p-block of defect 0, then there exists a p'-element x such that $C_G(x)$ is a p'-subgroup by the definition of the defect. Then $x \in O_{p'}(G)$ since G is p-nilpotent. It therefore suffices to show that G has a character of p-defect 0 under the hypotheses of the theorem. Let G be a minimal counterexample of the theorem.

LEMMA 3.1. *The following conditions hold.*

(i) $O^{p'}(G) = G$.

(ii) $p \mid |C_G(x)|$ *for* $\forall x \in O_{p'}(G)$.

(iii) *If* V *is a* p'*-subgroup of* G *with* $1 \neq V \lhd G$, *then* $O_p(G/V) \neq 1$.

Proof. (i) Let $\chi \in Irr(G)$ and let $\zeta \in Irr(O^{p'}(G))$ be a constituent of $\chi_{O^{p'}(G)}$. Then $\chi(1)/\zeta(1)$ divides $|G : O^{p'}(G)|$ by [3, Corollary 11.29]. Hence χ is a character of p-defect 0 if and only if ζ is a character of p-defect 0.

(ii) follows immediately from [5, Lemma 1].

(iii) Set $\bar{G} = G/V$. If $O_p(\bar{G}) = 1$, then $\bar{G}$ has a character of p-defect 0 by the minimality of G, and so has G. ∎

Let $\Phi(G)$ be the Frattini subgroup (the intersection of all maximal subgroups of G). By [6, Theorem 1.12], if G is solvable, then $F(G/\Phi(G)) = F(G)/\Phi(G)$ is a completely reducible and faithful $G/F(G)$-module (possibly of mixed characteristic). Furthermore, $G/\Phi(G)$ splits over $F(G)/\Phi(G)$.

LEMMA 3.2. $\Phi(G) = 1$. *In particular,* G *splits over* $F(G)$.

Proof. Since $O_p(G) = 1$, $F(G)$ is a p'-subgroup of G, and hence $F(G/\Phi(G)) = F(G)/\Phi(G)$ is a p'-group. Set $\bar{G} = G/\Phi(G)$. Then

$O_p(\bar{G}) = 1$. If $\Phi(G) \neq 1$, then $\bar{G}$ has a character of p-defect 0 by the minimality of G, and so has G, a contradiction. ∎

Let H be a complement of $F(G)$ in G. Then $G = F(G) \rtimes H$. We set $V = F(G)$.

LEMMA 3.3. *V is an irreducible H-module.*

Proof. From the statement above, V is a completely reducible and faithful H-module. If V is not an irreducible H-module, then there exist V_i $(i = 1, 2)$ such that $V = V_1 \times V_2$ and $1 \neq V_i \lhd G$. Set $\bar{G} = G/V_2$ and $\tilde{G} = G/V_1$. If $O_p(\bar{G}) = 1$, then $\bar{G}$ has a character of p-defect 0, and so has G, a contradiction. Hence $1 \neq O_p(\bar{G}) = \bar{P}_1$, where P_1 is a p-subgroup of G. In the same way, we have $1 \neq O_p(\tilde{G}) = \tilde{P}_2$, where P_2 is a p-subgroup of G. Since P_2 centralizes V_2, P_2 acts faithfully on V_1. Hence $P_2 \cap VP_1 = C_{P_2}(V_1) = 1$ since $[V_1, P_1] = 1$.

Next we reset $\bar{G} = G/VP_1$ and $\tilde{G} = G/V_2P_1$. Then

$$1 \neq \bar{P}_2 \subseteq O_p(\bar{G}). \tag{1}$$

Since $O_p(\tilde{G}) = 1$, $\tilde{G}$ has a character χ of p-defect 0 by induction. By (1), $\bar{G}$ has no characters of p-defect 0, and so $\operatorname{Ker}\chi \not\supseteq V_1$. Hence there exists a $1 \neq \varphi \in Irr(V_1)$ with $\varphi \mid \chi$. Since $\tilde{G} = \tilde{V} \rtimes \tilde{H}, I_{\tilde{H}}(\varphi)$ has a character of p-defect 0 by Lemma 2.1. Hence $O_p(I_{\tilde{H}}(\varphi)) = 1$. We set $T = I_G(\varphi) = \{g \in G \mid \varphi^g = \varphi\}$. Then

$$O_p(T/VP_1) = 1. \tag{2}$$

Hence $O_p(T/V_1) \subseteq VP_1/V_1$. Since P_1 acts faithfully on V_2, $O_p(T/V_1) = 1$, and hence T/V_1 has a character η of p-defect 0. Since $1 \neq P_1V/V \subseteq O_p(T/V)$, T/V have no characters of p-defect 0. Therefore $\operatorname{Ker}\eta \not\supseteq V_2$. So there exists $1 \neq \zeta \in Irr(V_2)$ with $\zeta \mid \eta_{V_2}$. Now, since $T = (T \cap H)V$,

$$O_p(I_{T\cap H}(\zeta)) = 1 \tag{3}$$

by Lemma 2.1. Then $I_H(\varphi\zeta) = I_H(\varphi) \cap I_H(\zeta) = I_{T\cap H}(\zeta)$, and hence $O_p(I_H(\varphi\zeta)) = 1$. By induction, $I_H(\varphi\zeta)$ has a character of p-defect 0. Hence G has a character of p-defect 0 by Lemma 2.1, a contradiction. ∎

By Lemma 3.3, V is an elementary abelian q-group for some prime $q \neq p$.

Let $W \rtimes L$ such that W, L are elementary abelian q-group and q'-group, q a prime, respectively. Furthermore, let $\varphi \in Irr(W)$ and let U_1, U_2 be subgroups of W such that $U_2 \subseteq U_1 \subseteq W$. Then we set $I_L(\varphi) = \{g \in L \mid \varphi^g = \varphi\}$ and $I_L(U_1/U_2) = \{g \in L \mid [U_1, g] \subseteq U_2\}$.

LEMMA 3.4. *Let H_1 be a subgroup of H and set $G_1 = VH_1$. Let $\varphi \in Irr(V)$. Then the following are equivalent.*

(i) *There exists* $\chi \in Irr(G_1)$ *such that* $\varphi \mid \chi_V$ *and* χ *is a character of* p*-defect* 0.

(ii) $O_p(I_{H_1}(\varphi)) = 1$.

Proof. By Lemma 2.1, (i) $\Leftrightarrow$ $I_{H_1}(\varphi)$ has a character of p-defect 0 $\Leftrightarrow$ $O_p(I_{H_1}(\varphi)) = 1$ (by induction).

An irreducible H-module V is called quasi-primitive if V_N is homogeneous for all $N \lhd H$. Then we shall first consider the following case. ∎

Case I

V is not a quasi-primitive H-module.

LEMMA 3.5. *There exists a subgroup* H_0 *of* H *with* $|H:H_0| = p$, $H_0 \lhd H$, *and* $V_{H_0} = V_1 \times \cdots \times V_p$, *where* $V_i, 1 \leq i \leq p$, *are the homogeneous components of* V *with respect to* H_0.

Proof. Choose $N \lhd H$ maximal such that V_N is not homogeneous. Write $V_N = V_1 \times \cdots \times V_k$, where V_i are the homogeneous components of V_N.

Let M/N be a chief factor of H. Since V_M is homogeneous, M transitively permutes the V_i (see [11, Lemma 1.6]). Since M/N is an abelian chief factor of G, M acts regularly on the V_i and $|M/N| = k$. Let $I = N_H(V_1)$, so that $MI = H$ and $M \cap I = N$. Let $C/N = C_{H/N}(M/N) \supseteq M/N$ and $B = C \cap I \lhd MI = H$. Then B fixes each V_i and V_B is not homogeneous. Thus $B = N$ and $C = M$. Hence M/N is the unique minimal normal subgroup of H/N. Set $\bar{H} = H/N$.

Suppose that $\bar{M} = M/N$ is a p-group. Since $\bar{H}$ is p-nilpotent, it has a normal Hall p'-subgroup. Hence $\bar{H}$ must be a p-group. Then $\bar{M} \subseteq Z(\bar{H})$ and so $M = H$. If we set $N = H_0$, then this lemma holds.

Next suppose that $\bar{M}$ is a p'-group. We set $I_1 = O_p(C_I(V_1))$. Since $\bar{M} \supseteq [\bar{I}_1, \bar{M}] \lhd \bar{H}$, $[\bar{I}_1, \bar{M}] = \bar{M}$ or 1.

If $[\bar{I}_1, \bar{M}] = 1$, then I_1 centralizes $O_{p'}(M)/O_{p'}(N)$. Since I_1 centralizes $O_{p'}(N), I_1$ centralizes $O_{p'}(M)$. On the other hand, for $i, 1 \leq i \leq p$, there exists $x_i \in O_{p'}(M)$ with $V_1^{x_i} = V_i$. Hence $I_1 = I_1^{x_i} = O_p(C_{I^{x_i}}(V_i)) \subseteq C_{I_1}(V_i)$. Therefore $I_1 \subseteq C_{I_1}(V)$, which implies that $O_p(C_I(V_1)) = I_1 = 1$. Then $O_p(V_1 I) = 1$. Therefore $V_1 I$ has a character ζ of p-defect 0. By Lemma 3.1(iii), $O_p(H) \neq 1$. If $O_p(H) \nsubseteq N$, then $\bar{M} \subseteq \overline{O_p(H)}$ by the minimality of $\bar{M}$. This contradicts that $\bar{M}$ is a p'-group. Hence $O_p(H) \subseteq N \subseteq I$, and so $1 \neq O_p(H) \subseteq O_p(I)$. Thus $1 \neq O_p(I)$. Therefore $V_1 I/V_1$ has no characters of p-defect 0. Hence $V_1 \nsubseteq \operatorname{Ker} \zeta$, and so there exists $1 \neq \varphi \in Irr(V_1)$ with $\varphi \mid \zeta_{V_1}$. Since $IV/V_2 \times \cdots \times V_p \simeq IV_1$, ζ can be regarded as a character of IV. Hence there exists a $\chi \in Irr(I_{IV}(\varphi))$ such that $\varphi \mid \chi_{V_1}$ and $\chi^{IV} = \zeta$. On the other hand, $I_G(\varphi) = I_{IV}(\varphi)$, and hence $\chi^G \in Irr(G)$. Then $\chi^G = \zeta^G$ is a character of p-defect 0, a contradiction.

Next suppose that $[\bar{I}_1, \bar{M}] = \bar{M}$. Then $[\bar{I}_1, \overline{O_{p'}(M)}] = \bar{M}$. Since $[I_1, O_{p'}(N)] \subseteq I_1 \cap O_{p'}(N) = 1, I_1 \subseteq C_H(O_{p'}(N)) \lhd H$. We set $M_1 = [I_1, O_{p'}(M)] \subseteq C_H(O_{p'}(N))$. Let P_0 be a Sylow p-subgroup of N. Then $[M_1, P_0] \subseteq M_1 \cap N$ since P_0 normalizes M_1. Thus P_0 centralizes $M_1/M_1 \cap N$. Since M_1 is a p'-group, $M_1 = C_{M_1}(P_0)(M_1 \cap N)$. Then $\bar{M} = \bar{M}_1 = \overline{C_{M_1}(P_0)}$ and $[C_{M_1}(P_0), N] = 1$ since $N = O_{p'}(N)P_0$. Hence V_i^x is isomorphic to V_i as an N-module, and so $V_i^x = V_i, 1 \le i \le k$, for $\forall x \in C_{M_1}(P_0)$. This contradicts the fact that M transitively permutes the V_i. ∎

LEMMA 3.6. $O_p(C_{H_0}(V_2 \times \cdots \times V_p)) \neq 1$.

Proof. Since $O_p(H) \neq 1$, $Z(P) \cap O_p(H) \neq 1$, where P is a Sylow p-subgroup of H. Let $z \in Z(P) \cap O_p(H)$ with $|z| = p$. Then $z \in Z(H)$ since H is p-nilpotent. Thus z acts regularly on $V^{\#}$. Since z fixes all homogeneous components of H_0, $z \in H_0$, in particular, $1 \neq z \in O_p(H) \cap H_0 \subseteq O_p(H_0)$.

We set $\overline{H_0(V_2 \times \cdots \times V_p)} = H_0(V_2 \times \cdots \times V_p)/C_{H_0}(V_2 \times \cdots \times V_p) \simeq H_0V/V_1C_{H_0}(V_2 \times \cdots \times V_p)$. By induction, $\overline{H_0(V_2 \times \cdots \times V_p)}$ has a character χ of p-defect 0 since $O_p(\overline{H_0(V_2 \times \cdots \times V_p)}) = 1$. Since $1 \neq \bar{z} \in \overline{O_p(H_0)}$, $\overline{H}_0$ has no characters of p-defect 0. Therefore $V_2 \times \cdots \times V_p \nsubseteq \operatorname{Ker}\chi$, and so there exists $\varphi \in Irr(V_2 \times \cdots \times V_p)$ with $1 \neq \varphi \mid \chi_{V_2 \times \cdots \times V_p}$. By Lemma 2.1, $O_p(I_{\overline{H}_0}(\varphi)) = 1$. Let $U_1 = \operatorname{Ker}\varphi$. Then $|V_2 \times \cdots \times V_p/U_1| = q$ and

$$
\begin{aligned}
I_{\overline{H}_0}(\varphi) &= I_{\overline{H}_0}\big((V_2 \times \cdots \times V_p)/U_1\big) \\
&= \big\{\bar{h} \in \overline{H}_0 \mid h \in H_0, [h, V_2 \times \cdots \times V_p] \subseteq U_1\big\}.
\end{aligned}
$$

Set $P_1 = O_p(I_{H_0}((V_2 \times \cdots \times V_p)/U_1))$. Since $\bar{P}_1 \subseteq O_p(I_{\bar{H}_0}((V_2 \times \cdots \times V_p)/U_1)) = 1$, $P_1 \subseteq C_{H_0}(V_2 \times \cdots \times V_p)$, and hence $P_1 \subseteq O_p(C_{H_0}(V_2 \times \cdots \times V_p))$. Therefore, if $O_p(C_{H_0}(V_2 \times \cdots \times V_p)) = 1$, then $P_1 = 1$. Let $g \in I_H(V/(V_1 \times U_1))$. If $g \notin H_0$, then $\langle g \rangle$ transitively permutes the V_i. This implies that $V_i \subseteq V_1 \times U_1$, $1 \le i \le p$, and hence $V \subseteq V_1 \times U_1$, which is a contradiction. Thus $I_H(V/(V_1 \times U_1)) = I_{H_0}(V/(V_1 \times U_1)) = I_{H_0}((V_2 \times \cdots \times V_p)/U_1)$. Let ζ be a linear character of V with $\operatorname{Ker}\zeta = V_1 \times U_1$. Then $O_p(I_H(\zeta)) = O_p(I_H(V/(V_1 \times U_1))) = O_p(I_{H_0}((V_2 \times \cdots \times V_p)/U_1)) = 1$. By induction, $I_H(\zeta)$ has a character of p-defect 0, and so has G by Lemma 2.1. This contradicts our choice of G. ∎

LEMMA 3.7. *V has a subgroup U_0 which satisfies the following conditions.*

(i) $|V : U_0| = q$.

(ii) $O_p(I_{H_0}(V/U_0)) = 1$ *and* $O_p(I_H(V/U_0)) = \langle x \rangle$ *for some* $x \in H$ *of order* p.

(iii) $I_H(V/U_0) = \langle x\rangle I_{H_0}(V/U_0)$, *where* x *is the element in* (ii).

(iv) $V_i \nsubseteq U_0$, $1 \le i \le p$.

Proof. Since $H_0V \subsetneq G$, H_0V has a character ξ of p-defect 0 by induction. By the first paragraph of the proof of Lemma 3.6, $1 \ne O_p(H_0)$. Hence H_0 has no characters of p-defect 0, and so $V \nsubseteq \operatorname{Ker}\xi$. Therefore there exists $1 \ne \lambda \in Irr(V)$ with $\lambda \mid \xi_V$. Let $U_0 = \operatorname{Ker}\lambda$. Then $|V : U_0| = q$. By Lemma 3.4, $1 = O_p(I_{H_0}(\lambda)) = O_p(I_{H_0}(V/U_0))$. On the other hand, G has no characters of p-defect 0, and hence $1 \ne O_p(I_H(\lambda)) = O_p(I_H(V/U_0))$. Since $|H : H_0| = p$, $|O_p(I_H(V/U_0))| = p$, and hence $O_p(I_H(V/U_0)) = \langle x\rangle$ for some $x \in H$ of order p. Then $I_H(V/U_0) = \langle x\rangle I_{H_0}(V/U_0)$. If $V_i \subseteq U_0$ for some i, then $V = V_i \times V_i^x \times \cdots \times V_i^{x^{p-1}} \subseteq U_0$, a contradiction. ∎

LEMMA 3.8. $I_{H_0}(V/U_0) = \bigcap_{i=1}^p I_{H_0}(V_i/W_i)$, *where* $W_i = U_0 \cap V_i$.

Proof. Let $h \in I_{H_0}(V/U_0)$. Then $[V, h] \subseteq U_0$, and hence $[V_i, h] \subseteq U_0 \cap V_i = W_i$. Thus $h \in I_{H_0}(V_i/W_i)$, $1 \le i \le p$. Conversely, let $h \in \bigcap_{i=1}^p I_{H_0}(V_i/W_i)$. Then $[V, h] = \Pi_{i=1}^p[V_i, h] \subseteq \Pi_{i=1}^p W_i \subseteq U_0$. ∎

Let $z \in Z(P) \cap O_p(H)$ with $|z| = p$, where P is a Sylow p-subgroup of H. Then $z \in Z(H)$ and $z \in H_0$ (see the proof of Lemma 3.6). We set $W_0 = \bigcap_{i=0}^{p-1} W_1^{z^i}$.

LEMMA 3.9. *Let* W^* *be a subgroup of* V_1 *such that* $V_1 \supseteq W^* \supseteq W_0$ *and* $|V_1 : W^*| = q$. *Then* $\bigcap_{i=0}^{p-1} W^{*z^i} = W_0$ *and* $I_{H_0}(V_1/W^*) = I_{H_0}(V_1/W_0)$.

Proof. Since $z \in Z(H)$, V_1 is a homogeneous $\langle z\rangle$-module. By Lemma 2.3, V_1/W_0 is an irreducible $\langle z\rangle$-module. Since $V_1 \supseteq \bigcap_{i=0}^{p-1} W^{*z^i} \supseteq W_0$, $\bigcap_{i=0}^{p-1} W^{*z^i} = W_0$. Next

$$\begin{aligned} I_{H_0}(V_1/W^*) &= \left(I_{H_0}(V_1/W^*)\right)^{z^i}, \qquad i = 0, \ldots, p-1 \\ &= I_{H_0}(V_1/W^{*z^i}), \\ &= \bigcap_{i=0}^{p-1} I_{H_0}(V_1/W^{*z^i}) \\ &= I_{H_0}(V_1/W_0). \end{aligned}$$

∎

LEMMA 3.10. $I_{H_0}(V_1/U_0) = \bigcap_{i=1}^p I_{H_0}(V_i/W_0^{x^{i-1}})$, *where* $V_1^{x^{i-1}} = V_i$.

Proof. By Lemmas 3.8 and 3.9, $I_{H_0}(V/U_0) \subseteq I_{H_0}(V_1/W_1) = I_{H_0}(V_1/W_0)$. Since $x \in I_H(V/U_0)$, $I_{H_0}(V/U_0) = I_{H_0}(V/U_0)^{x^{i-1}} \subseteq I_{H_0}(V_1^{x^{i-1}}/W_0^{x^{i-1}}) = I_{H_0}(V_i/W_0^{x^{i-1}})$. Thus $I_{H_0}(V/U_0) \subseteq \bigcap_{i=1}^p I_{H_0}(V_i/W_0^{x^{i-1}})$. On the other hand, since $\Pi_{i=1}^p W_0^{x^{i-1}} \subseteq U_0$, $\bigcap_{i=1}^p I_{H_0}(V_i/W_0^{x^{i-1}}) = I_{H_0}(V/\Pi_{i=1}^p W_0^{x^{i-1}}) \subseteq I_{H_0}(V_1/U_0)$. Therefore $I_{H_0}(V_1/U_0) = \bigcap_{i=1}^p I_{H_0}(V_i/W_0^{x^{i-1}})$. ∎

LEMMA 3.11. *Let U be a subgroup of V which satisfies the following conditions.*

(i) $|V/U| = q$.
(ii) $W_0 \times W_0^x \times \cdots \times W_0^{x^{p-1}} \subseteq U$.
(iii) $V_i \nsubseteq U$, $1 \leq i \leq p$.

Then $I_H(V/U) \subseteq N_H(W_0 \times W_0^x \times \cdots \times W_0^{x^{p-1}})$.

Proof. Let $y \in I_H(V/U)$. If $V_i^y = V_j$, then $(U \cap V_i)^y = U \cap V_j \supseteq W_0^a$, where $a = x^{j-1}$. Hence $V_1 \supseteq (U \cap V_i)^{ya^{-1}} \supseteq W_0$. By Lemma 3.9,

$$W_0 = \bigcap_{k=0}^{p-1} \left\{(U \cap V_i)^{ya^{-1}}\right\}^{z^k} = \left\{\bigcap_{k=0}^{p-1} (U \cap V_i)^{z^k}\right\}^{ya^{-1}}. \qquad (1)$$

On the other hand, $W_0^{x^{i-1}} \subseteq U \cap V_i$. Setting $b = x^{i-1}$, $W_0 \subseteq (U \cap V_i)^{b^{-1}} \subseteq V_1$. By Lemma 3.9, $\bigcap_{k=0}^{p-1} \{(U \cap V_i)^{b^{-1}}\}^{z^k} = W_0$. Hence $\{\bigcap_{k=0}^{p-1} (U \cap V_i)^{z^k}\}^{b^{-1}} = W_0$, and so $\bigcap_{k=0}^{p-1} (U \cap V_i)^{z^k} = W_0^b$. By (1), $W_0 = (W_0^b)^{ya^{-1}}$, and hence $W_0^{x^{j-1}} = (W_0^{x^{i-1}})^y$. This implies that $y \in N_H(W_0 \times W_0^x \times \cdots \times W_0^{x^{p-1}})$. ∎

We set $N = VN_H(W_0 \times W_0^x \times \cdots \times W_0^{x^{p-1}})$ and $\bar{N} = N/(W_0 \times W_0^x \times \cdots \times W_0^{x^{p-1}})$. Then $\bar{N} \triangleright \bar{V} = \bar{V}_1 \times \cdots \times \bar{V}_p$.

LEMMA 3.12. $O_p(\bar{N}) = 1$.

Proof. Suppose that $O_p(\bar{N}) \neq 1$. Let P_0 be a p-subgroup of $N \cap H$ with $\bar{P}_0 = O_p(\bar{N})$. For $\forall a \in P_0$, $\bar{V}_1^{\bar{a}} = \bar{V}_1$, and hence $a \in H_0$. This implies that $P_0 \subseteq H_0$. Furthermore, since $[\bar{P}_0, \bar{V}] = 1$, $[P_0, V] \subseteq W_0 \times W_0^x \times \cdots \times W_0^{x^{p-1}}$. Thus $P_0 \subseteq I_{H_0}(V/U_0) \subseteq N \cap H$ by Lemmas 3.7 and 3.11. Since $P_0 \lhd N \cap H$, $1 \neq P_0 \subseteq O_p(I_{H_0}(V/U_0)) = 1$, which is a contradiction. ∎

Let P_0 be a Sylow p-subgroup of H_0. By Lemma 3.6, $P_0 \triangleright O_p(C_{H_0}(V_2 \times \cdots \times V_p)) \neq 1$. Therefore $Z(P_0) \cap O_p(C_{H_0}(V_2 \times \cdots \times V_p))$ contains an element z_1 of order p.

LEMMA 3.13. $z_1 \in N$.

Proof. Since H_0 is p-nilpotent, $z_1 \in Z(H_0)$. If $z_1^x = z_1$, then $z_1 \in (C_{H_0}(V_2 \times \cdots \times V_p))^x = C_{H_0}(V_1 \times V_3 \times \cdots \times V_p)$, and hence $z_1 \in C_{H_0}(V_2 \times \cdots \times V_p) \cap C_{H_0}(V_1 \times V_3 \times \cdots \times V_p) = C_{H_0}(V) = 1$, which is a contradiction. Thus $z_1 \notin Z(H) \supseteq \langle z \rangle$. Therefore $\langle z_1 \rangle \times \langle z \rangle \subseteq Z(H_0)$. Since V_1 is a homogeneous H_0-module, V_1 is a homogeneous $\langle z_1 \rangle \times \langle z \rangle$-module. Setting $\overline{\langle z_1 \rangle \times \langle z \rangle} = \langle z_1 \rangle \times \langle z \rangle / C_{\langle z_1 \rangle \times \langle z \rangle}(V_1)$, then $\langle \bar{z}_1 \rangle = \langle \bar{z} \rangle$. Hence

$$W_0 = \bigcap_{i=0}^{p-1} W_1^{z^i} = \bigcap_{i=0}^{p-1} W_1^{z_1^i}.$$

This implies that $z_1 \in N_H(W_0) \cap C_{H_0}(V_2 \times \cdots \times V_p) \subseteq N$. ∎

LEMMA 3.14. *$G = \bar{N}$. Moreover, let W^* be a subgroup of V_i with $|V_i : W^*| = q$ for some i, $1 \le i \le p$. Then $I_{H_0}(V_i/W^*) = I_{H_0}(V_i)$.*

Proof. Let $\chi \in Irr(\bar{N})$ and let $\zeta \in Irr(\bar{V})$ with $\zeta \mid \chi_{\bar{V}}$. Suppose that $\mathrm{Ker}\,\zeta \supseteq \bar{V}_i$ for some i, $1 \le i \le p$. By considering $\zeta^{x^{1-i}}$, we may assume that $\mathrm{Ker}\,\zeta \supseteq \bar{V}_1$. Then z_1 centralizes $\bar{V}/\mathrm{Ker}\,\zeta$, and hence $\bar{z}_1 \in I_{\bar{N}}(\zeta)$. By Lemma 3.13, $z_1 \in O_p(H) \cap N \subseteq O_p(N)$. Therefore $\bar{z}_1 \in O_p(I_{\overline{N\cap H}}(\zeta))$, in particular, $O_p(I_{\overline{N\cap H}}(\zeta)) \neq 1$. By Lemma 3.4, χ is not a character of p-defect 0.

Next suppose that $\mathrm{Ker}\,\zeta \not\supseteq \bar{V}_i$ $(i = 1, 2, \dots, p)$. Let $\bar{U} = \mathrm{Ker}\,\zeta$ and let U be an inverse image of $\bar{U}$. By Lemma 3.11, $I_H(V/U) \subseteq N$ since $U \supseteq W_0 \times W_0^x \times \cdots \times W_0^{x^{p-1}}$. Thus $I_H(V/U) = I_{N\cap H}(V/U)$. ζ is regarded as a character of V. Then $I_H(\zeta) = I_H(V/U) = I_{N\cap H}(V/U) = I_{N\cap H}(\zeta)$. Since G has no characters of p-defect 0, $O_p(I_H(\zeta)) \neq 1$ by Lemma 3.4. Hence $O_p(I_{N\cap H}(\zeta)) \neq 1$. Thus $O_p(I_{\overline{N\cap H}}(\zeta)) \neq 1$, and hence χ is not a character of p-defect 0. Therefore $\bar{N}$ has no characters of p-defect 0. By Lemma 3.12, $O_p(\bar{N}) = 1$, and hence $G = \bar{N}$ by the minimality of G. In particular, $W_0 = 1$.

Next $I_{H_0}(V_i/W^*) = I_{H_0}(V_1^{x^{i-1}}/W^*) = I_{H_0}(V_1/(W^*)^{x^{1-i}})^{x^{i-1}} = I_{H_0}(V_1)^{x^{i-1}} = I_{H_0}(V_i)$ by Lemma 3.9. ∎

LEMMA 3.15. *For $\varphi, \lambda \in Irr(V_1)$ with $\varphi \neq 1 \neq \lambda$, there exists $h_1 \in C_{H_0}(V_2 \times \cdots \times V_{p-1})$ such that $\varphi_1^{h_1} = \lambda$ and $\overline{h_1^x} = \overline{h_1^{-1}}$ in $\bar{H}_0 = H_0/C_{H_0}(V_1)$.*

Proof. We set $W^* = \mathrm{Ker}\,\varphi$ and $W_1 = \mathrm{Ker}\,\lambda$. Then $|V_1 : W^*| = |V_1 : W_1| = q$. Let α be a primitive qth root of unity. Then there exist $v_1, w_1 \in V_1$ with $\varphi(v_1) = \alpha = \lambda(w_1)$. Setting $w_{i+1} = w_1^{x^i}$ $(i = 0, \dots, p-1)$, $w_{i+1} \in W_1^{x^i} = W_{i+1}$. Let $\bar{V} = V/(W^* \times W_2 \times \cdots \times W_p)$. Then $\bar{V} \simeq V_1/W^* \times \cdots \times V_p/W_p = \langle \bar{v}_1 \rangle \times \langle \bar{w}_2 \rangle \times \cdots \times \langle \bar{w}_p \rangle$, where $\bar{v}_1 \in V_1/W^*$ and $\bar{w}_i \in V_i/W_i$, $2 \le i \le p$. Thus we identify $\bar{V}$ with $V_1/W^* \times \cdots \times V_p/W_p$. Let $\bar{U} = \langle \bar{v}_1^{-1}\bar{w}_2 \rangle \times \langle \bar{w}_2^{-1}\bar{w}_3 \rangle \times \cdots \times \langle \bar{w}_{p-1}^{-1}\bar{w}_p \rangle \subseteq \langle \bar{v}_1 \rangle \times \langle \bar{w}_2 \rangle \times \cdots \times \langle \bar{w}_p \rangle$ and let U be the inverse image of $\bar{U}$ in V. Then $|V/U| = q$. Furthermore,

$$
\begin{aligned}
I_{H_0}(V/U) &= I_{H_0}(V_1/W^*) \cap I_{H_0}(V_2/W_2) \cap \cdots \cap I_{H_0}(V_p/W_p) \\
&= C_{H_0}(V_1) \cap C_{H_0}(V_2) \cap \cdots \cap C_{H_0}(V_p) = C_{H_0}(V) = 1
\end{aligned}
$$

by Lemma 3.14. This implies that $|I_H(V/U)| = p$. Let $x^i h \in I_H(V/U)$ with $h \in H_0$. By considering the powers of $x^i h$, we may assume that $i = 1$. Then $\widetilde{v_1^{xh}} = \tilde{v}_1$ in $\tilde{V} = V/U$, and hence $v_1^{-1}v_1^{xh} \in U$. Thus $\bar{v}_1^{-1}\overline{v_1^{xh}} \in \bar{U} \cap (\langle \bar{v}_1 \rangle \times \langle \bar{w}_2 \rangle) = \langle \bar{v}^{-1}\bar{w}_2 \rangle$. Hence $\bar{v}_1^{-1}\overline{v_1^{xh}} = \bar{v}^{-1}\bar{w}_2$, and so $\overline{v_1^{xh}} = \bar{w}_2 = \overline{w_1^x}$. Thus

$$
\overline{v_1^{xhx^{-1}}} = \bar{w}_1 \qquad \text{and} \qquad xhx^{-1} \in H_0. \tag{1}
$$

By a similar argument, we have $w_2^{-1}w_2^{xh} \in U$, and hence $\bar{w}_2^{-1}\overline{w_2^{xh}} = \bar{w}_2^{-1}\bar{w}_3$. This implies that $\overline{w_2^{xh}} = \bar{w}_3 = \overline{w_2^x}$, and so $\overline{w_2^{xhx^{-1}}} = \bar{w}_2$ and $xhx^{-1} \in$

$I_{H_0}(V_2/W_2) = C_{H_0}(V_2)$ by Lemma 3.14. Similarly, we have $\overline{w_i^{xhx^{-1}}} = \bar{w}_i$ for all i, $3 \le i \le p-1$. Hence

$$xhx^{-1} \in \bigcap_{i=2}^{p-1} I_{H_0}(V_i/W_i) = \bigcap_{i=2}^{p-1} C_{H_0}(V_i) = C_{H_0}(V_2 \times \cdots \times V_{p-1})$$

by Lemma 3.14. Furthermore, $\widetilde{w_p^{xh}} = \tilde{w}_p$ in $\tilde{V}$, and hence $w_p^{-1} w_p^{xh} \in U$. If

$$\begin{aligned}\bar{w}_p^{-1}\overline{w_p^{xh}} &= \left(\bar{v}_1^{-1}\bar{w}_2\right)^{i_1}\left(\bar{w}_2^{-1}\bar{w}_3\right)^{i_2}\cdots\left(\bar{w}_{p-1}^{-1}\bar{w}_p\right)^{i_{p-1}} \\ &= \bar{v}_1^{-i_1}\bar{w}_2^{(i_1-i_2)}\cdots\bar{w}_{p-1}^{(i_{p-2}-i_{p-1})}\bar{w}_p^{i_{p-1}},\end{aligned}$$

then $i_1 \equiv i_2 \equiv \cdots \equiv i_{p-1} \equiv -1 \bmod(q)$ and

$$\bar{v}_1 = \overline{w_p^{xh}} = \overline{w_1^h}. \tag{2}$$

Since $U^{xh} = U$, $(U \cap V_1)^{xh} = U \cap V_2$, and hence $(W^*)^{xh} = W_2 = W_1^x$. Let $h_1 = xhx^{-1}$. Then $(W^*)^{h_1} = (W^*)^{xhx^{-1}} = W_1$. Since $\varphi^{h_1}(w_1) = \varphi(v_1) = \alpha$ by (1), this implies that $\varphi^{h_1} = \lambda$. By (1) and (2), $\overline{v_1^{h_1}} = \bar{w}_1$ and $\bar{v}_1 = \overline{w_1^h}$. Hence $\overline{v_1} = \overline{v_1^{h_1 h}}$ in $\bar{V}_1 = V_1/W^*$. Since $(U \cap V_p)^{xh} = U \cap V_1$, $W_p^{xh} = W^*$, and so $W_1^h = W^*$.

This implies that $(W^*)^{h_1 h} = W_1^h = W^*$. Thus $h_1 h \in I_{H_0}(V_1/W^*) = C_{H_0}(V_1)$ by Lemma 3.14. Hence $\overline{h_1^x} = \bar{h} = \bar{h}_1^{-1}$ in $\bar{H}_0 = H_0/C_{H_0}(V_1)$. ∎

LEMMA 3.16. *Consider V_1 as the additive group of the finite field $GF(q^n)$. Let $\bar{H}_0 = H_0/C_{H_0}(V_1)$. Then $\overline{C_{H_0}(V_1 \times \cdots \times V_{p-1})} = \bar{H}_0$ and $\bar{H}_0$ is a cyclic group of order $q^n - 1$. Furthermore, $\bar{H}_0$ consists of all non-zero linear transformations.*

Proof. By Lemma 3.15, $C_{H_0}(V_2 \times \cdots \times V_{p-1})$ acts transitively on $Irr(V_1) - \{1_{V_1}\}$. Hence $C_{H_0}(V_2 \times \cdots \times V_{p-1})$ has two orbits on $Irr(V_1)$. By Brauer's permutation lemma, $C_{H_0}(V_2 \times \cdots \times V_{p-1})$ has two orbits on V_1 by conjugation. Thus $C_{H_0}(V_2 \times \cdots \times V_{p-1})$ acts transitively on $V_1^{\#}$.

By Lemmas 2.3(ii) and 3.9, $\langle z \rangle$ acts irreducibly on $V_1/W_0 \simeq V_1$ since $W_0 = 1$ (see Lemma 3.14). Since $z \in Z(H_0)$, $\bar{H}_0$ acts as scalar multiplications on V_1 by [8, Theorem 19.8], and hence $\bar{H}_0$ acts regularly on $V_1^{\#}$. By the transitivity of $C_{H_0}(V_2 \times \cdots \times V_{p-1})$ on $V_1^{\#}$, $\overline{C_{H_0}(V_1 \times \cdots \times V_{p-1})} = \bar{H}_0$ and $\bar{H}_0$ consists of all non-zero linear transformations. Thus $|H_0| = |V_1^{\#}| = q^n - 1$. ∎

LEMMA 3.17. *$F(p, q, n)$ is isomorphic to a subgroup of G.*

Proof. Let $\langle y\rangle$ be a cyclic group of order p and let $N = H_0/C_{H_0}(V_1) \times \cdots \times H_0/C_{H_0}(V_p)$ be the (outer) direct product. Next we define $(\bar{h}_1, \ldots, \bar{h}_p)^y = (\overline{h_p^x}, \overline{h_1^x}, \cdots, \overline{h_{p-1}^x}) \in H_0/C_{H_0}(V_1) \times \cdots \times H_0/C_{H_0}(V_p)$, and $(\bar{h}_1, \ldots, \bar{h}_p)^{y^i} = ((\bar{h}_1, \ldots, \bar{h}_p)^{y^{i-1}})^y$ inductively, where $h_j \in H_0, 1 \le j \le p$. Then $\langle y\rangle$ acts on $H_0/C_{H_0}(V_1) \times \cdots \times H_0/C_{H_0}(V_p)$. Since $V_1^{x^i} = V_{i+1}$ $(i = 0, \cdots, p-1)$, this definition is well defined. Let $(H_0/C_{H_0}(V_1) \times \cdots \times H_0/C_{H_0}(V_p)) \rtimes \langle y\rangle$ be the semi-direct product. Let f be a map of $H = H_0 \rtimes \langle x\rangle$ into $(H_0/C_{H_0}(V_1) \times \cdots \times H_0/C_{H_0}(V_p)) \rtimes \langle y\rangle$ which is defined by the rule $f(hx^i) = (\bar{h}, \ldots, \bar{h})y^i$, where $h \in H_0$. Then

$$\begin{aligned} f(hx^ikx^j) &= f(hx^ikx^{-i}x^{i+j}) \\ &= \left(\overline{hk^{x^{-i}}}, \ldots, \overline{hk^{x^{-i}}}\right)y^{i+j}. \end{aligned}$$

On the other hand,

$$\begin{aligned} f(hx^i)f(kx^j) &= (\bar{h}, \ldots, \bar{h})y^i(\bar{k}, \ldots, \bar{k})y^j \\ &= (\bar{h}, \ldots, \bar{h})(\bar{k}, \ldots, \bar{k})^{y^{-i}}y^{i+j} \\ &= \left(\overline{hk^{x^{-i}}}, \ldots, \overline{hk^{x^{-i}}}\right)y^{i+j}. \end{aligned}$$

Thus $f(hx^ikx^j) = f(hx^i)f(kx^j)$, which implies that f is a homomorphism. Let $\operatorname{Ker} f \ni hx^i$ with $h \in H_0$. Then $(\bar{h}, \ldots, \bar{h}) = (\bar{1}, \ldots, \bar{1}) \in H_0/C_{H_0}(V_1) \times \ldots \times H_0/C_{H_0}(V_p)$ and $y^i = 1$, and hence $h \in C_{H_0}(V) = 1$ and $x^i = 1$. This implies that $\operatorname{Ker} f = 1$.

By Lemma 3.16, there exists $h \in C_{H_0}(V_2 \times \ldots \times V_{p-1})$ with $\langle\bar{h}\rangle = \bar{H}_0 = H_0/C_{H_0}(V_1)$. Let $1 \ne \varphi \in Irr(V_1)$ and set $\lambda = \varphi^h$. By Lemma 3.15, there exists $h_1 \in C_{H_0}(V_2 \times \cdots \times V_{p-1})$ such that $\varphi^{h_1} = \varphi^h$ and $\overline{h_1^x} = \bar{h}_1^{-1}$ in $\bar{H}_0 = H_0/C_{H_0}(V_1)$. Setting $W^* = \operatorname{Ker}\varphi$, $h_1h^{-1} \in I_{H_0}(\varphi) = I_{H_0}(V_1/W^*) = C_{H_0}(V_1)$ by Lemma 3.14. Thus $\bar{h} = \bar{h}_1$ in $\bar{H}_0$. Now

$$\begin{aligned} f(h_1) &= (\bar{h}_1, \bar{h}_1, \ldots, \bar{h}_1) \\ &= (\bar{h}_1, \bar{1}, \ldots, \bar{1}, \bar{h}_1) \qquad \big(\text{since } h_1 \in C_{H_0}(V_2 \times \cdots \times V_{p-1})\big) \\ &= \left(\overline{(h_1^{-1})}^x, \bar{1}, \ldots, \bar{1}, \bar{h}_1\right) \\ &= \left(\bar{1}, \ldots, \bar{1}, \bar{h}_1\right)\left(\overline{(h_1^{-1})}^x, \bar{1}, \ldots, \bar{1}\right) \\ &= \left(\bar{1}, \ldots, \bar{1}, \bar{h}_1\right)\left(\bar{1}, \ldots, \bar{1}, \bar{h}_1^{-1}\right)^y \in [N, y]. \end{aligned}$$

Since $\bar{h} = \bar{h}_1$ in $\bar{H}_0 = H_0/C_{H_0}(V_1)$, $|\bar{h}_1| = |\bar{h}| = q^n - 1$, and hence $|f(h_1)| = q^n - 1$. Next we set $h_i = h_1^{x^{i-1}}$ $(i = 1, \cdots, p-1)$. Then $h_i \in C_{H_0}(V_1 \times \cdots \times V_{i-2} \times V_{i+1} \times \cdots \times V_p)$, and by the same argument as above $|\bar{h}_i| = q^n - 1$ in $H_0/C_{H_0}(V_i)$, $f(h_i) \in [N, y]$, and $|f(h_i)| = q^n - 1$. Furthermore, since $\bar{h}_2 =$

$\overline{h_1^x} = \bar{h}_1^{-1}$ in $H_0/C_{H_0}(V_1)$, $\bar{h}_{i+1} = \bar{h}_i^{-1}$ in $H_0/C_{H_0}(V_i)$ $(i = 1, \ldots, p-2)$. If $f(h_1)^{i_1} \cdots f(h_{p-1})^{i_{p-1}} = 1$, then

$$\big(\bar{h}_1^{i_1}, \bar{1}, \ldots, \bar{1}, \bar{h}_1^{i_1}\big)\big(\bar{h}_2^{i_2}, \bar{h}_2^{i_2}, \bar{1}, \ldots, \bar{1}\big) \cdots \big(\bar{1}, \ldots, \overline{h_{p-1}}^{i_{p-1}}, \overline{h_{p-1}}^{i_{p-1}}, \bar{1}\big) = 1.$$

Hence

$$\big(\bar{h}_1^{i_1}\bar{h}_2^{i_2}, \bar{h}_2^{i_2}\bar{h}_3^{i_3}, \ldots, \overline{h_{p-2}}^{i_{p-2}}\overline{h_{p-1}}^{i_{p-1}}, \bar{h}_1^{i_1}\big) = \big(\bar{1}, \ldots, \bar{1}\big).$$

Thus $\bar{h}_1^{i_1} = \bar{1}$ in $H_0/C_{H_0}(V_p)$. Therefore $\bar{1} = \overline{(h_1^{i_1})^x} = \overline{(h_1^x)}^{i_1} = \bar{h}_2^{i_1} = \bar{h}_1^{-i_1}$ in $H_0/C_{H_0}(V_1)$ since $V_p^x = V_1$, which implies that $q^n - 1 \mid i_1$. Next, since $\bar{h}_1^{i_1} = \bar{1}$ in $H_0/C_{H_0}(V_1)$, $\bar{h}_2^{i_2} = \bar{1}$ in $H_0/C_{H_0}(V_1)$. Therefore $\bar{1} = \overline{(h_2^{i_2})^x} = (\bar{h_2^x})^{i_2} = \bar{h}_3^{i_2} = \bar{h}_2^{-i_2}$ in $H_0/C_{H_0}(V_2)$, which implies that $q^n - 1 \mid i_2$. Similarly, we have $q^n - 1 \mid i_k$ $(k = 1, \ldots, p-1)$. Thus $\langle f(h_1), \ldots, f(h_{p-1})\rangle = \langle f(h_1)\rangle \times \cdots \times \langle f(h_{p-1})\rangle \subseteq [N, y]$. On the other hand, $|\langle f(h_1)\rangle \times \cdots \times \langle f(h_{p-1})\rangle| = (q^n - 1)^{p-1} = |N|/|C_N(y)| = |[N, y]|$, and hence $\langle f(h_1)\rangle \times \cdots \times \langle f(h_{p-1})\rangle = [N, y]$.

Now, $z_1 \in O_p(C_{H_0}(V_2 \times \cdots \times V_p))$ with $|z_1| = p$ (see Lemma 3.6). Then $f(z_1) = (\bar{z}_1, \ldots, \bar{z}_1) = (\bar{z}_1, \bar{1}, \ldots, \bar{1})$, and hence $f(H) \supseteq \langle f(z_1)\rangle \times \langle f(z_1^x)\rangle \times \cdots \times \langle f(z_1^{x^{p-1}})\rangle = \Omega_1(O_p(N))$. Therefore $f(H) \supseteq ([N, y]\Omega_1(O_p(N))) \rtimes \langle y\rangle$.

Let $V \ni v = v_1 \cdots v_p$, where $v_i \in V_i$, $1 \le i \le p$. For $(\bar{h}_1, \ldots, \bar{h}_p)y^i \in N \rtimes \langle y\rangle$ with $h_j \in H_0$ $(j = 1, \ldots, p)$, we define

$$v^{(\bar{h}_1, \ldots, \bar{h}_p)y^i} = v_1^{h_1 x^i} \cdots v_p^{h_p x^i}.$$

Then $N \rtimes \langle y\rangle$ acts on V. Furthermore, $v^{f(hx^i)} = v^{(\bar{h}, \ldots, \bar{h})y^i} = v_1^{hx^i} \cdots v_p^{hx^i} = v^{hx^i}$, where $h \in H_0$. Let $V \rtimes (N \rtimes \langle y\rangle)$ be the semi-direct product. Let $\tilde{f}$ be a map of $G = V \rtimes (H_0 \rtimes \langle x\rangle)$ into $V \rtimes (N \rtimes \langle y\rangle)$ which is defined by the rule

$$\tilde{f}(vhx^i) = v(\bar{h}, \ldots, \bar{h})y^i \ \big(= vf(hx^i)\big), \qquad \text{where } v \in V \text{ and } h \in H_0.$$

Then it is easily checked that $\tilde{f}$ is an injective homomorphism. Hence

$$\begin{aligned}\tilde{f}(G) = \tilde{f}(V \rtimes H) = V \rtimes f(H) &\supseteq V \rtimes \big(([N, y]\Omega_1(O_p(N))) \rtimes \langle y\rangle\big)\\ &\simeq F(p, q, n).\end{aligned}$$

∎

Case II

V is a quasi-primitive H-module.

In this case, if N is a normal abelian subgroup of H, then V_N is a faithful, completely reducible, and homogeneous module. Hence N is cyclic. Thus every normal subgroup of H is cyclic.

LEMMA 3.18. *Let $F = F(H)$ and let Z be the socle of the cyclic group $Z(F)$. Then F is a q'-group and there exist $E, T \lhd H$ with*

(i) *$F = ET$, $Z = E \cap T$, and $T = C_F(E)$.*

(ii) *$E/Z = E_1/Z \times \cdots \times E_r/Z$ for chief factors E_i/Z of H with $E_i \subseteq C_H(E_j)$ for $i \neq j$.*

(iii) *For each i, $Z(E_i) = Z$, $|E_i/Z| = p_i^{2n_i}$ for a prime p_i and an integer n_i, and $E_i = O_{p_i'}(Z)F_i$ for an extra-special group $F_i = O_{p_i}(E_i) \lhd H$ of order $p_i^{2n_i+1}$.*

(iv) *There exists $U \subseteq T$ of index at most 2 with U cyclic, $U \lhd H$, and $C_T(U) = U$.*

(v) *$T = C_H(E)$.*

Proof. Since V is a quasi-primitive H-module, $V_{O_q(H)}$ is homogeneous, and hence $[V, O_q(H)] = 1$, which implies that $O_q(H) = 1$.

(i) $\sim$ (v) follows from [6, Corollary 1.10]. ∎

LEMMA 3.19. *$O_{p'}(F_1 \cdots F_r) = 1$ or $O_{p'}(F_1 \cdots F_r) \simeq Q_8$, where Q_8 is a quaternion group of order* 8.

Proof. Suppose Lemma 3.19 is false. Therefore $O_{p'}(F_1 \cdots F_r) \neq 1$ and $O_{p'}(F_1 \cdots F_r) \not\simeq Q_8$. By re-numbering, we may assume that $O_{p'}(F_1 \cdots F_r) = F_1 \cdots F_k$ $(k \leq r)$. Set $\bar{F}_t = F_t/Z(F_t)$, $1 \leq t \leq k$. Then there exist hyperbolic pairs $\{u_1, v_1\} \cdots \{u_{n_t}, v_{n_t}\}$ with $(u_i, v_j) = \delta_{ij}$ and $(u_i, u_j) = (v_i, v_j) = 0$ (see Lemma 2.4). Let R_t be the inverse image of $\langle u_i, \ldots, u_{n_t} \rangle$ in F_t. Then R_t is an abelian subgroup of F_t of order $p_t^{n_t+1}$. Let $R = R_1 \cdots R_k$. Then R is a non-cyclic abelian subgroup of $F_1 \cdots F_k$. So, there exists a subgroup $1 \neq R_0$ of R such that R/R_0 is cyclic and $C_V(R_0) \neq 1$. Setting $V_0 = C_V(R_0)$, $N_H(R_0)$ acts on V_0 by conjugation.

We set $H_0 = N_H(R_0)$ and $\overline{H_0 V_0} = H_0 V_0 / C_{H_0}(V_0)$. Since $O_p(\overline{H_0 V_0}) = 1$ and $1 \neq R_0 \subseteq C_{H_0}(V_0)$, $\overline{H_0 V_0}$ has a character χ of p-defect 0 by induction. Since $1 \neq \overline{O_p(H)} \subseteq O_p(\bar{H}_0)$, $\operatorname{Ker} \chi \not\supseteq \bar{V}_0$. Therefore there exists $1 \neq \varphi \in Irr(\bar{V}_0)$ with $\varphi \mid \chi_{\bar{V}_0}$. By Lemma 2.1(ii), $I_{\bar{H}_0}(\varphi)$ has a character of p-defect 0, and hence $O_p(I_{\bar{H}_0}(\varphi)) = 1$. Let $\bar{V}_1 = \operatorname{Ker} \varphi$ with $V_1 \subseteq V_0$. Setting $I_0 = I_{H_0}(V_0/V_1)$, $\bar{I}_0 = I_{\bar{H}_0}(\varphi)$. Thus $O_p(\bar{I}_0) = 1$.

By Lemma 3.18, R_0 is a q'-group, and so $V = V_0 \times [V, R_0]$. We set $I = I_H(V/(V_1 \times [V, R_0]))$. Let $\zeta \in Irr(V)$ with $\operatorname{Ker} \zeta = V_1 \times [V, R_0]$. Then $I =$

$I_H(\zeta)$. If $O_p(I) = 1$, then there exists $\eta \in Irr(VI)$ such that $\zeta \mid \eta_V$ and η is a character of p-defect 0 by Lemma 3.4. Since $I_G(\zeta) = VI$, η^G is a character of p-defect 0. Thus $O_p(I) \neq 1$. Let $x \in O_p(I)$ with $|x| = p$. Then $[x, R_0] \subseteq O_p(I) \cap O_{p'}(H) = 1$. Thus $x \in C_H(R_0) \subseteq H_0$. On the other hand, since $I_0 \subseteq I$ and $O_p(\bar{I}_0) = 1$, $x \in O_p(I) \cap I_0 \subseteq O_p(I_0) \subseteq C_{H_0}(V_0) \subseteq I_0$. Thus $x \in O_p(C_{H_0}(V_0))$.

Since R normalizes $C_{H_0}(V_0)$, $[x, R] \subseteq O_p(C_{H_0}(V_0)) \cap O_{p'}(H) = 1$. Since $R = R_1 \cdots R_k$, $[x, R_i] = 1$, $1 \leq i \leq k$. Furthermore, since $R_i \supseteq Z(F_i)$ and $p_i \neq p$, $[x, F_i] = 1$ by Lemma 2.4. Thus $[x, O_{p'}(F_1 \cdots F_r)] = 1$.

Setting $M = O_p(F_1 \cdots F_r)$, M is an extra-special p-group by Lemma 3.18(iii). Since $[M, O_{p'}(H)] \subseteq O_p(H) \cap O_{p'}(H) = 1$ and H is p-nilpotent, $H/C_H(M)$ is a p-group. By Lemma 3.18(ii), $M/Z(M)$ is a completely reducible H-module, and hence H centralizes $M/Z(M)$. Let P be a Sylow p-subgroup of H with $x \in P$. By [2, Lemma 4.6, p. 195], $x = yz$ with $y \in C_P(M)$ and $z \in M$. Since $[x, O_{p'}(F_1 \cdots F_r)] = [z, O_{p'}(F_1 \cdots F_r)] = 1$, $[y, O_{p'}(F_1 \cdots F_r)] = 1$. Set $Z = Z(F(H))$. Since Z normalizes $C_{H_0}(V_0)$ and Z acts regularly on $V^{\#}$, $[x, Z] \subseteq C_{H_0}(V_0) \cap Z = C_Z(V_0) = 1$. Thus $[x, Z] = [z, Z] = 1$, and hence $[y, Z] = 1$. This implies that $[y, F_1 \cdots F_r Z] = [y, E] = 1$, where E is as in Lemma 3.18. By Lemma 3.18(v), $y \in C_H(E) = T \subseteq F(H)$. Since $z \in M \subseteq F(H)$, $x = yz \in F(H)$.

Since $V_{O_p(H)}$ is a faithful, completely reducible, and homogeneous module and $O_p(H) \subseteq C_H(R_0) \subseteq H_0$, V_0 is a faithful $O_p(H)$-module. Thus $C_{O_p(H)}(V_0) = 1$. On the other hand, $1 \neq x \in C_{O_p(H)}(V_0)$, which is a contradiction. ∎

LEMMA 3.20. *If $O_{p'}(F_1 \cdots F_r) \simeq Q_8$, then $G \simeq J$.*

Proof. We divide the proof of Lemma 3.20 into several steps. ∎

STEP 1. (i) *$p = 3$ and $H/F(H)$ is a p-group.*

(ii) *$F(H) \simeq Q \times Z_0$, where $Q \simeq Q_8$ and Z_0 is a cyclic group of odd order.*

Proof. Setting $Q = O_{p'}(F_1 \cdots F_r)$, $Q \simeq Q_8$. The hypotheses imply that $p \neq 2$. Since $H = O^{p'}(H) \subseteq O^2(H)$, $H = O^2(H)$. Since $Aut(Q) \simeq S_4$ (the symmetric group of degree 4) and $Q/Z(Q)$ is isomorphic to a subgroup of $H/C_H(Q)$, $H/C_H(Q) \simeq A_4$ (the alternating group of degree 4). In particular, $p = 3$.

Let T, U and $Z(F)$ be as in Lemma 3.18. If $T \neq U$, then $2 \mid |H/C_H(U)|$ since $C_T(U) = U$. Since U is cyclic, $H/C_H(U)$ is abelian, and hence $O^2(H) \subsetneq H$, which is a contradiction. Thus $T = U$. This implies that $T = Z(F)$.

Let K be a Hall p'-subgroup of H and P a Sylow p-subgroup of H. Since H is p-nilpotent, $H = PK$. Since $Z(F)$ is cyclic, $H/C_H(Z(F))$ is abelian.

Since $O^{p'}(H) = H$, $H/C_H(Z(F))$ is a p-group. Hence

$$K \subseteq C_H(Z(F)). \tag{1}$$

Since $F(K) \operatorname{char} K \lhd H$, $F(K) \subseteq F(H)$, and hence $F(K) \subseteq O_{p'}(F(H)) = QZ(F)$. Let L be a Hall $2'$-subgroup of K. Since $[L, Q] = 1$,

$$L \subseteq C_K(F(K)) \subseteq F(K). \tag{2}$$

Let S be a Sylow 2-subgroup of K. Since $H/C_H(Q) \simeq A_4$, $S \subseteq QC_H(Q)$. On the other hand, $Q \subseteq S$, and hence $S = QC_S(Q)$. By (1), $C_S(Q) \subseteq C_K(F(K)) \subseteq F(K)$. Thus

$$S = QC_S(Q) \subseteq F(K). \tag{3}$$

By (2) and (3), $K = F(K) \subseteq QZ(F)$. Then $H/F(H)$ is a p-group since $K = F(K) \subseteq F(H)$.

Next assume that $O_p(H)$ is non-abelian. By re-numbering, we may assume that F_1 (see Lemma 3.18) is a non-abelian p-group. By Lemma 3.18(ii), $F_1/Z(F_1)$ is an irreducible H-module. Since $[K, F_1] = 1$, $F_1/Z(F_1)$ is an irreducible P-module. Then, by [2, Lemma 4.6, p. 195], $P = C_P(F_1)F_1$. By [6, Corollary 1.3], F_1 has a non-cyclic normal abelian subgroup P_0 since $p = 3$. Then $P_0 \lhd H$, which is a contradiction. Thus $O_p(H) \subseteq Z(F)$, and hence $F(H) = QZ(F)$.

Let $Z(F) = Z_0 \times Z_1$, where Z_0 is a group of odd order and Z_1 is a 2-group. Since $H/C_H(Z_1)$ is a 2-group and $O^2(H) = H$, $H = C_H(Z_1)$. Let $\bar{H} = H/QZ_0$. Then $\bar{H} = \bar{P} \times \bar{Z}_1$. Since $O^2(H) = H$, $\bar{Z}_1 = \bar{1}$, and hence $F(H) = QZ_0Z_1 = QZ_0 = Q \times Z_0$. ∎

STEP 2. *The actions of H on $Irr(V)$ and V are permutation isomorphic.*

Proof. By Lemma 3.18, $(q, |F(H)|) = 1$. Since $H/F(H)$ are a p-group, $(q, |H|) = 1$. Then Step 2 follows from Lemma 2.2. ∎

STEP 3. *If $H_1 \subsetneq H$ and $1 \neq O_p(H_1)$, then there exists $v \in V$ with $C_{H_1}(v) = 1$.*

Proof. By induction, VH_1 has a character χ of p-defect 0 since $O_p(VH_1) = 1$. Since $1 \neq O_p(H_1)$, H_1 has no characters of p-defect 0, and hence $V \nsubseteq \operatorname{Ker} \chi$. So, there exists $1 \neq \varphi \in Irr(V)$ with $\varphi \mid \chi$. On the other hand, since $F(H)$ acts regularly on $V^{\#}$, $C_H(v)$ is a p-group for $\forall v \in V^{\#}$. Hence $I_H(\varphi)$ is a p-group by Step 2, and so is $I_{H_1}(\varphi)$. By Lemma 3.4, $I_{H_1}(\varphi) = 1$, and hence there exists $v \in V$ with $C_{H_1}(v) = 1$ by Step 2. ∎

STEP 4. *$H = H_0 \times Z_0$, where $H_0 \simeq SL(2,3)$ and $|Z_0| = 3$.*

Proof. Set $H_1 = C_H(Z_0)$ and suppose that $H_1 \subsetneq H$. Since $1 \neq O_p(H) \subseteq O_p(H_1)$, there exists $v \in V$ with $C_{H_1}(v) = 1$ by Step 3. Setting $V_0 = \langle v^{Z_0} \rangle$, V_0 is an irreducible Z_0-module by Lemma 2.3. Since $[C_H(V_0), Z_0] \subseteq C_H(V_0) \cap Z_0 = C_{Z_0}(V_0) = 1$, $C_H(V_0) \subseteq C_H(Z_0) = H_1$. Therefore $C_H(V_0) = C_{H_1}(V_0) \subseteq C_{H_1}(v) = 1$. Thus $O_p(V_0 N_H(V_0)) = 1$. Suppose that $V_0 \subsetneq V$. By induction, $V_0 N_H(V_0)$ has a character of p-defect 0. Since $1 \neq O_p(H) \subseteq Z_0 \subseteq N_H(V_0)$, $N_H(V_0)$ has no characters of p-defect 0. Setting $N = N_H(V_0)$, there exists $v_0 \in V_0^{\#}$ with $C_N(v_0) = 1$ by a similar argument to that in the proof of Step 3. By Step 1, $C_H(v_0)$ is a p-group. Hence there exists $x \in C_H(v_0)$ with $|x| = p$ by Lemma 3.4. Since V_0 is an irreducible Z_0-module, $\langle v_0^{Z_0} \rangle = V_0$. Since x normalizes $\langle v_0^{Z_0} \rangle$, $x \in N$, and hence $x \in C_N(v_0) = 1$, which is a contradiction. Hence $V (= V_0)$ is an irreducible Z_0-module. By [8, Prop. 19.8], $H \subseteq T(q^m)$ (defined in the Introduction). Since A_4 ($\simeq H/C_H(Q)$) is involved in H, H is not metacyclic. On the other hand, $T(q^m)$ is metacyclic and so is H, which is a contradiction. Thus $C_H(Z_0) = H$.

Now $O^{3'}(H) = H$ since $p = 3$, and so Z_0 is a cyclic 3-group. Furthermore, since $C_H(Q) = C_H(F(H)) \subseteq F(H)$, $|H/F(H)| = 3$. If a Sylow 3-subgroup of H is cyclic, then H acts regularly on $V^{\#}$. This contradicts Lemma 3.1(ii). Let $x \in H$ with $x \notin Z_0$ and $|x| = 3$. Setting $H_0 = Q\langle x \rangle$, $H = H_0 \times Z_0$ and $H_0 \simeq SL(2,3)$. Let $\langle z \rangle = Z_0$ and set $L = H_0 \times \langle z^3 \rangle$. Assume that $\langle z^3 \rangle \neq 1$. Since $L \subsetneq H$ and $1 \neq \langle z^3 \rangle \subseteq O_3(H)$, $C_L(v) = 1$ for some $v \in V^{\#}$ by Step 3. By Lemma 3.1(ii), $C_H(v) = \langle y \rangle$ with $|y| = 3$. Let $y = hu$ with $h \in H_0$ and $u \in Z_0$. Then $1 = y^3 = h^3 u^3 = u^3$. Hence $u \in \Omega_1(Z_0) \subseteq \langle z^3 \rangle \subseteq L$, and so $y \in C_L(v) = 1$, which is a contradiction. Thus $z^3 = 1$. Since $O_3(H) \neq 1$ by Lemma 3.1(iii), $|Z_0| = 3$. ∎

STEP 5. *$|V| = q^2$ and V is an irreducible Q-module.*

Proof. Let $V_0 \subseteq V$ be an irreducible Q-module. Let k be the field of q-elements and let kQ be a group ring. Since kQ is semisimple, $kQ \simeq \oplus_i M_{n_i}(D_i)$, where $M_{n_i}(D_i)$ is the ring of $n_i \times n_i$ matrices over the division ring D_i. Since $8 = \dim_k kQ = \sum_i \dim_k M_{n_i}(D_i) = 1 + 1 + 1 + 1 + 2^2$, the degree of every irreducible representation of Q over k is 1 or 2. Since $Q' = Z(Q) \nsubseteq C_Q(V_0)$, $\dim_k V_0 = 2$ and so $|V_0| = q^2$. Setting $N = N_H(V_0) \supseteq Q$, $N = Q$, $Q \times Z_0$, H, or $N \simeq SL(2,3)$. If $N = H$, then $V_0 = V$ since V is an irreducible H-module. Hence we may assume that $N \neq H$.

Next we shall prove that there exists a $v_0 \in V_0$ with $C_N(v_0) = 1$. Assume that $N = Q$ or $Q \times Z_0$. Then, since N acts regularly on $V_0^{\#}$, the assertion stated above holds. Next assume that $N \simeq SL(2,3)$. Let $x \in N$

with $|x| = 3$. If $C_{V_0}(x) = 1$, then $N = H_0 = Q\langle x\rangle$ acts regularly on $V_0^{\#}$. Hence we may assume that $C_{V_0}(x) \neq 1$. If $C_{V_0}(x) = V_0$, then $[Q, x] \subseteq Q \cap C_{H_0}(V_0) = C_Q(V_0) = 1$, which contradicts the fact that $H_0 \simeq SL(2, 3)$. Thus $|C_{V_0}(x)| = q$.

Let $v, w \in C_{V_0}(x)^{\#}$. Assume that v and w are conjugate in N. Let $w = v^y$ with $y \in Q$. Then $\langle x, x^y\rangle \subseteq C_N(w)$. Since Q acts regularly on $V_0^{\#}$, $\langle x, x^y\rangle = \langle x\rangle$, and hence $y \in Z(Q)$. Thus $w = v^{-1}$. Since $|C_N(u)| = 3$ for $\forall u \in V_0^{\#}$ and $C_N(u)$ is conjugate to $\langle x\rangle$ in N, $u^g \in C_{V_0}(x)$ for some $g \in N$. Thus each N-orbit of $V_0^{\#}$ contains an element of $C_{V_0}(x)$. Therefore N has exactly $\frac{q-1}{2}$ orbits on $V_0^{\#}$. Since each orbit contains exactly eight elements, $\frac{q-1}{2} \cdot 8 = q^2 - 1$. Hence $4 = q + 1$, and so $q = 3$, which is a contradiction since $p = 3$. Thus there exists a $v_0 \in V_0$ with $C_N(v_0) = 1$. By Lemma 3.1(ii), $C_H(v_0) \neq 1$. Let $1 \neq a \in C_H(v_0)$. Then a normalizes $\langle v_0^Q\rangle = V_0$ since V_0 is an irreducible Q-module. Thus $a \in C_N(v_0) = 1$, which is a contradiction. ∎

STEP 6. $G \simeq J$.

Proof. Let $x \in H_0$ with $|x| = 3$ and $\langle z\rangle = Z_0$. Then $\langle x\rangle \times \langle z\rangle$ is a Sylow 3-subgroup of $H = H_0 \times Z_0$. Now, $\langle x\rangle \times \langle z\rangle$ has four distinct subgroups of order 3. Let $\langle a\rangle$, $\langle b\rangle$, $\langle c\rangle$, and $\langle z\rangle$ be subgroups of $\langle x\rangle \times \langle z\rangle$ of order 3. Since $C_V(z) = 1$, $V = \langle C_V(a), C_V(b), C_V(c)\rangle$. Since V is a faithful H-module, $[V, a] \neq 1$, and hence $|C_V(a)|$ is 1 or q. Similarly, we have that $|C_V(b)|$ and $|C_V(c)|$ are 1 or q. Hence we may assume that $V = C_V(a) \times C_V(b)$. Then, if $C_V(c) = C_V(a)$, $C_V(a) = C_V(\langle c\rangle \times \langle a\rangle) = C_V(\langle x\rangle \times \langle z\rangle) \subseteq C_V(z) = 1$, which is a contradiction. Hence $C_V(c) \cap C_V(a) = 1$. Similarly, we have that $C_V(c) \cap C_V(b) = 1$. Hence c acts regularly on $C_V(a)^{\#}$ and $C_V(b)^{\#}$, and so c acts regularly on $V^{\#}$. Thus $C_V(c) = 1$.

Next we shall prove that two elements of $C_V(a)$ conjugate in H are already conjugate in $Z(Q) \times Z_0$. Let $v, w \in C_V(a)^{\#}$ and let $v^h = w$ with $h \in H$. Since $v^a = v$ and $v^{ha} = v^h$, $\langle a, hah^{-1}\rangle \subseteq C_H(v)$. Since $|C_H(v)| = 3$, $\langle a\rangle = \langle hah^{-1}\rangle$, and hence $h \in N_H(\langle a\rangle) = \langle a\rangle(Z(Q) \times Z_0)$. This proves the above assertion.

Let $v \in C_V(a)^{\#}$ and $w \in C_V(b)^{\#}$. Suppose that v is conjugate to w in H. Let $v^h = w$ with $h \in H$. Since $v^h \in C_V(b)$, $v \in C_V(b^{h^{-1}})$. Thus $\langle a, b^{h^{-1}}\rangle \subseteq C_H(v)$. Since $|C_H(v)| = 3$, $\langle a\rangle = \langle b\rangle^{h^{-1}}$. Then $[a, h] \in (\langle a\rangle \times \langle b\rangle) \cap H' = (\langle a\rangle \times \langle b\rangle) \cap Q = 1$. Thus $\langle a\rangle = \langle b\rangle$, contrary to our choice of $\langle a\rangle, \langle b\rangle$. So any element of $C_V(a)^{\#}$ can not be conjugate to an element of $C_V(b)^{\#}$ in H.

By Lemma 3.1(ii), each orbit on $V^{\#}$ contains an element of $C_V(a)^{\#}$ or $C_V(b)^{\#}$ since $C_V(c) = C_V(z) = 1$. By the previous argument, H has $\frac{q-1}{6} + \frac{q-1}{6} = \frac{q-1}{3}$ orbits on $V^{\#}$. Since each H-orbit contains exactly $8 \cdot 3$

elements, $\frac{q-1}{3} \cdot 8 \cdot 3 = q^2 - 1$. Hence $8 = q + 1$. This implies that $q = 7$. Since V is an elementary abelian, we may assume that $H \subseteq GL(2, 7)$. By Lemma 2.5, $G = VH \simeq J$. ∎

LEMMA 3.21. *If $O_{p'}(F_1 \cdots F_r) = 1$, then $E(p, q, n)$ is isomorphic to a subgroup of G.*

Proof. We divide the proof of Lemma 3.21 into three steps. ∎

STEP 1. *$O_{p'}(F(H))$ is cyclic and $H/O_{p'}(F(H))$ is a p-group.*

Proof. Let T, U be as in Lemma 3.18. Since $O_{p'}(F_1 \cdots F_r) = 1$, $O_{p'}(F(H)) = O_{p'}(T)$. If $T \neq U$, then $2 \mid |H/C_H(U)|$ since $C_T(U) = U$. Since U is cyclic, $H/C_H(U)$ is abelian. By Lemma 3.1(i), $O^{p'}(H) = H$, and hence $p = 2$. So, in this case, $O_{p'}(F(H))$ is cyclic. If $T = U$, then it is obvious that $O_{p'}(F(H))$ is cyclic. Thus, in each case, $O_{p'}(F(H))$ is cyclic.

Let K be a Hall p'-subgroup of H. Then $F(K) = O_{p'}(F(H))$ is cyclic. Setting $Z = O_{p'}(F(H))$, $H/C_H(Z)$ is abelian. Since $O^{p'}(H) = H$, $H/C_H(Z)$ is a p-group. Hence $K \subseteq C_K(Z) = C_K(F(K)) \subseteq F(K) = Z$. Thus $K = Z$ and Step 1 follows. ∎

STEP 2. *H is isomorphic to a subgroup of $T(q^m)$, where $|V| = q^m$.*

Proof. Let $Z = O_{p'}(F(H))$. By a similar argument to that in the proof of Step 3 of Lemma 3.20, the same assertion as Step 3 holds since Z is cyclic and H/Z is a p-group. Furthermore, in the proof of Step 4 of Lemma 3.20, if we reset Z instead of Z_0, then we can prove that $H \subseteq T(q^m)$ if $C_H(Z) \subsetneq H$.

Next we assume that $C_H(Z) = H$. Then, by Step 1, $H = P \times Z$, where P is a Sylow p-subgroup of H. Since $O^{p'}(H) = H$, $Z = 1$, and hence H is a p-group. Since every normal subgroup of H is cyclic, H is cyclic, generalized quaternion, dihedral, or semi-dihedral by [6, Corollary 1.3]. If H is cyclic or generalized quaternion, then H acts regularly on $V^{\#}$, which contradicts Lemma 3.1(ii). If H is dihedral or semi-dihedral, then there exists a normal cyclic subgroup U of H with $|H : U| = 2$ and $C_H(U) = U$. Then V_U is homogeneous. Let $1 \neq v \in V$. Then $C_H(v) \neq 1$ by Lemma 3.1(ii). Let $t \in C_H(v)$ with $|t| = 2$. Since U acts regularly on $V^{\#}$, $t \notin U$. By Lemma 2.3, $\langle v^U \rangle$ is an irreducible U-module. Since $v \in C_V(t)$, $\langle v^U \rangle$ is $U\langle t \rangle = H$-module. Hence $V = \langle v^U \rangle$ is an irreducible U-module. By [8, Prop. 19.8], $H \subseteq T(q^m)$. This completes the proof of Step 2. ∎

STEP 3. *$E(p, q, n)$ is isomorphic to a subgroup of G.*

Proof. By Step 2, we may assume that $H \subseteq T(q^m)$. Let $M = \{x \to \alpha x \mid \alpha \in GF(q^m)^{\#}\} \lhd T(q^m)$. Then $T(q^m)/M$ and M are cyclic. By Lemma 3.1(ii), H is non-cyclic, and hence $H \nsubseteq M$ and $H \cap M \neq 1$. Setting $\overline{T(q^m)} = T(q^m)/M$, $1 \neq \bar{H} \subseteq \overline{T(q^m)}$. Since $O^{p'}(H) = H$, $\bar{H}$ is a cyclic

p-group. Let f be the natural isomorphism from $H/(H \cap M)$ to $\bar{H}$, and let H_0 be the inverse image of $\Omega_1(\bar{H})$. Setting $G_0 = VH_0 \subseteq G = VH$, G/G_0 is a p-group, and hence $O_{p'}(G_0) = O_{p'}(G)$. For $\forall x \in O_{p'}(G)$, there exists $y \in C_G(x)$ with $|y| = p$ by Lemma 3.1(ii). Since G_0 contains all elements in G of order p, $C_{G_0}(x) \ni y$. By the definition of the defect, G_0 has no p-blocks of defect 0 since G_0 is a p-nilpotent. By the minimality of G, $G = G_0$, and hence $H_0 = H$. Thus we have $|\bar{H}| = p$. Let $\langle\sigma\rangle = Gal(GF(q^{np})/GF(q^n))$, where $m = np$. Then $H \subseteq M\langle\sigma\rangle$.

If $p \nmid q^n - 1$, then $q^n \equiv a (\mathrm{mod}\, p)$, where $2 \le a \le p-1$. Hence $q^{np} \equiv a^p \equiv a (\mathrm{mod}\, p)$. Thus $p \nmid q^{np} - 1$. Then $|H| = ps$ with $(p,s) = 1$. Since $O^{p'}(H) = H, H$ is a Frobenius group with kernel $O_{p'}(H)$ or $|H| = p$. If H is a Frobenius group, then H has a p-block of defect 0, and so has G, which is a contradiction. If $|H| = p$, then H acts regularly on $V^{\#}$, which contradicts Lemma 3.1(ii). Thus $p \mid q^n - 1$. Let $\langle\nu\rangle$ be a subgroup of the multiplicative group $GF(q^{np})^{\#}$ of order $(q^{np}-1)/(q^n-1)$. Set $N = \{x \to \alpha x \mid \alpha \in \langle\nu\rangle^{\#}\} \subseteq M$. By Lemma 3.1(ii), H has no regular orbits on V, and hence $N\langle\sigma\rangle \subseteq H \subseteq T(q^m)$ by [10, Prop. 1.4]. Hence $E(p,q,n) \simeq VN\langle\sigma\rangle \subseteq VH = G$. ∎

LEMMA 3.22. *We have a final contradiction.*

Proof. If V is not a quasi-primitive H-module, then G involves $F(p,q,n)$ by Lemma 3.17, which contradicts the hypotheses of the theorem.

Next suppose that V is a quasi-primitive H-module. By Lemma 3.19, $O_{p'}(F_1 \cdots F_r) = 1$ or $O_{p'}(F_1 \cdots F_r) \simeq Q_8$. By Lemmas 3.20 and 3.21, $G \simeq J$ or G involves $E(p,q,n)$, which contradicts the hypotheses of the theorem. Thus, in each case, we have a contradiction, and this completes the proof of the theorem. ∎

REFERENCES

1. W. Feit, "The Representation Theory of Finite Groups," North-Holland, Amsterdam, 1982.
2. D. Gorenstein, "Finite Groups," Harper & Row, New York, 1968.
3. I. M. Issacs, "Character Theory of Finite Groups," Academic Press, New York, 1976.
4. N. Ito, Some studies on group characters, *Nagoya Math. J.* **2** (1951), 17–28.
5. N. Ito, On the characters of solvable groups, *Nagoya Math. J.* **3** (1951), 31–48.
6. O. Manz and T. R. Wolf, "Representations of Solvable Groups," Cambridge Univ. Press, Cambridge, UK, 1993.
7. H. Nagao, "Groups and Designs" (in Japanese), Iwanami, Tokyo, 1974.
8. D. S. Passman, "Permutation Groups," Benjamin, New York, 1968.
9. Y. Tsushima, On the existence of characters of defect zero, *Osaka J. Math.* **11** (1974), 417–423.

10. A. Turull, Supersolvable automorphism groups of solvable groups, *Math. Z*. **183** (1983), 47–73.
11. T. R. Wolf, Defect groups and character heights in blocks of solvable groups, *J. Algebra* **72** (1981), 183–209.
12. J. R. Zhang, p-Regular orbits and p-blocks of defect zero, *Comm. Algebra* **21** (1993), 299–307.