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Complete metrizability of topologies of strong uniform convergence on bornologies

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ABSTRACT

We continue the study of topologies of strong uniform convergence on bornologies initiated by Beer and Levi (2009) [4]. In Beer and Levi (2009) [4] the metrizability of such topologies restricted to continuous functions was characterized and a compatible metric was displayed. We find necessary and sufficient conditions for completeness of the metric, for complete metrizability and for Polishness of topologies of strong uniform convergence on bornologies on continuous functions. Polishness of such topologies forces their coincidence with the compact-open topology.

1. Introduction

Topologies of strong uniform convergence on bornologies were introduced by Beer and Levi in their paper [4] and then studied also in [8] and [6]. In [8] the authors continued the study initiated in [4] and they characterized several main topological properties like separability, second countability, countable netweight, countable tightness, Frechetness, etc. of these function spaces in terms of topological properties of the domain and the bornology. In [4] the metrizability of such topologies restricted to continuous functions was characterized and a compatible metric was displayed. Using shields introduced in [2] we find necessary and sufficient conditions for completeness of the metric and for complete metrizability of topologies of strong uniform convergence on bornologies on continuous functions.

We also characterize Polishness of topologies of strong uniform convergence on bornologies on the space of continuous functions. Polishness of such topologies forces their coincidence with the compact-open topology τ_k.

2. Preliminaries

Let (X, d) and (Y, ρ) be metric spaces, let Y^X be the set of all functions from X to Y and let $C(X, Y)$ be the set of all continuous functions from X to Y.

All metric spaces are assumed to contain at least two points. We denote by $\mathcal{P}_0(X)$ the set of all nonempty subsets of X, by $\mathcal{C}_0(X)$ the set of all nonempty closed subsets of X, by $\mathcal{K}_0(X)$ the set of all nonempty compact subsets of X and by $\mathcal{F}_0(X)$ the set of all nonempty finite sets in X.

If $x_0 \in X$ and $\epsilon > 0$, we write $S(x_0, \epsilon)$ for the open ϵ-ball with center x_0 and $B^\epsilon = \bigcup_{x \in B} S(x, \epsilon)$.

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If $A, B \in \mathcal{P}_0(X)$, we say that A and B are near provided for each $\epsilon > 0$ $A^\epsilon \cap B \neq \emptyset$. The sets A and B are called far provided they are not near. If A and B are far, we define the gap $D(A, B)$ between them by the formula

$$D(A, B) = \sup\{\epsilon > 0:\ A^\epsilon \cap B = \emptyset\},$$

whereas if A and B are near, we put $D(A, B) = 0$.

Let $\mathcal{B}$ be an ideal of nonempty subsets of (X, d). $\mathcal{B}$ is a bornology if it forms a cover of X [9,4,5,12]. The smallest bornology on X is the family $\mathcal{F}_0(X)$ and the largest is the family $\mathcal{P}_0(X)$.

Let $\mathcal{B}$ be a bornology on X. By a base $\mathcal{B}_0$ for $\mathcal{B}$, we mean a subfamily of $\mathcal{B}$ that is cofinal with respect to inclusion: for every $B \in \mathcal{B}$ there is $B_0 \in \mathcal{B}_0$ with $B \subseteq B_0$. A base is called closed (compact) if each of its members is a closed (compact) subset of X.

Given a bornology $\mathcal{B}$ on X, the classical uniformity for the topology of uniform convergence $\tau_{\mathcal{B}}$ on $\mathcal{B}$ for Y^X [13] has as a base all sets of the form

$$[B; \epsilon] = \{(f, g):\ \forall x \in B,\ \rho(f(x), g(x)) < \epsilon\} \quad (B \in \mathcal{B},\ \epsilon > 0).$$

Denote by $\Delta_{\mathcal{B}}$ the uniformity on Y^X generated by the family $\{[B; \epsilon]:\ B \in \mathcal{B},\ \epsilon > 0\}$.

Definition 2.1. (See [4].) Let (X, d) and (Y, ρ) be metric spaces and let $\mathcal{B}$ be a bornology with a closed base on X. Then the topology of strong uniform convergence $\tau^s_{\mathcal{B}}$ on $\mathcal{B}$ is determined by a uniformity on Y^X having as a base all sets of the form

$$[B; \epsilon]^s = \{(f, g):\ \exists \delta > 0,\ \forall x \in B^\delta,\ \rho(f(x), g(x)) < \epsilon\} \quad (B \in \mathcal{B},\ \epsilon > 0).$$

Denote by $\Delta^s_{\mathcal{B}}$ the uniformity on Y^X generated by the family $\{[B; \epsilon]^s:\ B \in \mathcal{B},\ \epsilon > 0\}$.

This topology is in general finer than the classical topology of uniform convergence $\tau_{\mathcal{B}}$ on $\mathcal{B}$, but reduces to it on the class of functions that are strongly uniformly continuous on $\mathcal{B}$.

Definition 2.2. (See [4].) Let (X, d) and (Y, ρ) be metric spaces and let B be a subset of X. We say that a function $f : X \to Y$ is strongly uniformly continuous on B if for every $\epsilon > 0$ there is $\delta > 0$ such that if $d(x, w) < \delta$ and $\{x, w\} \cap B \neq \emptyset$, then $\rho(f(x), f(w)) < \epsilon$.

Strong uniform continuity of f on $\{x\}$ amounts to continuity of f at x while strong uniform continuity of f on X amounts to uniform continuity of f. Of course if $f \in C(X, Y)$ and K is a compact subset of X, then f is strongly uniformly continuous on K.

We now write for $f \in Y^X$,

$$\mathcal{B}^f = \{B \in \mathcal{P}_0(X):\ f \restriction B \text{ is uniformly continuous}\},$$
$$\mathcal{B}_f = \{B \in \mathcal{P}_0(X):\ f \text{ is strongly uniformly continuous on } B\}.$$

Evidently $\mathcal{B}_f \subseteq \mathcal{B}^f$, $B \in \mathcal{B}_f \Rightarrow \overline{B} \in \mathcal{B}_f$. $\mathcal{B}_f$ is a bornology if and only if $f \in C(X, Y)$. The bornology $\mathcal{B}_f$ for $f \in C(X, Y)$ has a closed base and contains $\mathcal{K}_0(X)$. When $\mathcal{B}$ is an arbitrary bornology on X, we put

$$C^s_{\mathcal{B}}(X, Y) = \{f \in C(X, Y):\ \mathcal{B} \subseteq \mathcal{B}_f\}.$$

It was proved in [4] that $C^s_{\mathcal{B}}(X, Y)$ is closed in $(Y^X, \tau^s_{\mathcal{B}})$ and that restricted to $C^s_{\mathcal{B}}(X, Y)$, $\tau^s_{\mathcal{B}} = \tau_{\mathcal{B}}$.

Beer and Levi initiated the study of topologies of strong uniform convergence on bornologies on the space of continuous functions in their paper [4]. In fact they characterized metrizability and 1st countability of such topologies. They proved that if $\mathcal{B}$ is a bornology on (X, d) with a closed base and (Y, ρ) a metric space, then $(C(X, Y), \tau^s_{\mathcal{B}})$ is metrizable iff the bornology $\mathcal{B}$ has a countable base. In their paper [4] they displayed a compatible metric for the metrizable space $(C(X, Y), \tau^s_{\mathcal{B}})$ in terms of a countable base $\{B_k : k \in N\}$ as follows. For each positive integer k let d_k be an extended real-valued pseudometric on $C(X, Y)$ defined by

$$d_k(f, g) = \inf_{\delta > 0} \sup_{x \in B_k^\delta} \rho(f(x), g(x)) \quad (f, g \in C(X, Y)).$$

Since $\mathcal{B}$ is a cover, whenever $f \neq g$, there exists $k \in N$ with $d_k(f, g) > 0$. As a result, $d^s_{\mathcal{B}} : C(X, Y) \times C(X, Y) \to [0, \infty)$ defined by

$$d^s_{\mathcal{B}}(f, g) = \Sigma_{k \in N} 2^{-k} \min\{1, d_k(f, g)\}$$

is a metric on $C(X, Y)$ which is compatible with $\tau^s_{\mathcal{B}}$ and the uniformity generated by $d^s_{\mathcal{B}}$ coincides with $\Delta^s_{\mathcal{B}}$.

Of course a natural question arises to find necessary and sufficient conditions on the bornology $\mathcal{B}$ under which $d^s_{\mathcal{B}}$ is complete. In our paper we will study this question.

3. Complete metrizability of $\tau^s_{\mathcal{B}}$

A subset A of a space X is called relatively pseudocompact if $f(A)$ is bounded in R for all $f \in C(X,R)$. In metric spaces relative pseudocompactness is the same as relative compactness.

Lemma 3.1. *Let (X,d) be a metric space and $\mathcal{B}$ be a bornology with a closed base. If $(C(X,R),\tau_{\mathcal{B}})$ $((C(X,R),\tau^s_{\mathcal{B}}))$ is of the second Baire category, then every relatively pseudocompact subset of X is contained in $\mathcal{B}$.*

Proof. We will prove the lemma for $(C(X,R),\tau^s_{\mathcal{B}})$. Let A be a relatively pseudocompact subset of X. For every $n \in N$ put $W_n = \{f \in C(X,R)\colon |f(x)| \leqslant n$ for every $x \in A\}$. Of course every W_n is a closed set in $(C(X,R),\tau^s_{\mathcal{B}})$ and $C(X,R) = \bigcup_{n\in N} W_n$. Since $(C(X,R),\tau^s_{\mathcal{B}})$ is of the second Baire category there must exist $n \in N$ and a nonempty open set $G \subset W_n$. Thus there is a closed element $B \in \mathcal{B}$, $f \in G$ and $\epsilon > 0$ such that $[B;\epsilon]^s(f) \subset G \subset W_n$. We claim that $A \subset B$. If not, let $x_0 \in A \setminus B$. There is $\delta > 0$ such that $S(x_0,\delta) \cap B^\delta = \emptyset$. Let $g \in C(X,R)$ be such that $g(x) = f(x)$ for every $x \notin S(x_0,\delta)$ and $g(x_0) = 3n$. Then $g \in G$, but $g \notin W_n$, a contradiction. □

Corollary 3.1. *Let (X,d) be a metric space and $\mathcal{B}$ be a bornology with a closed base. If $(C(X,R),\tau_{\mathcal{B}})$ $((C(X,R),\tau^s_{\mathcal{B}}))$ is of the second Baire category, then $\tau_k \subset \tau_{\mathcal{B}}$, where τ_k is the compact-open topology.*

Remark 3.1. Topologies $\tau_{\mathcal{B}}$ on $C(X,R)$ were studied in [13] only for $\mathcal{B}$ with a compact base. By the same idea as in [13] we can prove that if $\mathcal{B}$ is a bornology with a closed base, then $(C(X,R),\tau_{\mathcal{B}})$ is metrizable if and only if $\mathcal{B}$ has a countable base. Of course if $\mathcal{B}$ has a countable base then also the uniformity $\Delta_{\mathcal{B}}$ is metrizable.

Theorem 3.1. *Let (X,d) be a metric space and $\mathcal{B}$ be a bornology with a closed base. The following are equivalent*:

(1) *$(C(X,R),\tau_{\mathcal{B}})$ is completely metrizable*;
(2) *$(C(X,R),\Delta_{\mathcal{B}})$ is complete and metrizable*;
(3) *Every nonempty compact set in X belongs to $\mathcal{B}$ and $\mathcal{B}$ has a countable base*;
(4) *For every complete metric space (Y,ρ), $(C(X,Y),\Delta_{\mathcal{B}})$ is complete and metrizable*;
(5) *For every complete metric space (Y,ρ), $(C(X,Y),\tau_{\mathcal{B}})$ is completely metrizable.*

Proof. (1) ⇒ (2) $(C(X,R),\tau_{\mathcal{B}})$ is a topological group; we can use results from [11] to guarantee the completeness of $(C(X,R),\Delta_{\mathcal{B}})$. Concerning metrizability of $(C(X,R),\Delta_{\mathcal{B}})$ we use Remark 3.1.

(2) ⇒ (3) By Lemma 3.1 every nonempty compact set in X belongs to $\mathcal{B}$ and by Remark 3.1 the metrizability of $(C(X,R),\Delta_{\mathcal{B}})$ implies that $\mathcal{B}$ has a countable base.

(3) ⇒ (4) Let (Y,ρ) be a complete metric space. Let $\{B_k : k \in N\}$ be a countable base of $\mathcal{B}$; i.e. $\{B_k : k \in N\}$ is a cofinal subfamily of the bornology $\mathcal{B}$ with respect to inclusion. Then a countable family of entourages that forms a base for the defining uniformity $\Delta_{\mathcal{B}}$ is $\{[B_k;1/n] : (k,n) \in N \times N\}$, which gives metrizability of $\Delta_{\mathcal{B}}$ on $C(X,Y)$. We will prove that $\Delta_{\mathcal{B}}$ is complete. Let $\{f_n\colon n \in N\}$ be a $\Delta_{\mathcal{B}}$-Cauchy sequence in $C(X,Y)$. Since (Y,ρ) is complete, for every $x \in X$ there is a limit of the sequence $\{f_n(x)\colon n \in N\}$. Let $f : X \to Y$ be a function defined by $f(x) = \lim\{f_n(x)\colon n \in N\}$. It is easy to verify that $\{f_n \restriction B\colon n \in N\}$ converges uniformly to $f \restriction B$ for every $B \in \mathcal{B}$. Since $\mathcal{B}$ contains compact sets, f is continuous on every compact set. Since X is a metric space, f is continuous and $\{f_n\colon n \in N\}$ $\tau_{\mathcal{B}}$-converges to f. Thus $(C(X,Y),\Delta_{\mathcal{B}})$ is complete.

(4) ⇒ (5) and (5) ⇒ (1) are trivial. □

Definition 3.1. (See [2].) A bornology $\mathcal{B}$ on a metric space (X,d) is said to be shielded from closed sets if for every $E \in \mathcal{B}$ there is $B \in \mathcal{B}$ such that $E \subseteq B$ and whenever $C \in \mathcal{C}_0(X)$, $C \cap B = \emptyset$, then $D(E,C) > 0$.

Theorem 3.2. *Let (X,d) be a metric space and $\mathcal{B}$ be a bornology with a closed base. The following are equivalent*:

(1) *$(C(X,R),\tau^s_{\mathcal{B}})$ is completely metrizable*;
(2) *$(C(X,R),\Delta^s_{\mathcal{B}})$ is complete and metrizable*;
(3) *Every nonempty compact set in X belongs to $\mathcal{B}$ and $\mathcal{B}$ is shielded from closed sets and has a countable base*;
(4) *For every complete metric space (Y,ρ), $(C(X,Y),\Delta^s_{\mathcal{B}})$ is complete and metrizable*;
(5) *For every complete metric space (Y,ρ), $(C(X,Y),\tau^s_{\mathcal{B}})$ is completely metrizable.*

Proof. (1) ⇒ (2) Since $(C(X,R),\tau^s_{\mathcal{B}})$ is a topological group, we can use [11] to guarantee the completeness of $(C(X,R),\Delta^s_{\mathcal{B}})$ and the proof of Theorem 7.1 in [4] to guarantee the metrizability of $(C(X,R),\Delta^s_{\mathcal{B}})$.

(2) ⇒ (3) By Lemma 3.1 every compact set in X belongs to $\mathcal{B}$. The metrizability of $(C(X,R),\Delta^s_{\mathcal{B}})$ implies that $\mathcal{B}$ has a countable base [4]. Let $\{B_n\colon n \in N\}$ be a countable base of $\mathcal{B}$. Without loss of generality we can suppose that B_n is closed for every $n \in N$.

Now we prove that $\mathcal{B}$ is shielded from closed sets. We use an idea from Proposition 4.8 of [6]. Suppose $B_0 \in \mathcal{B}$ has no shield in $\mathcal{B}$. Then neither has its closure, and so we may assume that B_0 is closed because the bornology has a closed base. For each $n \in N$, put

$$C_n = B_0 \cup \bigcup_{1 \leqslant j \leqslant n} B_j.$$

Evidently $\{C_n\colon\ n \in N\}$ is an increasing cofinal sequence of closed sets in the bornology. Since each C_n is not a shield for B_0, there exists for each $n \in N$ a closed subset F_n of $X \setminus C_n$ with $D(F_n, B_0) = 0$. Pick $x_1 \in F_1$ with $d(x_1, B_0) < 1$ and then $\delta_1 \in (0,1)$ with $S(x_1, \delta_1) \cap B_0 = \emptyset$. Choose $n_1 \in N$ such that $x_1 \in C_{n_1}$. Next choose $x_2 \in F_{n_1}$ with $d(x_2, B_0) < 1/2$, and then choose $\delta_2 > 0$ such that $S(x_2, \delta_2) \cap C_{n_1} = \emptyset$. Since C_{n_1} contains B_0 we see that $\delta_2 < 1/2$. Now choose $n_2 > n_1$ with $x_2 \in C_{n_2}$. Continuing in this way we produce $n_1 < n_2 < n_3 < \cdots$ and sequences $\{x_n\}$ in X and $\{\delta_n\}$ in $(0,1)$ with these properties:

(1) for every $j \in N$, $x_j \in C_{n_j}$;
(2) for every $j \in N$, $x_{j+1} \in F_{n_j}$;
(3) for every $j \in N$, $d(x_j, B_0) < 1/j$;
(4) for every $j \in N$, $S(x_{j+1}, \delta_{j+1}) \cap C_{n_j} = \emptyset$;
(5) for every $j \in N$, $\delta_j < 1/j$.

We first observe that $\{x_n\}$ can have no cluster point in X. Suppose $\{x_n\}$ has a cluster point $p \in X$. Then there is a subsequence $\{x_{l_i}\}$ of the sequence $\{x_n\}$ which converges to p. Put $K = \{p\} \cup \{x_{l_i}\colon\ i \in N\}$. Then K is a compact set and thus $K \in \mathcal{B}$. There is $j \in N$ such that $K \subset C_{n_j}$, a contradiction. As a result of this and property (5), the family $\{S(x_j, \delta_j)\colon\ j \in N\}$ is locally finite. There is a sequence $\{\eta_j\colon\ j \in N\}$ of positive real numbers such that $\eta_j < \delta_j$ for each $j \in N$ and the family $\{S(x_j, \eta_j)\colon\ j \in N\}$ is pairwise disjoint. For every $n \in N$ we will define a continuous real function f_n on X with values in $[0,1]$ as follows:

$$f_n(x) = 1 - 2d(x, x_j)/\eta_j, \quad \text{if}\, d(x, x_j) < \eta_j/2, \quad j \leqslant n,\ j \in N,$$

and let $f_n(x) = 0$ if $x \notin \bigcup_{1 \leqslant j \leqslant n} S(x_j, \eta_j/2)$. The function f_n is a Lipschitz function, since it is a supremum of n Lipschitz functions. So f_n is uniformly continuous.

Now we will show that the sequence $\{f_n\}$ is Cauchy in $(C(X,R), \Delta^s_{\mathcal{B}})$. Let $[B;\epsilon]^s \in \Delta^s_{\mathcal{B}}$. Since $\{C_{n_j}\colon\ j \in N\}$ is cofinal in $\mathcal{B}$ there is $k \in N$ such that $B \subset C_{n_k}$. For every $m, l \geqslant k$ we have $f_m(x) = f_l(x)$ for every $x \in C_{n_k}$. Thus for every $m, l \geqslant k$, $(f_m, f_l) \in [B;\epsilon]^s$ since all functions from the sequence $\{f_n\}$ are uniformly continuous. Since $(C(X,R), \Delta^s_{\mathcal{B}})$ is complete, there must exist a limit point $f \in C(X,R)$. Of course $\{f_n\}$ pointwise converges to f, where f is the following function:

$$f(x) = 1 - 2d(x, x_j)/\eta_j, \quad \text{if}\, d(x, x_j) < \eta_j/2,$$

and $f(x) = 0$ if $x \notin \bigcup_{j \in N} S(x_j, \eta_j/2)$. Thus $f(x) = 0$ for every $x \in B_0$ and $f(x_j) = 1$ for every $j \in N$ and $D(B_0, \{x_j\colon\ j \in N\}) = 0$.

However $f_n \notin [B_0; 1]^s(f)$ for every $n \in N$, a contradiction. (For every $n \in N$ there is $0 < \alpha_n < \eta_n/2$ such that $f_n(x) = 0$ for every $x \in B_0^{\alpha_n}$.)

(3) ⇒ (4) If $\mathcal{B}$ is shielded from closed sets then by Theorem 4.1 in [6] $\Delta_{\mathcal{B}} = \Delta^s_{\mathcal{B}}$ on $C(X,Y)$. Thus by Theorem 3.1 we have that $(C(X,Y), \Delta^s_{\mathcal{B}})$ is complete and metrizable.

(4) ⇒ (5) and (5) ⇒ (1) are trivial. □

Corollary 3.2. *Let (X,d) be a metric space and $\mathcal{B}$ be a bornology with a countable closed base. The following are equivalent*:

(1) $(C(X,R), d^s_{\mathcal{B}})$ *is complete*;
(2) *Every nonempty compact set in X belongs to $\mathcal{B}$ and $\mathcal{B}$ is shielded from closed sets*;
(3) *For every complete metric space (Y,ρ), $(C(X,Y), d^s_{\mathcal{B}})$ is complete.*

Of course the bornology of nonempty relatively compact subsets of a metric space (X,d) has a closed base, is shielded from closed sets and contains nonempty compact sets.

It was proved in [6] that the family of UC subsets forms a bornology $\mathcal{B}^{uc}(X)$ with closed base which is shielded from closed sets and contains nonempty compact sets. A nonempty subset A of a metric space (X,d) is called a UC subset [4] if whenever $\{a_n\colon\ n \in N\}$ is a sequence in A with $\lim_{n\to\infty} d(a_n, X \setminus \{a_n\}) = 0$, then $\{a_n\colon\ n \in N\}$ has a cluster point in X.

Also the family of CC subsets is a bornology $\mathcal{B}^{cc}(X)$ with closed base which is shielded from closed sets and contains nonempty compact sets [3]. A nonempty subset A of a metric space (X,d) is called a cofinally complete subset or a CC-subset [1] if each sequence $\{a_n\colon\ n \in N\}$ in A with $\lim_{n\to\infty} \nu(a_n) = 0$ clusters in X, where $\nu(x) = \sup\{\epsilon > 0\colon\ \overline{S(x,\epsilon)}$ is compact $\}$ and $\nu(x) = 0$ otherwise.

Let (X,d) be a metric space and $\mathcal{B}$ be a bornology on X. $\mathcal{B}$ is local if $\mathcal{B}$ contains as a member a neighborhood of each $x \in X$ [4,7,10].

Lemma 3.2. *Let (X, d) be a metric space and $\mathcal{B}$ be a bornology on X with a countable base that contains the nonempty compact subsets of X. Then $\mathcal{B}$ is local.*

Proof. Let $x \in X$ and $\{S(x, 1/n)\colon n \in N\}$ be a base of neighborhoods of x. Let $\{B_k\colon k \in N\}$ be a countable base of $\mathcal{B}$; i.e. $\{B_k\colon k \in N\}$ is a cofinal subfamily of the bornology $\mathcal{B}$ with respect to the inclusion. We claim that there is $n \in N$ with

$$S(x, 1/n) \subset B_n.$$

Suppose that this is not true. Thus for every $n \in N$ there is

$$x_n \in S(x, 1/n) \setminus B_n.$$

Of course $\{x_n\colon n \in N\}$ converges to x. Put $L = \{x\} \cup \{x_n\colon n \in N\}$. By the assumption $L \in \mathcal{B}$. There must exist $k \in N$ such that $L \subset B_k$, a contradiction. □

Note that Lemma 3.2 works also in first countable topological spaces. In such a case it generalizes the well-known fact that a first countable hemicompact space is locally compact.

Also if the bornology $\mathcal{B}^{tb}(X)$ of nonempty d-totally bounded subsets of a metric space (X, d) has a countable base, then $\mathcal{B}^{tb}(X)$ is local, as was proved in [6, Proposition 4.11]. The same holds also for bornologies $\mathcal{B}^{uc}(X)$, $\mathcal{B}^{cc}(X)$ and for the bornology of nonempty relatively compact subsets of a metric space (X, d).

Let (X, d) be a metric space and $\mathcal{B}$ be a bornology on X. $\mathcal{B}$ is stable under small enlargements [4] if

$$\forall B \in \mathcal{B}, \quad \exists \epsilon > 0 \text{ with } B^{\epsilon} \in \mathcal{B}.$$

Obviously a bornology that is stable under small enlargements is shielded from closed sets and is local, and as shown in [2, Theorem 5.18], a bornology that is both local and shielded from closed sets is stable under small enlargements. Thus we have the following result.

Proposition 3.1. *Let (X, d) be a metric space and $\mathcal{B}$ be a bornology with a countable base. The following are equivalent*:

(1) *$\mathcal{B}$ is stable under small enlargements*;
(2) *$\mathcal{B}$ is shielded from closed sets and $\mathcal{K}_0(X) \subset \mathcal{B}$.*

Thus if bornologies $\mathcal{B}^{uc}(X)$, $\mathcal{B}^{cc}(X)$ and the bornology of nonempty relatively compact subsets have a countable base, they are stable under small enlargements.

We have the following variant of our main result.

Theorem 3.3. *Let (X, d) be a metric space and $\mathcal{B}$ be a bornology with a closed base. The following are equivalent*:

(1) *$(C(X, R), \tau^s_{\mathcal{B}})$ is completely metrizable*;
(2) *$(C(X, R), \Delta^s_{\mathcal{B}})$ is complete and metrizable*;
(3) *Every nonempty compact set in X belongs to $\mathcal{B}$ and $\mathcal{B}$ is shielded from closed sets and has a countable base*;
(4) *$\mathcal{B}$ is stable under small enlargements and has a countable base*;
(5) *For every complete metric space (Y, ρ), $(C(X, Y), \Delta^s_{\mathcal{B}})$ is complete and metrizable*;
(6) *For every complete metric space (Y, ρ), $(C(X, Y), \tau^s_{\mathcal{B}})$ is completely metrizable.*

Notice that with the hypotheses of Theorem 3.3 the topologies of uniform convergence and of strong uniform convergence agree (Theorem 6.2 of [4]).

Because of Hu's theorem [10] if $\mathcal{B}$ is a bornology on a metric space (X, d) with a countable closed base and stable under small enlargements, $\mathcal{B}$ is the bornology of bounded sets for some equivalent metric on X (see [6, p. 13]). However the converse is not true (see Example 4.10 in [6]).

Corollary 3.3. *Let (X, d) be a metric space. Then we have*:

(1) *$(C(X, R), \tau^s_{\mathcal{B}^{uc}(X)})$ is completely metrizable iff $\mathcal{B}^{uc}(X)$ has a countable base.*
(2) *$(C(X, R), \tau^s_{\mathcal{B}^{cc}(X)})$ is completely metrizable iff $\mathcal{B}^{cc}(X)$ has a countable base.*
(3) *$(C(X, R), \tau^s_{\mathcal{B}^{tb}(X)})$ is completely metrizable iff $\mathcal{B}^{tb}(X)$ has a countable base.*

Proof. To prove (3) we use Proposition 4.11 in [6], which claims that if the bornology $\mathcal{B}^{tb}(X)$ of nonempty d-totally bounded sets has a countable base, then $\mathcal{B}^{tb}(X)$ is stable under small enlargements. □

Corollary 3.4. *Let (X, d) be a metric space and $\mathcal{B}$ be a bornology with a countable closed base. The following are equivalent*:

(1) $(C(X, R), d^s_{\mathcal{B}})$ *is complete*;
(2) $\mathcal{B}$ *is stable under small enlargements*;
(3) *For every complete metric space* (Y, ρ), $(C(X, Y), d^s_{\mathcal{B}})$ *is complete.*

4. Polishness

A completely metrizable space which is also separable is called a Polish space.

Theorem 4.1. *Let (X, d) be a metric space and $\mathcal{B}$ be a bornology with a closed base. The following are equivalent*:

(1) $(C(X, R), \tau^s_{\mathcal{B}})$ *is Polish*;
(2) $\tau^s_{\mathcal{B}} = \tau_k$ *on* $C(X, R)$ *and X is hemicompact*;
(3) $\tau^s_{\mathcal{B}} = \tau_k$ *on* $C(X, R)$ *and X is locally compact and separable.*

Proof. (2) $\Rightarrow$ (1) is trivial, since hemicompactness of X guarantees Polishness of $(C(X, R), \tau_k)$ [13].

(1) $\Rightarrow$ (2) By Lemma 3.1 the complete metrizability of $(C(X, R), \tau^s_{\mathcal{B}})$ implies that every compact set in X belongs to $\mathcal{B}$. The separability of $(C(X, R), \tau^s_{\mathcal{B}})$ implies (by [8]) that every closed element from $\mathcal{B}$ is compact. Thus $\tau_k = \tau^s_{\mathcal{B}}$. Even the metrizability of $(C(X, R), \tau_k)$ implies the hemicompactness of X [13].

(2) $\Leftrightarrow$ (3) It is the well-known fact that a metric space is hemicompact if and only if it is locally compact and separable. □

Theorem 4.2. *Let (X, d) be a metric space and $\mathcal{B}$ be a bornology with a closed base. The following are equivalent*:

(1) $(C(X, R), \tau_{\mathcal{B}})$ *is Polish*;
(2) $\tau_{\mathcal{B}} = \tau_k$ *on* $C(X, R)$ *and X is hemicompact*;
(3) $\tau_{\mathcal{B}} = \tau_k$ *on* $C(X, R)$ *and X is locally compact and separable.*

Proof. (2) $\Rightarrow$ (1) is trivial, since hemicompactness of X guarantees Polishness of $(C(X, R), \tau_k)$ [13].

(1) $\Rightarrow$ (2) By Lemma 3.1 the complete metrizability of $(C(X, R), \tau_{\mathcal{B}})$ implies that every compact set in X belongs to $\mathcal{B}$. Using the same idea as in the proof of Theorem 3.5 in [8] we can show that the separability of $(C(X, R), \tau_{\mathcal{B}})$ implies that every closed element from $\mathcal{B}$ is compact. Thus $\tau_k = \tau_{\mathcal{B}}$. Even the metrizability of $(C(X, R), \tau_k)$ implies the hemicompactness of X [13].

(2) $\Leftrightarrow$ (3) It is the well-known fact that a metric space is hemicompact if and only if it is locally compact and separable. □

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References

[1] G. Beer, Between compactness and completeness, Topology Appl. 155 (2008) 503–519.
[2] G. Beer, C. Costantini, S. Levi, When is bornological convergence topological?, preprint.
[3] G. Beer, G. Di Maio, The bornology of cofinally complete subsets, preprint.
[4] G. Beer, S. Levi, Strong uniform continuity, J. Math. Anal. Appl. 350 (2009) 567–589.
[5] G. Beer, S. Levi, Total boundedness and bornologies, Topology Appl. 156 (2009) 1271–1288.
[6] G. Beer, S. Levi, Uniform continuity, uniform convergence, and shields, preprint.
[7] A. Caterino, T. Panduri, C. Vipera, Boundedness, one-point extensions, and B-extensions, Math. Slovaca 58 (2008) 101–114.
[8] A. Caserta, G. Di Maio, Ľ. Holá, Arzela's theorem and strong uniform convergence on bornologies, J. Math. Anal. Appl. 371 (2010) 384–392.
[9] H. Hogbe-Nlend, Bornologies and Functional Analysis, North-Holland, Amsterdam, 1977.
[10] S.T. Hu, Boundedness in a topological space, J. Math. Pures Appl. 228 (1949) 287–320.
[11] J.L. Kelley, General Topology, D. Van Nostrand Co., Princeton, 1955.
[12] A. Lechicki, S. Levi, A. Spakowski, Bornological convergences, J. Math. Anal. Appl. 297 (2004) 751–770.
[13] R.A. McCoy, I. Ntantu, Topological Properties of Spaces of Continuous Functions, Lecture Notes in Math., vol. 1315, Springer-Verlag, 1988.