Available online at www.sciencedirect.com
ScienceDirect
Journal of Functional Analysis 252 (2007) 42–67
JOURNAL OF Functional Analysis
www.elsevier.com/locate/jfa

From multiplicative unitaries to quantum groups II ☆

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Received 5 June 2007; accepted 9 July 2007

Available online 4 September 2007

Communicated by Alain Connes

Abstract

It is shown that all important features of a C*-algebraic quantum group (A, Δ) defined by a modular multiplicative W depend only on the pair (A, Δ) rather than the multiplicative unitary operator W. The proof is based on thorough study of representations of quantum groups. As an application we present a construction and study properties of the universal dual of a quantum group defined by a modular multiplicative unitary—without assuming existence of Haar weights.

Keywords: Multiplicative unitary; Quantum group; Representation; Duality

1. Introduction

Building on the pioneering work of Baaj and Skandalis [1], S.L. Woronowicz introduced in [20] the class of *manageable multiplicative unitary operators.* Such multiplicative unitaries were shown to give rise to very interesting objects. Every such operator W acting on $\mathcal{H} \otimes \mathcal{H}$ (where $\mathcal{H}$ is some separable Hilbert space) gives rise to a C*-algebra $A \subset \mathrm{B}(\mathcal{H})$ with comultiplication Δ and a lot of additional structure [20, Theorem 1.5]. This extra structure comes in the form of the reduced dual $\widehat{A}$, the position of $W \in \mathrm{M}(\widehat{A} \otimes A)$, the coinverse κ, unitary coinverse R and the scaling group $(\tau_t)_{t\in\mathbb{R}}$. Moreover A comes naturally with an embedding into $\mathrm{B}(\mathcal{H})$, so it

☆ Research supported by KBN grants Nos. 115/E-343/SPB/6.PRUE/DIE50/2005-2008 and 1P03A03626.
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0022-1236/$ – see front matter
doi:10.1016/j.jfa.2007.07.006

inherits the ultraweak topology from the latter space. All this structure is defined with direct use of W.

The importance of manageability was emphasized with appearance the famous paper [8] in which Kustermans and Vaes gave a very satisfactory definition of a locally compact quantum group. They showed that every such object gives rise to a manageable multiplicative unitary. In a more recent paper [2] it is shown that the conditions of *regularity* and *semi-regularity* are not satisfied by multiplicative unitaries related to quantum groups.

Meanwhile, in [21,23] (and later in [12]) new examples of quantum groups were constructed using the theory developed in [20]. Only later, in [14] it was shown that they fitted into the scheme of locally compact quantum groups. Moreover the multiplicative unitaries used to define them were not manageable, but only *modular*. The difference between the latter notions is superfluous as was later explained in [13]. Still the possibility that two *different* multiplicative unitary operators gave rise to quantum groups described by *isomorphic* C^*-algebras with comultiplication (preserved by the isomorphism) remained unexplored until S.L. Woronowicz noticed in [22] that a formula for a right invariant weight on a quantum group defined by a modular multiplicative unitary could be expressed by one of the operators involved in the definition of modularity (cf. Definition 1). This formula might give a weight which is infinite on all non-zero positive elements, but if we choose the multiplicative unitary correctly we may find the Haar measure for our quantum group.

This development prompted the following question: if we can use different multiplicative unitaries to give rise to the same pair (A, Δ), does the additional structure on A depend on the choice of the multiplicative unitary? In this paper we give an answer to this question. The rich structure consisting of the coinverse, unitary coinverse, scaling group, reduced dual, the reduced bicharacter and the ultraweak topology on A are determined uniquely by the pair (A, Δ) in the sense that they do not depend on the choice of the multiplicative unitary giving rise to (A, Δ).

Let us briefly describe the contents of the paper. In the next section we will recall the definition of a modular multiplicative unitary and state the most important consequences of the definition. We will define what we mean by a quantum group and give a precise formulation of our main result together with its classical interpretation.

Section 3 is devoted to developing the representation theory of quantum groups. This is the main tool in the proof of our main result. We will define and study strongly continuous representations of a quantum group. Constructions of direct sums, tensor products and contragradient representations will be presented. The crucial notions of intertwining operators, equivalence, quasi equivalence and algebras generated by representations will be discussed. Section 4 contains the proof of our main theorem. The reasoning is based very firmly on the facts explained in Section 3.

As one application of Theorem 5 we will give, in Section 5, a detailed account of the construction of the universal dual of a given quantum group. We will reproduce some of the results of Kustermans ([7]) in the more general setting of quantum groups arising from multiplicative unitaries. Again the main tool will be the theory of representations of quantum groups developed in Section 3. The notion of a universal quantum group C^*-algebra will be introduced and properties of this object will be studied.

Throughout the paper we will freely use the language of C^*-algebras developed for use in the theory of quantum groups. We refer the reader to papers [9,18,19] for notions of multiplier algebras, morphisms of C^*-algebras, C^*-algebras generated by quantum families of multipliers, etc.

2. Definitions and results

Let us recall the definition of a modular multiplicative unitary. We shall use the complex conjugate Hilbert space $\overline{\mathcal{H}}$ of a given Hilbert space $\mathcal{H}$. Its precise definition is given in Section 3.3.

Definition 1. (See [13, Definition 2.1].) Let $\mathcal{H}$ be a Hilbert space. A unitary operator $W \in \mathrm{B}(\mathcal{H} \otimes \mathcal{H})$ is a *modular multiplicative unitary* if it is a multiplicative unitary and there exist two positive self-adjoint operators $\widehat{Q}$ and Q on $\mathcal{H}$ with zero kernels and a unitary operator $\widetilde{W} \in \mathrm{B}(\overline{\mathcal{H}} \otimes \mathcal{H})$ such that

$$W(\widehat{Q} \otimes Q)W^* = \widehat{Q} \otimes Q$$

and

$$(x \otimes u|W|z \otimes y) = \bigl(\bar{z} \otimes Qu|\widetilde{W}|\bar{x} \otimes Q^{-1}y\bigr)$$

for all $x, z \in \mathcal{H}$, $u \in \mathrm{D}(Q)$ and $y \in \mathrm{D}(\widehat{Q})$.

Theorem 2. ([13,20]) *Let $\mathcal{H}$ be a separable Hilbert space and let $W \in \mathrm{B}(\mathcal{H} \otimes \mathcal{H})$ be a modular multiplicative unitary. Let*

$$A = \bigl\{(\omega \otimes \mathrm{id})W\colon\ \omega \in \mathrm{B}(\mathcal{H})_*\bigr\}^{\|\cdot\|\text{-closure}}, \tag{2.1}$$

$$\widehat{A} = \bigl\{(\mathrm{id} \otimes \omega)(W^*)\colon\ \omega \in \mathrm{B}(\mathcal{H})_*\bigr\}^{\|\cdot\|\text{-closure}}. \tag{2.2}$$

Then

(1) *A and $\widehat{A}$ are non-degenerate* C^**-subalgebras of* $\mathrm{B}(\mathcal{H})$;
(2) $W \in \mathrm{M}(\widehat{A} \otimes A)$;
(3) *there exists a unique $\Delta \in \mathrm{Mor}(A, A \otimes A)$ such that*

$$(\mathrm{id} \otimes \Delta)W = W_{12}W_{13};$$

moreover Δ is coassociative and the sets

$$\bigl\{(a \otimes I)\Delta(b)\colon\ a, b \in A\bigr\} \quad \textit{and} \quad \bigl\{\Delta(a)(I \otimes b)\colon\ a, b \in A\bigr\}$$

are linearly dense subsets of $A \otimes A$;
(4) *there exists a unique closed linear operator κ on the Banach space A such that the set $\{(\omega \otimes \mathrm{id})W\colon\ \omega \in \mathrm{B}(\mathcal{H})_*\}$ is a core for κ and*

$$\kappa\bigl((\omega \otimes \mathrm{id})W\bigr) = (\omega \otimes \mathrm{id})(W^*);$$

furthermore for any $a, b \in \mathrm{Dom}(\kappa)$ the product $ab \in \mathrm{Dom}(\kappa)$ and $\kappa(ab) = \kappa(b)\kappa(a)$, the image of κ coincides with $\mathrm{Dom}(\kappa)^*$ and $\kappa(\kappa(a)^*)^* = a$ for any $a \in \mathrm{Dom}(\kappa)$;
(5) *there exists a unique one parameter group $(\tau_t)_{t\in\mathbb{R}}$ of $*$-automorphisms of A and a unique ultraweakly continuous involutive $*$-anti-automorphism R of A such that $R \circ \tau_t = \tau_t \circ R$ for all $t \in \mathbb{R}$ and $\kappa = R \circ \tau_{i/2}$.*

The objects κ, (τ_t) and R appearing in statement (5) of Theorem 2 are referred to as the *coinverse*, *scaling group* and *unitary coinverse*.

All results of the fundamental paper [20] have been formulated for multiplicative unitaries acting on separable Hilbert spaces. For this reason we shall restrict our attention solely to such spaces. In other words, from now on all Hilbert spaces are assumed to be separable. Moreover if existence of a certain Hilbert space is a part of statement of a theorem (see e.g. Proposition 13) then it can be shown that this Hilbert space is (or can be chosen) separable. Still, many of our results are also true if the Hilbert spaces are of arbitrary dimension.

We shall consider pairs (A, Δ) consisting of a C^*-algebra A and a morphism $\Delta \in \mathrm{Mor}(A, A \otimes A)$. We say that two such pairs (A, Δ_A) and (B, Δ_B) are isomorphic if there is an isomorphism $\Phi \in \mathrm{Mor}(A, B)$ such that

$$\Delta_B \circ \Phi = (\Phi \otimes \Phi) \circ \Delta_A. \tag{2.3}$$

Definition 3. Let A be a C^*-algebra and $\Delta \in \mathrm{Mor}(A, A \otimes A)$. We say that a the pair $\mathbb{G} = (A, \Delta)$ is a *quantum group* if there exists a modular multiplicative unitary such that (A, Δ) is isomorphic to the C^*-algebra with comultiplication associated to W in the way described in Theorem 2. In such a case we shall say that W is a modular multiplicative unitary *giving rise* to the quantum group $\mathbb{G}$.

The following definition has been proposed e.g. in [11, p. 237].

Remark 4. Let us note that the results of [13] guarantee that $\mathbb{G} = (A, \Delta)$ is a quantum group if and only if there exists a *manageable* multiplicative unitary [20, Definition 1.2] giving rise to $\mathbb{G}$.

The aim of this paper is to provide justification for Definition 3.

A modular multiplicative unitary W on a Hilbert space $\mathcal{H}$ gives rise to a quantum group $\mathbb{G} = (A, \Delta)$ as described in Theorem 2, but it also produces another quantum group called the *reduced dual* of $\mathbb{G}$. This is the quantum group $\widehat{\mathbb{G}} = (\widehat{A}, \widehat{\Delta})$, where $\widehat{A}$ is the C^*-subalgebra of $\mathrm{B}(\mathcal{H})$ described in Theorem 2 and $\widehat{\Delta}$ is given by

$$\widehat{\Delta}(x) = \sigma\bigl(W^*(I \otimes x)W\bigr), \tag{2.4}$$

where σ is the flip on the tensor product $\widehat{A} \otimes \widehat{A}$. One possible modular multiplicative unitary giving rise to $\widehat{\mathbb{G}}$ is $\widehat{W} = \Sigma W^* \Sigma$, where Σ is the flip on $\mathcal{H} \otimes \mathcal{H}$. It is clear that the reduced dual of $\widehat{\mathbb{G}}$ is isomorphic to $\mathbb{G}$. Note that Eq. (2.4) and Theorem 2(3) show that the element $W \in \mathrm{M}(\widehat{A} \otimes A)$ is a *bicharacter*, i.e. it satisfies

$$(\mathrm{id} \otimes \Delta)W = W_{12}W_{13} \quad \text{and} \quad (\widehat{\Delta} \otimes \mathrm{id})W = W_{23}W_{13}. \tag{2.5}$$

The quantum group $\widehat{\mathbb{G}}$ and the bicharacter $W \in \mathrm{M}(\widehat{A} \otimes A)$ are a priori defined in terms of the modular multiplicative unitary which gives rise to $\mathbb{G}$, rather than $\mathbb{G}$ itself.

The elements of the polar decomposition of the coinverse κ are also determined a priori by the multiplicative unitary. For example the scaling group (τ_t) is given by

$$\tau_t(a) = Q^{2it} a Q^{-2it},$$

where Q is one of the operators involved in the definition of modularity. In addition any modular multiplicative unitary $W \in \mathrm{B}(\mathcal{H} \otimes \mathcal{H})$ giving rise to $\mathbb{G}$ provides a topology on A, namely the restriction of the ultraweak topology from $\mathrm{B}(\mathcal{H})$ to A. The anti-automorphism R and the automorphisms $(\tau_t)_{t\in\mathbb{R}}$ (also defined through W) are continuous for this topology.

We shall prove that all necessary data of a quantum group $\mathbb{G}$ are independent of the choice of modular multiplicative unitary giving rise to $\mathbb{G}$.

Theorem 5. *Let $\mathbb{G} = (A, \Delta)$ be a quantum group. Choose a Hilbert space $\mathcal{H}$ and a modular multiplicative unitary $W \in \mathrm{B}(\mathcal{H} \otimes \mathcal{H})$ giving rise to $\mathbb{G}$ and use Theorem* 2 *to construct the embedding $A \subset \mathrm{B}(\mathcal{H})$ and the objects $\widehat{A}$, κ, R and $(\tau_t)_{t\in\mathbb{R}}$. Let $\widehat{\Delta}$ be the comultiplication on $\widehat{A}$ given by* (2.4). *Then*

(1) *The ultraweak topology on A inherited from $\mathrm{B}(\mathcal{H})$ is independent of the choice of $\mathcal{H}$ and W.*
(2) *The coinverse κ, its domain and all elements of polar decomposition are independent of the choice of $\mathcal{H}$ and W.*
(3) *$\widehat{\mathbb{G}} = (\widehat{A}, \widehat{\Delta})$ and the bicharacter $W \in \mathrm{M}(\widehat{A} \otimes A)$ are defined uniquely* (*up to isomorphism*) *by $\mathbb{G}$. They do not depend on the choice of $\mathcal{H}$ and $W \in \mathrm{B}(\mathcal{H} \otimes \mathcal{H})$.*

The proof of Theorem 5 will be given in Section 4.

Note that one way to interpret Theorem 5 is to say that a quantum group $\mathbb{G} = (A, \Delta)$ is naturally endowed with an analog of the *class* of the Haar measure. This is because the ultraweak topology on A (determined uniquely by $\mathbb{G}$) fixes the von Neumann algebra A'' which is the noncommutative analog of $L^\infty(\mathbb{G})$. Also the set of all ultraweakly continuous functionals on A plays the role of $L^1(\mathbb{G})$.

3. Representations of quantum groups

Throughout this section let us fix a quantum group $\mathbb{G} = (A, \Delta)$.

Definition 6. A *strongly continuous unitary representation* U of $\mathbb{G}$ acting on a Hilbert space H is a unitary element $U \in \mathrm{M}(\mathcal{K}(H) \otimes A)$ such that $(\mathrm{id} \otimes \Delta)U = U_{12}U_{13}$. The class of all strongly continuous unitary representations of $\mathbb{G}$ will be denoted by $\mathrm{Rep}(\mathbb{G})$.

We shall use the symbol H_U for the Hilbert space on which the representation U acts: $U \in \mathrm{M}(\mathcal{K}(H_U) \otimes A)$.

Recall that $\mathbb{G}$ is defined as the pair (A, Δ) arising from some modular multiplicative unitary $W \in \mathrm{B}(\mathcal{H} \otimes \mathcal{H})$ for some Hilbert space $\mathcal{H}$ (cf. Section 2). It follows from statements (1) and (2) of Theorem 2 that $W \in \mathrm{M}(\mathcal{K}(\mathcal{H}) \otimes A)$ and so by the first part of (2.5) W is a strongly continuous unitary representation of $\mathbb{G}$ on $\mathcal{H}$.

If H is a Hilbert space then the element $\mathbb{I}_H = I_H \otimes I_A \in \mathrm{M}(\mathcal{K}(H) \otimes A)$ clearly is a strongly continuous unitary representation of $\mathbb{G}$. Such representations are called *trivial.*

In what follows we shall most of the time omit the words “strongly continuous unitary” and speak simply about representations of $\mathbb{G}$.

3.1. Intertwining operators

Let $U, V \in \mathrm{Rep}(\mathbb{G})$ and let $t \in \mathrm{B}(H_U, H_V)$. We say that t *intertwines* U and V if

$$(t \otimes I)U = V(t \otimes I). \tag{3.1}$$

The above equation may be understood in several contexts. If the C*-algebra A is faithfully represented on a Hilbert space $\mathcal{H}$ then U and V become elements of $\mathrm{B}(H_U \otimes \mathcal{H})$ and $\mathrm{B}(H_V \otimes \mathcal{H})$, respectively. In this situation (3.1) means that

$$(t \otimes I_{\mathcal{H}})U = V(t \otimes I_{\mathcal{H}}).$$

Equivalently $t \in \mathrm{B}(H_U, H_V)$ intertwines U and V if and only if for any $\omega \in A^*$ we have

$$t(\mathrm{id} \otimes \omega)(U) = (\mathrm{id} \otimes \omega)(V)t.$$

Finally we can identify of $\mathrm{M}(\mathcal{K}(H_U) \otimes A)$ and $\mathrm{M}(\mathcal{K}(H_U) \otimes A)$ with the C*-algebras $\mathcal{L}(H_U \otimes A)$ and $\mathcal{L}(U_V \otimes A)$ of adjointable maps of the Hilbert A-modules $H_U \otimes A$ and $H_V \otimes A$, respectively [9, pp. 10, 37]. Then $t \in \mathrm{B}(H_U, H_V)$ intertwines U and V if and only if

$$(t \otimes I_A)U = V(t \otimes I_A)$$

as elements of $\mathcal{L}(H_U \otimes A, H_V \otimes A)$.

Let $U, V \in \mathrm{Rep}(\mathbb{G})$. The set of operators intertwining U and V will be denoted by $\mathrm{Hom}(U, V)$.

The following properties follow immediately from the definition of $\mathrm{Hom}(U, V)$:

(1) $\mathrm{Hom}(U, V)$ is a subspace of $B(H_U, H_V)$ closed in the weak operator topology;
(2) for any $t \in \mathrm{Hom}(U, V)$ we have $t^* \in \mathrm{Hom}(V, U)$;
(3) $I_{H_U} \in \mathrm{Hom}(U, U)$;
(4) if T is another representation of $\mathbb{G}$ then for any $t \in \mathrm{Hom}(U, V)$ and $s \in \mathrm{Hom}(V, T)$ we have $st \in \mathrm{Hom}(U, T)$ and composition of intertwining operators is bilinear for the vector space structures on $\mathrm{Hom}(U, V)$ and $\mathrm{Hom}(V, T)$, in particular $\mathrm{Hom}(U, U)$ is a $*$-algebra with unit;
(5) for any $t \in \mathrm{Hom}(U, V)$ the composition t^*t is a positive element of the $*$-algebra $\mathrm{Hom}(U, U)$, i.e. there exists $x \in \mathrm{Hom}(U, U)$ such that $t^*t = x^*x$.

All this shows that the class $\mathrm{Rep}(\mathbb{G})$ of strongly continuous unitary representations of $\mathbb{G}$ with intertwining operators as morphisms forms a *concrete* W*-*category* as defined in [6, Definitions 1.1 and 2.1].

3.2. Equivalence and quasi-equivalence

Definition 7. Let U and V be representations of $\mathbb{G}$.

(1) U is a *subrepresentation* of V if $\mathrm{Hom}(U, V)$ contains an isometry;
(2) U and V are *equivalent* if $\mathrm{Hom}(U, V)$ contains an invertible operator;
(3) U and V are *disjoint* if $\mathrm{Hom}(U, V) = \{0\}$;

(4) U and V are *quasi-equivalent* if no subrepresentation of U is disjoint from V and no subrepresentation of V is disjoint from U.

Let $U, V \in \mathrm{Rep}(\mathbb{G})$. We write $U \approx V$ if U and V are equivalent. Clearly "$\approx$" is an equivalence relation. One can show that quasi-equivalence is also an equivalence relation (cf. e.g. Proposition 13).

Remark 8. Let $U, V \in \mathrm{Rep}(\mathbb{G})$. By [6, Corollary 2.7] U and V are equivalent if and only if $\mathrm{Hom}(U, V)$ contains a unitary operator. Similarly U and V are equivalent if $\mathrm{Hom}(U, V)$ contains an operator with trivial kernel and dense range. Moreover equivalence is the same thing as isomorphism in the W*-category $\mathrm{Rep}(\mathbb{G})$.

3.3. Operations on representations

3.3.1. Direct sums

Let (U_α) be a family of representations of $\mathbb{G}$. The C*-algebra

$$\bigoplus_\alpha \big(\mathcal{K}(H_{U_\alpha}) \otimes A\big)$$

is contained in

$$\mathcal{K}\Big(\bigoplus_\alpha H_{U_\alpha}\Big) \otimes A.$$

Moreover the inclusion is a morphism of C*-algebras ([15]). Therefore we have

$$\mathrm{M}\Big(\bigoplus_\alpha \big(\mathcal{K}(H_{U_\alpha}) \otimes A\big)\Big) \subset \mathrm{M}\Big(\mathcal{K}\Big(\bigoplus_\alpha H_{U_\alpha}\Big) \otimes A\Big).$$

The C*-algebra on the left hand side consists of norm bounded families (T_α) of multipliers of the C*-algebras $\mathcal{K}(H_{U_\alpha}) \otimes A$. Therefore the family $U = (U_\alpha)$ is a unitary element of $\mathrm{M}(\mathcal{K}(\bigoplus_\alpha H_{U_\alpha}) \otimes A)$. It is not difficult to see that it is a strongly continuous unitary representation of $\mathbb{G}$. So defined U is called the *direct sum* of the representations (U_α). Of course $H_U = \bigoplus_\alpha H_{U_\alpha}$. Moreover for any α the canonical injection $H_\alpha \hookrightarrow H$ and projection $H \to H_\alpha$ belong to $\mathrm{Hom}(U_\alpha, U)$ and $\mathrm{Hom}(U, U_\alpha)$, respectively. In particular each U_α is a subrepresentation of U.

Remark 9. Let (U_α) be a family representations of $\mathbb{G}$ and let U be the direct sum of (U_α). Then for any $\omega \in A_*$ and any α we have

$$\big\|(\mathrm{id} \otimes \omega)U\big\| \geqslant \big\|(\mathrm{id} \otimes \omega)U_\alpha\big\|.$$

In what follows we shall restrict attention to countable direct sums, so that our Hilbert spaces remain separable.

3.3.2. Tensor products

Let $U, V \in \mathrm{Rep}(\mathbb{G})$. Then the element

$$U \circledT V = U_{13}V_{23} \in \mathrm{M}\big(\mathcal{K}(H_U) \otimes \mathcal{K}(H_V) \otimes A\big) = \mathrm{M}\big(\mathcal{K}(H_U \otimes H_V) \otimes A\big)$$

is a representation of $\mathbb{G}$. The representation $U \circledT V$ is the *tensor product* of representations U and V.

If T is another representation of $\mathbb{G}$ then we have $(U \circledT V) \circledT T \approx U \circledT (V \circledT T)$, so the tensor product of any finite number of representations of $\mathbb{G}$ is associative up to equivalence. Note that the operation of taking tensor product is not, in general, commutative. In the worst case $U \circledT V$ is not equivalent to $V \circledT U$. However, even if $U \circledT V \approx V \circledT U$, then in general, the flip $\Sigma : H_U \otimes H_V \to H_V \otimes H_U$ does not intertwine $U \circledT V$ with $V \circledT U$: $\Sigma \notin \mathrm{Hom}(U \circledT V, V \circledT U)$.

The operation of taking tensor products endows $\mathrm{Rep}(\mathbb{G})$ with the structure of a monoidal W*-category [17, p. 39].

3.3.3. Contragradient representations

Let H be a Hilbert space. The complex conjugate space $\overline{H}$ is defined as the set of elements $\bar{x}$, where $x \in H$. The vector space structure on $\overline{H}$ is given by $\bar{x} + \bar{y} = \overline{x + y}$ and $\zeta \bar{x} = \overline{\bar{\zeta} x}$ for $\bar{x}, \bar{y} \in \overline{H}$ and $\zeta \in \mathbb{C}$. The Hilbert space structure on $\overline{H}$ is obtained by setting

$$(\bar{x} \mid \bar{y}) = (y \mid x),$$

where on the right-hand side we use the scalar product in H. We have the natural operation of *transposition* taking operators on H to operators on $\overline{H}$. This operation will be denoted by $m \mapsto m^\top$: for any closed operator m on K the operator $m^\top$ is defined by

$$\left(\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} \in \mathrm{Graph}\, m^\top\right) \quad \Longleftrightarrow \quad \left(\begin{pmatrix} x \\ y \end{pmatrix} \in \mathrm{Graph}\, m^*\right).$$

When restricted to bounded operators, the transposition becomes an anti-isomorphism of C*-algebras $\mathrm{B}(H) \to \mathrm{B}(\overline{H})$.

In what follows we shall denote by R the unitary coinverse of the quantum group $\mathbb{G}$. This is the "unitary" part of the polar decomposition of the coinverse κ (cf. Section 2, [13,20]).

Proposition 10. *Let $U \in \mathrm{Rep}(\mathbb{G})$. Then the element*

$$U^{\mathrm{c}} = U^{\top \otimes R} \in \mathrm{M}\big(\mathcal{K}(\overline{H_U}) \otimes A\big)$$

is a strongly continuous unitary representation of $\mathbb{G}$ acting on $H_{U^{\mathrm{c}}} = \overline{H_U}$.

Proof. Denote by σ the flip map on $A \otimes A$. Using [13, Theorem 2.3(5)] and remembering that $\top$ and R are anti-isomorphisms we obtain

$$\begin{aligned}(\mathrm{id} \otimes \Delta)U^{\mathrm{c}} &= (\mathrm{id} \otimes \Delta)(\top \otimes R)U = (\top \otimes \Delta \circ R)U \\ &= \big(\top \otimes [\sigma \circ (R \otimes R) \circ \Delta]\big)U \\ &= \big(\top \otimes [\sigma \circ (R \otimes R)]\big)(\mathrm{id} \otimes \Delta)U\end{aligned}$$

$$
\begin{aligned}
&= (\mathrm{id} \otimes \sigma)(\top \otimes R \otimes R)(\mathrm{id} \otimes \Delta)U \\
&= (\mathrm{id} \otimes \sigma)(\top \otimes R \otimes R)(U_{12}U_{13}) \\
&= (\mathrm{id} \otimes \sigma)\big(\big[(\top \otimes R)U\big]_{13}\big[(\top \otimes R)U\big]_{12}\big) \\
&= (\mathrm{id} \otimes \sigma)\big(U^{\mathrm{c}}_{13}U^{\mathrm{c}}_{12}\big) = U^{\mathrm{c}}_{12}U^{\mathrm{c}}_{13}. \qquad \square
\end{aligned}
$$

Definition 11. Let U be a strongly continuous unitary representation of $\mathbb{G}$. The *contragradient representation* of U is the strongly continuous unitary representation U^{c} of $\mathbb{G}$ defined in Proposition 10.

Remark 12. In contrast to existing definitions of contragradient representations in literature (e.g. [16, Section 3]) we have $(U^{\mathrm{c}})^{\mathrm{c}} = U$ for any strongly continuous unitary representation U of $\mathbb{G}$.

The operation of taking contragradient representation is well compatible with tensor products. In fact if U and V are representations of $\mathbb{G}$ then identifying $\overline{H_U \otimes H_V}$ with $\overline{H_V} \otimes \overline{H_U}$ via the unitary map

$$\overline{H_V} \otimes \overline{H_U} \ni \bar{y} \otimes \bar{x} \longmapsto \overline{x \otimes y} \in \overline{H_U \otimes H_V}$$

we have

$$(U \operatorname{\textcircled{\scriptsize T}} V)^{\mathrm{c}} = V^{\mathrm{c}} \operatorname{\textcircled{\scriptsize T}} U^{\mathrm{c}}. \tag{3.7}$$

3.4. Quasi-equivalence and tensor products

Proposition 13. *Let U and V be representations of $\mathbb{G}$. Then the following conditions are equivalent*:

(1) *U and V are quasi-equivalent*;
(2) *There exists a Hilbert space Z such that $\mathbb{I}_Z \operatorname{\textcircled{\scriptsize T}} U$ and $\mathbb{I}_Z \operatorname{\textcircled{\scriptsize T}} V$ are equivalent.*

Note that this result can be formulated without the notion of tensor product of representations. This is because tensor product with a trivial representation is expressible as a direct sum. We omit the proof of this proposition as it follows the lines of proofs of analogous results for representations of C*-algebras ([5]). Moreover, in this paper we shall exclusively use condition (2) of Proposition 13 as the definition of quasi-equivalence. Equivalence of (1) and (2) will not be used.

3.5. Algebras generated by representations

In this subsection we shall describe the algebras generated by representations of $\mathbb{G}$. We shall use the notion of a C*-algebra generated by a quantum family of affiliated elements [19, Definition 4.1].

Let $U \in \mathrm{Rep}(\mathbb{G})$. Then there exists a unique C*-algebra B_U acting non-degenerately on H_U such that $U \in \mathrm{M}(B_U \otimes A)$ and B_U is generated by U. Indeed, by Remark 4 there is a Hilbert space $\mathcal{H}$ and a manageable multiplicative unitary $W \in \mathrm{B}(\mathcal{H} \otimes \mathcal{H})$ giving rise to $\mathbb{G}$. Then cf. [20, Theorems 1.6 and 1.7] one may take

$$B_U = \left\{(\mathrm{id} \otimes \omega)(U^*)\colon\ \omega \in \mathrm{B}(\mathcal{H})_*\right\}^{\|\cdot\|\text{-closure}}. \tag{3.8}$$

Uniqueness of B_U follows from the remark after [19, Definition 4.1]. In particular B_U is independent of the multiplicative unitary W and Hilbert space $\mathcal{H}$ entering (3.8). Note that B_U is unique not only as a C*-algebra, but also as a subset of $\mathrm{B}(H_U)$.

The next proposition states some basic facts about C*-algebras generated by representations. We omit the simple proof.

Proposition 14. *Let $\mathbb{G}$ be a quantum group and let $U, V \in \mathrm{Rep}(\mathbb{G})$. Then*

(1) *if U is a subrepresentation of V then there exists a $\Phi \in \mathrm{Mor}(B_V, B_U)$ such that $(\Phi \otimes \mathrm{id})V = U$. This morphism maps B_V onto B_U and is continuous for the ultraweak topologies inherited by B_V and B_U from $\mathrm{B}(H_V)$ and $\mathrm{B}(H_U)$, respectively*;
(2) *if $U \approx V$ then there is a spatial isomorphism $\Phi \in \mathrm{Mor}(B_U, B_V)$ such that $(\Phi \otimes \mathrm{id})U = V$*;
(3) *if Z is a Hilbert space and $V = \mathbb{I}_Z \textcircled{\scriptsize T} U$ then there is an isomorphism $\Phi \in \mathrm{Mor}(B_U, B_V)$ such that $(\Phi \otimes \mathrm{id})U = V$. Moreover Φ is a homeomorphism for the ultraweak topologies inherited by B_U and B_V from $\mathrm{B}(H_U)$ and $\mathrm{B}(H_V)$, respectively.*

From Proposition 14(2), (3) and Proposition 13 we immediately get

Corollary 15. *Let $\mathbb{G}$ be a quantum group and let U, V be representations of $\mathbb{G}$. Assume that U and V are quasi equivalent. Then there is an isomorphism $\Phi \in \mathrm{Mor}(B_U, B_V)$ such that*

$$(\Phi \otimes \mathrm{id})U = V.$$

Moreover Φ is a homeomorphism for the ultraweak topologies inherited by B_U and B_V from $\mathrm{B}(H_U)$ and $\mathrm{B}(H_V)$, respectively.

At the end of this section let us mention an important result about matrix elements of representations. In it we shall use the strict closure of the operator κ defined on the strictly dense subset $\mathrm{Dom}(\kappa)$ of $\mathrm{M}(A)$ (cf. [20, p. 133]).

Proposition 16. *Let U be a representation of $\mathbb{G}$. Then for any $\eta \in \mathrm{B}(H_U)_*$ the element $(\eta \otimes \mathrm{id})U$ belongs to the domain of κ and we have*

$$\kappa\big((\eta \otimes \mathrm{id})U\big) = (\eta \otimes \mathrm{id})(U^*).$$

This proposition is a direct consequence of [20, Theorems 1.7 and 1.6(4)] and the fact that for any quantum group $\mathbb{G}$ there is a *manageable* multiplicative unitary giving rise to $\mathbb{G}$ (cf. [13]).

3.6. Absorbing representations

Definition 17. Let $\mathbb{G}$ be a quantum group and let U be a representation of $\mathbb{G}$.

(1) U is called *right absorbing* if for any representation V of $\mathbb{G}$ we have

$$V \textcircled{\scriptsize T} U \approx \mathbb{I}_{H_V} \textcircled{\scriptsize T} U.$$

(2) U is called *left absorbing* if for any representation V of $\mathbb{G}$ we have

$$U \textcircled{\scriptsize T} V \approx U \textcircled{\scriptsize T} \mathbb{I}_{H_V}.$$

Remark 18. Let $\mathbb{G}$ be a quantum group. By (3.7) a representation U of $\mathbb{G}$ is right absorbing if and only if U^{c} is left absorbing.

Proposition 19. *Let $\mathbb{G} = (A, \Delta)$ be a quantum group and let $U \in \mathrm{Rep}(\mathbb{G})$. Let π be a representation of A on the Hilbert space H_U which is covariant in the sense that for any $a \in A$*

$$U\big(\pi(a) \otimes I\big)U^* = (\pi \otimes \mathrm{id})\Delta(a). \tag{3.9}$$

Then U is right absorbing.

Proof. Let V be a representation of V. Applying $(\mathrm{id} \otimes \pi \otimes \mathrm{id})$ to both sides of the equation $(\mathrm{id} \otimes \Delta)V = V_{12}V_{13}$ we obtain

$$U_{23}\big[(\mathrm{id} \otimes \pi)V\big]_{12}U_{23}^* = \big[(\mathrm{id} \otimes \pi)V\big]_{12}V_{13}.$$

Therefore

$$\big[(\mathrm{id} \otimes \pi)V\big]_{12}^* U_{23}\big[(\mathrm{id} \otimes \pi)V\big]_{12} = V_{13}U_{23}. \tag{3.10}$$

The right-hand side of (3.10) is by definition equal to $V \textcircled{\scriptsize T} U$ while the left hand side is equivalent to $U_{23} = \mathbb{I}_{H_V} \textcircled{\scriptsize T} U$. This means that U is right absorbing. □

Note that if $\mathcal{H}$ is a Hilbert space and $W \in \mathrm{B}(\mathcal{H} \otimes \mathcal{H})$ is a modular multiplicative unitary giving rise to $\mathbb{G}$ then W, viewed as an element of $\mathrm{M}(\mathcal{K}(\mathcal{H}) \otimes A)$, is a representation of $\mathbb{G}$ and the embedding of A into $B(\mathcal{H})$ given by W is a covariant representation of A. In particular we have

Corollary 20. *Let $\mathcal{H}$ be a Hilbert space and let $W \in \mathrm{B}(\mathcal{H} \otimes \mathcal{H})$ be a modular multiplicative unitary giving rise to $\mathbb{G}$. Then the representation $W \in \mathrm{M}(\mathcal{K}(\mathcal{H}) \otimes A)$ of $\mathbb{G}$ on $\mathcal{H}$ is right absorbing.*

Proposition 21. *Let $\mathbb{G}$ be a quantum group. Then any two right absorbing representations are quasi-equivalent.*

Proof. Let U and V be right absorbing representations of $\mathbb{G}$ and let T be a left absorbing representation of $\mathbb{G}$ (one can take e.g. $T = U^{\mathrm{c}}$, cf. Remark 18). We have

$$\mathbb{I}_{H_T} \textcircled{\scriptsize T} V \approx T \textcircled{\scriptsize T} V \approx T \textcircled{\scriptsize T} \mathbb{I}_{H_V}$$

and

$$T \, ⓣ \, \mathbb{I}_{H_U} \approx T \, ⓣ \, U \approx \mathbb{I}_{H_T} \, ⓣ \, U.$$

Clearly $T \, ⓣ \, \mathbb{I}_{H_U}$ and $T \, ⓣ \, \mathbb{I}_{H_V}$ are quasi equivalent. $\square$

4. Proof of the main theorem

Let $\mathbb{G} = (A, \Delta)$ be a quantum group. Let U be a right absorbing representation of $\mathbb{G}$. Then There is a unique comultiplication $\widehat{\Delta}_U$ on B_U such that

$$(\widehat{\Delta}_U \otimes \mathrm{id})U = U_{23}U_{13}. \tag{4.11}$$

To see this let $\mathcal{H}$ be a Hilbert space and let $W \in \mathrm{B}(\mathcal{H} \otimes \mathcal{H})$ be a modular multiplicative unitary giving rise to $\mathbb{G}$. the second part of (2.5) tells us that

$$(\widehat{\Delta} \otimes \mathrm{id})W = W_{23}W_{13}.$$

Now both U and $W \in \mathrm{M}(\mathcal{K}(\mathcal{H}) \otimes A)$ are right absorbing representations of $\mathbb{G}$ (by Corollary 20) and by Proposition 21 and Corollary 15 there is an isomorphism $\Phi \in \mathrm{Mor}(B_U, \widehat{A})$ which is a homeomorphism for the ultraweak topologies on $B_U \subset \mathrm{B}(H_U)$ and $\widehat{A} \subset \mathrm{B}(\mathcal{H})$ and

$$(\Phi \otimes \mathrm{id})U = W. \tag{4.12}$$

Therefore setting $\widehat{\Delta}_U = (\Phi \otimes \Phi)^{-1} \circ \widehat{\Delta} \circ \Phi$ we obtain a comultiplication on B_U satisfying (4.11).

From what we have seen so far it is clear that $(B_U, \widehat{\Delta}_U)$ is a quantum group isomorphic to $\widehat{\mathbb{G}} = (\widehat{A}, \widehat{\Delta})$, i.e. the reduced dual of $\mathbb{G}$ defined by W.

Now U could have been any other modular multiplicative unitary giving rise to $\mathbb{G}$. It follows that the reduced dual $(\widehat{A}, \widehat{\Delta})$ is independent of the multiplicative unitary giving rise to $\mathbb{G}$. Moreover, the ultraweak topology on $\widehat{A}$ is independent of W.

Repeating the above reasoning for the quantum group $\widehat{\mathbb{G}}$ we see that the ultraweak topology on A (which is $\widehat{\widehat{A}}$) is independent of the modular multiplicative unitary giving rise to G. This proves statement (1) of Theorem 5.

We have already shown that the reduced dual $\widehat{\mathbb{G}}$ is independent of the choice of modular multiplicative unitary giving rise to $\mathbb{G}$. The position of W in $\mathrm{M}(\widehat{A} \otimes A)$ is also fixed uniquely. Indeed, for any right absorbing representation U we have the isomorphism $\Phi \in \mathrm{Mor}(B_U, \widehat{A})$ satisfying (4.12). This proves statement (3) of Theorem 5.

Statement (2) of Theorem 5 follows from statements (1) and (3). To see this notice that given a modular multiplicative unitary $W \in \mathrm{B}(\mathcal{H} \otimes \mathcal{H})$, the core of κ is determined by the ultraweak topology inherited by $\widehat{A}$ from $\mathrm{B}(\mathcal{H})$. Now this topology is independent of W while the action of κ on this core depends only on the position of W in $\mathrm{M}(\widehat{A} \otimes A)$ (cf. Theorem 2(4)). The uniqueness of the polar decomposition of κ guarantees that R and $(\tau_t)_{t \in \mathbb{R}}$ are independent of the choice of modular multiplicative unitary giving rise to $\mathbb{G}$.

From now on we shall write A_* for the space of functionals on A continuous for the ultraweak topology on A coming from representation of A defined by any modular multiplicative unitary. The image of any right absorbing representation in $\mathrm{M}(\widehat{A} \otimes A)$ will be called *the reduced bicharacter* for $(\mathbb{G}, \widehat{\mathbb{G}})$. In what follows the reduced bicharacter will be denoted by the letter W. By a *realization* of W on a Hilbert space $\mathcal{H}$ we shall mean any modular multiplicative unitary acting on $\mathcal{H} \otimes \mathcal{H}$ giving rise to $\mathbb{G}$.

5. Universal dual of a quantum group

The aim of this section is to define and analyze the universal dual object of a given quantum group $\mathbb{G} = (A, \Delta)$. Such objects were already considered in [15, Section 3] under the name "Pontryagin dual."

5.1. Maximal representations and universal C*-*algebra*

Proposition 22. *There exists a strongly continuous representation $\mathbb{W}$ of $\mathbb{G}$ such that for any $U \in \mathrm{Rep}(\mathbb{G})$ and any $\omega \in A_*$ we have*

$$\left\|(\mathrm{id}\otimes\omega)\mathbb{W}\right\| \geqslant \left\|(\mathrm{id}\otimes\omega)U\right\|. \tag{5.13}$$

Proof. Let $(\omega_n)_{n\in\mathbb{N}}$ be a sequence of elements of A_* which is dense in A_* and each element is repeated infinitely many times. For any $n \in \mathbb{N}$ there exists a representation U_n of $\mathbb{G}$ such that

$$\left\|(\mathrm{id}\otimes\omega_n)U_n\right\| \geqslant \sup_U \left\|(\mathrm{id}\otimes\omega_n)U\right\| - \frac{1}{n}$$

where the supremum is taken over all strongly continuous unitary representations of $\mathbb{G}$.

We define $\mathbb{W}$ to be the direct sum of $(U_n)_{n\in\mathbb{N}}$. Formula (5.13) follows immediately from the definition of $\mathbb{W}$. Indeed, given a representation U of $\mathbb{G}$, $\omega \in A_*$ and $\varepsilon > 0$ we can find n such that $\|\omega - \omega_n\| < \frac{\varepsilon}{3}$ and $n > \frac{3}{\varepsilon}$. Then

$$\left\|(\mathrm{id}\otimes\omega_n)U_n\right\| \geqslant \left\|(\mathrm{id}\otimes\omega_n)U\right\| - \frac{\varepsilon}{3}$$

and

$$\begin{aligned}
\left|\left\|(\mathrm{id}\otimes\omega)\mathbb{W}\right\| - \left\|(\mathrm{id}\otimes\omega_n)\mathbb{W}\right\|\right| &\leqslant \left\|(\mathrm{id}\otimes\omega)\mathbb{W} - (\mathrm{id}\otimes\omega_n)\mathbb{W}\right\| < \frac{\varepsilon}{3},\\
\left|\left\|(\mathrm{id}\otimes\omega)U\right\| - \left\|(\mathrm{id}\otimes\omega_n)U\right\|\right| &\leqslant \left\|(\mathrm{id}\otimes\omega_n)U - (\mathrm{id}\otimes\omega)U\right\| < \frac{\varepsilon}{3}.
\end{aligned}$$

Therefore

$$\begin{aligned}
\left\|(\mathrm{id}\otimes\omega)\mathbb{W}\right\| &\geqslant \left\|(\mathrm{id}\otimes\omega_n)\mathbb{W}\right\| - \frac{\varepsilon}{3} \geqslant \left\|(\mathrm{id}\otimes\omega_n)U_n\right\| - \frac{\varepsilon}{3}\\
&\geqslant \left\|(\mathrm{id}\otimes\omega_n)U\right\| - \frac{2\varepsilon}{3} \geqslant \left\|(\mathrm{id}\otimes\omega)U\right\| - \varepsilon
\end{aligned}$$

(cf. Remark 9). □

Definition 23. Let $\mathbb{G}$ be a quantum group. A representation $\mathbb{W}$ fulfilling the condition of Proposition 22 is called *maximal*.

Lemma 24. *Let $\mathbb{W}$ be a maximal representation of $\mathbb{G}$ and let $V \in \mathrm{Rep}(\mathbb{G})$. If $\Phi \in \mathrm{Mor}(B_V, B_{\mathbb{W}})$ is such that $(\Phi\otimes\mathrm{id})V = \mathbb{W}$ then Φ is an isomorphism.*

Proof. The C*-algebras B_V and $B_{\mathbb{W}}$ are closures of the sets of right slices of V and $\mathbb{W}$, respectively. Therefore Φ maps a dense subset of B_V onto a dense subset of $B_{\mathbb{W}}$. By the maximality of $\mathbb{W}$ the map Φ increases norm:

$$\left\|\Phi\big((\mathrm{id}\otimes\omega)V\big)\right\| = \left\|(\mathrm{id}\otimes\omega)\mathbb{W}\right\| \geqslant \left\|(\mathrm{id}\otimes\omega)V\right\|$$

for any $\omega\in A_*$. It follows that Φ is isometric and consequently an isomorphism. □

Theorem 25. *Let $\mathbb{W}$ be a maximal representation of $\mathbb{G}$ and let*

$$\widehat{A}_{\mathrm{u}} = \big\{(\mathrm{id}\otimes\omega)\mathbb{W}\colon\ \omega\in A_*\big\}^{\|\cdot\|\text{-closure}}. \tag{5.14}$$

Then

(1) $\widehat{A}_{\mathrm{u}}$ *is a non-degenerate separable* C*-*subalgebra of* $\mathrm{B}(H_{\mathbb{W}})$ *and* $\mathbb{W}\in\mathrm{M}(\widehat{A}_{\mathrm{u}}\otimes A)$.
(2) *For any representation U of $\mathbb{G}$ there exists a unique $\Phi_U\in\mathrm{Mor}(\widehat{A}_{\mathrm{u}}, B_U)$ such that*

$$(\Phi_U\otimes\mathrm{id})\mathbb{W} = U. \tag{5.15}$$

(3) *For any pair $(B,\mathbb{U})$ such that $\mathbb{U}\in\mathrm{Rep}(\mathbb{G})$ and B is a non-degenerate* C*-*subalgebra of $\mathrm{B}(H_{\mathbb{U}})$ such that $\mathbb{U}\in\mathrm{M}(B\otimes A)$ and such that for any $U\in\mathrm{Rep}(\mathbb{G})$ there exists a unique $\Phi_U\in\mathrm{Mor}(B,B_U)$ such that $(\Phi_U\otimes\mathrm{id})\mathbb{U}=U$, there exists an isomorphism $\Psi\in\mathrm{Mor}(\widehat{A}_{\mathrm{u}}, B)$ such that $(\Psi\otimes\mathrm{id})\mathbb{W}=\mathbb{U}$.*

Proof. Clearly we have $\widehat{A}_{\mathrm{u}} = B_{\mathbb{W}}$ and so statement (1) is just a reformulation of the remarks at the beginning of Section 3.5 (cf. [20, Theorem 1.6]).

AD (2). To prove existence of Φ_U notice that both U and $\mathbb{W}$ are subrepresentations of $U\oplus\mathbb{W}$. Therefore, by Proposition 14(1) there exist $\Phi_1\in\mathrm{Mor}(B_{U\oplus\mathbb{W}}, B_{\mathbb{W}})$ and $\Phi_2\in\mathrm{Mor}(B_{U\oplus\mathbb{W}}, B_U)$ such that

$$\mathbb{W} = (\Phi_1\otimes\mathrm{id})(U\oplus\mathbb{W}),\qquad U = (\Phi_2\otimes\mathrm{id})(U\oplus\mathbb{W}).$$

By Lemma 24 the morphism Φ_1 is an isomorphism. Then

$$U = \big(\big[\Phi_2\circ\Phi_1^{-1}\big]\otimes\mathrm{id}\big)\mathbb{W}.$$

We let $\Phi_U = \Phi_2\circ\Phi_1^{-1}$.

Uniqueness of Φ_U is clear: if $\Phi'\in\mathrm{Mor}(\widehat{A}_{\mathrm{u}}, B_U)$ satisfies

$$(\Phi'\otimes\mathrm{id})\mathbb{W} = U$$

then for any $\omega\in A_*$ we have

$$\Phi'\big((\mathrm{id}\otimes\omega)\mathbb{W}\big) = (\mathrm{id}\otimes\omega)U = \Phi_U\big((\mathrm{id}\otimes\omega)\mathbb{W}\big).$$

Thus, in view of (5.14) we have $\Phi' = \Phi_U$.

AD (3). This is a standard consequence of the universal property of $(\widehat{A}_{\mathrm{u}}, \mathbb{W})$. □

Remark 26. Note that the unique morphism Φ_U described in Theorem 25(2) is a surjection onto B_U. In particular its image does not contain multipliers of B_U which are not in B_U (cf. the proof of Lemma 24).

Proposition 27. *Let* $\mathbb{W}$ *be a maximal representation of* $\mathbb{G}$*. Then for any* C^**-algebra* D *and any unitary* $U \in \mathrm{M}(D \otimes A)$ *such that* $(\mathrm{id} \otimes \Delta)U = U_{12}U_{13}$ *there exists a unique* $\Phi_U \in \mathrm{Mor}(\widehat{A}_{\mathrm{u}}, D)$ *such that* $(\Phi_U \otimes \mathrm{id})\mathbb{W} = U$.

Proof. We can assume that the C^*-algebra D is faithfully and non-degenerately represented on a Hilbert space H_U. Then $U \in \mathrm{M}(\mathcal{K}(H_U) \otimes A)$ is a representation of $\mathbb{G}$ and by Theorem 25(2) there exists a unique $\pi_U \in \mathrm{Mor}(\widehat{A}_{\mathrm{u}}, B_U)$ such that (5.15) holds. Since B_U is generated by U and $U \in \mathrm{M}(D \otimes A)$, the identity map is a morphism from B_U to D. Moreover this is the only morphism from B_U to D which leaves U unchanged. □

Corollary 28. *For any* $U \in \mathrm{Rep}(\mathbb{G})$ *there exists a unique non-degenerate representation* π_U *of* $\widehat{A}_{\mathrm{u}}$ *on the Hilbert space* H_U *such that*

$$(\pi_U \otimes \mathrm{id})\mathbb{W} = U.$$

Moreover the association $U \leftrightarrow \pi_U$ *establishes a bijective correspondence between* $\mathrm{Rep}(\mathbb{G})$ *and the class of all non-degenerate representations of the* C^**-algebra* $\widehat{A}_{\mathrm{u}}$.

Proof. Let $U \in \mathrm{Rep}(\mathbb{G})$. Then $U \in \mathrm{M}(\mathcal{K}(H_U) \otimes A)$ and by Proposition 27 there exists a unique $\Phi_U \in \mathrm{Mor}(\widehat{A}_{\mathrm{u}}, \mathcal{K}(H_U))$ such that we have (5.15). We let π_U be Φ_U considered as a map from $\widehat{A}_{\mathrm{u}}$ to $\mathrm{B}(H_U)$. Of course π_U is a non-degenerate representation of $\widehat{A}_{\mathrm{u}}$.

Conversely, for any non-degenerate representation π of $\widehat{A}_{\mathrm{u}}$ on a Hilbert space H, the unitary element $U = (\pi \otimes \mathrm{id})\mathbb{W} \in \mathrm{M}(\mathcal{K}(H) \otimes A)$ is a strongly continuous unitary representation of $\mathbb{G}$. □

Remark 29. Let us note that the correspondence between representations of $\mathbb{G}$ and representations of $\widehat{A}_{\mathrm{u}}$ described in Corollary 28 is a functor from the W^*-category $\mathrm{Rep}(\mathbb{G})$ to the W^*-category of non-degenerate representations of $\widehat{A}_{\mathrm{u}}$. In fact it is an equivalence of categories preserving direct sums and tensor products.

Definition 30. The C^*-algebra $\widehat{A}_{\mathrm{u}}$ defined in Theorem 25 is called the *universal quantum group* C^**-algebra* of $\mathbb{G}$. The representation $\mathbb{W} \in \mathrm{M}(\widehat{A}_{\mathrm{u}} \otimes A)$ is called the *universal representation* of $\mathbb{G}$.

5.2. The universal dual

Proposition 31. *Let* $\widehat{A}_{\mathrm{u}}$ *be the universal quantum group* C^**-algebra of* $\mathbb{G}$ *and let* $\mathbb{W} \in \mathrm{M}(\widehat{A}_{\mathrm{u}} \otimes A)$ *be the universal representation. Then*

(1) *There exists a unique* $\widehat{\Delta}_{\mathrm{u}} \in \mathrm{Mor}(\widehat{A}_{\mathrm{u}}, \widehat{A}_{\mathrm{u}} \otimes \widehat{A}_{\mathrm{u}})$ *such that*

$$(\widehat{\Delta}_{\mathrm{u}} \otimes \mathrm{id})\mathbb{W} = \mathbb{W}_{23}\mathbb{W}_{13}. \tag{5.16}$$

The morphism $\widehat{\Delta}_{\mathrm{u}}$ *is coassociative and*

$$\left\{\widehat{\Delta}_{\mathrm{u}}(x)(I_{\widehat{A}_{\mathrm{u}}}\otimes y):\ x,y\in\widehat{A}_{\mathrm{u}}\right\},\qquad \left\{(x\otimes I_{\widehat{A}_{\mathrm{u}}})\widehat{\Delta}_{\mathrm{u}}(y):\ x,y\in\widehat{A}_{\mathrm{u}}\right\} \tag{5.17}$$

are linearly dense subsets of $\widehat{A}_{\mathrm{u}}\otimes\widehat{A}_{\mathrm{u}}$.

(2) *There exists a unique* $\hat{e}^{\mathrm{u}}\in\mathrm{Mor}(\widehat{A}_{\mathrm{u}},\mathbb{C})$ *such that*

$$\left(\hat{e}^{\mathrm{u}}\otimes\mathrm{id}\right)\mathbb{W}=I_A. \tag{5.18}$$

The morphism $\hat{e}^{\mathrm{u}}$ *has the following property*:

$$\left(\mathrm{id}\otimes\hat{e}^{\mathrm{u}}\right)\circ\widehat{\Delta}_{\mathrm{u}}=\left(\hat{e}^{\mathrm{u}}\otimes\mathrm{id}\right)\circ\widehat{\Delta}_{\mathrm{u}}=\mathrm{id}. \tag{5.19}$$

Proof. The unitary element

$$\begin{aligned}\mathbb{W}_{23}\mathbb{W}_{13}&\in\mathrm{M}(\widehat{A}_{\mathrm{u}}\otimes\widehat{A}_{\mathrm{u}}\otimes A)\\&\subset\mathrm{M}\big(\mathcal{K}(H_{\mathbb{W}})\otimes\mathcal{K}(H_{\mathbb{W}})\otimes A\big)=\mathrm{M}\big(\mathcal{K}(H_{\mathbb{W}}\otimes H_{\mathbb{W}})\otimes A\big)\end{aligned}$$

is a strongly continuous unitary representation of $\mathbb{G}$. Moreover $\mathbb{W}_{23}\mathbb{W}_{13}$ is a quantum family of elements affiliated with $B_{\mathbb{W}_{23}\mathbb{W}_{13}}$ generating this algebra. Therefore the identity map is a morphism from $B_{\mathbb{W}_{23}\mathbb{W}_{13}}$ to $\widehat{A}_{\mathrm{u}}\otimes\widehat{A}_{\mathrm{u}}$.

By the universal property of $(\widehat{A}_{\mathrm{u}},\mathbb{W})$, there exists a unique Morphism $\Phi\in\mathrm{Mor}(\widehat{A}_{\mathrm{u}},B_{\mathbb{W}_{23}\mathbb{W}_{13}})$ such that

$$(\Phi\otimes\mathrm{id})\mathbb{W}=\mathbb{W}_{23}\mathbb{W}_{13}.$$

Let $\widehat{\Delta}_{\mathrm{u}}$ be the composition of Φ with the identity on $B_{\mathbb{W}_{23}\mathbb{W}_{13}}$ considered as a morphism from $B_{\mathbb{W}_{23}\mathbb{W}_{13}}$ to $\widehat{A}_{\mathrm{u}}\otimes\widehat{A}_{\mathrm{u}}$. Then $\widehat{\Delta}_{\mathrm{u}}\in\mathrm{Mor}(\widehat{A}_{\mathrm{u}},\widehat{A}_{\mathrm{u}}\otimes\widehat{A}_{\mathrm{u}})$ satisfies (5.16).

To obtain coassociativity of $\widehat{\Delta}_{\mathrm{u}}$ we compute:

$$\begin{aligned}\big(\big[(\widehat{\Delta}_{\mathrm{u}}\otimes\mathrm{id})\circ\widehat{\Delta}_{\mathrm{u}}\big]\otimes\mathrm{id}\big)\mathbb{W}&=(\widehat{\Delta}_{\mathrm{u}}\otimes\mathrm{id}\otimes\mathrm{id})(\widehat{\Delta}_{\mathrm{u}}\otimes\mathrm{id})\mathbb{W}\\&=(\widehat{\Delta}_{\mathrm{u}}\otimes\mathrm{id}\otimes\mathrm{id})(\mathbb{W}_{23}\mathbb{W}_{13})\\&=\mathbb{W}_{34}\mathbb{W}_{24}\mathbb{W}_{14}\\&=(\mathrm{id}\otimes\widehat{\Delta}_{\mathrm{u}}\otimes\mathrm{id})(\mathbb{W}_{23}\mathbb{W}_{13})\\&=(\mathrm{id}\otimes\widehat{\Delta}_{\mathrm{u}}\otimes\mathrm{id})(\widehat{\Delta}_{\mathrm{u}}\otimes\mathrm{id})\mathbb{W}\\&=\big(\big[(\mathrm{id}\otimes\widehat{\Delta}_{\mathrm{u}})\circ\widehat{\Delta}_{\mathrm{u}}\big]\otimes\mathrm{id}\big)\mathbb{W}.\end{aligned}$$

Now we can take right slice with any $\omega\in A_*$ and coassociativity of $\widehat{\Delta}_{\mathrm{u}}$ follows. The fact that the sets (5.17) are contained in $\widehat{A}_{\mathrm{u}}\otimes\widehat{A}_{\mathrm{u}}$ and their linear density of in $\widehat{A}_{\mathrm{u}}\otimes\widehat{A}_{\mathrm{u}}$ is proved in the same way as [20, Proposition 5.1] (the crucial ingredient being (5.16)).

AD (2). Take $U=1\otimes I_A\in\mathrm{M}(\mathcal{K}(\mathbb{C})\otimes A)$. Then U is a strongly continuous unitary representation of $\mathbb{G}$ and by the universal property of $(\widehat{A}_{\mathrm{u}},\mathbb{W})$, there exists a unique $\hat{e}^{\mathrm{u}}\in\mathrm{Mor}(\widehat{A}_{\mathrm{u}},\mathbb{C})$ satisfying (5.18). Notice that it follows from (5.16) and (5.18) that

$$\begin{aligned}\big(\big[\big(\mathrm{id}\otimes\hat{e}^{\mathrm{u}}\big)\widehat{\Delta}_{\mathrm{u}}\big]\otimes\mathrm{id}\big)\mathbb{W}&=\big(\mathrm{id}\otimes\hat{e}^{\mathrm{u}}\otimes\mathrm{id}\big)(\widehat{\Delta}_{\mathrm{u}}\otimes\mathrm{id})\mathbb{W}\\&=\big(\mathrm{id}\otimes\hat{e}^{\mathrm{u}}\otimes\mathrm{id}\big)(\mathbb{W}_{23}\mathbb{W}_{13})\\&=\big(I_{\widehat{A}_{\mathrm{u}}}\otimes\big[\big(\hat{e}^{\mathrm{u}}\otimes\mathrm{id}\big)\mathbb{W}\big]\big)\mathbb{W}=\mathbb{W}.\end{aligned}$$

In particular, for any $\omega\in A_*$ we obtain

$$\big(\hat{e}^{\mathrm{u}}\otimes\mathrm{id}\big)\widehat{\Delta}_{\mathrm{u}}\big((\mathrm{id}\otimes\omega)\mathbb{W}\big)=(\mathrm{id}\otimes\omega)\big(\big[\big(\mathrm{id}\otimes\hat{e}^{\mathrm{u}}\big)\widehat{\Delta}_{\mathrm{u}}\big]\otimes\mathrm{id}\big)\mathbb{W}=(\mathrm{id}\otimes\omega)\mathbb{W}$$

and the first part of (5.19) follows. The second part is proved analogously. $\square$

Definition 32. Let $\widehat{A}_{\mathrm{u}}$ be the universal quantum group C*-algebra of $\mathbb{G}$ and let $\widehat{\Delta}_{\mathrm{u}}$ be the morphism defined in Proposition 31(1). The pair $(\widehat{A}_{\mathrm{u}},\widehat{\Delta}_{\mathrm{u}})$ will be called the *universal dual* of $\mathbb{G}$.

Remark 33. The universal dual of a quantum group is not, in general, a quantum group. Nevertheless, as we will see, it retains a lot of structure, such as the coinverse, scaling group and unitary coinverse.

Proposition 34. *Let $W\in\mathrm{M}(\widehat{A}\otimes A)$ be the reduced bicharacter for $\mathbb{G}$ and $\widehat{\mathbb{G}}=(\widehat{A},\widehat{\Delta})$. Then there exists a unique $\widehat{\Lambda}\in\mathrm{Mor}(\widehat{A}_{\mathrm{u}},\widehat{A})$ such that*

$$(\widehat{\Lambda}\otimes\mathrm{id})\mathbb{W}=W.$$

The morphism $\widehat{\Lambda}$ satisfies

$$(\widehat{\Lambda}\otimes\widehat{\Lambda})\circ\widehat{\Delta}_{\mathrm{u}}=\widehat{\Delta}\circ\widehat{\Lambda}.\tag{5.20}$$

Proof. Existence and uniqueness of $\widehat{\Lambda}$ follows from Theorem 25(2).

To prove property (5.20), notice first that

$$\begin{aligned}(\widehat{\Delta}\otimes\mathrm{id})(\widehat{\Lambda}\otimes\mathrm{id})\mathbb{W}&=(\widehat{\Delta}\otimes\mathrm{id})W=W_{23}W_{13}\\&=\big[(\widehat{\Lambda}\otimes\mathrm{id})\mathbb{W}\big]_{23}\big[(\widehat{\Lambda}\otimes\mathrm{id})\mathbb{W}\big]_{13}\\&=(\widehat{\Lambda}\otimes\widehat{\Lambda}\otimes\mathrm{id})(\mathbb{W}_{23}\mathbb{W}_{13})\\&=(\widehat{\Lambda}\otimes\widehat{\Lambda}\otimes\mathrm{id})(\widehat{\Delta}_{\mathrm{u}}\otimes\mathrm{id})\mathbb{W}.\end{aligned}$$

Therefore for $\omega\in A_*$ we have

$$\begin{aligned}\widehat{\Delta}\big(\widehat{\Lambda}\big((\mathrm{id}\otimes\omega)\mathbb{W}\big)\big)&=(\mathrm{id}\otimes\mathrm{id}\otimes\omega)(\widehat{\Delta}\otimes\mathrm{id})(\widehat{\Lambda}\otimes\mathrm{id})\mathbb{W}\\&=(\mathrm{id}\otimes\mathrm{id}\otimes\omega)(\widehat{\Lambda}\otimes\widehat{\Lambda}\otimes\mathrm{id})(\widehat{\Delta}_{\mathrm{u}}\otimes\mathrm{id})\mathbb{W}\\&=(\widehat{\Lambda}\otimes\widehat{\Lambda})\widehat{\Delta}_{\mathrm{u}}\big((\mathrm{id}\otimes\omega)\mathbb{W}\big).\qquad\square\end{aligned}$$

Definition 35. The unique $\widehat{\Lambda}\in\mathrm{Mor}(\widehat{A}_{\mathrm{u}},\widehat{A})$ defined in Proposition 34 is called the *reducing morphism.*

Remark 36. The reducing morphism clearly plays the role analogous to the regular representation of a group C*-algebra. We chose to name it differently in order not to confuse it with the established notion of a regular representation of a locally compact quantum group ([8]).

Proposition 37. *Assume that there exists a character $\hat{e}$ of $\widehat{A}$ such that*

$$(\mathrm{id} \otimes \hat{e})\widehat{\Delta} = \mathrm{id} \tag{5.21}$$

and let $W \in \mathrm{M}(\widehat{A} \otimes A)$ be the reduced bicharacter. Then

(1) $(\hat{e} \otimes \mathrm{id})W = I_A$.
(2) $\hat{e}^{\mathrm{u}} = \hat{e} \circ \widehat{\Lambda}$.
(3) $\widehat{\Lambda}$ *is an isomorphism.*

Proof. Statement (1) is a consequence of

$$\begin{aligned} W &= (\mathrm{id} \otimes \hat{e} \otimes \mathrm{id})(\widehat{\Delta} \otimes \mathrm{id})W \\ &= (\mathrm{id} \otimes \hat{e} \otimes \mathrm{id})(W_{23}W_{13}) = \bigl(I_{\widehat{A}} \otimes \bigl[(\mathrm{id} \otimes \hat{e})W\bigr]\bigr)W \end{aligned}$$

and the unitarity of W.

Once this is established, (2) follows because

$$\bigl([\hat{e} \circ \widehat{\Lambda}] \otimes \mathrm{id}\bigr)\mathbb{W} = (\hat{e} \otimes \mathrm{id})(\widehat{\Lambda} \otimes \mathrm{id})\mathbb{W} = (\hat{e} \otimes \mathrm{id})W = I_A$$

which is the defining property of $\hat{e}^{\mathrm{u}}$.

We will now show that the reduced bicharacter is a maximal representation of $\mathbb{G}$. Then $\widehat{\Lambda}$ will be an isomorphism by Lemma 24. The argument used here has already appeared in [4, p. 177]. We will use it in the version similar to that of [3, p. 875].

The first observation is that it follows from the formula in statement (1) that for any $\omega \in A_*$ we have

$$\bigl|\omega(I_A)\bigr| = \bigl|\hat{e}\bigl((\mathrm{id} \otimes \omega)W\bigr)\bigr| \leqslant \bigl\|(\mathrm{id} \otimes \omega)W\bigr\|. \tag{5.22}$$

Let $U \in \mathrm{Rep}(\mathbb{G})$ and let us realize the reduced bicharacter W as a manageable multiplicative unitary on a Hilbert space $\mathcal{H}$. By [20, Theorem 1.7] we have $W_{23}U_{12}W_{23}^* = U_{12}U_{13}$ as elements of $\mathrm{M}(\mathcal{K}(H_U) \otimes \mathcal{K}(\mathcal{H}) \otimes A)$. Let $\widehat{U} = \sigma_{\mathcal{K}(H_U),A}(U)^* \in \mathrm{M}(A \otimes \mathcal{K}(H_U))$, where $\sigma_{\mathcal{K}(H_U),A} \in \mathrm{Mor}(\mathcal{K}(H_U) \otimes A, A \otimes \mathcal{K}(H_U))$ is the flip. It follows that

$$\widehat{U}_{23}^* W_{12} = \widehat{U}_{13} W_{12} \widehat{U}_{13}^*.$$

For each $\omega \in A_*$ and $\eta \in \mathrm{B}(H_U)_*$ we define $\omega_\eta \in A_*$ by

$$\omega_\eta(x) = (\omega \otimes \eta)\bigl(\widehat{U}^*(x \otimes I_{H_U})\bigr).$$

We have

$$\begin{aligned}(\mathrm{id}\otimes\omega_\eta)W &= (\mathrm{id}\otimes\omega\otimes\eta)\big(\widehat{U}_{23}^*W_{12}\big)\\ &= (\mathrm{id}\otimes\omega\otimes\eta)\big(\widehat{U}_{13}W_{12}\widehat{U}_{13}^*\big)\\ &= (\mathrm{id}\otimes\eta)\big[\widehat{U}\big[\big((\mathrm{id}\otimes\omega)W\big)\otimes I\big]\widehat{U}^*\big].\end{aligned}\tag{5.23}$$

Also

$$\begin{aligned}\eta\big((\mathrm{id}\otimes\omega)U\big) &= \eta\big((\omega\otimes\mathrm{id})\big(\widehat{U}^*\big)\big)\\ &= (\omega\otimes\eta)\big(\widehat{U}^*\big) = \omega_\eta(I_A).\end{aligned}\tag{5.24}$$

Now using (5.24), (5.22) and (5.23) we have

$$\begin{aligned}\big|\eta\big((\mathrm{id}\otimes\omega)U\big)\big| &= \big|\omega_\eta(I_A)\big| \leqslant \big\|(\mathrm{id}\otimes\omega_\eta)W\big\|\\ &= \big\|(\mathrm{id}\otimes\eta)\big[\widehat{U}\big[\big((\mathrm{id}\otimes\omega)W\big)\otimes I\big]\widehat{U}^*\big]\big\| \leqslant \|\eta\|\,\big\|(\mathrm{id}\otimes\omega)W\big\|.\end{aligned}$$

Since for any $t\in \mathrm{B}(H_U)$ we have $\|t\| = \sup\{|\eta(t)|\colon\ \eta\in\mathrm{B}(H_U)_*,\ \|\eta\|=1\}$, we conclude that for any $\omega\in A_*$

$$\big\|(\mathrm{id}\otimes\omega)U\big\| \leqslant \big\|(\mathrm{id}\otimes\omega)W\big\|. \qquad \square$$

Remark 38. The assumption (5.21) in Proposition 37 is in fact equivalent to the formula in statement (1) of that proposition (cf. the proof of Proposition 31(2)).

Proposition 39. *Let $(\widehat{A}_{\mathrm{u}},\mathbb{W})$ be the universal quantum group* C*-*algebra of and the universal representation of $\mathbb{G}$. Then*

(1) *for any $t\in\mathbb{R}$ there exists a unique $\hat{\tau}_t^{\mathrm{u}}\in\mathrm{Mor}(\widehat{A}_{\mathrm{u}},\widehat{A}_{\mathrm{u}})$ such that*

$$\big(\hat{\tau}_t^{\mathrm{u}}\otimes\mathrm{id}\big)\mathbb{W} = (\mathrm{id}\otimes\tau_{-t})\mathbb{W}.\tag{5.25}$$

Moreover $(\hat{\tau}_t^{\mathrm{u}})_{t\in\mathbb{R}}$ is a one parameter group of automorphisms of $\widehat{A}_{\mathrm{u}}$.

(2) *For any $x\in\widehat{A}_{\mathrm{u}}$ the map $\mathbb{R}\ni t\mapsto\hat{\tau}_t^{\mathrm{u}}(x)\in\widehat{A}_{\mathrm{u}}$ is continuous.*

(3) *For any $t\in\mathbb{R}$ we have*

$$\big(\hat{\tau}_t^{\mathrm{u}}\otimes\hat{\tau}_t^{\mathrm{u}}\big)\circ\widehat{\Delta}_{\mathrm{u}} = \widehat{\Delta}_{\mathrm{u}}\circ\hat{\tau}_t^{\mathrm{u}}\tag{5.26}$$

and $\hat{e}^{\mathrm{u}}\circ\hat{\tau}_t^{\mathrm{u}} = \hat{e}^{\mathrm{u}}$.

(4) *If $(\hat{\tau}_t)_{t\in\mathbb{R}}$ is the scaling group of $\widehat{\mathbb{G}}$ and $\widehat{\Lambda}\in\mathrm{Mor}(\widehat{A}_{\mathrm{u}},\widehat{A})$ is the reducing morphism then for any $t\in\mathbb{R}$ we have*

$$\hat{\tau}_t\circ\widehat{\Lambda} = \widehat{\Lambda}\circ\hat{\tau}_t^{\mathrm{u}}.\tag{5.27}$$

Proof. AD (1). For any $t\in\mathbb{R}$ the element $(\mathrm{id}\otimes\tau_{-t})\mathbb{W}\in\mathrm{M}(\mathcal{K}(H_{\mathbb{W}})\otimes A)$ is a representation of $\mathbb{G}$. Therefore, by the universal property of $(\widehat{A}_{\mathrm{u}},\mathbb{W})$, there exists a unique $\hat{\tau}_t^{\mathrm{u}}\in\mathrm{Mor}(\widehat{A}_{\mathrm{u}},\widehat{A}_{\mathrm{u}})$ such that (5.25) holds. It is easy to see that $(\hat{\tau}_t^{\mathrm{u}})_{t\in\mathbb{R}}$ is a one parameter group of automorphisms of $\widehat{A}_{\mathrm{u}}$.

AD (2). Take $x \in \widehat{A}_{\mathrm{u}}$ and $\varepsilon > 0$. There exists a functional $\omega \in A_*$ such that

$$\|x - (\mathrm{id} \otimes \omega)\mathbb{W}\| < \frac{\varepsilon}{3}$$

and the map $\mathbb{R} \ni t \mapsto \omega \circ \tau_{-t}$ is norm continuous. Therefore for $t, s \in \mathbb{R}$

$$\begin{aligned}\|\hat{\tau}^{\mathrm{u}}_t(x) - \hat{\tau}^{\mathrm{u}}_s(x)\| \leqslant{}& \|\hat{\tau}^{\mathrm{u}}_t(x) - \hat{\tau}^{\mathrm{u}}_t\big((\mathrm{id} \otimes \omega)\mathbb{W}\big)\| \\ &+ \|\hat{\tau}^{\mathrm{u}}_t\big((\mathrm{id} \otimes \omega)\mathbb{W}\big) - \hat{\tau}^{\mathrm{u}}_s\big((\mathrm{id} \otimes \omega)\mathbb{W}\big)\| \\ &+ \|\hat{\tau}^{\mathrm{u}}_s\big((\mathrm{id} \otimes \omega)\mathbb{W}\big) - \hat{\tau}^{\mathrm{u}}_s(x)\|.\end{aligned}$$

The first and third terms are each smaller than $\frac{\varepsilon}{3}$ and the middle term

$$\begin{aligned}&\|\hat{\tau}^{\mathrm{u}}_t\big((\mathrm{id} \otimes \omega)\mathbb{W}\big) - \hat{\tau}^{\mathrm{u}}_s\big((\mathrm{id} \otimes \omega)\mathbb{W}\big)\| \\ &\quad = \|(\mathrm{id} \otimes \omega)\big((\hat{\tau}^{\mathrm{u}}_t \otimes \mathrm{id})\mathbb{W} - (\hat{\tau}^{\mathrm{u}}_s \otimes \mathrm{id})\mathbb{W}\big)\| \\ &\quad = \|(\mathrm{id} \otimes \omega)\big((\mathrm{id} \otimes \tau_{-t})\mathbb{W} - (\mathrm{id} \otimes \tau_{-s})\mathbb{W}\big)\| \\ &\quad = \|\big(\mathrm{id} \otimes [\omega \circ \tau_{-t} - \omega \circ \tau_{-s}]\big)\mathbb{W}\| \\ &\quad \leqslant \|\omega \circ \tau_{-t} - \omega \circ \tau_{-s}\|\end{aligned}$$

is smaller than $\frac{\varepsilon}{3}$ for s sufficiently close to t.

AD (3). Just as in the proof of formula (5.20) (Proposition 34) we can easily show that

$$\big((\widehat{\Delta}_{\mathrm{u}} \circ \hat{\tau}^{\mathrm{u}}_t) \otimes \mathrm{id}\big)\mathbb{W} = \big(\big[(\hat{\tau}^{\mathrm{u}}_t \otimes \hat{\tau}^{\mathrm{u}}_t) \circ \widehat{\Delta}_{\mathrm{u}}\big] \otimes \mathrm{id}\big)\mathbb{W}$$

and (5.26) follows. Similarly for any $t \in \mathbb{R}$ and $\omega \in A_*$

$$\begin{aligned}\hat{e}^{\mathrm{u}}\big(\hat{\tau}^{\mathrm{u}}_t\big((\mathrm{id} \otimes \omega)\mathbb{W}\big)\big) &= \hat{e}^{\mathrm{u}}\big((\mathrm{id} \otimes \omega)\big(\hat{\tau}^{\mathrm{u}}_t \otimes \mathrm{id}\big)\mathbb{W}\big) \\ &= \hat{e}^{\mathrm{u}}\big((\mathrm{id} \otimes \omega)(\mathrm{id} \otimes \tau_{-t})\mathbb{W}\big) \\ &= \omega\big(\tau_{-t}\big(\big(\hat{e}^{\mathrm{u}} \otimes \mathrm{id}\big)\mathbb{W}\big)\big) \\ &= \omega\big(\tau_{-t}(I_A)\big) = \omega(I_A) \\ &= \omega\big(\big(\hat{e}^{\mathrm{u}} \otimes \mathrm{id}\big)\mathbb{W}\big) \\ &= \hat{e}^{\mathrm{u}}\big((\mathrm{id} \otimes \omega)\mathbb{W}\big)\end{aligned}$$

proves the other formula.

AD (4). Let W be the reduced bicharacter for $(\mathbb{G}, \widehat{\mathbb{G}})$. First let us see that for any $t \in \mathbb{R}$ we have

$$(\hat{\tau}_t \otimes \mathrm{id})W = (\mathrm{id} \otimes \tau_{-t})W. \tag{5.28}$$

Indeed, we can realize W as a modular multiplicative unitary on some Hilbert space $\mathcal{H}$. Then ([13,20]) the scaling group of $\mathbb{G}$ is given by $\tau_t(a) = Q^{2it} a Q^{-2it}$ where Q is one of the two positive self adjoint operators appearing in the definition of modularity of W. Similarly the scaling group of $\widehat{\mathbb{G}}$ is $\hat{\tau}_t(x) = \widehat{Q}^{2it} x \widehat{Q}^{-2it}$, where $\widehat{Q}$ is the other positive self-adjoint operator. Now since W commutes with $\widehat{Q} \otimes Q$, we obtain (5.28).

Now for $t \in \mathbb{R}$ we can compute

$$\begin{aligned}(\widehat{\Lambda} \otimes \mathrm{id})\big(\hat{\tau}_t^{\mathrm{u}} \otimes \mathrm{id}\big)\mathbb{W} &= (\widehat{\Lambda} \otimes \mathrm{id})(\mathrm{id} \otimes \tau_{-t})\mathbb{W} \\ &= (\mathrm{id} \otimes \tau_{-t})(\widehat{\Lambda} \otimes \mathrm{id})\mathbb{W} \\ &= (\mathrm{id} \otimes \tau_{-t})W = (\hat{\tau}_t \otimes \mathrm{id})W \\ &= \big(\hat{\tau}_t^{\mathrm{u}} \otimes \mathrm{id}\big)(\widehat{\Lambda} \otimes \mathrm{id})\mathbb{W}.\end{aligned}$$

As before, the resulting formula

$$\big(\big[\widehat{\Lambda} \circ \hat{\tau}_t^{\mathrm{u}}\big] \otimes \mathrm{id}\big)\mathbb{W} = \big([\hat{\tau}_t \circ \widehat{\Lambda}] \otimes \mathrm{id}\big)\mathbb{W}$$

suffices to have (5.27). $\square$

Lemma 40. *Let $W \in \mathrm{M}(\widehat{A} \otimes A)$ be the reduced bicharacter. Then*

$$W^{\widehat{R} \otimes R} = W.$$

Proof. Let $\omega \in \widehat{A}_*$ and $\mu \in A_*$ be analytic for (the transpose of) $(\hat{\tau}_t)$ and (τ_t), respectively. Using the fact that for any $\nu \in \widehat{A}_*$ and $\lambda \in A_*$

$$\begin{aligned}\kappa\big((\nu \otimes \mathrm{id})W\big) &= (\nu \otimes \mathrm{id})(W^*), \\ \hat{\kappa}\big((\lambda \otimes \mathrm{id})\widehat{W}\big) &= (\lambda \otimes \mathrm{id})\big(\widehat{W}^*\big)\end{aligned}$$

we have

$$\begin{aligned}\big([\omega \circ \hat{\tau}_{\frac{i}{2}}] \otimes [\mu \circ \tau_{\frac{i}{2}}]\big)\big(W^{\widehat{R} \otimes R}\big) &= \big([\omega \circ \hat{\tau}_{\frac{i}{2}} \circ \widehat{R}] \otimes [\mu \circ \tau_{\frac{i}{2}} \circ R]\big)W \\ &= \big([\omega \circ \hat{\tau}_{\frac{i}{2}} \circ \widehat{R}] \otimes \mu\big)(W^*) \\ &= \big(\mu \otimes [\omega \circ \hat{\tau}_{\frac{i}{2}} \circ \widehat{R}]\big)\widehat{W} \\ &= (\mu \otimes \omega)(\widehat{W}^*) = (\omega \otimes \mu)W \\ &= \big([\omega \circ \hat{\tau}_{\frac{i}{2}}] \otimes [\mu \circ \tau_{\frac{i}{2}}]\big)W,\end{aligned}$$

where in the last step we used holomorphic continuation of

$$(\hat{\tau}_t \otimes \tau_t)W = W$$

which follows from (5.28). The conclusion follows from the fact that functionals of the form $([\omega \circ \hat{\tau}_{\frac{i}{2}}] \otimes [\mu \circ \tau_{\frac{i}{2}}])$ separate points of $\widehat{A} \otimes A$. $\square$

Before the next proposition let us state a remark which will come handy in the proof.

Remark 41. Let C be a C*-algebra and let S be an anti-morphism from $\widehat{A}_{\mathrm{u}}$ to C, i.e. $S \in \mathrm{Mor}(\widehat{A}_{\mathrm{u}}, C^{\mathrm{op}})$. Then for any $\omega \in A_*$ we have

$$(\mathrm{id} \otimes \omega) \circ (S \otimes R) = S \circ \bigl(\mathrm{id} \otimes [\omega \circ R]\bigr). \tag{5.29}$$

Proposition 42. *Let $(\widehat{A}_{\mathrm{u}}, \mathbb{W})$ be the universal quantum group* C**-algebra and the universal representation of $\mathbb{G}$. Then*

(1) *There exists a unique anti-automorphism $\widehat{R}^{\mathrm{u}}$ of $\widehat{A}_{\mathrm{u}}$ such that for any $\omega \in A_*$*

$$\widehat{R}^{\mathrm{u}}\bigl((\mathrm{id} \otimes \omega)\mathbb{W}\bigr) = \bigl((\mathrm{id} \otimes \omega)\mathbb{W}^{\mathrm{c}}\bigr)^{\top}. \tag{5.30}$$

(2) *$\widehat{R}^{\mathrm{u}}$ is involutive.*
(3) *Let $\hat{\sigma}$ be the flip on $\widehat{A}_{\mathrm{u}} \otimes \widehat{A}_{\mathrm{u}}$. Then*

$$\hat{\sigma} \circ \bigl(\widehat{R}^{\mathrm{u}} \otimes \widehat{R}^{\mathrm{u}}\bigr) \circ \widehat{\Delta}_{\mathrm{u}} = \widehat{\Delta}_{\mathrm{u}} \circ \widehat{R}^{\mathrm{u}}. \tag{5.31}$$

Moreover if $(\hat{\tau}^{\mathrm{u}}_t)_{t\in\mathbb{R}}$ is the one parameter group of automorphisms of $\widehat{A}_{\mathrm{u}}$ defined in Proposition 39 *then for any $t \in \mathbb{R}$ we have*

$$\hat{\tau}^{\mathrm{u}}_t \circ \widehat{R}^{\mathrm{u}} = \widehat{R}^{\mathrm{u}} \circ \hat{\tau}^{\mathrm{u}}_t. \tag{5.32}$$

(4) *If $\widehat{R}$ is the unitary coinverse of $\widehat{\mathbb{G}}$ and $\widehat{\Lambda} \in \mathrm{Mor}(\widehat{A}_{\mathrm{u}}, \widehat{A})$ is the reducing morphism then we have*

$$\widehat{R} \circ \widehat{\Lambda} = \widehat{\Lambda} \circ \widehat{R}^{\mathrm{u}}.$$

Proof. AD (1). The contragradient representation $\mathbb{W}^{\mathrm{c}}$ of $\mathbb{W}$ is a strongly continuous unitary representation of $\mathbb{G}$. Therefore, by the universal property of $(\widehat{A}_{\mathrm{u}}, \mathbb{W})$, there exists a unique morphism $\theta \in \mathrm{Mor}(\widehat{A}_{\mathrm{u}}, B_{\mathbb{W}^{\mathrm{c}}})$ such that

$$(\theta \otimes \mathrm{id})\mathbb{W} = \mathbb{W}^{\mathrm{c}} \tag{5.33}$$

(cf. Theorem 25(2)). Clearly $B_{\mathbb{W}^{\mathrm{c}}} = (\widehat{A}_{\mathrm{u}})^{\top}$ and we can define a map

$$\widehat{R}^{\mathrm{u}} : \widehat{A}_{\mathrm{u}} \ni x \longmapsto \theta(x)^{\top} \in \widehat{A}_{\mathrm{u}}.$$

It is easy to see that so defined $\widehat{R}^{\mathrm{u}}$ is an anti-morphism of $\widehat{A}_{\mathrm{u}}$ to itself which satisfies (5.30) which determines this anti-morphism uniquely. The map $\widehat{R}^{\mathrm{u}}$ is an anti-automorphism of $\widehat{A}_{\mathrm{u}}$. This follows for example from the fact that $(\widehat{R}^{\mathrm{u}})^2 = \mathrm{id}$ established below.

AD (2). Let us take contragradient representations of both sides of (5.33). Then

$$\bigl(\widehat{R}^{\mathrm{u}} \otimes R\bigr)\mathbb{W} = \mathbb{W}. \tag{5.34}$$

Now applying $(\widehat{R}^{\mathrm{u}} \otimes R)$ to both sides of (5.34) and then using this equation we arrive at

$$\bigl((\widehat{R}^{\mathrm{u}})^2 \otimes \mathrm{id}\bigr)\mathbb{W} = \bigl(\widehat{R}^{\mathrm{u}} \otimes R\bigr)\mathbb{W} = \mathbb{W}.$$

Thus for any $\omega \in A_*$ we have

$$\big(\widehat{R}^{\mathrm{u}}\big)^2\big((\mathrm{id}\otimes\omega)\mathbb{W}\big) = (\mathrm{id}\otimes\omega)\big(\big(\widehat{R}^{\mathrm{u}}\big)^2\otimes\mathrm{id}\big)\mathbb{W} = (\mathrm{id}\otimes\omega)\mathbb{W}$$

and it follows $(\widehat{R}^{\mathrm{u}})^2 = \mathrm{id}$.

AD (3). Let us begin with (5.32). For any $t \in \mathbb{R}$ we have:

$$\begin{aligned}\big(\big[\widehat{R}^{\mathrm{u}}\circ\hat{\tau}^{\mathrm{u}}_t\big]\otimes R\big)\mathbb{W} &= \big(\widehat{R}^{\mathrm{u}}\otimes R\big)\big(\hat{\tau}^{\mathrm{u}}_t\otimes\mathrm{id}\big)\mathbb{W} = \big(\widehat{R}^{\mathrm{u}}\otimes R\big)(\mathrm{id}\otimes\tau_{-t})\mathbb{W}\\ &= \big(\widehat{R}^{\mathrm{u}}\otimes[R\circ\tau_{-t}]\big)\mathbb{W} = \big(\widehat{R}^{\mathrm{u}}\otimes[\tau_{-t}\circ R]\big)\mathbb{W}\\ &= (\mathrm{id}\otimes\tau_{-t})\big(\widehat{R}^{\mathrm{u}}\otimes R\big)\mathbb{W} = (\mathrm{id}\otimes\tau_{-t})\mathbb{W}\\ &= \big(\hat{\tau}^{\mathrm{u}}_t\otimes\mathrm{id}\big)\mathbb{W} = \big(\hat{\tau}^{\mathrm{u}}_t\otimes\mathrm{id}\big)\big(\widehat{R}^{\mathrm{u}}\otimes R\big)\mathbb{W}\\ &= \big(\big[\hat{\tau}^{\mathrm{u}}_t\circ\widehat{R}^{\mathrm{u}}\big]\otimes R\big)\mathbb{W}.\end{aligned}$$

Now let us take $\omega \in A_*$. Then with information from the above computation and using formula (5.29) twice (once with $S = \widehat{R}^{\mathrm{u}}\circ\hat{\tau}^{\mathrm{u}}_t$ and then with $S = \hat{\tau}^{\mathrm{u}}_t\circ\widehat{R}^{\mathrm{u}}$) we get

$$\begin{aligned}\big(\widehat{R}^{\mathrm{u}}\circ\hat{\tau}^{\mathrm{u}}_t\big)\big(\big(\mathrm{id}\otimes[\omega\circ R]\big)\mathbb{W}\big) &= \widehat{R}^{\mathrm{u}}\big(\big(\mathrm{id}\otimes[\omega\circ R]\big)\big(\big(\hat{\tau}^{\mathrm{u}}_t\otimes\mathrm{id}\big)\mathbb{W}\big)\big)\\ &= (\mathrm{id}\otimes\omega)\big(\big[\widehat{R}^{\mathrm{u}}\circ\hat{\tau}^{\mathrm{u}}_t\big]\otimes R\big)\mathbb{W}\\ &= (\mathrm{id}\otimes\omega)\big(\big[\hat{\tau}^{\mathrm{u}}_t\circ\widehat{R}^{\mathrm{u}}\big]\otimes R\big)\mathbb{W}\\ &= \big(\hat{\tau}^{\mathrm{u}}_t\circ\widehat{R}^{\mathrm{u}}\big)\big(\big(\mathrm{id}\otimes[\omega\circ R]\big)\mathbb{W}\big).\end{aligned}$$

Now since $\{\omega\circ R\colon\ \omega\in A_*\} = A_*$ and $\widehat{A}_{\mathrm{u}}$ is the closure of the set of right slices of $\mathbb{W}$ we get (5.32).

For the proof of (5.31) let us first notice that the defining property of $\widehat{\Delta}_{\mathrm{u}}$ and (5.34) imply that

$$\begin{aligned}\big(\widehat{R}^{\mathrm{u}}\otimes\widehat{R}^{\mathrm{u}}\otimes R\big)(\widehat{\Delta}_{\mathrm{u}}\otimes\mathrm{id})\mathbb{W} &= \big(\widehat{R}^{\mathrm{u}}\otimes\widehat{R}^{\mathrm{u}}\otimes R\big)(\mathbb{W}_{23}\mathbb{W}_{13})\\ &= \big[\big(\widehat{R}^{\mathrm{u}}\otimes R\big)\mathbb{W}\big]_{13}\big[\big(\widehat{R}^{\mathrm{u}}\otimes R\big)\mathbb{W}\big]_{23}\\ &= \mathbb{W}_{13}\mathbb{W}_{23}.\end{aligned}$$

Therefore

$$\begin{aligned}\big(\big[\hat{\sigma}\circ\big(\widehat{R}^{\mathrm{u}}\otimes\widehat{R}^{\mathrm{u}}\big)\circ\widehat{\Delta}_{\mathrm{u}}\big]\otimes R\big)\mathbb{W} &= \mathbb{W}_{23}\mathbb{W}_{13} = (\widehat{\Delta}_{\mathrm{u}}\otimes\mathrm{id})\mathbb{W}\\ &= (\widehat{\Delta}_{\mathrm{u}}\otimes\mathrm{id})\big(\widehat{R}^{\mathrm{u}}\otimes\widehat{R}^{\mathrm{u}}\big)\mathbb{W}\\ &= \big(\big[\widehat{\Delta}_{\mathrm{u}}\circ\widehat{R}^{\mathrm{u}}\big]\otimes R\big)\mathbb{W}\end{aligned}$$

and in the same way as in the proof of (5.32) we obtain (5.31). This time formula (5.29) must also be used twice: once with $S = \hat{\sigma}\circ(\widehat{R}^{\mathrm{u}}\otimes\widehat{R}^{\mathrm{u}})\circ\widehat{\Delta}_{\mathrm{u}}$ and then with $S = \widehat{\Delta}_{\mathrm{u}}\circ\widehat{R}^{\mathrm{u}}$.

AD (4). Let $W \in \mathrm{M}(\widehat{A}\otimes A)$ be the reduced bicharacter. Using (5.34) in the second and Lemma 40 in the fourth step we have

$$
\begin{aligned}
\big([\widehat{\Lambda}\circ\widehat{R}^{\mathrm{u}}]\otimes R\big)\mathbb{W} &= (\widehat{\Lambda}\otimes\mathrm{id})\big(\widehat{R}^{\mathrm{u}}\otimes R\big)\mathbb{W}\\
&= (\widehat{\Lambda}\otimes\mathrm{id})\mathbb{W} = W = W^{\widehat{R}\otimes R}\\
&= \big(\widehat{R}^{\mathrm{u}}\otimes R\big)(\widehat{\Lambda}\otimes\mathrm{id})\mathbb{W} = \big([\widehat{R}\circ\widehat{\Lambda}]\otimes R\big)\mathbb{W}.
\end{aligned}
$$

Again, as in proofs of (2) and (3) we can use (5.29) once with $S=\widehat{\Lambda}\circ\widehat{R}^{\mathrm{u}}$ and then with $S=\widehat{R}\circ\widehat{\Lambda}$ and appeal to the fact that R is a homeomorphism for the ultraweak topology on A. □

Proposition 43. *Let $(\widehat{A}_{\mathrm{u}},\mathbb{W})$ be the universal quantum group* C**-algebra and the universal representation of $\mathbb{G}$. Then there exists a unique closed linear operator $\hat{\kappa}^{\mathrm{u}}$ on the Banach space $\widehat{A}_{\mathrm{u}}$ such that*

$$\big\{(\mathrm{id}\otimes\omega)(\mathbb{W}^*)\colon\ \omega\in A_*\big\} \tag{5.35}$$

is a core for $\hat{\kappa}^{\mathrm{u}}$ and

$$\hat{\kappa}^{\mathrm{u}}\big((\mathrm{id}\otimes\omega)(\mathbb{W}^*)\big) = (\mathrm{id}\otimes\omega)\mathbb{W}.$$

Moreover

(1) *the domain of $\hat{\kappa}^{\mathrm{u}}$ is an algebra and $\hat{\kappa}^{\mathrm{u}}$ is anti-multiplicative*: $\hat{\kappa}^{\mathrm{u}}(xy)=\hat{\kappa}^{\mathrm{u}}(y)\hat{\kappa}^{\mathrm{u}}(x)$ *for all* $x,y\in\mathrm{Dom}(\hat{\kappa}^{\mathrm{u}})$.
(2) *For any $x\in\mathrm{Dom}(\hat{\kappa}^{\mathrm{u}})$ the element $\hat{\kappa}^{\mathrm{u}}(x)^*$ belongs to* $\mathrm{Dom}(\hat{\kappa}^{\mathrm{u}})$ *and* $\hat{\kappa}^{\mathrm{u}}(\hat{\kappa}^{\mathrm{u}}(x)^*)^*=x$.
(3) $\hat{\kappa}^{\mathrm{u}}=\widehat{R}^{\mathrm{u}}\circ\hat{\tau}^{\mathrm{u}}_{\frac{i}{2}}$.

Proof. Take $\eta\in\mathrm{B}(H_{\mathbb{W}})_*$ and $\omega\in A_*$. Then with repeated use of a variation of formula (5.29) we compute:

$$
\begin{aligned}
\big(\eta\circ\widehat{R}^{\mathrm{u}}\circ\hat{\tau}^{\mathrm{u}}_t\big)\big[\big(\mathrm{id}\otimes[\omega\circ R]\big)(\mathbb{W}^*)\big] &= \big(\big[\eta\circ\widehat{R}^{\mathrm{u}}\circ\hat{\tau}^{\mathrm{u}}_t\big]\otimes[\omega\circ R]\big)(\mathbb{W}^*)\\
&= \big(\big[\eta\circ\widehat{R}^{\mathrm{u}}\big]\otimes[\omega\circ R]\big)\big[\big(\hat{\tau}^{\mathrm{u}}_t\otimes\mathrm{id}\big)(\mathbb{W}^*)\big]\\
&= \big(\big[\eta\circ\widehat{R}^{\mathrm{u}}\big]\otimes[\omega\circ R]\big)\big[(\mathrm{id}\otimes\tau_{-t})(\mathbb{W}^*)\big]\\
&= \big(\big[\eta\circ\widehat{R}^{\mathrm{u}}\big]\otimes[\omega\circ R\circ\tau_{-t}]\big)(\mathbb{W}^*)\\
&= (\omega\circ R\circ\tau_{-t})\big[\big(\big[\eta\circ\widehat{R}^{\mathrm{u}}\big]\otimes\mathrm{id}\big)(\mathbb{W}^*)\big]\\
&= \omega\big[(R\circ\tau_{-t})\big(\big[\eta\circ\widehat{R}^{\mathrm{u}}\big]\otimes\mathrm{id}\big)(\mathbb{W}^*)\big].
\end{aligned}
$$

We shall now take the limit $t\to\frac{i}{2}$. By Proposition 16 and properties of analytic generators of one parameter groups, the last term above converges to

$$
\begin{aligned}
\omega\big[\big(\big[\eta\circ\widehat{R}^{\mathrm{u}}\big]\otimes\mathrm{id}\big)\mathbb{W}\big] &= (\omega\circ R)\big[\big(\big[\eta\circ\widehat{R}^{\mathrm{u}}\big]\otimes R\big)\mathbb{W}\big]\\
&= \big(\eta\otimes[\omega\circ R]\big)\big[\big(\widehat{R}^{\mathrm{u}}\otimes R\big)\mathbb{W}\big] = \big(\eta\otimes[\omega\circ R]\big)\mathbb{W}
\end{aligned}
$$

(cf. (5.34)). Since functionals of the form $\omega \circ R$ fill up all of A_*, we find that for any $\omega \in A_*$ the element $(\mathrm{id} \otimes \omega)(\mathbb{W}^*)$ belongs to the domain of the analytic extension of the group $(\hat{\tau}^{\mathrm{u}}_t)_{t \in \mathbb{R}}$ to the point $t = \frac{i}{2}$ and we have

$$\big(\widehat{R}^{\mathrm{u}} \circ \hat{\tau}^{\mathrm{u}}_{\frac{i}{2}}\big)\big((\mathrm{id} \otimes \omega)(\mathbb{W}^*)\big) = (\mathrm{id} \otimes \omega)\mathbb{W}.$$

Moreover the set (5.35) is dense in $\widehat{A}_{\mathrm{u}}$ and is $(\hat{\tau}^{\mathrm{u}}_t)_{t \in \mathbb{R}}$-invariant. Indeed, recall that all automorphisms $(\tau_t)_{t \in \mathbb{R}}$ are ultraweakly continuous, so

$$\begin{aligned}
\hat{\tau}^{\mathrm{u}}_t\big((\mathrm{id} \otimes \omega)(\mathbb{W}^*)\big) &= (\mathrm{id} \otimes \omega)\big(\hat{\tau}^{\mathrm{u}}_t \otimes \mathrm{id}\big)(\mathbb{W}^*) \\
&= (\mathrm{id} \otimes \omega)(\mathrm{id} \otimes \tau_{-t})(\mathbb{W}^*) \\
&= \big(\mathrm{id} \otimes [\omega \circ \tau_{-t}]\big)(\mathbb{W}^*)
\end{aligned}$$

belongs to (5.35). It is a simple observation (cf. e.g. [10, Proposition F.5]) that this implies that (5.35) must be a core for $\hat{\tau}^{\mathrm{u}}_{\frac{i}{2}}$. If we now put $\hat{\kappa}^{\mathrm{u}} = \widehat{R}^{\mathrm{u}} \circ \hat{\tau}^{\mathrm{u}}_{\frac{i}{2}}$ then $\hat{\kappa}^{\mathrm{u}}$ is a closed operator on $\widehat{A}_{\mathrm{u}}$. Clearly there at most one operator with a given core and prescribed action on this core. Of course, w have (3). Properties (1) and (2) are well-known facts from the theory analytic extensions of one parameter groups of automorphisms ([24]). □

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