Linear Maps Preserving Reduced Norms*

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ABSTRACT

If a linear map between central simple algebras preserves reduced norms, it is an isomorphism or antiisomorphism followed by multiplication by an element of reduced norm 1.

Let K be a field, and suppose for the moment that $\mathrm{char}(K) \neq 2$. For nonzero a, b in K one can define the (generalized) quaternion algebra $A = K[i, j]$ with $i^2 = a$ and $j^2 = b$ and $ij = -ji$. The (reduced) norm of an element $x_0 + x_1 i + x_2 j + x_3 ij$ in A is $x_0^2 - ax_1^2 - bx_2^2 + abx_3^2$, and in this way a quadratic form is associated with the algebra A. It is a well-known theorem in the theory of quadratic forms [5, p. 146] that these forms distinguish the quaternion algebras: that is, two such forms are equivalent only when the algebras are isomorphic. What I want to show is that one can extend this theorem from quaternion algebras to central simple algebras of arbitrary dimension.

Recall that if A is a central simple algebra of dimension n^2 over a field K, then the norm function $N(a)$ (the determinant of the left multiplication map $x \mapsto ax$) always satisfies the formal identity $N(a) = [\mathrm{RN}(a)]^n$ for a suitable function RN called the reduced norm. On $n \times n$ matrix algebras $\mathrm{RN}(a)$ is of course simply $\det(a)$, and in general RN can be defined by choosing a Galois extension L of K so that $A \otimes_k L \simeq M_n(L)$ and proving that $\det_L(a \otimes 1)$ lies in K and is independent of the choice of L. The fact that determinants are unchanged by taking transposes implies that RN is unchanged if we reverse the order of multiplication; that is, RN is the same on A and on the opposite algebra A^{op}. But apart from this we will see that the reduced norm distinguishes the algebra.

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It turns out that what we need for the proof, besides this general information on central simple algebras [1], is just one result from linear algebra, one which in fact will be a particular case of our theorem: it is the characterization of those linear maps on matrices that preserve the determinant. This result, proved in various versions by Frobenius [3], Dieudonné [2], Marcus and Moyls [4], and others, says that if K is a field and $\varphi: M_n(K) \to M_n(K)$ is a K-linear map satisfying $\det \varphi(X) = \det X$ for all X in $M_n(K)$, then φ has the form $\varphi(X) = PXQ$ or $\varphi(X) = PX^tQ$, where X^t denotes the transpose of X, and P and Q are matrices with $\det(PQ) = 1$.

THEOREM. *Let A and B be central simple algebras of the same dimension over a field K, and let* RN *denote reduced norm. Let $\varphi: A \to B$ be a K-linear map with*

$$\mathrm{RN}_{B/K}(\varphi(X)) = \mathrm{RN}_{A/K}(X)$$

for all X in A. Then $\varphi(X)$ can be written uniquely in the form $b\psi(X)$ where b is an element with $\mathrm{RN}_{B/K}(b) = 1$ and $\varphi: A \to B$ is an isomorphism or an antiisomorphism of algebras.

Proof. Look first at the case $A = B = M_n(K)$, in which case the reduced norm is simply the determinant. We know then $\varphi(X)$ is PXQ or PX^tQ. Rewriting this as $(PQ)(Q^{-1}XQ)$ or $(PQ)(Q^{-1}X^tQ)$, we see that φ has the form required in the theorem. Furthermore, all automorphisms of $M_n(K)$ are inner, and $PXQ \equiv P_1XQ_1$ only when $P_1 = cP$ and $Q_1 = c^{-1}Q$ for some scalar c, so the element $b = PQ$ and the (anti)isomorphism ψ are uniquely determined. The uniqueness that we thus get by rewriting the result for matrix algebras is the crucial fact needed to derive the general theorem. What follows is a straightforward descent argument, fortunately one simple enough that it can be understood without any previous knowledge of descent theory.

Recall first that all finite division algebras are commutative, so for finite K the algebras A and B are in fact matrix algebras, and no further argument is needed. Thus we may assume K is infinite. Now the reduced norm is a polynomial function—more concretely, if $\{a_i\}$ is a K-basis of A, then $\mathrm{RN}_{A/K}(\Sigma x_i a_i)$ is a polynomial in the x_i. Similarly $\mathrm{RN}_{B/K}(\varphi(\Sigma x_i a_i))$ is a polynomial. These two polynomials agree for all values of the x_i in the infinite field K. Hence they must be identically the same, and consequently they will still agree when we allow the x_i to take values in some extension field L of K. Thus if φ_L is the L-linear extension of φ to $A \otimes_K L$, we will have

$$\mathrm{RN}_{B\otimes L/L} \circ \varphi_L = \mathrm{RN}_{A\otimes L/L}.$$

Central simple algebras always have separable splitting fields. Hence we can find a finite Galois extension L of K splitting our two algebras, so that $A \otimes L \simeq M_n(L) \simeq B \otimes L$. Each element g in $\mathrm{Gal}(L/K)$ acts on $A \otimes L$ and $B \otimes L$ by acting on the second factor; these actions are ring isomorphisms, though not L-algebra maps. Writing everything in terms of bases over K, one sees that an element b in $B \otimes L$ is actually in B $(= B \otimes K)$ iff $g(b) = b$ for all g. Similarly, an L-linear map $\psi_L : A \otimes L \to B \otimes L$ is the extension of a K-linear $\psi : A \to B$ iff $g \psi_L g^{-1} = \psi_L$ for all g. (Concretely, this just says that when we write out the matrix of ψ_L in terms of K-bases of A and B, the matrix entries are in K.) Now we know that our φ_L still preserves reduced norms, and we have chosen L so that the reduced norms are simply determinants. Hence we know that $\varphi_L(X) = b\psi_L(X)$ for a uniquely determined b in $B \otimes L$ and a unique L-algebra isomorphism or antiisomorphism $\psi_L : A \otimes L \to B \otimes L$. We have however $\varphi_L = g \varphi_L g^{-1}$ for each g in $\mathrm{Gal}(L/K)$, and hence

$$\varphi_L(X) = g\varphi_L(g^{-1}X) = g\big(b\psi_L(g^{-1}X)\big) = g(b)g\psi_L(g^{-1}X).$$

The map $g\psi_L g^{-1}$ is again L-linear, and is still a ring isomorphism or antiisomorphism. The uniqueness therefore gives $b = g(b)$ and $\psi_L = g\psi_L g^{-1}$. As this is true for all g, we know b lies in B and ψ_L is the extension of some K-linear map $\psi : A \to B$. It remains only to show that $\varphi = b\psi$, that b has reduced norm 1, that ψ is an algebra isomorphism or antiisomorphism, and that the expression is unique; and all of these statements follow trivially from the truth of the corresponding statements over L. ■

COROLLARY. *The reduced norm form determines a central simple algebra up to isomorphism or antiisomorphism.*

Quaternion algebras are isomorphic to their own opposite algebras, so for them we have recovered the theorem mentioned at the beginning of the paper.

ADDENDUM

After submitting this paper, I found that the corollary and (in essence) the theorem were derived earlier by Nathan Jacobson. Specifically, the corollary for $\mathrm{char}(K) \neq 2, 3$ is Theorem 12 in his "Generic norm of an algebra," *Osaka J. Math.* 15:25–50 (1953), and for all infinite K it appears as Theorem 10 in "Structure groups and Lie algebras of Jordan algebras of symmetric elements of associative algebras with involution," *Adv. in Math.* 20:106–150 (1976).

But his arguments depend on reducing the result to theorems on structure groups and isotopy of Jordan algebras, so I think my proof is still of some interest.

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