Generalized Springer correspondence and Green functions for type B/C graded Hecke algebras

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Abstract

This paper presents a combinatorial approach to the tempered representation theory of a graded Hecke algebra $\hat{H}$ of classical type $B$ or $C$, with arbitrary parameters. We present various combinatorial results which together give a uniform combinatorial description of what becomes the Springer correspondence in the classical situation of equal parameters. More precisely, by using a general version of Lusztig’s symbols which describe the classical Springer correspondence, we associate to a discrete series representation of $\hat{H}$ with central character $W_0^c$, a set $\Sigma(W_0^c)$ of $W_0$-characters (where $W_0$ is the Weyl group). This set $\Sigma(W_0^c)$ is shown to parametrize the central characters of the generic algebra which specialize into $W_0^c$. Using the parabolic classification of the central characters of $\hat{H}(\mathbb{R})$ on the one hand, and a truncated induction of Weyl group characters on the other hand, we define a set $\Sigma(W_0^c)$ for any central character $W_0^c$ of $\hat{H}(\mathbb{R})$, and show that this property is preserved. We show that in the equal parameter situation we retrieve the classical Springer correspondence, by considering a set $\mathcal{U}$ of partitions which replaces the unipotent classes of $SO_{2n+1}(\mathbb{C})$ and $Sp_{2n}(\mathbb{C})$, and a bijection between $\mathcal{U}$ and the central characters of $\hat{H}(\mathbb{R})$. We end with a conjecture, which basically states that our generalized Springer correspondence determines $\hat{H}(\mathbb{R})$ exactly as the classical Springer correspondence does in the equal label case. In particular, we conjecture that $\Sigma(W_0^c)$ indexes the modules in $\hat{H}(\mathbb{R})$ with central character $W_0^c$, in the following way. A module $M$ in $\hat{H}(\mathbb{R})$ has a natural grading for the action of $W_0$, and the $W_0$-representation $\chi(M)$ in its top degree is irreducible. When $M$ runs through the modules in $\hat{H}(\mathbb{R})$ with central character

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Let $W_0$, $\chi(M)$ runs through $\Sigma(W_0c)$. Moreover, still in analogy with the equal parameter case, we conjecture that the $W_0$-structure of the modules in $\hat{H}^\dagger(\mathbb{R})$ can be computed using (generalized) Green functions.

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1. Introduction

In this article we study combinatorics which conjecturally describe the irreducible tempered representations with real central character, of (specialized) graded Hecke algebras of type $B$ and $C$. The graded Hecke algebra was introduced by Lusztig in [10], where he showed it to play an important role in the representation theory of affine Hecke algebras.

Let $\mathcal{R} = (R_0, X, \tilde{R}_0, Y, \Pi_0)$ be a root datum. Let $W$ be the associated affine Weyl group, and $S$ the set of simple reflections of $W$. Given a number $q > 0$ and for every $s \in S$ a number $f_s \in \mathbb{R}$ such that $f_s = f_s'$ if $s \sim_W s'$, put $q(s) = q^{f_s}$ for $s \in S$. To these data, one can associate a unital associative $\mathbb{C}$-algebra $H = H(W, q)$ called the affine Hecke algebra. It has a $\mathbb{C}$-basis $\{T_w \mid w \in W\}$, subject to the relations $T_w T_{w'} = T_{ww'}$ if $l(w) + l(w') = l(ww')$ and $(T_s + 1)(T_s - q(s)) = 0$ for all $s \in S$. It is well known that such an object exists and is unique. The affine Hecke algebra is equipped with a natural trace determined by $T_w \mapsto \delta_{w,1}$.

These algebras arise naturally in the representation theory of a reductive $p$-adic group $\mathcal{G}$. In this setting, the best-known case is when $\mathcal{G}$ is the group of $F$-rational points of a split connected reductive group defined over $F$, $f_s = 1$ for all $s$ and $q = q$, the cardinality of the residue field of the $p$-adic field $F$. Then the irreducible representations of $\mathcal{H}$ are in bijection with the irreducible admissible representations of $\mathcal{G}$ having an Iwahori-fixed vector. Moreover, there exists a Plancherel measure preserving bijection between the Iwahori-spherical part of $\hat{\mathcal{G}}$ and $\hat{\mathcal{H}}$ (cf. [3]).

More generally, affine Hecke algebras $\mathcal{H}$ with non-equal labels $q(s)$ arise if one considers unipotent rather than Iwahori-spherical representations of $\mathcal{G}$ (see [13]). In this case as well, one is led to the question for a parametrization of the irreducible tempered representations, as well as for the Plancherel measure (the measure on $\hat{\mathcal{H}}$ giving the decomposition of $\tau$ of $\mathcal{H}$).

In the equal label case where $\mathcal{G}$ is of split adjoint type, a parametrization of $\hat{\mathcal{H}}$ was obtained by Kazhdan and Lusztig in [7], including criteria to select the tempered representations among $\hat{\mathcal{H}}$. For affine Hecke algebras with arbitrary labels $q(s)$, Opdam has determined the Plancherel formula in [17], as well as the classification of the central characters in the support of the Plancherel measure. It remains an open question, however, to determine the set of irreducible tempered representations of $\mathcal{H}$ which have a given central character.
A first step is to consider the case of real central character. In this case, by results of Lusztig, one can solve the problem of finding the irreducible tempered representations of $\mathcal{H}$ by solving the corresponding problem for a related (correctly specialized) graded Hecke algebra associated to the same root system. The graded Hecke algebra contains parameters $k_\alpha$, for simple $\alpha \in R_0$, depending on the parameters $q(s)$ (see (2.8) below).

Let $\mathbb{H}$ denote a graded Hecke algebra associated to $\mathcal{R}$ and parameters $k_\alpha$. That is, if $t^* = X \otimes Z C$, define $\mathbb{H} = C[W_0] \otimes S[t^*]$ with cross relations $x \cdot s_\alpha - s_\alpha \cdot s_\alpha(x) = k_\alpha(x, \tilde{x})$ for $x \in t^*$, and $\alpha$ simple. If all $k_\alpha = k$, an important role in the classification of the irreducible tempered representations of $\mathbb{H}$ is played by the Springer modules. Indeed, then the set $\hat{\mathbb{H}}^f(R)$ of equivalence classes of irreducible tempered $\mathbb{H}$-representations with real central character is given by a set of representing modules $\{M_\chi | \chi \in \hat{W}_0\}$, such that

1. The module $M_\chi$ is a naturally graded $W_0$-module whose top degree is $\chi$;
2. The bijection $\chi \mapsto M_\chi$ is completely determined by requiring that $\chi$ occurs in $M_\chi | W_0$ for all $\chi$;
3. $M_\chi$ and $M_{\chi'}$ have the same central character if and only if $\chi$ and $\chi'$ are Springer correspondents of the same unipotent class of $\hat{G}$, the complex simply connected reductive group with root system $R_0$;
4. Let $M_i^\chi$ be the degree-$i$ part of $M_\chi | W_0$. The multiplicity $(M_i^\chi, \psi)$ is given by (a coefficient of) a Green function.

In this article, we conjecture an analogue of (1)–(4) for a graded Hecke algebra attached to a root system $R_0$ of type $B_n$ (or, equivalently, $C_n$) but with unequal labels. Suppose that $k_1$ is the label of the long roots and $k_2$ the label of the short roots. We will always assume that $k_1 \neq 0$. If $k_2 = m k_1$ for $m \in \frac{1}{2} \mathbb{Z}$, the conjecture basically states that (1)–(4) still hold, although of course in (3) the set of unipotent class must be replaced by a combinatorial analogue. If $k_2$ is arbitrary, the conjecture explains how to retrieve $\hat{\mathbb{H}}^f(R)$ from the corresponding description for $k_2 = m k_1$, where $m \in \frac{1}{2} \mathbb{Z}$ and $|k_2/k_1 - m| < \frac{1}{2}$.

We could have also stated this conjecture for the affine Hecke algebra, but it is more natural to state it for the graded algebra. Indeed, by theorems of Lusztig (cf. [10]), the irreducible representations of $\mathcal{H}$ with given central character are in bijection with the irreducible representations of a certain graded Hecke algebra. We may obtain isomorphic graded Hecke algebras for central characters of differently labelled affine Hecke algebras.

The structure of this article is as follows. In Section 2, we explain in more detail the above-mentioned relation between the affine and the graded Hecke algebra obtained by Lusztig. We recall the notion of residual subspace (for $\mathbb{H}$), resp. residual coset (for $\mathcal{H}$) introduced by Heckman and Opdam, resp. Opdam. We are then able to state the characterization, obtained by Opdam, of the set of central characters of irreducible tempered representations of the affine Hecke algebra as the union of tempered forms of real residual cosets. Using Lusztig’s theorems we transfer this characterization to the graded Hecke algebra. From then on we focus on the graded algebra.
In Section 3, we explain the determination of $\hat{H}^f(R)$ in the equal label cases, using the Springer correspondence and Green functions.

In Section 4, we generalize the combinatorial description of this machinery to apply to arbitrary $k_2 = mk_1$. In this description we use a generalization of Lusztig’s symbols, which we call $m$-symbols, and of the Green functions. These are a particular choice of the ones used by Shoji in [20] in the context of complex reflection groups. We use the $m$-symbols to associate a set of Springer correspondents $\Sigma_m(W_0c)$ to a central character $W_0c$ of $\hat{H}^f(R)$. We then show that $\Sigma_m(W_0c)$ is in bijection with the set of central characters of the generic algebra which specialize into $W_0c$, if we choose $k_2 = mk_1$. This bijection yields a combinatorial explanation of the $m$-symbols in terms of Young tableaux. It supports the belief that for generic parameters, the modules in $\hat{H}^f(R)$ are separated by their central character (for type $B$ and $C$). Then we define a set of partitions $U_m(n)$ which replaces the set of unipotent conjugacy classes of $G$, and a (Springer-like) map associating a set of Weyl group characters to each $\lambda \in U_m(n)$. We show that, as in the equal label case, our “unipotent classes” are in bijection with the set of central characters of $\hat{H}^f(R)$. Therefore, as in the equal label case, we may also view the Springer correspondents as attached to a unipotent class, instead of to a central character.

In Section 5, we state our conjecture on the parametrization of $\hat{H}^f(R)$, and how the generalized Green functions compute the $W_0$-structure of the modules in $\hat{H}^f(R)$.

We end in Section 6 by giving some explicit examples of the combinatorial constructions and a particular case of the conjectures.

2. The graded and the affine Hecke algebra

In this section we recall the definitions of both types of Hecke algebras, in order to describe the set of central characters of irreducible tempered representations of the graded Hecke algebra. This is a translation of the corresponding description, due to Opdam in [17], for the affine Hecke algebra. This translation is done by using Lusztig’s results in [10], which describe the relation between the (representation theory of) the graded and the affine Hecke algebra.

2.1. The graded Hecke algebra

We begin by fixing some notation and introducing the graded Hecke algebra.

2.1.1. Definition

Let $\mathcal{R} = (R_0, X, \tilde{R}_0, Y, \Pi_0)$ be a reduced root datum. This means that $X$ and $Y$ are finitely generated free abelian groups with a perfect pairing $\langle \cdot, \cdot \rangle$. The set $R_0 \subset X$ forms a reduced root system in the real vector space $a^* = X \otimes_{\mathbb{Z}} \mathbb{R}$. Let $a^*$ have inner product $\langle \cdot, \cdot \rangle$ and let $Q = \mathbb{Z}R_0 \subset X$. Then $a^* = a^*_0 \oplus (a^*_0)^\perp$ where $a^*_0 = Q \otimes_{\mathbb{Z}} \mathbb{R}$. The set of coroots $\tilde{R}_0 \subset Y$ satisfies $\langle x, \tilde{\beta} \rangle = 2(x, \beta) / \beta \beta$ for $x \in R_0$ and $\langle x, \tilde{\beta} \rangle = 0$ for $x \in (a^*_0)^\perp$. Suppose $\text{rank}(Q) = \dim(a^*_0) = n$. The set $\Pi_0 \subset R_0$ (the simple roots)
is a choice \( z_1, \ldots, z_n \) such that every \( z \in R_0 \) can be written as a non-negative or non-positive \( \mathbb{Z} \)-linear combination of the \( z_i \in \Pi_0 \). We define the degenerate root datum associated to \( \mathcal{R} \) to be \( \mathcal{R}^{\text{deg}} = (R_0, \alpha^*, \tilde{R}_0, \alpha, \Pi_0) \).

Let \( t^* = X \otimes_{\mathbb{Z}} \mathbb{C} \) and \( t = Y \otimes_{\mathbb{Z}} \mathbb{C} \). Notice that \( t^* \) (resp. \( t \)) has distinguished real form \( \alpha^* \) (resp. \( \alpha = Y \otimes_{\mathbb{Z}} \mathbb{R} \)). This real form plays a crucial role in the definition of tempered representations (see below).

Every \( z \in R_0 \) determines a reflection \( s_z : X \to X : x \mapsto x - \langle x, \tilde{z} \rangle z \). The Weyl group \( W_0 \) defined as \( W_0 = \langle s_z \mid z \in R_0 \rangle \) is known to be generated by the simple reflections, i.e., \( W_0 = \langle s_z \mid z \in \Pi_0 \rangle \). Put \( I = \{ 1, 2, \ldots, n \} \).

Choose formal parameters \( k_z \) for \( z \in \Pi_0 \) such that \( k_z = k_\beta \) if \( z \) and \( \beta \) are conjugate under \( W_0 \). We denote by \( k \) the function \( z \mapsto k_z \). Then the graded (also called degenerate) Hecke algebra \( \mathcal{H} = \mathcal{H}(\mathcal{R}^{\text{deg}}, k) \) is by definition the cross product of algebras

\[
\mathcal{H} = \mathbb{C}[W_0] \otimes S[t^*] \otimes \mathbb{C}[k_z]. \tag{2.1}
\]

subject to the cross relations that the \( k_z \) are central and that

\[
x \cdot s_z - s_z \cdot x = k_z(x, \tilde{z}) \tag{2.2}
\]

for all \( x \in t^*, z \in \Pi_0 \).

Notice that \( \mathcal{H} \) is a graded algebra if we put \( t^* \) and the \( k_z \) in degree one and \( \mathbb{C}[W_0] \) in degree zero.

We will be concerned with certain specializations of this algebra. Choose \( k_z \in \mathbb{R} \) such that \( k_z = k_\beta \) if \( z \) is \( W_0 \)-conjugate to \( \beta \). Let \( k \) denote the function \( z \mapsto k_z \). Then we define

\[
\mathcal{H}(\mathcal{R}^{\text{deg}}, k) = \mathcal{H}(\mathcal{R}^{\text{deg}}, k) \otimes_{\mathbb{C}[k_z]} \mathbb{C}_k, \tag{2.3}
\]

where \( \mathbb{C}_k \) denotes the one-dimensional evaluation module on which \( k_z \) acts by \( k_z \). It is common to refer to this algebra as graded Hecke algebra as well. The goal of this article is to make a conjecture on the tempered representation theory of \( \mathcal{H}(\mathcal{R}^{\text{deg}}, k) \).

### 2.1.2. Parabolic algebras

Let \( P \subset I \) generate the standard parabolic root system \( R_P \subset R_0 \) and let \( \alpha_P \subset \alpha \) be the real span of \( \tilde{R}_P \). Then we define two degenerate root data \( \mathcal{R}^{\text{deg}}_P = (R_P, \alpha^*, \tilde{R}_P, \alpha_P, \Pi_P) \) and \( \mathcal{R}^{P,\text{deg}} = (R_P, \alpha^*, \tilde{R}_P, \alpha, \Pi_P) \). Let \( k_P \) denote the restriction of \( k \) to \( R_P \). Given \( \mathcal{H} = \mathcal{H}(\mathcal{R}^{\text{deg}}, k) \), we define \( \mathcal{H}_P = \mathcal{H}(\mathcal{R}^{\text{deg}}_P, k_P) \) and \( \mathcal{H}^P = \mathcal{H}(\mathcal{R}^{P,\text{deg}}, k_P) \).

### 2.1.3. Tempered representations

Let us review some well-known facts about the representation theory of the specialized algebra \( \mathcal{H} = \mathcal{H}(\mathcal{R}^{\text{deg}}, k) \).
If $V$ is a finite-dimensional $\mathbb{H}$-module, then the abelian algebra $S[t^*]$ induces a weight space decomposition

$$V = \bigoplus_{\lambda \in \mathfrak{t}^*} V_\lambda,$$

where

$$V_\lambda = \{ v \in V \mid \forall x \in t^*, \ (x - \lambda(x))^N v = 0 \text{ for some } N \in \mathbb{N} \}.$$

The $\lambda$ for which $V_\lambda \neq \{0\}$ are called the weights of $V$. If $V$ is irreducible, then by Dixmier’s version of Schur’s lemma, the center $Z$ of $\mathbb{H}$ acts by a character. Using the cross relations one can show [10] that $Z = S[t^*]W_0$. It follows that any irreducible representation is finite-dimensional, since $\mathbb{H}$ is finitely generated over $Z$. Let $\lambda \in \mathfrak{t}$ be such that for all $p \in Z$ and $v \in V$, we have $p \cdot v = p(\lambda)v$. Notice that $\lambda$ is determined up to $W_0$-orbit only. We call $W_0\lambda$ (or any of its elements) the central character of $V$.

We can now define the notion of temperedness. We define

$$a^{**} = \{ \lambda \in a^* \mid \langle \lambda, \tilde{z}_i \rangle \geq 0 \text{ for all } i \in I \},$$

and its antidual

$$a_- = \{ \xi \in a \mid \langle \lambda, \xi \rangle \leq 0 \text{ for all } \lambda \in a^{**} \}.$$

Then we define

**Definition 2.1.** Let $V$ be a finite dimensional $\mathbb{H}$-module.

(i) $V$ is called tempered if every weight $\gamma$ of $V$ satisfies $\text{Re}(\gamma) \in a_-$.  
(ii) $V$ is called a discrete series representation if, for every weight $\gamma$ of $V$, $\text{Re}(\gamma)$ lies in the interior of $a_-$. 

It follows that discrete series representations can only exist if $\text{rank}(R_0) = \text{rank}(X)$. We will be interested in the following set:

**Definition 2.2.** If $(\pi, V)$ is an irreducible representation of $\mathbb{H}$ with central character $W_0\lambda$, then we say that $(\pi, V)$ has real central character if $W_0\lambda \subset a$. We denote the set of equivalence classes of irreducible tempered representations of $\mathbb{H}$ which have real central character by $\hat{\mathbb{H}}^t_\mathbb{R}(\mathbb{R})$.

**Remark 2.3.** Notice that $\mathbb{H}_P = \mathbb{H}_P \otimes S[t^*]$, where $t^P$ is the orthogonal complement in $t$ of $t_P = Y_P \otimes \mathbb{C}$. In particular, the tempered representations of $\mathbb{H}_P$ are obtained as $U \otimes \mathbb{C}_v$ where $U$ is a tempered representation of $\mathbb{H}_P$ and $\mathbb{C}_v$ is the one-dimensional $t^P$-module belonging to $v \in i a^P$. 


Remark 2.4. It is easy to see that for all $\lambda \in \mathbb{C}^*$, the linear map $\psi_\lambda : \mathbb{H}(R^\text{deg}, k) \rightarrow \mathbb{H}(R^\text{deg}, \lambda k)$ which is uniquely determined by $x \mapsto \lambda^{-1} x$ and $w \mapsto w$, is in fact an algebra isomorphism.

We will only use such scaling isomorphisms in the case when they are well-behaved with respect to the notion of temperedness, i.e., when $\alpha^{*+}$ is preserved. This means that in the study of the tempered dual of $\mathbb{H}(R^\text{deg}, k)$, we only need to consider $k$ up to scaling by positive integers.

In the case where $R_0$ contains an irreducible component which has two root lengths, we can multiply the short roots in this component with a unique integer $p \in \{2, 3\}$ such that we again obtain a root system $R'_0$. If we define $k'$ by multiplication of the corresponding root label by $p$ then we can easily check that the identity map (on the level of vector spaces) $\psi : \mathbb{H}(R^\text{deg}, k) \rightarrow \mathbb{H}(R^\text{deg}, k')$ is in fact an algebra isomorphism. For example, the graded Hecke algebra of type $B_n$ with $k_1 = k_{e_i} \pm e_j$ and $k_2 = k_{e_i}$ is isomorphic to the graded Hecke algebra of type $C_n$ and labels $k_{e_i} \pm e_j = k_1$ and $k_{2e_i} = 2k_2$. Thus, the graded Hecke algebra of type $B_n$ and labels $k_2 = \frac{1}{2} k_1$ is isomorphic to the graded Hecke algebra of type $C_n$ with equal labels.

Remark 2.5. Let $\tau$ be a linear character of $W_0$. Define $\tau(k)$ by $\tau(k_2) = \tau(s_2) k_{2\tau}$. Define $\psi_\tau : \mathbb{H}(R^\text{deg}, k) \rightarrow \mathbb{H}(R^\text{deg}, \tau(k))$ induced by $\psi_\tau(w) = \tau(w)$ and $\psi_\tau(x) = x$. It is easy to check that this defines an isomorphism. Moreover one easily checks that this isomorphism respects the notion of temperedness and discrete series representation.

Notice that if $M$ is a tempered module for $\mathbb{H}(R, k)$ which has $W_0$-decomposition $M_{|W_0} = \sum_{\chi \in W_0} n_{\chi} \mathcal{H}$, then $M$ is also a tempered module for $\mathbb{H}(R, \tau(k))$, with $W_0$-decomposition $\sum_{\chi \in W_0} n_{\chi} (\mathcal{H} \otimes \tau)$, and vice versa.

For example, let $\tau = e$ be the sign character of $W_0$, suppose rank$(R_0) = \text{rank}(X)$ and consider the trivial representation $M$ of $\mathbb{H}(R^\text{deg}, k)$ defined by $s_i \mapsto 1$, $z_i \mapsto k_{z_i}$ (for simple $z_i$). Then $M$ is also a module for $\mathbb{H}(R^\text{deg}, -k)$ with $s_i \mapsto -1$; $z_i \mapsto k_{z_i} = -(-k_{z_i})$. If all $k_\chi < 0$, then these are discrete series modules. Thus, for $k_\chi < 0$, the trivial representation of $W_0$ is the unique one-dimensional representation of $W_0$ which extends to a discrete series representation of $\mathbb{H}$. If all $k_\chi > 0$, then this statement is true for the sign representation.

2.1.4. Residual subspaces

In our approach towards a classification of the tempered irreducible representations of the graded Hecke algebra, we use (following [4,17]) certain affine subspaces $L \subset \alpha$ which describe their central characters. These subspaces are defined as follows. Let $L \subset \alpha$ be an affine subspace. Define the parabolic root subsystem $R_L \subset R_0$ by

$$R_L = \{ \alpha \in R_0 \mid \alpha(L) = \text{constant} \}.$$  

Then we call $L$ a residual subspace if and only if

$$|\{ \alpha \in R_L \mid \alpha(L) = k_\chi \}| = |\{ \alpha \in R_L \mid \alpha(L) = 0 \}| + \text{codim}(L).$$  (2.4)
In particular, \( a \) itself is residual. It is clear that the notion of residual subspace is \( W_0 \)-invariant, since \( w(R_L) = R_{wL} \). The following property was proved in [4].

**Lemma 2.6.** If \( L \subset a \) is an affine subspace with codim(\( L \)) = rank(\( R_L \)) then, putting \( a_L = \tilde{\mathfrak{R}}_L \otimes \mathbb{Z} \mathbb{R} \), we have \( L = c_L + a_L^\perp \), where \( c_L = L \cap a_L \), and \( a_L^\perp = a_L^\perp \). We call \( c_L \) the center of \( L \). Then \( L \) is a residual subspace if and only if codim(\( L \)) = rank(\( R_L \)) and its center \( c_L \) is a residual point with respect to the graded Hecke algebra \( \mathbb{H}_L = \mathbb{H}(\mathcal{R}_L^{\text{deg}}, k_L) \), attached to the data \( \mathcal{R}_L^{\text{deg}} = (R_L, a_L^*, \tilde{\mathfrak{R}}_L, a_L, \Pi_L) \) and labeling function \( k_L \), the restriction of \( k \) to \( R_L \).

Therefore the classification of residual subspaces boils down to the classification of residual points. This classification has been done for all root systems in [4]. We remark that by Lemma 7.10 of [17] and induction on the rank of \( R_0 \), it is not hard to see that there are only finitely many residual subspaces.

The importance of the residual subspaces lies in the following fact, that we will prove below using the affine Hecke algebra. For a residual subspace \( L = c_L + a_L \), let \( L^{\text{temp}} = c_L + i a_L^\perp \subset t = a_C \) be the corresponding **tempered form** of \( L \), which we call a **tempered residual subspace**.

The following theorem, which we will prove in the next paragraph, is the starting point for the combinatorics in chapter 4.

**Theorem 2.7.** The collection \( \cup_L L^{\text{temp}} \) of all tempered residual subspaces of \( \mathbb{H} \) is equal to the set of central characters of irreducible tempered representations of \( \mathbb{H} \). Moreover, the set of central characters of discrete series representations is equal to the set of \( W_0 \)-orbits of residual points.

Notice that in particular, since \( k \in \mathbb{R} \), all discrete series representations have real central character.

### 2.2. The affine Hecke algebra

The proof of Theorem 2.7 relies on its analogue (Theorem 2.11) for the affine Hecke algebra, due to Opdam and Delorme-Opdam, and on a theorem of Lusztig (Theorem 2.12) which establishes an important connection between the graded and the affine Hecke algebra.

**2.2.1. Definitions**

Recall the root datum \( \mathcal{R} = (R_0, X, \tilde{\mathfrak{R}}_0, Y, \Pi_0) \). The elements of \( X \) act on \( X \) by translation, and it is easy to see that this action is normalized by the action of \( W_0 \). We define the extended affine Weyl group to be the product \( W = W_0 \times X \). Clearly \( W \) contains the normal subgroup \( W^a = W_0 \times Q \) generated by the affine reflections \( s_a \), where \( a = (\tilde{z}, k) \in R := \tilde{\mathfrak{R}}_0 \times \mathbb{Z} \), defined by

\[
s_{(\tilde{z}, k)}(x) = x - (\langle x, \tilde{z} \rangle + k)z.
\]
We call $R$ the affine root system. Let $\check{R}_0^{\min}$ denote the set of minimal coroots with respect to the dominance ordering. Define the simple affine roots to be the elements of $\Pi = \{(\check{\alpha}, 0) \mid \alpha \in \Pi_0\} \cup \{(\check{\alpha}, 1) \mid \check{\alpha} \in \check{R}_0^{\min}\}$. Then $W^a$ is a Coxeter group on generators $\{s_a \mid a \in \Pi\}$. We call $W^a$ the affine Weyl group. We have a decomposition $W = W^a \rtimes \Omega$, where $\Omega \cong X/Q$.

Notice that the Dynkin diagram of the affine root system is the affine extension of the Dynkin diagram of the coroot system of each irreducible component of $R_0$. In particular, if the root system $R_0$ is irreducible of type $B_n$, then the affine root system $R$ has Dynkin diagram of type $C_n^{\text{aff}}$. In this case, depending on the choice of $X$ one has two or three orbits of simple affine roots under the action of $W$; if $\check{\theta} \in 2Y$ then there are three orbits.

The affine Hecke algebra is a deformation of the group algebra of $W$. In order to define it we choose positive real numbers $q_a$ for every $a \in R$, such that $q_a = q_{wa}$ for every $a \in R, w \in W$. Let $l$ be the length function of $W$; that is, the extension of the length function of the Coxeter group $W^a$ to $W$ by putting $l(\omega) = 0$ for all $\omega \in \Omega$. Define, for a simple reflection $s_a$ of $W$, the number $q(s_a) = q_{a+1}$. In view of the assumptions on the $q_a$, there exists a unique $q : W \to \mathbb{R}_{>0}$ such that $q(ww') = q(w)q(w')$ if $l(ww') = l(w) + l(w')$, and $q(\omega) = 1$ for all $\omega \in \Omega$.

**Definition 2.8.** We denote by $H = H(R, q)$ the affine Hecke algebra associated to $(R, q)$, i.e., the unique complex associative algebra with $C$-basis $(T_w)_{w \in W}$ satisfying the following relations:

$$
\begin{align*}
\text{If } l(ww') = l(w) + l(w') \text{ then } T_{ww'} &= T_w T_{w'}, \\
\text{If } a \in \Pi, s = s_a, \text{ then } (T_s + 1)(T_s - q(s)) &= 0.
\end{align*}
$$

(2.5)

It is well-known that such an object exists and is indeed unique. It admits a decomposition $H = \mathcal{H}_0 \otimes \mathcal{A}$, where $\mathcal{H}_0$ is the algebra spanned by $T_w, w \in W_0$ and $\mathcal{A}$ is a commutative algebra which is isomorphic to $\mathbb{C}[X]$ through $x \mapsto \theta_x \in \mathcal{A}$. The cross relations are known as the Bernstein–Zelevinsky–Lusztig relations. To write them down, it is convenient to define root labels for the possibly non reduced root system $R_{nr}$, where

$$
R_{nr} = R_0 \cup \{2\alpha \mid \check{\alpha} \in \check{R}_0 \cap 2Y\}.
$$

(2.6)

We then label the coroots of $R_{nr}$. If $\check{\alpha} \in \check{R}_{nr} \cap \check{R}_0$ then we put $q_{\check{\alpha}} = q(\check{\alpha}, 0)$ and if $\check{\alpha}/2 \in \check{R}_{nr} \setminus \check{R}_0$ then we define

$$
q_{\check{\alpha}/2} = \frac{q(\check{\alpha}, 1)}{q(\check{\alpha}, 0)}.
$$

(2.7)
If \( x \in R_0, 2x \notin R_{nr} \) then we define \( q_{\frac{1}{2}}/a = 1 \). With this notation, we have for \( s = s_x, x \in \Pi_0 \) and \( x \in X \):

\[
\theta_s T_s - T_s \theta_{s(x)} = ((q_{\frac{1}{2}}/a - 1) + q_{\frac{1}{2}}/a (q_{\frac{1}{2}}/a - 1) - 1) \theta_{x - \theta_s(x)}.
\]

By an unpublished result of Bernstein (see [10]), the center \( Z \) of \( H \) can be identified to be \( Z = A W_0 \).

Let \( T = \text{Hom}(X, \mathbb{C}^*) \), then \( \text{Spec}(Z) = T / W_0 \). In an irreducible representation \((\pi, V)\) of \( H \), there exists a \( t_t \in T \) such that any \( z \in Z \) acts by \( (t_t) \). We call \( W_0 t_t \), or by abuse of terminology, any of its elements, the central character of \((\pi, V)\).

The torus \( T \) has polar decomposition \( T = T_u T_{rs} = \text{Hom}(X, S^1) \text{Hom}(X, R_0^{>0}) \).

**Definition 2.9.** Let \((\pi, V)\) be a finite dimensional representation of \( H \).

(i) We call \( V \) a tempered representation if all \( A \)-weights \( t \) of \( V \) satisfy \( |t(x)\| \leq 1 \) for all \( x \in X^+ = \{ x \in X \mid \langle x, \tilde{x} \rangle \geq 0 \ \forall \ x \in \Pi_0 \} \).

(ii) We call \( V \) a discrete series representation if all \( A \)-weights of \( V \) satisfy \( |t(x)| < 1 \) for all \( 0 \neq x \in X^+ \).

**Definition 2.10.** Let \( \tilde{\mathcal{H}}^t(\mathbb{R}) \) denote the set of equivalence classes of irreducible tempered representations of \( H \), whose central character lies in \( T_{rs} \).

### 2.2.2. Residual cosets

By theorems of Opdam (cf. [17]) and Delorme–Opdam (cf. [2]), the set of central characters of irreducible tempered representations of \( H \) can be described in terms of the affine analog of the residual subspaces for \( \mathbb{H} \). Let \( L \subset T \) be a coset of a subtorus \( T^L \subset T \). Put \( R_L = \{ x \in R_0 \mid \alpha(T^L) = 1 \} \) and \( W_L = W_0(R_L) \). Define

\[
R^p_L = \{ x \in R_L \mid \alpha(L) = -q_{\frac{1}{2}}/a \text{ or } \alpha(L) = q_{\frac{1}{2}}/a q_{\frac{1}{2}}/a \}
\]

and

\[
R^c_L = \{ x \in R_L \mid \alpha(L) = \pm 1 \}.
\]

By definition, we call \( L \) a residual coset if and only if

\[
|R^p_L| = |R^c_L| + \text{codim}(L).
\]

A zero-dimensional residual coset is called a residual point. Sometimes we will also say “residual point” for “\( W_0 \)-orbit of residual points”. By Theorem 3.29 and Corollary 3.30 in [17], a residual point is the common central character of a non-empty set of irreducible discrete series representations of \( H \). Conversely, the central character of an irreducible discrete series representation is always a residual point.
For tempered representations in general, we need the notion of tempered form of a residual coset. Suppose that $L$ is a residual coset, and let $T_L$ be the subtorus of $T$ whose Lie algebra is spanned by $\mathbf{R}_L$. Then we may choose $r_L$ in $L$ such that $L = r_L T_L$ and $r_L \in L \cap T_L$. We call $r_L$ the center of $L$. The tempered form of $L$ is defined to be $L_{\text{temp}} = r_L T_L$. Let $\mathcal{L}$ be the set of residual cosets (it is not hard to see that $\mathcal{L}$ is finite). We have

**Theorem 2.11 (Opdam, Delorme–Opdam).**

$$\bigcup_{L \in \mathcal{L}} L_{\text{temp}} = \{\text{central characters of irr. tempered representations of } \mathcal{H}\}.$$  

**Proof.** The inclusion $\subset$ follows from [17, Theorems 3.29 and 4.23]. The other follows from [2] where it is shown that every irreducible tempered representation occurs in the support of the Plancherel measure for $\mathcal{H}$. □

For a residual coset $L$ with center $r_L$, it is not hard to see that in the polar decomposition $r_L = s_L c_L \in T_u T_{rs}$, the real part $c_L$ is independent of the choice of $r_L$. We call a residual coset real if we can choose $s_L = 1$.

Let $L = r_L T_L$ be a real residual coset, and let $c_L = \log(r_L) \in \text{Lie}(T_L, rs)$. Then $c_L + \text{Lie}(T_{rs}^L)$ is a residual subspace for $\mathfrak{h}(\mathcal{R}_\text{deg}, k)$ with root labels $k_x$ as in (2.8) below, and vice versa.

### 2.2.3. Parabolic subalgebras

The proof of Theorem 2.7 uses the parabolic nature of the classification of residual subspaces. Therefore we fix some more notation here. Let $R_L$ be a standard parabolic root subsystem of $R_0$ with simple roots $\Pi_L \subset \Pi_0$. Then $\mathcal{R}_L = (R_L, X, \mathbf{R}_L, Y, \Pi_L)$ is a root datum. Define also

$$Y_L = Y \cap \mathbb{R} \mathbf{R}_L \text{ and } X_L = X / (X \cap (\mathbf{R}_L)^\perp).$$

Then $\mathcal{R}_L = (R_L, X_L, \mathbf{R}_L, Y_L, \Pi_L)$ is a root datum satisfying $\text{rank}(R_L) = \text{rank}(X_L)$. On $R_{L, \text{nr}} := \mathbb{Q} R_L \cap R_{\text{nr}}$ we define root labels $q_{L, x} = q_{x}^{L}$ by restricting $q$ from $R_{\text{nr}}$ to $R_{L, \text{nr}}$. We extend $q_L$, resp. $q_L^{L}$, to $W_0(R_L) \ltimes X_L$, resp. $W_0(R_L) \ltimes X$. Then we define $\mathcal{H}_L = \mathcal{H}(\mathcal{R}_L^L, q^L)$ and $\mathcal{H}_L = \mathcal{H}(\mathcal{R}_L, q_L)$. Note that $\mathcal{H}_L$ can naturally be identified with a subalgebra of $\mathcal{H}$.

### 2.2.4. Lusztig’s reduction theorem

In [10], Lusztig establishes the connection between the representation theory of the affine and graded Hecke algebra. Since these reductions play an essential role, we will explain them in some detail. It will be necessary to slightly adapt his constructions. Suppose the parameters of $\mathbb{H}$ depend on those of $\mathcal{H}$ as follows (in view of [17, (7.8)]):

$$k_x = \log(q^{3/2}q^{1/2}_x).$$  

(2.8)
Now fix a real central character $W_0t$ of an irreducible representation of $\mathcal{H}$. Denote the corresponding maximal ideal of $Z$ by $I_t$, and let $\hat{Z}_t$ be the $I_t$-adic completion of $Z$. Furthermore, we put

$$\hat{\mathcal{H}}_t = \mathcal{H} \otimes_Z \hat{Z}_t.$$  

Denote the set of irreducible representations of $\mathcal{H}$ with central character $W_0t$ by $\text{Irr}_t(\mathcal{H})$. Then

$$\text{Irr}_t(\mathcal{H}) \leftrightarrow \text{Irr}(\hat{\mathcal{H}}_t), \quad (2.9)$$

which follows from the fact that

$$\mathcal{H}/I_t\mathcal{H} \cong \hat{\mathcal{H}}_t/\hat{I}_t\hat{\mathcal{H}}_t,$$

where $\hat{I}_t$ denotes the maximal ideal of $\hat{Z}_t$ corresponding to $I_t$.

Analogously, for $W_0\gamma$ the $W_0$-orbit of $\gamma \in \alpha$, after carrying out the analogous constructions for $\hat{\mathcal{H}}$ we have

$$\text{Irr}_\gamma(\mathcal{H}) \leftrightarrow \text{Irr}(\hat{\mathcal{H}}_\gamma). \quad (2.10)$$

Let $\hat{\mathcal{H}}(\mathbb{R})$ denote the set of equivalence classes of irreducible representations of $\mathcal{H}$ with real central character, and analogously for $\hat{\mathcal{H}}(\mathbb{R})$. We can now prove the following:

**Theorem 2.12 (Lusztig).** There exists a natural bijection

$$\{\text{irr. representations in } \hat{\mathcal{H}}(\mathbb{R})\} \leftrightarrow \{\text{irr. representations in } \hat{\mathcal{H}}(\mathbb{R})\}.$$  

**Proof.** This is basically Lusztig’s second reduction theorem (Theorem 9.3 in [10]). We explain how to adapt his construction, since he works with the different assumptions that the root labels $q_\lambda$ are of the form $q_\lambda = q^{n_\lambda}$ for some $q \in \mathbb{C}^*$ and $n_\lambda \in \mathbb{N}$, and moreover that for all $t' \in W_0t$, one has $\alpha(t') \in \langle q \rangle$ (the group in $\mathbb{C}^*$ generated by $q$) if $\tilde{\alpha} \notin 2Y$, and $\alpha(t') \in \pm\langle q \rangle$ if $\tilde{\alpha} \in 2Y$.

Fix a real central character $W_0t \subset T_{rs}$. Since the exponential map, restricted to $\alpha$, yields a $W_0$-equivariant isomorphism $\exp : \alpha \to T_{rs}$, it gives rise to a bijection $W_0\gamma \to W_0t$, where $\gamma \in \alpha$ is such that $\exp(\gamma) = t$. In Lusztig’s notation, this means that $t_0 = 1$. Our assumption that all $q_\lambda > 1$ then implies that Lemma 9.5 still holds. Hence, Theorem 9.3 still holds: the algebras $\hat{Z}_t$ and $\hat{Z}_\gamma$ are isomorphic, and moreover

$$\hat{\mathcal{H}}_t \cong \hat{\mathcal{H}}_\gamma.$$
This isomorphism is compatible with the \( \hat{\mathbb{Z}} \cong \hat{\mathbb{Z}} \)-structures. We therefore find a natural bijection between the irreducible representations of \( \hat{\mathcal{H}}_t \) and the irreducible representations of \( \hat{\mathcal{H}}_{t/R} \). Combining with (2.9), (2.10) and the fact that this holds for any real central character \( W_0 \), we find the desired result. □

It is clear from the constructions that if the \( \mathcal{H} \)-module \( V \) corresponds to the \( \mathcal{H} \)-module \( U \) under this bijection, then \( V \) is a tempered (resp. discrete series) module if and only if \( U \) is a tempered (resp. discrete series) module. Therefore we obtain in particular a bijection

\[
\hat{\mathcal{H}}_t (\mathbb{R}) \leftrightarrow \hat{\mathcal{H}}'_t (\mathbb{R}).
\]  

(2.11)

Lemma 2.13. Let \( V \) be an irreducible discrete series representation of \( \mathcal{H} \) with central character \( \gamma \). Then \( \gamma \in a \), i.e., \( V \) has real central character.

Proof. We apply the analogue of Lusztig’s first reduction theorem ([10, Theorem 8.6]) to \( \mathcal{H} \) (instead of \( \mathcal{H} \)). We adapt his definition of the root subsystem \( R_c \), and use the parabolic root system \( a^*_R \) of [17], (4.15) instead. Explicitly, \( R_c = R_0 \cap a^*_R \), where \( a^*_R \) is the \( \mathbb{R} \)-span of \( \{ \alpha \in R_0 \mid \alpha(\gamma) \in \{0, \pm k \} \} \). We choose \( \gamma \) in its \( W_0 \)-orbit such that \( R_c = R_P \), a standard parabolic subsystem and put \( c = W_P \gamma \). Let \( \Gamma \subset W_0 \) be the group \( \{ w \in W_0 \mid w(c) = c \text{ and } w(R_P^+) = R_P^+ \} \), and \( n = |W_0\gamma| / |W_P \gamma| \). Then \( \Gamma \) acts on \( \hat{\mathcal{H}}_P \) and on the completed algebra \( \hat{\mathcal{H}}_P \) by means of diagram automorphisms. We define \( \hat{\mathcal{H}}_P [\Gamma] = \oplus_{g \in \Gamma} \hat{\mathcal{H}}_P \cdot g \) with multiplication \( (a, g)(a', g') = (ag(a'), gg') \). According to [10] (see also [17, Theorem 4.10]) we have

\[
\hat{\mathcal{H}}_P \cong (\hat{\mathcal{H}}_P [\Gamma])_n,
\]

(2.12)

where \( \hat{\mathcal{H}}_P \) is the completed algebra as in (2.10), and the right-hand side denotes the algebra of \( n \times n \)-matrices with entries in \( \hat{\mathcal{H}}_P [\Gamma] \). The upshot is that \( V = \text{Ind}_{\hat{\mathcal{H}}_P[\Gamma]}^{\mathcal{H}} U \), where \( U \) is an irreducible \( \mathcal{H} [\Gamma] \)-module. Furthermore, by Lusztig’s explicit description of the isomorphism (2.12), we obtain the following formula for the set \( \text{Wt}(V) \) of weights of \( V \):

\[
\text{Wt}(V) = \bigcup_{w \in W_P} w \cdot \text{Wt}(U),
\]

(2.13)

where \( W_P \) denotes the set of coset representatives of \( W_0 / W_P \) of minimal length.

Suppose that \( \gamma \notin a \), then \( R_c(= R_P) \neq R_0 \). The assumption that \( V \) is a discrete series module implies that \( \gamma' \in \text{Wt}(U) \) satisfies \( \text{Re}(\gamma') \in \sum_{i \in I} \mathbb{R} < 0 \hat{z}_i \). Let \( w_P \) (resp. \( w_P^+ \)) be the longest element of \( W_P \) (resp. \( W_P^+ \)), then \( \text{Re}(w_P \gamma') \neq \sum_{i \in I} \mathbb{R} < 0 \hat{z}_i \) while on the other hand by (2.13), \( w_P^+ \gamma' \in \text{Wt}(V) \). This is a contradiction, so the Lemma follows. □
2.2.5. Proof of Theorem 2.7

Let $L_{\text{temp}} = \gamma_L + ia_L$ be a tempered residual subspace of $\mathbb{H}$, and let $\gamma_L + ix \in L_{\text{temp}}$. Then $r_L = \exp(\gamma_L)$ is a residual point for $\mathcal{H}_L$, and by [17] it is then the central character of a non-empty set $\Delta_{W_L}r_L$ of discrete series representations of $\mathcal{H}_L$. Since $r_L$ is real, we can invoke 2.12 to see that $\gamma_L$ is the central character of a set of discrete series representations of $\mathbb{H}_L$, which is in natural bijection with $\Delta_{W_L}r_L$. It follows that $\gamma_L + ix$ is the central character of a tempered representation of $\mathbb{H}_L$, and (by induction) also of $\mathbb{H}$.

Conversely, let $V$ be an irreducible tempered representation of $\mathbb{H}$ with central character $W_0\gamma$. Let the set of weights of $V$ be $\text{Wt}(V)$. We then define, analogous to the analysis in [2] of (weak) constant terms of tempered representations of the affine Hecke algebra, the sets

$$E_P^0(V) = \{\lambda \in \text{Wt}(V) \mid \text{Re}(\lambda) \in \mathbb{R}\bar{\mathbb{C}}} P\},$$

where $P \subset I$. Now let $P$ be minimal such that $E_P^0(V) \neq \emptyset$. Then it is easy to see that

$$V_P = \sum_{\lambda \in E_P^0(V)} V_{\lambda}$$

is a non-zero tempered $\mathbb{H}_P$-module, and minimality of $P$ implies that for all $\lambda \in \text{Wt}(V_P) = E_P^0(V)$, we have $\lambda \in \sum_{x \in \Pi_P} \mathbb{R}_{<0} x$. Now take an irreducible subquotient $U$ of $V_P$, then $\text{Wt}(U) \subset \text{Wt}(V_P)$, so $U$ is an irreducible, tempered $\mathbb{H}_P = \mathbb{H}_P \otimes S[t,P^*]$-module, which implies (see Remark 2.3) that $U = U_P \otimes \mathbb{C}_V$, where $U_P$ is an irreducible discrete series module of $\mathbb{H}_P$. Since $U$ is a tempered $\mathbb{H}_P$-module, $\text{Re}(v)|_{t,P^*} = 0$, so $v \in ia_P$. By Lemma 2.13, $U_P$ has real central character $W_P\lambda$. We choose $\gamma$ in its $W_0$-orbit such that $\gamma = \lambda + v$. Finally, we apply Lusztig’s Theorem 2.12, combined with [17, Theorem 3.29], to conclude that $\lambda$ is a residual point for $\mathbb{H}_P$. It follows that $\gamma \in \lambda + ia_P$, i.e., indeed $\gamma$ lies in a tempered residual subspace of $\mathbb{H}$. □

Corollary 2.14. The set $\hat{\mathbb{H}}^I(\mathbb{R})$ is finite.

Proof. There are only finitely many equivalence classes of irreducible discrete series representations of $\mathbb{H}$ whose central character is a prescribed residual point. Therefore, the number of residual points being finite, there are only finitely many equivalence classes of irreducible discrete series representations of $\mathbb{H}$. Secondly, every irreducible tempered representation $V$ of $\mathbb{H}$ occurs as a summand of the unitary induction of a representation $U \otimes \mathbb{C}_v$ of some $\mathbb{H}_P$, where $U$ is a discrete series representation of $\mathbb{H}_P$. If $V$ has real central character, then $v = 0$. So the finitely many summands of $\text{Ind}_{\mathbb{H}_P}^{\mathbb{H}}(U_P \otimes \mathbb{C}_0)$ (with $P \subset I$ and $U_P$ an irreducible discrete series representation of $\mathbb{H}_P$) exhaust the set $\hat{\mathbb{H}}^I(\mathbb{R})$. □
3. $\hat{\mathcal{H}}'(\mathbb{R})$ and Springer correspondence in the equal label case

In this section, we recall the description of $\hat{\mathcal{H}}'(\mathbb{R})$ in the classical case where the root labels $k_\alpha$ are all equal. In this case, there are $|\hat{W}_0|$ such modules. They are in fact none other than the Springer modules, hence they have a natural grading as $W_0$-module. One can compute their $W_0$-structure in every degree with Green functions.

3.1. The Springer correspondence

In the equal label case, $\hat{\mathcal{H}}'$ has been determined by Kazhdan and Lusztig in [7]. They assume that $X = P$. In view of Lusztig’s theorem and in particular the bijection (2.11), we give their description for $\mathcal{H}$ rather than for $\mathcal{H}_L$. Let $\mathcal{H}$ have labels $q_\alpha = q > 1$ for all $\alpha$ (notice that $X = P$ implies that all $q_{\alpha/2} = 1$, put $k = \log(q)$ (in view of (2.8)) and let $\mathcal{H}$ be the associated graded Hecke algebra with $k_\alpha = k$ for all $\alpha$.

Let $\hat{G}$ be the complex reductive group whose root system is $(R_0, X, \hat{\mathcal{R}}_0, Y, \Pi_0)$. For a unipotent $u \in \hat{G}$, let $B_u$ be the variety of Borel subgroups of $\hat{G}$ containing $u$. Putting $\dim(B_u) = d_u$, the highest non-vanishing cohomology group of $H(B_u)$ is $H^{2d_u}(B_u)$. Springer has shown that $H(B_u)$ is a $W_0$-module, and that the action of $W_0$ respects the grading. The group $A(u) = C_{\hat{G}}(u)/Z_{\hat{G}}C_{\hat{G}}(u)$ also acts on $H(B_u)$, such that the actions of $W_0$ and $A(u)$ commute. Let, for $\rho \in \hat{A}(u)$, $H(B_u)_\rho$ be the $\rho$-isotypic part in $H(B_u)$. Springer has shown that (i) if $H^{2d_u}(B_u)_\rho \neq 0$, then it is an irreducible $W_0$-module (whose character we will denote by $\iota_{u, \rho}$); (ii) If $\iota_{u, \rho} = \iota_{u', \rho'}$ then $(u, \rho)$ and $(u', \rho')$ are conjugate under $\hat{G}$; (iii) for every $\chi \in \hat{W}_0$ there exists a pair $(u, \rho)$ such that $\chi = \iota_{u, \rho}$.

Given $u$, we denote by $\hat{A}(u)_0$ the set of irreducible characters $\rho \in \hat{A}(u)$ for which $H(B_u)_\rho \neq 0$. In general, $\hat{A}(u)_0$ is strictly smaller than $\hat{A}(u)$, but never empty: it is known to always contain the trivial representation. We write

$$\mathcal{I}_0 := \{(u, \rho) \mid u \in \hat{G} \text{ unipotent}, \rho \in \hat{A}(u)_0\},$$

(3.1)

where we identify conjugate pairs. Then $\mathcal{I}_0 \leftrightarrow \hat{W}_0$; this bijection is called the Springer correspondence. Given a unipotent class $C \subset \hat{G}$, we denote the set of its Springer correspondents by

$$\Sigma(C) = \{\iota_{u, \rho} \mid \rho \in \hat{A}(u)_0, u \in C\} \subset \hat{W}_0.$$

**Theorem 3.1** (Kazhdan and Lusztig [7]). $\hat{\mathcal{H}}'(\mathbb{R})$ is naturally parametrized by $\mathcal{I}_0$ (and hence also by $\hat{W}_0$). Let $M_{u, \rho}$ be the $\mathcal{H}$-module which corresponds to $(u, \rho) \in \mathcal{I}_0$. Then $M_{u, \rho}$ and $M_{u', \rho'}$ have the same central character if and only if $u = u'$.

We will also write $M_{\chi}$ for $M_{u, \rho}$ if, through the Springer correspondence, $\chi = \iota_{u, \rho}$. In fact, the restrictions to $W_0$ of the $M_{\chi}$ yield the Springer modules:
Theorem 3.2 (Lusztig [11,14]). In the above setting, we have

$$M_{\mu, \rho} |_{w_0} = \varepsilon \otimes H(B_u)_{\rho}.$$  \hspace{1cm} (3.2)

In particular the elements of $\hat{\mathfrak{h}}'(\mathbb{R})$, viewed as $W_0$-modules, are graded (by homological degree).

It follows from Theorems 2.7 and 3.1 that in the equal label cases, the central characters of $\hat{H}_t(R)$ are indexed both by the $W_0$-orbits $W_0cL$ of the centers of the residual subspaces $L$ of $\mathbb{H}$, and by the unipotent conjugacy classes in $\hat{G}$. Therefore, there exists a bijection between these two sets, such that if $W_0cL$ corresponds to the unipotent class $C \subset \hat{G}$, then the modules in $\hat{H}_t(R)$ with central character $W_0cL$ are the modules $M_{\mu, \rho}$ such that $u \in C$. From [17, Appendix] we recall the description of this bijection. It is the map

$$u \mapsto W_0 \gamma(u) \subset \alpha, \hspace{1cm} (3.3)$$

where $\gamma(u)$ is determined from $u$ as follows: suppose that the weighted Dynkin diagram of $u$ is labeled $(x_1, \ldots, x_n)$, then $\gamma(u) \in \alpha$ is determined by the equations $z_i(\gamma(u)) = \frac{k}{2} x_i$.

This map is such that if $u$ corresponds via the Bala–Carter classification (cf. [1, Theorem 5.9.5]) to the pair $(L, P_L)$ where $L$ is a Levi subgroup of $\hat{G}$ and $P_L$ is a distinguished parabolic subgroup of the semisimple part of $L$, then $W_0 \gamma(u) = W_0c_M$ where $M$ is the residual subspace with root system $R_M = R_L$ and center $c_M$. In particular, we obtain a bijection between distinguished unipotent classes of $\hat{G}$ and $W_0$-orbits of residual points of $H$.

3.2. Green functions

Let $q = p^s$ for a prime $p$. Let $k$ be an algebraically closed field of characteristic $p$. Let $G$ be a connected reductive algebraic group defined over $k$, with a split $\mathbb{F}_q$-structure, and corresponding Frobenius map $F$. Then $G^F$ is a finite group of Lie type. We assume that $G$ and $G^F$ have root system $(R_0, X, \tilde{R}_0, Y, \Pi_0)$ of classical type.

Let $C$ be a unipotent class in $G$, then, since we assume that $G$ is split, $C$ is $F$-stable. It is known (by [18]) that there exists a representative $u \in C^F$, unique up to $G^F$-conjugacy modulo the center of $G$, such that $F$ acts trivially on the component group $A(u) = C_G(u)/C_G(u)^0$, and the set of $G^F$-conjugacy classes in $C^F$ is in bijective correspondence with the set $A(u)/\sim$ of conjugacy classes of $A(u)$. We denote by $u_a$ a representative of the $G^F$-conjugacy class in $C^F$ represented by $a \in A(u)/\sim$. Finally, let $B_u$ denote the variety of Borel subgroups of $G$ containing $u$. Since $\hat{G}$ and $G$ have the same root datum, we have $H(B_u) = H(\mathbb{B}_u)$ (where for simplicity $u$ also denotes the corresponding unipotent element in $\hat{G}$ which has the same Jordan decomposition.
as \( u \in G \). For \( w \in W_0 \) we define the Green function \( Q_w \) as

\[
Q_w(g) = \sum_{l=0}^{d_u} \text{Tr}((w, a), H^{2l}(B_u))q^l,
\]

if \( g \in G_{\text{uni}}^F \) is \( G^F \)-conjugate to \( u_a \).

In [9], Lusztig finds an algorithm to compute the Green functions as solutions of a matrix equation. We briefly review it here. Let \( \chi \in \hat{W}_0 \), and let \( \chi \leftrightarrow i \in I_0 \) under the Springer correspondence. Then one defines a function \( Q_i = Q_{\chi} \), also called Green function, by

\[
Q_{\chi} = \frac{1}{|W_0|} \sum_{w \in W_0} \chi(w)Q_w.
\]

Define for each \( i = (u, \rho) \in I_0 \), a \( G^F \)-invariant function \( Y_i \) on \( G_{\text{uni}}^F \) by

\[
Y_i(g) = \begin{cases} 
\rho(a) & \text{if } g \text{ is } G^F - \text{conjugate to } u_a, \\
0 & \text{if } g \notin C^F,
\end{cases}
\]

where \( u \in C \). It has been shown by Lusztig [9] that there exist \( \pi_{ji} \) such that

\[
Q_i = \sum_{j \in I_0} \pi_{ji}Y_j. \tag{3.5}
\]

**Proposition 3.3** (Lusztig [12], Shoji [19]). There exist polynomials \( \pi_{ij} \) in \( \mathbb{Z}[t] \), which are independent of \( q \), such that \( \pi_{ij} = \pi_{ij}(q) \).

**Proof.** In view of (3.4) and (3.5), if \( j = (u, \rho) \) and \( Q_i = Q_{\chi} \) then the entry \( \pi_{ji} \) can be expressed as

\[
\pi_{ji} = \sum_{l \geq 0} \langle H^{2l}(B_u), \chi \otimes \rho \rangle q^l, \tag{3.6}
\]

where \( \langle H^{2l}(B_u), \chi \otimes \rho \rangle \) denotes the multiplicity of \( \chi \otimes \rho \) in \( H^{2l}(B_u) \). Moreover, it is known that the structure of \( H^{2l}(B_u) \) as a \( W_0 \times A(u) \)-module is independent of \( p \). \( \square \)

Given a unipotent \( u \in G \), we denote the unipotent class in which it lies by \( C_u \). We define a pre-ordering on \( I_0 \) such that \( i = (u, \rho) \preceq i' = (u', \rho') \) if \( C_u \subset C_{u'} \). Let \( \sim \) be the associated equivalence relation \( i \sim i' \iff C_u = C_{u'} \), and refine the pre-ordering into a total one.
Let $t$ be an indeterminate. For a class function $f$ on $W_0$, we define $R(f)$ by

$$R(f) = (t - 1)^n P_0(t) \frac{1}{|W_0|} \sum_{w \in W_0} \varepsilon(w) f(w) \det(t \text{id} - w),$$

where $P_0(t)$ is the Poincaré polynomial, $\varepsilon$ is the sign character of $W_0$ and the determinant is taken in the natural reflection representation of $W_0$. For an irreducible character $\chi \in \hat{W}_0$, $R(\chi)$ is the fake degree of $\chi$. We define the matrix $\Omega = (\omega_{ij})_{i,j \in \mathcal{I}_0}$ to be the matrix whose entries are

$$\omega_{ij} = t^N R(\chi_i \otimes \chi_j \otimes \varepsilon),$$

where $N$ is the number of positive roots in $R_0$.

**Theorem 3.4 (Lusztig [9]).** Let $P = (P_{ij})_{i,j \in \mathcal{I}_0}$ and $\Lambda = (\Lambda_{ij})_{i,j \in \mathcal{I}_0}$ be matrices of unknowns, subject to

$$\begin{cases} 
\Lambda_{ij} = 0 & \text{unless } i \sim j, \\
P_{ij} = 0 & \text{unless either } i < j \text{ and } i \not\sim j, \text{ or } i = j, \\
P_{ii} = t^{d_v}, i = (u, \rho), \\
P \Lambda \ t P = \Omega.
\end{cases}$$ (3.7)

This equation has unique solution matrices. The matrix $P$ satisfies $P_{ij}(q) = \pi_{ij}$. In particular, $P_{ij} \in \mathbb{Z}[t]$.

**4. Combinatorial generalization of the Springer correspondence**

For a root system of classical type, the Springer correspondence and Eq. (3.7) can all be written in terms of combinatorial objects. In this section we review this description and then generalize it, for a graded Hecke algebra of type $B$ or $C$ with arbitrary labels. Note that a graded Hecke algebra of type $A$ or $D$ only exists with equal labels.

First we establish some notation. For $m \geq 0$, let $[m]$ be the smallest integer at least equal to $m$ and let $\lfloor m \rfloor$ be the biggest integer at most equal to $m$. We write $\lambda \vdash n$ to denote that $\lambda$ is a partition of $n$. We write $\mathcal{P}_{n,2}$ for the set of double partitions of $n$, that is, the set of pairs $(\xi, \eta)$ where $\xi$ and $\eta$ are partitions such that $|\xi| + |\eta| = n$. Let $\mathcal{P}_n$ be the set of partitions of $n$. We will usually write the parts of a partition in increasing order; parts of length zero are allowed.

Let $W_0$ be the Weyl group of type $B_n$ and $\hat{W}_0$ the set of its irreducible characters. It is well known that $\hat{W}_0$ is in bijection with $\mathcal{P}_{n,2}$. We adopt the explicit choice of this bijection, where the representation indexed by $(\xi, \eta)$ has a basis indexed by standard Young tableaux of shape $(\xi, \eta)$, as in [5]. In particular, the trivial representation is indexed by $(n, -)$ and the sign representation by $(-, 1^n)$. We denote an irreducible
representation of \( W_0 \) by \((\pi_A, V_A)\) for \( A \in \mathcal{P}_{n,2} \), and its character by \( \chi_A \) or sometimes simply by \( A \). We will sometimes also write \( \varepsilon \) for \( \chi_{(-,1^n)} \), the sign character.

4.1. Special and generic parameters

One may also use \((\ref{eq:2.4})\) to define residual subspaces for the generic algebra \( \mathcal{H}(\mathcal{R}^{\text{deg}}, k) \). Upon specialization \( k_{\alpha} \mapsto k_{\alpha} \), we distinguish between the following possibilities.

**Definition 4.1.** Let \( \mathcal{H} = \mathcal{H}(\mathcal{R}, k) \). For \( P \subset I \), let \( \mathcal{H}_P = \mathcal{H}(\mathcal{R}^{\text{deg}}_P, k_P) \) and let the generic algebra \( \mathcal{H}_P^{\text{deg}}(\mathcal{R}^{\text{deg}}_P, k_P) \) have residual points \( \{ W_P c_P, 1^{(k)} \}, W_P c_P, 2^{(k)} ,\ldots, W_P c_P, i^{(k)} \} \). Then we call the parameters \( k_{\alpha} \) generic if, for all \( P \subset I \), the evaluation map \( W_P c_P, i^{(k)} \mapsto W_P c_P, i^{(k)} \) is a bijection onto the residual points of \( \mathcal{H}_P \). Otherwise, we call them special.

**Example 4.2.** We give the special parameters for the root systems of type \( B_n, C_n \).

Let \( V \) be an \( n \)-dimensional real vector space with orthonormal basis \( e_1, \ldots, e_n \) and consider \( \mathcal{R}_0 = \{ \pm e_i \pm e_j \} \cup \{ \pm e_j \} \), the root system of type \( B_n \). Let the set \( \Pi_0 \) of simple roots consist of \( \alpha_i = e_i - e_{i+1} \) for \( i = 1, \ldots, n-1 \) and \( \alpha_n = e_n \). Choose root labels \( k_1 = k_{\pm e_i, \pm e_j} \) and \( k_2 = k_{\pm e_i} \).

The parameters \( k_1, k_2 \) are shown to be special in \([4]\) if

\[
2(n-1) k_1 k_2 \prod_{j=1}^{2(n-1)} (2k_2 - jk_1)(2k_2 + jk_1) = 0, \tag{4.1}
\]

and generic otherwise.

In particular, notice that in view of Section 2.4, we may view these special values as the union of the integral relations \( k_2 = jk_1 \), resp. \( k_2' = jk_1' \) (\( j = 1, \ldots, n-1 \)), for the graded Hecke algebras of type \( B_n \), resp. \( C_n \).

Since the choices \( k_1 = k_2 \) resp. \( k_1 = 2k_2 \) give rise to a graded Hecke algebra with equal labels (of type \( B \) resp. \( C \)), we call any of these cases an equal label case.

4.2. Symbols

The Springer correspondence for classical groups of type \( B, C \) has been described by Lusztig (in \([8]\)) in terms of certain objects which he calls symbols. These have been generalized by Malle (see \([16]\)) and Shoji (see \([20]\)) in order to apply to complex reflection groups. We will use a certain type of these generalized symbols.

Fix \( n \) and let \( m_1 \geq m_2 \geq n \) be positive integers. Furthermore choose two non-negative integers \( r, s \). Define \( \Lambda_1 = (0, r, 2r, \ldots, (m_1 - 1)r) \in \mathbb{R}^{m_1} \) and \( \Lambda_2 = (r + s, 2r + s, \ldots, (m_2 - 1)r + s) \in \mathbb{R}^{m_2} \). Let \( (\xi, \eta) \in \mathcal{P}_{n,2} \), and let (by adding zeroes if necessary) \( l(\xi) = m_1 \) and \( l(\eta) = m_2 \). Then we denote by \( \Lambda^{m_1,m_2}_{\alpha,\beta}(\xi, \eta) \) the array with two rows, whose upper row consists of the entries of \( \Lambda_1 + \xi \) and whose lower row consists of
the entries of $\Lambda_2 + \eta$; the entry of $\eta_i$ is written to the right of the entry of $\xi_i$. For example, if $(\xi, \eta) = (1^3, 2)$, then

$$\Lambda^{2,1}_{6,5}(\xi, \eta) = \begin{pmatrix} 0 & 2 & 4 & 7 & 9 & 11 \\ 1 & 3 & 5 & 7 & 11 \end{pmatrix}.$$  

In fact we are not so much interested in these symbols, but rather in their equivalence classes generated by the shift operation $(m_1, m_2) \rightarrow (m_1 + 1, m_2 + 1)$. Putting $m = m_1 - m_2$, we denote the equivalence class of $\Lambda^{r,s}_{m_1,m_2}(\xi, \eta)$ by $\tilde{\Lambda}_m^{r,s}(\xi, \eta)$. We call this class, or any of its elements, the symbol of $(\xi, \eta)$. We will use the following specific choices of symbols:

**Definition 4.3.** For $(\xi, \eta) \in P_{n,2}$ and $m \in \frac{1}{2} \mathbb{Z}_{\geq 0}$, we define

$$\sigma_m(\xi, \eta) = \begin{cases} \Lambda^{2,0}_{n+m,n}(\xi, \eta) & \text{if } m \in \mathbb{Z} \\ \Lambda^{2,1}_{n+m',n}(\xi, \eta) & \text{if } m \not\in \mathbb{Z} \end{cases} \quad \text{and} \quad \tilde{\sigma}_m(\xi, \eta) = \begin{cases} \Lambda^{2,0}_{m}(\xi, \eta) & \text{if } m \in \mathbb{Z} \\ \Lambda^{2,1}_{m'}(\xi, \eta) & \text{if } m \not\in \mathbb{Z} \end{cases},$$

where, for $m \not\in \mathbb{Z}$, we define $m' = m + \frac{1}{2}$.

We will call $\tilde{\sigma}_m(\xi, \eta)$ or any of its representatives the $m$-symbol of $(\xi, \eta)$. The $m$-symbols generate an equivalence relation on $\tilde{W}_0$. For $A, B \in P_{n,2}$, we say that $A \sim_m B$ if $\tilde{\sigma}_m(A)$ and $\tilde{\sigma}_m(B)$ have representatives which contain the same entries with the same multiplicities. Denote the equivalence class of $A$ by $[A]_m$. We transfer this equivalence relation to $\tilde{W}_0$ in the obvious way.

The Springer correspondence for SO$^{2n+1}(k)$ (resp. Sp$^{2n}(k)$) (for char($k$)$\neq 2$) has been described by Lusztig [8] in terms of $m$-symbols with $m = 1$ (resp. $m = \frac{1}{2}$). In this description, the sets of Springer correspondents of the various unipotent classes are the equivalence classes under $\sim_m$ in $\tilde{W}_0$.

**Definition 4.4.** Let $(\xi, \eta) \in P_{n,2}$, with $\xi = (0 < \xi_1 \leq \xi_2 \leq \cdots \leq \xi_l)$, $\eta = (0 \leq \eta_1 \leq \eta_2 \leq \cdots \leq \eta_l)$. Denote the entry of $\xi_i$ in $\sigma_m(\xi, \eta)$ by $e_m(\xi_i)$, and analogously for $e_m(\eta_j)$.

Notice that $e_m$ does not descend to a function on $\tilde{\sigma}_m(\xi, \eta)$. However, we will be only interested in differences of the form $e_m(\xi_i) - e_m(\eta_j)$, which are independent of the choice of representative of $\tilde{\sigma}_m(\xi, \eta)$.

### 4.3. The a-function and truncated induction

An important tool related to symbols is the $a$-function. For $(\xi, \eta) \in P_{n,2}$, $\sigma_m(\xi, \eta)$ can be written as $\sigma_m(-,-) + (\xi, \eta)$. Then we define

$$a_m(\xi, \eta) = \sum_{x,y \in \sigma_m(\xi,\eta)} \min(x, y) - \sum_{x,y \in \sigma_m(-,-)} \min(x, y).$$
This function is invariant for the shift operation, and thus it induces a function on the \(m\)-symbols \(\overline{a}_m(\xi, \eta)\). We denote it by \(a_m\) as well. Clearly, \(a_m\) is constant on similarity classes. We also transport \(a_m\) to \(\hat{W}_0\).

We use \(a_m\) to define an ordering \(\succ_m\) on \(\hat{W}_0\). First we choose a partial order on \(\hat{W}_0\) by demanding that \(\chi \succ_m \chi'\) implies that \(a_m(\chi) \leq a_m(\chi')\), and that similarity classes form intervals. Then we arbitrarily refine this partial ordering into a total one.

We use the \(a\)-function to define a truncated induction that we will need later. Let \(W' \subset W_0\) be a subgroup and let \(\chi_0\) be a character of \(W'\). Let

\[
\text{Ind}_{W_0}^{W'}(\chi_0) = \sum_{\chi \in \hat{W}_0} n_{\chi', \chi} \xi,
\]

and suppose that \(\chi_0 \in \hat{W}_0\) satisfying \(n_{\chi', \chi_0} > 0\) is such that \(n_{\chi', \chi} > 0\) implies \(a_m(\chi) \leq a_m(\chi_0)\). Then we define

\[
\text{tr}_m - \text{Ind}_{W_0}^{W'}(\chi_0) = \sum_{\chi \cdot a_m(\chi) = a_m(\chi_0)} n_{\chi', \chi} \xi.
\]

4.4. Residual points

If \(R_0\) is of type \(B_n\) then a standard parabolic root subsystem \(R_L \subset R_0\) contains irreducible factors of type \(A\) and at most one of type \(B\). Therefore, we first review the classification of the residual points for these types. This has been done in [4]. First consider type \(A_{n-1}\). Let \(V\) be an \(n\)-dimensional real vector space with root system \(R_0 = \{e_i - e_j; i, j = 1, \ldots, n\}\). Let \(k_x = k\) for all \(x\). Then the only residual point in \(V\) has coordinates (up to Weyl group action)

\[
c_n(k) = \left( -\frac{n-1}{2}, -\frac{n-3}{2}, \ldots, \frac{n-3}{2}, \frac{n-1}{2} \right) k.
\]

Now consider a root system of type \(B_n\) with labels \(k_1, k_2\) as in Section 4.1. Then, according to [4, Proposition 4.3], the \(W_0\)-orbits of residual points in \(V\) are in bijection with set \(P_n\) of partitions of \(n\). Let \(\lambda \in P_n\). We consider its Young diagram \(T_\lambda\). The content of a square \(\Box\) with coordinates \((i, j)\) is defined to be \(c(\Box) = i - j\). An example of a Young tableau where each square is filled with its content is given in Fig. 1.

Let \(c(\lambda, k_1, k_2)\) be the point in \(V\) with coordinates \(c(\Box)k_1 + k_2\) where \(\Box\) runs over the squares of \(T_\lambda\). Then, by [4], for generic parameters the points \(c(\lambda, k_1, k_2)\) where \(\lambda\) runs through \(P_n\) form a complete set of representatives for the \(W_0\)-orbits of residual points of \(\mathbb{H}\). At special parameters, one may have \(c(\lambda, k_1, k_2) = c(\mu, k_1, k_2)\) even if \(\lambda \neq \mu\), and the \(c(\lambda, k_1, k_2)\) need not be residual any more. Let \(k_2 = mk_1\) with \(m \in \frac{1}{2}\mathbb{Z}\).
Since \( W_0 \) acts by permutations and sign changes, we may write \( c(\lambda, k, mk) \) as

\[
\begin{pmatrix} pk, \ldots, pk, (p-1)k, \ldots, k, 0, \ldots, 0 \end{pmatrix} \in \mathbb{R}^n
\]

\( M_p \) times \( M_0 \) times

if \( m \) is integer, or as

\[
\begin{pmatrix} pk, \ldots, pk, (p-1)k, \ldots, k, \frac{1}{2}k, \ldots, \frac{1}{2}k \end{pmatrix} \in \mathbb{R}^n
\]

\( M_p \) times \( M_{\frac{1}{2}} \) times

if \( m \) is not integer. The conditions under which such a point remains residual have been investigated in [4]. We list the results here:

**Condition 4.5** (Heckman and Opdam [4]). Let \( k \neq 0 \) and \( m \in \frac{1}{2} \mathbb{Z} \geq 0 \).

- If \( m = 0 \) then the point (4.5) is residual if and only if (i) \( M_p = 1 \), (ii) \( M_l \in \{M_l+1, M_l+1+1\} \) for \( l > 0 \), and (iii) \( M_0 = \left\lfloor \frac{M_1+1}{2} \right\rfloor \).
- If \( 0 \neq m \in \mathbb{Z} \) then \( p \geq m \) and the point (4.5) is residual if (i) \( M_p = 1 \), (ii) \( M_l \in \{M_l+1, M_l+1+1\} \) for all \( l \geq m \), (iii) \( M_l \in \{M_l+1, M_l+1-1, M_l+1\} \) for \( l = 1, 2, \ldots, m-1 \), and finally (iv) \( M_0 = \left\lfloor \frac{M_1}{2} \right\rfloor \);
- If \( m \notin \mathbb{Z} \) then \( p \geq m \) and the point (4.6) is residual if (i) \( M_p = 1 \), (ii) \( M_l \in \{M_l+1, M_l+1+1\} \) for all \( l \geq m \), (iii) \( M_l \in \{M_l+1, M_l+1-1, M_l+1\} \) for \( l = \frac{1}{2}, \frac{3}{2}, \ldots, m-1 \).

Moreover, the residual points among these specialized points exhaust the \( W_0 \)-orbits of residual points for \( \mathbb{H} \) with parameters \( k_2 = mk_1, k_1 = k \neq 0 \).

**4.5. The \( m \)-tableau**

We translate this algebraic condition to a condition on Young tableaux. Define

**Definition 4.6.** For \( m \geq 0 \) and \( \lambda \vdash n \), let \( T_m(\lambda) \) be the Young tableau of \( \lambda \) where square \( \square \) is filled with \(|c(\square) + m|\). We call \( T_m(\lambda) \) the \( m \)-tableau of \( \lambda \).
An example is given in Fig. 2. The entries of $T_m(\lambda)$ arranged in non-increasing order, multiplied with $k$, form the vector $c(\lambda, k, mk)$ in the form (4.5) or (4.6).

**Definition 4.7.** Let $\lambda < n, m \in \frac{1}{2}Z \geq 0$. We define the extremities of $T_m(\lambda)$ to be the numbers in the following list. If $m \in Z$ (resp. $m \notin Z$), the extremities are the entries of every last square of the rows of $T_m(\lambda)$ which lies on or above the zero-diagonal (resp. the upper $\frac{1}{2}$-diagonal), and the entries of every last square of the columns of $T_m(\lambda)$ which lies on or below the zero-diagonal (resp. the lower $\frac{1}{2}$-diagonal).

Notice that it may happen that a number occurs twice as extremity. In particular, zero can only occur twice as extremity.

On the other hand, let $c$ be a residual point chosen in its $W_0$-orbit to take the form (4.5) or (4.6). Then $c$ is clearly determined by the following numbers:

**Definition 4.8.** Let $k \neq 0, m \in \frac{1}{2}Z \geq 0$ and let $W_0 c$ be residual such that $c$ is in form (4.5) or (4.6). We define the jumps of $c$ to be those $l \geq m$ such that $M_l = M_{l+1} + 1$, and those $l \in \{1, 2, \ldots, m-1\}$ (resp. in $\{\frac{1}{2}, \frac{3}{2}, \ldots, m-1\}$) such that $M_l = M_{l+1}$. It will be convenient to have $[m] + 2r$ jumps for some $r \in Z \geq 0$. Therefore, we add 0 (for $m \in Z$) or $-\frac{1}{2}$ (for $m \notin Z$) as a jump if necessary. We write $J(W_0 c)$ for the set of jumps of $c$, and $J^+(W_0 c)$ for the set of positive jumps of $c$. We write the jumps in increasing order: $j_1 < j_2 \cdots < j_{[m]+2r}$.

For this definition to make sense, it remains to be seen that there are enough positive jumps.

**Lemma 4.9.** In the notation of Definition 4.8, we have $|J^+(W_0 c)| = M_1 + m - 1$ if $1 \leq m \in Z$, $|J^+(W_0 c)| = M_1 + m - \frac{1}{2}$ if $m \notin Z$, and $|J^+(W_0 c)| = M_1$ if $m = 0$.

**Proof.** (i) Let $m = 0$. Then $j \in J^+(W_0 c)$ gives rise to a sequence $\{j, j-1, \ldots, 1\}$ in (4.5).

(ii) If $m \in Z$ then $m \leq j \in J(W_0 c)$ corresponds to a subsequence $\{j, j-1, \ldots, m\}k$ in (4.5). Therefore $M_m = |\{j \in J(W_0 c) \mid j \geq m\}|$. Since $l < m$ is a jump if and only if $M_{l+1} = M_l$, it follows that $M_m - M_1 = |\{i \in \{1, 2, \ldots, m-1\} \mid i \notin J(W_0 c)\}|$, i.e.
\[ |J^+(W_0c)| = |\{ j \in J(W_0c) \mid j \geq m \}| + |\{ 1 \leq j \leq m - 1 \mid j \in J(W_0c) \}| = M_m + (m + 1 - M_m + M_1) = m + 1 + M_1. \]

(iii) If \( m \notin \mathbb{Z} \) then \( m \leq j \in J(W_0c) \) corresponds to a subsequence \( \{ j, j - 1, \ldots, m \} \) in (4.6). Therefore \( M_m = |\{ j \in J(W_0c) \mid j \geq m \}|. \) Since \( l < m \) is a jump if and only if \( M_{l+1} = M_l \), it follows that \( M_m - M_{1/2} = |\{ i \in \{ \frac{1}{2}, \frac{3}{2}, \ldots, m - 1 \} \mid i \notin J(W_0c) \}|, \) i.e., \( |J^+(W_0c)| = |\{ j \in J(W_0c) \mid j \geq m \}| + |\{ \frac{1}{2} \leq j \leq m - 1 \mid j \in J(W_0c) \}| = M_m + (m + 1 - M_m + M_{1/2}) = m + 1 + M_{1/2}. \]

Therefore we can indeed, by defining 0 or \(- \frac{1}{2}\) to be a jump, arrange to have \( \lceil m \rceil + 2r \) jumps. Note that if \( m \neq 0 \), then 0 (resp. \(- \frac{1}{2}\)) occurs as jump if and only if \( M_1 \) (resp. \( M_{1/2} \)) is even. Notice also that if \( \frac{1}{2} \mathbb{Z} \ni m > n - 1 \), then \( |J(W_0c)| = \lfloor m \rfloor \).

**Lemma 4.10.** Let \( m \in \frac{1}{2} \mathbb{Z} \geq 0 \) and \( \lambda + n \). Let \( M_l \) be the multiplicity of \( l \) in \( T_m(\lambda) \). Then we have

(i) \( M_p \in \{ M_{p+1}, M_{p+1} + 1, M_{p+1} + 2 \} \) for \( p \geq m \);

(ii) \( M_p \in \{ M_{p+1} - 1, M_{p+1}, M_{p+1} + 1 \} \) for \( p = 1, 2, \ldots, m - 1 \) resp. \( p = \frac{1}{2}, \frac{3}{2}, \ldots, m - 1 \);

(iii) If \( 0 \neq m \in \mathbb{Z} \), then \( M_0 \in \{ \lfloor \frac{M_1}{2} \rfloor, \lceil \frac{M_1}{2} \rceil \} \);

(iv) If \( m = 0 \), then \( M_0 \in \{ \lfloor \frac{M_1 + 1}{2} \rfloor, \lceil \frac{M_1 + 1}{2} \rceil \} \).

**Proof.** (i) This follows because \( p \) and \( p + 1 \) occur in two diagonals, an upper and a lower one:

\[
\begin{array}{c}
p \quad p+1 \\
p \quad p+1 \\
p \quad p+1 \\
p \quad ?
\end{array}
\begin{array}{c}
p \quad p+1 \\
p \quad p+1 \\
p \quad p+1 \\
p \quad ?
\end{array}
\]

In the squares marked with ?, we may or may not have a \( p + 1 \).

(ii) Here, things change for the upper diagonal as we always have a \( p + 1 \) above the first \( p \):

\[
\begin{array}{c}
p \quad p+1 \\
p \quad p+1 \\
p \quad p+1 \\
p \quad ?
\end{array}
\begin{array}{c}
p \quad p+1 \\
p \quad p+1 \\
p \quad p+1 \\
p \quad ?
\end{array}
\]
(iii) This follows from the fact that the zeroes occur as indicated:

```
  1
 0 1
 1 0
```

(iv) As in case (iii), but the top square containing a one is not present now. □

**Corollary 4.11.** Let $\lambda \vdash n$ such that $c(\lambda, k, mk)$ is residual. Then for all $x$ occurring as entry in $T_m(\lambda)$ we have: $x$ is a jump $\iff$ $x$ is an extremity.

**Proof.** (i) Suppose $x > 0$. Since $c(\lambda, k, mk)$ is residual, the sequence covered by $T_m(\lambda)$ satisfies Condition 4.5, and therefore in the figures of the proof of parts (i) and (ii) of Lemma 4.10, at least one of the squares containing a ? are occupied by $T_m(\lambda)$. But then we have that $x$ is a jump $\iff$ only one square containing a ? is occupied $\iff$ there is an extremity square having entry $x$.

(ii) Suppose $x = 0$. Then there is an extremity square with entry zero $\iff$ only one square filled with ? in the proof of Proposition 4.10(iii) is occupied $\iff$ $m \neq 0$ and $M_1$ is even, or $m = 0$ and $M_1$ is odd. □

**Corollary 4.12.** Let $\lambda \vdash n$ and $m \in \frac{1}{2} \mathbb{Z} \geq 0$. Then $c(\lambda, k, mk)$ is residual if and only if $T_m(\lambda)$ has distinct extremities.

**Proof.** This follows from Condition 4.5, Lemma 4.11 and Corollary 4.12. □

It follows again that our notion of special parameters coincides with the one in [4]:

**Corollary 4.13.** Let $m \in \mathbb{R}$. The points $c(\lambda, k, mk)$ are all residual and distinct if and only if $m \notin \pm\{0, \frac{1}{2}, 1, \ldots, n - \frac{3}{2}, n - 1\}$.

**Proof.** In view of the preceding Corollary, it suffices to check for which $m$ every $T_m(\lambda)$ has distinct extremities. Indeed, it is easy to see that if there are $\lambda, \mu \vdash n$ such that $c(\lambda, k, mk)$ and $c(\mu, k, mk)$ are both residual and $c(\lambda, k, mk) = c(\mu, k, mk)$, then there is also a $v \vdash n$ such that $c(v, k, mk)$ is not residual. Let $m \in \mathbb{R} \geq 0$ and write $m = l + \varepsilon$, with $l \in \mathbb{Z} \geq 0$ and $0 \leq \varepsilon < 1$. Then $T_m(\lambda)$ is drawn on squared paper filled
as follows:

<table>
<thead>
<tr>
<th>l+ε</th>
<th>l+1+ε</th>
<th>l+2+ε</th>
<th>l+3+ε</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1+ε</td>
<td>1+1+ε</td>
<td>1+2+ε</td>
</tr>
<tr>
<td>l</td>
<td>l+1+ε</td>
<td>l+1+1+ε</td>
<td>l+1+2+ε</td>
</tr>
<tr>
<td>1+ε</td>
<td>1+1+ε</td>
<td>l+1+1+ε</td>
<td>l+1+2+ε</td>
</tr>
</tbody>
</table>

There are no repetitions in \( l \) + \( f + \varepsilon \), \( l + (f-1) + \varepsilon, \ldots, 1 + \varepsilon, 1 - \varepsilon, 2 - \varepsilon, \ldots \), unless \( l + f + \varepsilon = d - \varepsilon \) for some \( f, d \). This means that \( \varepsilon \in \{0, \frac{1}{2}\} \). Clearly, we can reduce to the case where the multiple extremity is either 0 or \( \frac{1}{2} \). But such an extremity can occur precisely for \( m \in \pm\{0, 1, \ldots, n - \frac{3}{2}, n - 1\} \). □

4.6. Specialization of residual points

In this section we show that a generically residual point specializes for all parameter values into the central character of a tempered representation.

First we describe, for \( P \subset I \), the residual points of \( H_P \). Suppose \( \text{Res}_P \) has type \( A_{k_1-1} \times \cdots \times A_{k_r-1} \times B_l = A_{k} \times B_l \). Then \( H_P = H(A_{k_1-1}) \otimes \cdots \otimes H(A_{k_r-1}) \otimes H(B_l) \). Thus, a residual point for \( H_P \) is specified by choosing a residual point in each factor separately. A graded Hecke algebra of type \( A_{k-1} \) with root label \( k_1 \neq 0 \) has only one residual point, namely \( c(k_1) \) as in (4.4). Therefore up to \( W_P \)-action the residual points of \( H_P \) are given as

\[
c_{k,\mu}(k_1, k_2) = c_{k_1}(k_1) \times \cdots \times c_{k_r}(k_1) \times c(\mu; k_1, k_2), \tag{4.7}
\]

where \( \mu \vdash l \) is such that \( c(\mu; k_1, k_2) \) is residual. By this notation, we mean that the first \( k_1 \) coordinates of \( c_{k,\mu}(k_1, k_2) \) are those of \( c_{k_1}(k_1) \) with \( k = k_1 \), its next \( k_2 \) coordinates are those of \( c_{k_2}(k_1) \), etc, until the last \( l \) coordinates which are those of \( c(\mu; k_1, k_2) \).

**Definition 4.14.** Let \( H \) have parameters \( k_2 = mk_1 \) and \( k_1 \neq 0 \). Let \( \mathcal{L}_m(n) \) denote the set of \( W_0 \)-orbits of residual subspaces of \( H \). We say that \( W_0L \in \mathcal{L}_m(n) \) has type \((k, \mu) \in \mathcal{P}_{n,2}\), if we can choose \( c_L \) in \( W_Lc_L \) such that it takes form (4.7). Notice
that for special parameters, \( \mu \) need not be uniquely determined by \( L \) since several residual points \( c(v, k_1, k_2) \) may coincide with each other for \( k_2 = mk_1 \). We denote by \( \mathcal{C}_m(n) = \{ W_0cL \mid W_0L \in \mathcal{L}_m(n) \} \) the set of central characters of \( \hat{h}^f(\mathbb{R}) \).

Sometimes we will abuse notation and write \( L \in \mathcal{L}_m(n) \) instead of \( W_0L \in \mathcal{L}_m(n) \), if it is clear that we are only interested in \( L \) up to \( W_0 \)-orbit.

**Proposition 4.15.** Let \( \lambda \vdash n \) and let \( k \neq 0, m \in \frac{1}{2} \mathbb{Z} \geq 0 \). Then there exists \( W_0L \in \mathcal{L}_m(n) \) such that

\[
W_0c(\lambda, k, mk) = W_0cL \in \mathcal{C}_m(n).
\]

Let \( L \) have type \((\mu, v)\). Then \( \mu \) consists of the lengths of the hooks in \( T_m(\lambda) \) ending on equal extremities, and \( v \) is obtained by deleting these hooks from \( \lambda \). In particular, the parts of \( \mu \) are all different. If \( m \) is integer they are odd, otherwise they are even. We write \( \mathrm{sp}_m(\lambda) = (\mu, v) \).

**Proof.** This boils down to showing that the entries of \( T_m(\lambda) \) can be written as the union of the entries of a residual tableau \( T_m(\mu) \) for some partition \( \mu \) whose Young diagram is contained in that of \( \lambda \), and a number of sequences of the form \((l, l, l - 1, l - 1, \ldots, 1, 1, 0)\) (resp. \((l, l, l - 1, l - 1, \ldots, \frac{1}{2}, \frac{1}{2})\)) if \( m \) is integer (resp. not integer), with \( l \geq 0 \). We assume that \( m \) is integer, the other case being analogous. The entries of \( T_m(\lambda) \) form the vector

\[
\left(\begin{array}{cccc}
p, & \ldots, & p, & (p - 1), \ldots, 1, 0, \ldots, 0 \\
M_p \text{ times} & & M_0 \text{ times}
\end{array}\right) \in \mathbb{R}^n
\]

We want to prove the proposition by induction on \( n \). If \( n = 1 \), the assertion is trivial. Now suppose \( n > 1 \), and the proposition holds for all partitions of weight less than \( n \). Take the maximal \( l \), for which \( ml \) and \( ml + 1 \) do not satisfy Condition 4.5. There are three cases:

(i) We have \( l \geq m \), and \( M_l = M_{l+1} + 2 \). Then both squares with a \( ? \) in Lemma 4.10 are not occupied, and this means that there are both a row and column ending on \( l \). Together, they contain the sequence \((l, l, l - 1, l - 1, \ldots, 1, 1, 0)\) if \( m \in \mathbb{Z} \), and \((l, l, l - 1, l - 1, \ldots, \frac{1}{2}, \frac{1}{2})\) otherwise. We remove this row and column, and push the disconnected piece one square northwest to reconnect it. This does not change the content of any square, and we find the \( m \)-tableau of a smaller partition. The claim follows by induction.

(ii) We have \( l \in \{1, 2, \ldots, m - 1\} \) and \( M_l = M_{l+1} + 1 \). This means that both squares containing \( ? \) in part (ii) of Lemma 4.10 are not occupied. Again, this means there is both a row and a column ending on \( l \), and we can remove them as in the case (i).

(iii) If \( m \in \mathbb{Z}, m \neq 0 \) (resp. \( m = 0 \)) and \( M_0 = \lceil \frac{M_l}{2} \rceil \) (resp. \( M_0 = \lceil \frac{M_{l+1}}{2} \rceil \)), then both squares containing a \( ? \) in case (iii) (resp. (iv)) of Lemma 4.10 are not occupied. We can therefore remove the last square containing a zero. \( \square \)
In view of this Proposition, we will often encounter strips of length $t$, whose squares contain entries \((t-1)/2, (t-3)/2, \ldots, 1, 0, 1, \ldots, (t-3)/2, (t-1)/2\) (if $t$ is integer) or \((t-1)/2, (t-3)/2, \ldots, \frac{1}{2}, \frac{1}{2}, \ldots, (t-3)/2, (t-1)/2\) (if $t$ is not integer). We call such a strip an $A$-strip (of length $t$).

The following fact allows us to focus on the centers of the residual subspaces, i.e., on the central characters of $\mathbb{R}^l (\mathbb{R})$, instead of on the residual subspaces.

**Proposition 4.16.** For all parameters $k_1, k_2$, the map

$$\mathcal{L}_m(n) \to \mathcal{C}_m(n) : W_0L \to W_0c_L$$

is a bijection.

**Proof.** First we remark that this statement is known to hold for the equal label case, for any root system (cf. [4, Remark 3.12]).

Suppose that the parameters are generic. Then $W_0L$ is indexed by $\mathcal{P}_{n,2}$ by associating to $(\kappa, \mu)$ the orbit of residual subspaces $W_0L_{\kappa,\mu}$ such that $L_{\kappa,\mu}$ has standard parabolic root system $R_{L_{\kappa,\mu}}$ of type $A_k \times B_{|\mu|}$ and center $c_{\kappa,\mu}(k_1, k_2)$. It remains to check that the $c_{\kappa,\mu}$ are all distinct. Notice that the coordinates in $c_{\kappa,\mu}$ corresponding to the $A$-factors are given by $c_{K_1}(k_1) \times \cdots \times c_{K_1}(k_1)$, independently of $k_2$. Therefore, if $k_2 \notin \frac{1}{2}\mathbb{Z}k_1$, the required bijection follows from Corollary 4.13 in combination with the corresponding fact for type $A$ root systems. If $k_2 = mk_1$ with $m > n - 1$ and $m \in \frac{1}{2}\mathbb{Z}$, then consider $c_{\kappa,\mu}$. Let $|\mu| = l$. Then $k_i \leq n - l$. Therefore, the coordinates of $c_\kappa(k_1)$ are all at most equal to $\frac{n-l}{2}$, whereas those corresponding to $c(\mu, k, mk)$ are all at least equal to $n-l+1$. This means that, again using Corollary 4.13 that we can reconstruct $\kappa$ and $\mu$ from $c_{\kappa,\mu}$ and the claim follows.

Next let $k_2 = mk_1$ with $m \in \frac{1}{2}\mathbb{Z}$. If $k_1 = 0$, then all orbits $W_0c(\lambda_0, k_2)$ coincide with each other into $W_0(k_2, \ldots, k_2)$, which is not residual. A generically residual point in $\mathbb{R}^l_L$ with $R_L = A_\lambda \times B_I$ and $l \neq 0$ therefore does not specialize into a residual point. If $R_L = A_\lambda$ then its center specializes into $c_L = 0$, which is also not residual. Thus $\mathcal{C}_m(n)$ is empty and, using Lemma 2.6, $\mathcal{L}_m(n)$ as well.

Finally suppose that the parameters are special with $k_2 = mk_1$, $k_1 \neq 0$ with $m \in \frac{1}{2}\mathbb{Z}$. Put $k = k_1$. We suppose for simplicity that $m \in \mathbb{Z}$, the other case being analogous. Suppose that $W_0L_1, W_0L_2$ are such that $W_0c_{L_1} = W_0c_{L_2} =: W_0c$. We choose $c$ in its $W_0$-orbit to take form (4.5). Let $L_i$ have type $(\lambda^{(i)}, \mu^{(i)})$. We may assume that $\lambda^{(1)}$ and $\lambda^{(2)}$ have no parts in common. Suppose that for $p \in \mathbb{Z}_{\geq 0}$, $pk$ occurs $M_p$ times as coordinate of $c$. Recall that $c$ is a residual point if and only if Condition 4.5 holds for $M = (M_0, M_1, \ldots, M_p)$. In this case, we are done. Otherwise, since $W_0c \in \mathcal{C}_m(n)$ it follows from Condition 4.5 and the fact that every non-zero entry occurs twice in the coordinates of $c$ which correspond to an $A$-factor, that there is exactly one way to decompose $M$ into the coordinates of a residual point, together with series of the form $(l, l, \ldots, 1, 1, 0)$ corresponding to type $A$-factors. Therefore, $\lambda^{(1)} = \lambda^{(2)}$ and $c(\mu^{(1)}, k, mk) = c(\mu^{(2)}, k, mk)$. This implies that $W_0L_1 = W_0L_2$. $\square$
Remark that this bijection only holds on the level of $W_0$-orbits, and not on the level of residual subspaces and their centers. Indeed, in general one may have $1 \neq w \in W_0$ such that $w(cL) = cL$ but $w(L) \neq L$.

We expect that for generic parameters, the representations in $\hat{H}^f(R)$ are separated by their central characters, and that there are $|\hat{W}_0|$ such. Therefore we remark

**Lemma 4.17.** Let $k_1, k_2$ be generic and put $m = k_2/k_1$. Then $|C_m(n)| = |\hat{W}_0|$.

**Proof.** Associate to $(\lambda, \mu) \in P_{n,2}$ the residual subspace of type $(\lambda, \mu)$. Since the parameters are generic, these are distinct. Now use Proposition 4.16. □

It remains to be shown that generically, for every residual point $c$, there is exactly one irreducible discrete series representation of $H$ with central character $W_0(c)$. For regular $c$, this is not hard (see [21]), but for singular $c$, it remains an open problem.

### 4.7. Classical Springer correspondence, combinatorially

We recall the combinatorial description of the Springer correspondence in the equal label cases. The set of unipotent conjugacy classes of $SO_{2n+1}(\mathbb{C})$ is parametrized by the elementary divisors partitions

$$U_1(n) = \{ \lambda = (1^{r_1}2^{r_2}\ldots)|2n + 1 \text{ if } i \text{ is even} \}.$$ 

Similarly for $Sp_{2n}(\mathbb{C})$, we have the set

$$U_2(n) = \{ \lambda = (1^{r_1}2^{r_2}\ldots)|2n \text{ if } i \text{ is odd} \}.$$ 

The distinguished unipotent classes among these are the ones parametrized by partitions which consist of distinct parts.

Lusztig has defined maps, which we denote by $\phi_m: U_m(n) \rightarrow P_{n,2}$ such that $\Sigma_m(C_{\lambda}) = [\phi_m(\lambda)]_m$. These have the following form. Let $\lambda \in U_m(n)$. If $m = \frac{1}{2}$ we make sure that $\lambda$ has an even number of parts by adding a part equal to zero if necessary. Then we define a partition $\lambda^*$ by putting $\lambda_i^* = \lambda_i + (i - 1)$. We define two partitions $\xi^*$ and $\eta^*$ by letting the odd parts of $\lambda^*$ be $2\xi_1^* + 1 < 2\xi_2^* + 1 < \ldots$ and by letting the even parts of $\lambda^*$ be $2\eta_1^* < 2\eta_2^* < \ldots$. Finally we define $\xi_i = \xi_i^* - (i - 1)$ and $\eta_i = \eta_i^* - (i - 1)$. Then by definition $\phi_m(\lambda) = (\xi, \eta)$.

We now describe the bijection $U_m(n) \leftrightarrow L_m(n)$ of (3.3), by giving explicitly the map $f_m^{BC}: U_m(n) \rightarrow L_m(n)$ implementing it (BC standing for Bala–Carter).

**Lemma 4.18.** If $u \in \hat{G}$ lies in a distinguished unipotent class $C_{\lambda}$, then the jumps of $\gamma(u)$ as in (3.3) are equal to $\frac{\lambda_i - 1}{2}$.

**Proof.** This follows easily from the definition of the weighted Dynkin diagram of $C_{\lambda}$ (cf. [1, p. 395]). □
It follows (cf. the description in [6]) that \( f_{m}^{BC} \) is as follows: for \( \lambda \in U_{m}(n) \), let \( l_{1} < l_{2} < \cdots \) be the parts that occur an odd number of times. If we remove each part \( l_{i} \) from \( \lambda \) we obtain a partition in which all parts have even multiplicity. Remove each second part and let the remaining partition be \( \kappa \). Then \( f_{m}^{BC}(\lambda) = W_{0}c_{\kappa,k}(k,mk) \) (as in (4.7)) where \( \mu \) is chosen such that the residual point \( c(\mu,k,mk) \) has jumps \( \frac{l_{i}}{2} \).

We can thus transfer the Springer correspondents from the unipotent classes \( U_{m}(n) \) to the central characters \( C_{m}(n) \) by

\[
\Sigma_{m}(f_{m}^{BC}(\lambda)) = [\phi_{m}(\lambda)]_{m}. \tag{4.8}
\]

\[4.8. \text{Generalized definition of Springer correspondents}\]

Let \( m \in \left\{ \frac{1}{2}, 1 \right\} \) and suppose that \( W_{0}c = f_{m}^{BC}(\lambda) \) for a distinguished unipotent conjugacy class \( C_{\lambda} \). Then \( (\xi(W_{0}c), \eta(W_{0}c)) = \phi_{m}(\lambda) \).

**Proof.** This follows from Lemma 4.18. \( \square \)

**Definition 4.20.** Let \( \mathfrak{H} \) have parameters \( k_{2} = mk_{1} \) with \( k_{1} \neq 0 \) and \( m \in \left\{ \frac{1}{2}, 1 \right\} \).
(i) We define the set of Springer correspondents of a residual point \( W_{0}c \in C_{m}(n) \) to be

\[
\Sigma_{m}(W_{0}c) = [(\xi(W_{0}c), \eta(W_{0}c))]_{m}.
\]

(ii) Let \( W_{0}L \in L_{m}(n) \) with \( R_{L} = A_{\kappa} \times B_{l} \) be such that \( c_{L} = c_{\kappa_{1}}(k_{1}) \times \cdots \times c_{\kappa_{r}}(k_{1}) \times c \), and \( c \) is a residual point for \( \mathfrak{H}(B_{l}) \). Then we define the set of Springer correspondents of \( W_{LcL} \) as

\[
\Sigma_{m}(W_{LcL}) = (\kappa_{1}) \otimes \cdots \otimes (\kappa_{r}) \otimes \Sigma_{m}(W_{0}(B_{l})c) = \text{triv}_{\kappa} \otimes \Sigma_{m}(W_{0}(B_{l})c).
\]
viewed as representations of $W_0(R_L)$. In other words, we tensor the Springer correspondents in each component of $c_L$.

(iii) In the setting of (ii), we define $\Sigma_m(W_0c_L)$ to be the set of all irreducible characters which appear in

$$\text{tr}_m - \text{Ind}_{W_0R_L}^{W_0}(\Sigma_m(W_Lc_L)).$$

That is, we truncatedly induce the Springer correspondents of the residual point $W_Lc_L$ to obtain those of $W_0c_L$.

In the equal label cases, the representations in $\hat{H}_t(R)$ with central character $W_0c_L = f_{mBC}(\lambda)$ are in bijection with the set $\Sigma_m(C_{\lambda})$ of Springer correspondents of $C_{\lambda}$. We will show that the number of centers of generically residual cosets which specialize into $W_0c_L$ is equal to the numbers of Springer correspondents of $C_{\lambda}$. Therefore we define the sets, for $m \in \frac{1}{2} \mathbb{Z}_{\geq 0}$ and $L \in \mathcal{L}_m(n)$,

$$C_m(W_0c_L) = \{W_0c_L' \mid L' \text{ gen. res. subspace s.t. } W_0c_L' = W_0c_L \text{ if } k_2 = mk_1\}.$$ 

Notice that, by 4.15 and 4.17, we have

$$\bigcup_{L \in \mathcal{L}_m(n)} |C_m(W_0c_L)| = |\hat{W}_0|.$$

We will show that $\Sigma_m(W_0c_L)$ is a similarity class for $\sim_m$ in $\hat{W}_0$ which is in bijection with $C_m(W_0c_L)$ (natural up to at most an involution), and that in the equal label cases we retrieve the already existing Springer correspondence, i.e., (4.8) holds.

4.9. Splitting and joining

Let

$$\mathcal{P}_m(W_0c) = \{\lambda + n \mid W_0c(\lambda, k, mk) = W_0c\}.$$ 

We identify $\mathcal{P}_m(W_0c)$ with $C_m(W_0c)$. We now define a map $\mathcal{S}_m : \mathcal{P}_m(W_0c) \to \mathcal{P}_{n,2}$ which will implement, using this identification, a natural bijection between $C_m(W_0c)$ and $\Sigma_m(W_0c)$.

**Definition 4.21.** Let $m \geq 0, \lambda \vdash n$ and consider $T_m(\lambda)$. We divide $T_m(\lambda)$ into blocks as follows. Locate the maximal entry of $T_m(\lambda)$, and draw a block from its square up to the top left square. This is either a horizontal or a vertical block. Subsequently, look for the maximal remaining entry and enclose it in vertical or horizontal block, such that the formed blocks form the diagram of a partition. This induces a splitting of $T_m(\lambda)$ into horizontal and vertical blocks. A $1 \times 1$-block is considered as horizontal if
Fig. 3. Examples: $S_1(1344) = (244, 2)$ and $S_1(1^33^3) = (222, 6)$.

(24,24):
\[
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
0 & 1 & 2 & 3 \\
1 & 0 & 1 & 2 \\
2 & 1 & 3 & 4 \\
\end{array}
\]

(22,26):
\[
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
0 & 1 & 2 & 3 \\
1 & 0 & 1 & 2 \\
2 & 1 & 3 & 4 \\
\end{array}
\]

Fig. 4. Examples: $J_1(24, 24) = (12234)$, $J_1(22, 26) = (112233)$.

it lies above the zero (resp. on or above the upper $\frac{1}{2}$)-diagonal, and as vertical if it lies below the zero-diagonal (resp. on or below the lower $\frac{1}{2}$)-diagonal. Let $\xi$, resp. $\eta$, be the partition whose parts are the lengths of the horizontal, resp. the vertical, blocks. Then we define $S_m(\lambda) = (\xi, \eta)$. Examples are given in Fig. 3.

**Lemma 4.22.** The procedure $S_m$ is well-defined on $\lambda + n$ if and only if the point $c(\lambda, k, m)$ is residual.

**Proof.** In view of Corollary 4.12 it remains to check that for $m \in \mathbb{Z}$ we do not get $1 \times 1$-blocks containing a zero. Such a block can only arise as the last block under $S_m$. But then in the situation of Lemma 4.10(iii)–(iv), the squares marked with ? do not belong to $T_m(\lambda)$, which implies that $c(\lambda, k, m)$ is not residual. □

We also define the inverse “joining” map $J_m$.

**Definition 4.23.** Let $m \geq 0$ and $(\xi, \eta) \in \mathcal{P}_{n,2}$. We place horizontal blocks of length $\xi_i$ and vertical blocks of length $\eta_j$ such that we obtain the $m$-tableau of a partition, in the following way. Each time, we look for the block which can reach the maximal number in the $m$-tableau. The blocks are nested in the sense that after every step the placed blocks form the diagram of a partition. We repeat this process until we have placed all blocks. The end result is the $m$-tableau $T_m(\lambda)$ of the partition $\lambda := J_m(\xi, \eta)$. Two examples for $m = 1$ are provided in Fig. 4.

This map is also well-defined:
Lemma 4.24. For arbitrary parameter values \( k_2 = mk_1 \) and \( k_1 \neq 0 \), \( J_m \) is well defined on \( \Sigma_m(W_0c) \) for every residual point \( c \).

Proof. (i) For generic parameters, this follows analogously to Corollary 4.13.

(ii) Now suppose that \( m \in \frac{1}{2} \mathbb{Z}_{>0} \). First we observe that it follows from the definition of \( \Sigma_m(W_0c) \) that all the entries in the \( m \)-symbols of \( \Sigma_m(W_0c) \) are mutually distinct.

(iia) First we assume \( m \in \mathbb{Z} \). Let \( J(W_0c) = \{ j_1, \ldots, j_m+2r \} \). Then \( J_m \) is not well-defined if after placing \( h \) horizontal and \( v \) vertical blocks, both the horizontal block of length \( \xi_{r+m-h} \) and the vertical block of length \( \eta_{r-v} \) can reach the same number \( x \) in the \( m \)-tableau. After \( h \) horizontal and \( v \) vertical blocks, the next block will start on a square containing \( |m-h+v| \). Suppose first that \( m \leq h-v \). Then the square under consideration lies above the zero diagonal and hence we find that we have \( \xi_{r+m-h} = x-(m-h+v)+1 \) and \( \eta_{r-v} = x+(m-h+v)+1 \), i.e., that \( \xi_{r+m-h} = \eta_{r-v} - 2(m-h+1) \). But then \( e_m(\xi_{r+m-h}) = e_m(\eta_{r-v}) \), which is a contradiction. The case \( m < h-v \) is treated along the same lines.

(iib) Now suppose that \( m \notin \mathbb{Z} \). Let \( J(W_0c) = \{ j_1, \ldots, j_m+2r \} \). Again, \( J_m \) is not well defined if after placing \( h \) horizontal and \( v \) vertical blocks, both the horizontal block of length \( \xi_{r+m+h} \) and the vertical block of length \( \eta_{r-v} \) can reach the same number \( x \) in the \( m \)-tableau. The next placed blocks starts on a square containing \( |m-h+v| \). Suppose for the moment that \( m > h-v \), then this square lies on or above the upper \( \frac{1}{2} \)-diagonal, and we find that the block lengths \( \xi_{r+m+h} \) and \( \eta_{r-v} \) satisfy \( \xi_{r+m+h} = x-(m-h+v)+1 \) and \( \eta_{r-v} = x+(m-h+v)+1 \). But again, this means that \( e_m(\xi_{r+m+h}) = e_m(\eta_{r-v}) \). The case where \( m < h-v \) can be treated in the same way. \( \square \)

4.10. Confluence of residual points

We want to show that \( S_m(P_m(W_0c)) = \Sigma_m(W_0c) \). First we show the existence of a canonical “balanced” \( m \)-tableau of a partition \( \lambda(W_0c) \in P_m(W_0c) \).

Lemma 4.25. Let \( m \in \frac{1}{2} \mathbb{Z}_{>0} \), and let \( W_0c \) be a residual point for \( \mathbb{H} \) with \( k_2 = mk_1 \), \( k_1 \neq 0 \). Then \( \lambda(W_0c) = J_m(\xi(W_0c), \eta(W_0c)) \in P_m(W_0c) \).

Proof. (i) Suppose \( m \in \mathbb{Z} \) and let \( c \) have jumps \( j_1 < j_2 < \cdots < j_{m+2r} \). The lengths of the parts of \( (\xi(W_0c), \eta(W_0c)) \) are \( j_1 < j_2+1 \leq \cdots \leq j_{2r-1} < j_{2r} + 1 \leq j_{2r+1} \leq j_{2r+2} - 1 \leq \cdots \leq j_{r+m} \). This means that if we carry out the map \( J_m \) (see Fig. 5), we first place a horizontal block of length \( j_{2r+m} - (m-1) \), which contains \( (m, m+1, m+2, \ldots, j_{m+2r}) \). Next will be a horizontal block of length \( j_{m+2r-1} - (m-2) \) containing \( (m-1, m, m+1, \ldots, j_{2r+m-1}) \). If \( r = 0 \) the procedure stops after having placed at most \( m-1 \) horizontal blocks. Suppose now that \( r > 0 \). Then, after having placed the \( m-1 \) horizontal blocks, we still have to place \( r + 1 \) horizontal blocks of length \( j_1, j_3, \ldots, j_{2r+1} \) and \( r \) vertical blocks of length \( j_2+1, j_4+1, \ldots, j_{2r}+1 \) on a 1-tableau. We first get a horizontal block of length \( j_{2r+1} \). This is clear since we can reach \( j_{2r+1} \) with a horizontal block and only \( j_{2r} - 1 < j_{2r+1} \) with a vertical block. The \( (m+1) \)th
block will have length $j_{2r} + 1$, which is placed vertically and contains $(0, 1, \ldots, j_{2r})$. For the $(m + 2)$th block, we find after a similar consideration that we get a horizontal block of length $j_{2r - 1}$, containing $(1, 2, \ldots, j_m)$. Notice that as $j_{2r - 1} + 1 < j_{2r + 1}$, the intermediate diagram we have obtained is still that of a partition. In the same way the $(m + 3)$th block is vertical of length $j_{2r - 2}$, with content $(0, 1, 2, \ldots, j_{2r - 2})$. We continue in this way to place all blocks. In the end, we have a diagram covering a sequence with jumps $j_1, j_2, \ldots, j_m$, together with a zero for every second block containing a one, which is the right amount.

(ii) Now suppose that $m \notin \mathbb{Z}$. Then the idea of the proof for integer $m$ carries over without modification: if $r = 0$ we need at most $m - \frac{1}{2}$ horizontal blocks, and if $r > 0$ we find the same alternating pattern of horizontal and vertical blocks after the first $m - \frac{1}{2}$ horizontal ones. □

This gives us a starting point for the 1-1 correspondence, because from Fig. 5 it is clear that also $S_m(\lambda(W_0c)) = (\xi(W_0c), \eta(W_0c))$.

Let $\lambda \in \mathcal{P}_m(W_0c)$. From $\lambda$ we can construct another partition $\mu$ in $\mathcal{P}_m(W_0c)$ by cutting off a piece of $\lambda$ and reattaching it elsewhere. Recall that the extremities of $\lambda$ are jumps for $c$. Therefore, consider the case where it is possible to cut from $\lambda$ a block which contains the sequence $(j_{p-1} + 1, j_{p-1} + 2, \ldots, j_p)$ in $T_m(\lambda)$, and that it can be reattached elsewhere. Then there is a unique other position where it may be reattached: to the extremity of $T_m(\lambda)$ containing $j_{p-1}$. Let the new partition thus obtained be $\mu$. Since $\mu$ is uniquely determined by $\lambda$ and $p$, we write $\mu = F_p(\lambda)$.

We show that these flips do not change the similarity class of the split tableaux. Denote, for a block $\xi_i$ which appears in $S_m(\lambda)$, by $j(\xi_i)$ the extremity on which block $\xi_i$ ends.
Fig. 6. Example of flip.

Proposition 4.26. Let $m \geq 0$ and $W_0 c$ a residual point as before. Let $\lambda \in \mathcal{P}_m(W_0 c)$. If $\mathcal{F}_p(\lambda)$ exists, then $S_m(\lambda) \sim_m S_m(\mathcal{F}_p(\lambda))$.

Proof. Denote $S_m(\lambda) = (\xi, \eta)$ and $S_m(\mathcal{F}_p(\lambda)) = (\xi', \eta')$. Notice that $\mathcal{F}_p(\lambda)$ can only exist if $n - 1 > m \in \frac{1}{2}\mathbb{Z}_{\geq 0}$. We therefore only need to consider such $m$.

(i) Suppose $m \in \mathbb{Z}$. We consider the flip of a part of length $l = j_p - j_{p-1}$ from $\xi_i$ to $\eta_j$. Since $j(\xi_i) = j_p$ and $f(\eta_j) = j_{p-1}$, it is clear that $\xi_i$ and $\eta_j$ are neighbors in the sense that block $\eta_j$ is selected by $S_m$ immediately after block $\xi_i$. Since there are $m + r - i$ horizontal and $r - j$ vertical blocks above $\eta_j$, it follows that $\xi_i$ starts on a square containing $x := |m - (m + r - i) - (r - j)| = |i - j|$. Suppose for the moment that $i \leq j$, the case $i > j$ being similar. Then the starting square of $\xi_i$ lies to the left of the zero-diagonal, and the flip is visualized in Fig. 6.

We see that we have $\xi_i = x + j_p + 1, \eta_j = j_{p-1} - x$ and $\xi'_i = x + j_{p-1}, \eta'_j = j_p - x + 1$. In order for the $m$-symbol to be preserved, we need to show that $\xi_i + 2(i - 1) = \eta'_j + 2(j - 1)$. But this is equivalent to $x + j_p + 1 + 2(i - 1) = j_p - x + 1 + 2(j - 1) \iff 2x = 2(j - i)$ which is the case. The case where $j < i$ is treated with a similar calculation, which we omit here.

(ii) Suppose that $m \notin \mathbb{Z}$. The proof carries over directly. □

Next we show that using these flips, we can construct any partition in $\mathcal{P}_m(W_0 c)$, starting from $\lambda(W_0 c)$.

Proposition 4.27. Let $\lambda \in \mathcal{P}_m(W_0 c)$. Then there is a sequence $i_1, i_2, \ldots, i_f$ such that $\lambda = \prod_{a=1}^f \mathcal{F}_{i_a}(\lambda(W_0 c))$. 
Proof. If the parameters are generic, then $\lambda = \lambda(W_0c)$. We thus only need to consider $n - 1 > m \in \frac{1}{2} \mathbb{Z}_{\geq 0}$. We treat the case where $m \in \mathbb{Z}$, the other case being analogous. Let $(\xi, \eta) = S_m(\lambda)$. Let the blocks of $(\xi, \eta)$ emerge from the splitting procedure in the order $b_1, b_2, \ldots, b_{m+2r}$. Notice that it may happen that some blocks are empty, i.e., there are less than $m + 2r$ blocks. The non-empty blocks all end on a jump. We put $j(b_i)$ to be the jump on which block $b_i$ ends, where $j(b_i) := 0$ if $b_i$ is empty. The diagram $\lambda(W_0c)$ is the unique diagram for which

$$j(\xi_1) \leq j(\eta_1) \leq j(\xi_2) \leq \cdots \leq j(\xi_r) \leq j(\eta_r) \leq j(\xi_{r+1}) \leq j(\eta_{r+1}) \leq \cdots \leq j(\xi_{r+m}).$$

If we look at the corresponding series for $\lambda \neq \lambda(W_0c)$, this series of inequalities will be violated in a number of places. However, we always have $j(\xi_1) \leq j(\xi_2) \leq \cdots \leq j(\xi_{m+r})$ and $j(\eta_1) \leq j(\eta_2) \leq \cdots \leq j(\eta_r)$. Now take the first violated inequality, for simplicity we assume that

$$j(\xi_1) \leq j(\eta_1) \leq j(\xi_2) \leq \cdots \leq j(\xi_{i-1}) \leq j(\eta_{i-1}) \leq j(\xi_i) \not\leq j(\eta_i), \quad (4.9)$$

the case where the first violated inequality is of the kind $j(\eta_i) \not\leq j(\xi_{i+1})$ admitting a similar consideration.

This means that the part $\xi_i$ is too big. We now show that $F_i(\lambda)$ exists. Say $j(\xi_i) = j_p$, then $\xi_i$ ends on a series of squares containing $(j_p-1, j_p-1+2, \ldots, j_p)$, and we can cut off these squares while still maintaining the graph of a partition if $j(\xi_{i-1}) \neq j_p-1$. But this inequality holds, since if $j(\xi_{i-1}) = j_p-1$, we would have $j_p-1 \leq j(\eta_{i-1}) \leq j_p$, which is impossible. So we cut off the rectangle. Since $j_p-1$ is a jump, it follows from Lemma 4.11 that there are now two extremity squares containing $j_p-1$, unless $j_p-1$ does not appear in $T_m(\lambda)$. But if that were true, then $j_p$ is the smallest jump appearing as extremity, giving a contradiction to (4.9). Thus, after cutting off the rectangle, we have two extremities equal to $j_p-1$, one in the former block $\xi_i$ and one other. This latter one cannot belong to a horizontal block, since in that case again we would have $j(\xi_i) = j_{p-1}$ for some $l < i$, and we would find an earlier violated inequality. So we attach the rectangle vertically to this square, which is the last square of $\eta_i$ for some $l$. If $l < i$, then $j_{p-1} = j(\eta_l) < j(\eta_i) = j_t$, so $p-1 < t$. On the other hand, we also have $j_p = j(\xi_i) > j(\eta_l) = j_t$, so $p > t$. This is a contradiction and so we see that $l \geq i$. After the flip, we still have the diagram of a partition if $j_s = j(\eta_{l+1}) > j_p$, i.e., if $s > p$. But before the flip we had $j_s = j(\eta_{l+1}) > j(\eta_l) = j_{p-1}$, so $s > p-1$. Since $s \neq p$, it follows that indeed we still have the diagram of a partition after this flip, and moreover the inequality we were looking at, is less or no longer violated by $F_i(\lambda)$. We keep applying this procedure until we reach $\lambda(W_0c)$. □

In $\Sigma_m(W_0c)$, we can do something similar. First we define

**Definition 4.28.** Consider a $m$-tableau $T_m(\lambda)$ such that $S_m(\lambda)$ has blocks $b_i$. We denote by $e_m(b_i)$ the entry of block $b_i$ in the $m$-symbol of $S_m(\lambda)$. 


Notice that $e_m$ depends on the chosen lengths of the rows of the symbol. However, we will only be interested in the difference of two entries, which is independent of this choice.

It now turns out that the $m$-symbol of $(\xi, \eta) \in \Sigma_m(W_0c)$ determines the order in which the parts of $\xi$ and $\eta$ are placed when applying $J_m$:

**Proposition 4.29.** If we apply $J_m$ to $(\xi, \eta)$, then the blocks are laid down in the order of decreasing entries in the $m$-symbol, i.e., block $b_i$ is placed before block $b_j$ if $e_m(b_i) > e_m(b_j)$.

**Proof.** Again, for simplicity we only treat the case where $m \in \mathbb{Z}$ is integer. We use induction on the number of blocks already placed. Suppose this is zero, then we either place $\xi_{r+m}$ or $\eta_r$, and $\xi_{r+m}$ will be selected if and only if it can reach a higher number than $\eta_r$, which is the case if and only if $m + \xi_{r+m} - 1 > \eta_r - m - 1 \iff \xi_{r+m} + 2(r + m - 1) > \eta_r + 2(r - 1)$, which is what we wanted.

Now suppose that we have placed already $k$ horizontal blocks $\xi_{r+m}, \xi_{r+m-1}, \ldots, \xi_{r+m-k+1}$ and $l$ vertical blocks $\eta_r, \eta_{r-1}, \ldots, \eta_{r-l+1}$. Then the next block starts on a square containing $|m - k + l|$. First we assume that $m \geq k - l$, i.e., the square under consideration is above the zero-diagonal. Then the next block to be placed will either be $\xi_{r+m-k}$ or $\eta_{r-l}$, and it will be $\xi_{r+m-k}$ if and only if

$$m - k + l + \xi_{r+m-k} - 1 > \eta_{r-l} - m + k - l - 1$$

$$\iff \xi_{m+m-k} + 2(m - k) > \eta_{r-l} - 2l$$

$$\iff \xi_{r+m-k} + 2(r + m - k - 1) > \eta_{r-l} - 2(r - l - 1).$$

Second, assume that $m < k - l$, then the starting square for the next block is below the zero-diagonal. In this case, the next block will be $\xi_{r+m-k}$ if and only if

$$\xi_{r+m-k} - (-m + k - l) - 1 > \eta_{r-l} - m + k - l - 1$$

$$\iff \xi_{r+m-k} + 2(m - k) > \eta_{r-l} - 2l$$

$$\iff \xi_{r+m-k} + 2(r + m - k - 1) > \eta_{r-l} - 2(r - l - 1).$$

This proves the induction step and therefore the claim. □

Then a flip in the $m$-symbol this corresponds to the following:

**Lemma 4.30.** In the situation of Fig. 6 we have $e_m(\xi_i) > e_m(\eta_i)$ and $e_m(\xi_i) < e_m(\eta_j) < e_m(\xi_{i+1})$. Therefore the flip interchanges $e_m(\xi_i)$ with the unique entry on the other row of the $m$-symbol with which it can be interchanged.

**Proof.** In the notation of the proof of Proposition 4.26, $\xi_i = j - i + j_p + 1$ and $\eta_j = j_{p-1} - j + i$, therefore $e_m(\xi_i) - e_m(\eta_j) = \xi_i + 2(i - 1) - \eta_j - 2(j - 1) \geq 2$. For $\eta_{j+1}$ we know that $\eta_{j+1} \geq j_{p+1} - (x + 1) + 1$, and therefore $e_m(\eta_{j+1}) - e_m(\xi_i) \geq j_{p+1} - j_p + 1 \geq 2$.

Finally, since $j \geq i$, $\xi_i \geq \eta_j \geq \eta_i$ and so also $e_m(\xi_i) \geq e_m(\eta_i)$. □
On the other hand, given any \emph{m}-symbol of a partition in \( \Sigma_m(W_0c) \), we can change it into the similar symbol with the same, but increasing entries, i.e., where \( e_m(\xi_1) < e_m(\eta_1) < e_m(\xi_2) < \cdots < e_m(\eta_{r-1}) < e_m(\xi_r) < e_m(\xi_{r+1}) < \cdots < e_m(\xi_{m+r}) \). We can for example achieve this by finding the first violated inequality, e.g. say \( e_m(\xi_i) > e_m(\eta_i) \), and then interchanging \( e_m(\xi_i) \) with \( e_m(\eta_j) \) for the unique \( j \) such that \( e_m(\eta_j) < e_m(\xi_i) < e_m(\eta_{j+1}) \). We denote the permutation where \( e_m(\xi_i) \) is moved to the bottom row, where it takes the place of entry \( e_m(\eta_j) \), by \( F^{i\downarrow}(\xi, \eta) = F^{\uparrow j}(\xi, \eta) \).

On the level of \emph{m}-tableaux, such a permutation of the \emph{m}-symbol corresponds to a flip:

**Lemma 4.31.** Let \( (\xi, \eta) \in \Sigma_m(W_0c) \) and let \( \lambda = J_m(\xi, \eta) \). Suppose \( e_m(\xi_1) < e_m(\eta_1) < e_m(\xi_2) < \cdots < e_m(\eta_{i-1}) < e_m(\xi_i) \) but \( e_m(\xi_i) > e_m(\eta_i) \). Then we have \( J_m(F^{i\downarrow}(\xi, \eta)) = F_p(\lambda) \) if the block of \( \xi_i \) ends on jump \( j_p \) in \( T_m(\lambda) \).

**Proof.** Let \( \eta_j \) be such that \( e_m(\eta_j) < e_m(\xi_i) < e_m(\eta_{j+1}) \). Then clearly \( j \geq i \). From Proposition 4.29 it follows that \( \eta_j \) is placed after \( \xi_i \). Since \( e_m(\xi_{i-1}) < e_m(\eta_j) \) it follows that \( \eta_j \) is placed immediately after \( \xi_i \). Say block \( \xi_i \) ends on \( x = j_p \) and block \( \eta_j \) ends on \( y < x \), then we can cut a rectangle containing \( y+1, y+2, \ldots, x \) off \( \xi_i \) and attach it to \( \eta_j \) provided \( j(\xi_{i-1}) \neq y \) and \( j(\eta_{j+1}) \neq x \). But this would contradict the fact that \( J_m \) is well defined on \( \Sigma_m(W_0c) \). As we have seen in the proof of Proposition 4.27, this flip of \( \xi_i \) to \( \eta_j \) corresponds with the interchange of \( e_m(\xi_i) \) and \( e_m(\eta_j) \) in the \emph{m}-symbol. □

Finally, we can show

**Theorem 4.32.** Let \( m \in \frac{1}{2}Z_{\geq 0} \) and let \( W_0c \) be a residual point for \( H \) with parameters \( k_2 = mk_1, k_1 \neq 0 \). Then \( S_m \) and \( J_m \) implement a bijection \( P_m(W_0c) \leftrightarrow \Sigma_m(W_0c) \).

**Proof.** Let \( \lambda \in P_m(W_0c) \), then it follows from 4.25 and 4.27 that \( S_m(\lambda) \in \Sigma_m(W_0c) \).

Since it is easy to see that \( S_m \) is injective it remains to show that the image of \( S_m \) is all of \( \Sigma_m(W_0c) \). Take \( (\xi, \eta) \in \Sigma_m(W_0c) \), then there is a sequence of permutations \( F^{i\downarrow j} \) such that

\[
(\xi, \eta) = \prod_j F^{i\downarrow j}(\xi(W_0c), \eta(W_0c)).
\]

Therefore, by Lemma 4.31, there is a sequence of jumps \( t_j \) such that

\[
(\xi, \eta) = \prod_j F^{i\downarrow j}(\xi(W_0c), \eta(W_0c)) = \prod_j F^{i\downarrow j}(S_m(\lambda(W_0c)))
\]

\[
= S_m\left( \prod_j F_{i\downarrow j}(\lambda) \right) \in S_m(P_m(W_0c)).
\]

This proves the claim. □
4.11. Transitivity of truncated induction

We want to prove the transitivity of the truncated induction $\text{tr}_m - \text{Ind}$. We first consider its analogue on $\mathcal{P}_n$. For a partition $\lambda \vdash n$, the analogue of the $m$-symbol is $\lambda$ itself. In this section only, until we go back to $\text{tr}_m - \text{Ind}$, we will write (as is more common) the parts of a partition in decreasing order. The $a$-function reduces to the well-known

$$n(\lambda) = \sum_i (i-1)\lambda_i = \sum_i \left(\frac{\lambda_i}{2}\right), \quad (4.10)$$

where $\lambda'$ denotes the partition conjugate to $\lambda$.

In this case, the partial ordering defined by the $a$-function refines the dominance ordering, in which by definition $\lambda \succeq \mu$ if

$$\lambda_1 + \lambda_2 + \cdots + \lambda_i \succeq \mu_1 + \mu_2 + \cdots + \mu_i \text{ for all } i \in \{1, 2, \ldots, n\}. \quad (4.11)$$

**Lemma 4.33.** For all $\lambda, \mu \vdash n$, $\lambda \succeq \mu \Rightarrow n(\lambda) \leq n(\mu)$.

**Proof.** $\lambda \succeq \mu \iff \lambda' \preceq \mu' \iff \sum_i \left(\frac{\lambda'_i}{2}\right) \leq \sum_i \left(\frac{\mu'_i}{2}\right) \iff n(\lambda) \leq n(\mu). \quad \square$

Now let us see how the $a$-function behaves under induction. Recall the Littlewood–Richardson rule (cf. [15] for a proof). For a partition $\lambda \vdash n$ we denote by $V_\lambda$ the corresponding irreducible $S_n$-module. If $\mu \vdash k$ and $\nu \vdash l$, then

$$\text{Ind}_{S_k \times S_l}^{S_{k+l}}(V_\mu \otimes V_\nu) = \sum_{\lambda \vdash n} c^\lambda_{\mu \nu} V_\lambda. \quad (4.12)$$

In this formula $c^\lambda_{\mu \nu}$ is the number of ways the Young tableau of $\mu$ can be made into the Young tableau of $\lambda$ by means of a strict $\nu$-expansion. This means the following. We add $|\nu|$ squares to the tableau of $\mu$ to obtain the tableau of $\lambda$, by first adding $\nu_1$ squares filled with a 1, then adding $\nu_2$ squares filled with a 2, etc. until finally we add $\nu_n$ squares filled with $n$. After each of these steps, we must still have the diagram of a partition, and no two squares in the same column may have the same entry. It now remains to explain when such an expansion is called strict. Suppose we read the added squares from right to left, and from top to bottom. We then have a sequence of numbers $a_1 a_2 \ldots a_l$. Now the expansion is called strict if for any intermediate sequence $a_1 a_2 \ldots a_i$ ($1 \leq i \leq l$), any of the occurring integers $1 \leq i \leq n$ occurs at least as many times as the next integer $k + 1$.

**Lemma 4.34.** Let $\mu \vdash k$ and $\nu \vdash l$. Then

$$\text{tr} - \text{Ind}_{S_k \times S_l}^{S_{k+l}}(V_\mu \otimes V_\nu) = V_{\mu \cup \nu}.$$
Proof. First we check that $\lambda = \mu \cup v$ indeed occurs in the induction, by constructing its Young tableau as a strict $v$-expansion of the Young tableau of $\mu$. Fill the part $v_i$ in $\lambda$ with $i$'s. Then we obtain a series of $l = l(v)$ horizontal bars in $\lambda$, filled with 1 up to $l$. Now move the contents of the filled squares as far down as possible. Then the unfilled squares make up a tableau of shape $\mu$ in $\lambda$, so that we obtain $\lambda$ as a $v$-expansion of $\mu$. This expansion is strict since if we have a square filled with $i$, then $i - 1$ occurs more often than $i$ in the preceding blocks of the expansion, by construction. Thus indeed $\lambda := \mu \cup v$ occurs in (4.12). Now we show that its $a$-value is maximal. This follows from 4.33, since

$$n(\lambda) = n(\mu \cup v) = \sum_i \binom{(\mu \cup v)_i}{2} = \sum_i \binom{\mu_i' + v_i'}{2},$$

but for any $\kappa \neq \lambda$ occurring in (4.12), $\kappa' < \mu' \cup v'$. □

Corollary 4.35. The truncated induction is transitive in the sense that for all $\lambda \vdash k$, $\mu \vdash l$, $v \vdash m$, we have

$$\text{tr} - \text{Ind}_{S_l \times S_m}^{S_{l+m}} (V_\lambda \otimes \text{tr} - \text{Ind}_{S_l \times S_m}^{S_{l+m}} (V_\mu \otimes V_\nu)) = \text{tr} - \text{Ind}_{S_l \times S_m}^{S_{l+m}} (V_\lambda \otimes V_\mu \otimes V_\nu).$$

Proof. The argument of the previous Lemma generalizes immediately to any number of partitions, which implies that both sides are equal to $V_{\lambda \cup \mu \cup \nu}$. □

We want to prove that this also holds for $\text{tr}_m - \text{Ind}$. Let us first see how the $a$-value of a partition is related to $a_m(\zeta, \eta)$.

Lemma 4.36. Let $\lambda$ and $\mu$ be such that $n(\lambda) \leq n(\mu)$. Then $a_m(\lambda, \kappa) \leq a_m(\mu, \kappa)$ and $a_m(\kappa, \lambda) \leq a_m(\kappa, \mu)$ for all $m \in \frac{1}{2} \mathbb{Z}_{\geq 0}$ and all partitions $\kappa$.

Proof. Notice that $a_m(\zeta, \eta) = n(v)$ where $v$ is the partition whose parts are the entries of the $m$-symbol of $(\zeta, \eta)$. The lemma therefore follows from the fact that $n(\lambda \cup \kappa) \leq n(\mu \cup \kappa)$ for all $\kappa$, if $n(\lambda) \leq n(\mu)$. □

We check what happens in case $\lambda$ has only one part:

Proposition 4.37. Let $k_2 = mk_1$ with $k_1 \neq 0$, $m \in \frac{1}{2} \mathbb{Z}_{\geq 0}$. For $(\zeta, \eta) \in \mathcal{P}_{n,2}$,

$$\text{tr}_m - \text{Ind}_{W_0(B_{k_1})}^{W_0(B_{k_2}) \times S_t} ((\zeta, \eta) \otimes (t)) = \sum_i (\zeta \cup t_{i,1}, \eta \cup t_{i,2}),$$

where $i$ runs over an indexing set containing one or two elements, i.e., for one or two decompositions $t_{i,1} + t_{i,2} = t$. 
Proof. Suppose that in a constituent $\mathcal{A}$ of maximal $a_m$-value, $t_{1,1}$ squares are added to $\xi$ and $t_{1,2}$ to $\eta$. Then by Lemmas 4.34 and 4.36, $A = (\xi \cup t_{1,1}, \eta \cup t_{1,2})$, and thus $\mathcal{A}$ occurs with multiplicity one in the induction. The entries $e_m(t_{1,j})$ corresponding to $t_{1,1}, t_{1,2}$ in the $m$-symbol of $(\xi \cup t_{1,1}, \eta \cup t_{1,2})$ are equal or have a difference of one. Suppose we are in the latter case and $e_m(t_{1,1}) = e_m(t_{1,2}) + 1$. Then, if $\mathcal{B}$ with $B = (\xi \cup (t_{1,1} - 1), \eta \cup (t_{1,2} + 1))$ occurs in the induction, we have $a_m(\mathcal{A}) = a_m(\mathcal{B})$. Put $t_{2,1} = t_{1,1} - 1$ and $t_{2,2} = t_{1,2} + 1$. □

Now we investigate the transitive behaviour of $\text{tr}_m - \text{Ind}$:

Lemma 4.38. Let $(\xi, \eta) \in \mathcal{P}_{l,2}$, and $a, b \in \mathbb{N}$. Suppose that $(\xi \cup a_1, \eta \cup a_2)$ occurs in $\text{tr}_m - \text{Ind}_S^W(a) \otimes (\xi, \eta)$ and that $(\xi \cup b_1, \eta \cup b_2)$ occurs in $\text{tr}_m - \text{Ind}_S^W(b) \otimes (\xi, \eta)$. Then $(\xi, \eta) \cup (a_1, a_2) \cup (b_1, b_2)$ occurs in

$$\text{tr}_m - \text{Ind}_S^W((a) \otimes (b) \otimes (\xi, \eta)).$$

(4.13)

Conversely, any constituent of (4.13) is of this form.

Proof. We may suppose that $a \leq b$. Let $A(a, b) = \{(x_1, x_2, y_1, y_2) \mid x_1 \leq y_1, x_2 \leq y_2, x_1 + x_2 \leq a, y_1 + y_2 \geq b\}$. Then we have

$$\text{Ind}_S^W((a) \otimes (b)) = \sum_{(x_1, x_2, y_1, y_2) \in A(a, b)} n_{(x_1, x_2, y_1, y_2)}((x_1 y_1), (x_2 y_2)),$$

where the multiplicity $n_{(x_1, x_2, y_1, y_2)}$ can be computed with the Littlewood–Richardson rule. Therefore,

$$\text{Ind}((a) \otimes (b) \otimes (\xi, \eta)) = \sum_{(x_1, x_2, y_1, y_2) \in A(a, b)} n_{(x_1, x_2, y_1, y_2)}(((x_1 y_1), (x_2 y_2)) \otimes (\xi, \eta)).$$

Now we claim that

$$\text{tr}_m - \text{Ind}((x_1 y_1), (x_2 y_2)) \otimes (\xi, \eta) = ((\xi \cup x_1 \cup y_1, \eta \cup x_2 \cup y_2).$$

Indeed, from Corollary 4.35 and Lemma 4.36 it follows that $a_m(\xi', \kappa) \leq a_m(\xi \cup x_1 \cup y_1, \kappa)$ for all $\xi' \vdash |\xi| + x_1 + y_1$ which is a strict $(x_1 y_1)$-expansion of $\xi$. In particular, we have $a_m(\xi', \eta') \leq a_m(\xi \cup x_1 \cup y_1, \eta')$ for every partition $\eta' \vdash |\eta| + x_2 + y_2$ which is a strict $(x_2, y_2)$-expansion of $\eta$, and also that $a_m(\xi', \eta \cup x_2 \cup y_2) \leq a_m(\xi \cup x_1 \cup y_1, \eta \cup x_2 \cup y_2)$. Similarly, it follows that $a_m(\xi', \eta') \leq a_m(\xi', \eta \cup x_2 \cup y_2)$ and $a_m(\xi \cup x_1 \cup y_1, \eta') \leq a_m(\xi \cup x_1 \cup y_1, \eta \cup x_2 \cup y_2)$. Together, it follows that $a_m(\xi', \eta') = a_m(\xi \cup x_1 \cup y_1, \eta \cup x_2 \cup y_2).$ It is easy to see that equality only holds for $(\xi', \eta') = (\xi \cup x_1 \cup y_1, \eta \cup x_2 \cup y_2)$. This proves the claim.
Thus, we need to know for which \((x_1, x_2, y_1, y_2) \in A(a, b)\) the \(a_m\)-value of \((\zeta \cup x_1 \cup y_1, \eta \cup x_2 \cup y_2)\) is maximal. It is easy to see that for such \((x_1, x_2, y_1, y_2)\) we have \(x_1 + x_2 = a, y_1 + y_2 = b\). Moreover, the entries of \(y_1, y_2\) in the \(m\)-symbol of \((\zeta \cup x_1 \cup y_1, \eta \cup x_2 \cup y_2)\) will satisfy \(|e_m(y_1) - e_m(y_2)| \leq 1\), and likewise \(|e_m(x_1) - e_m(x_2)| \leq 1\). This implies that \((\zeta \cup x_1, \eta \cup x_2)\) occurs in the induction \(\text{Ind}((\zeta, \eta) \otimes (a))\) and that \((\zeta \cup y_1, \eta \cup y_2)\) occurs in the induction of \(\text{Ind}((b) \otimes (\zeta, \eta))\).

Suppose that \(a_m(\zeta \cup x_1, \eta \cup x_2)\) is not maximal in the induction \(\text{Ind}((a) \otimes (\zeta, \eta))\). Then, since we have seen that \(|e_m(y_1) - e_m(y_2)| \leq 1\), it would follow that also \(a_m(\zeta \cup x_1 \cup y_1, \eta \cup x_2 \cup y_2)\) would not be maximal. This is a contradiction. We reason similarly w.r.t. \(a_m(\xi \cup y_1, \eta \cup y_2)\). It follows that for any representation which occurs in the truncated induction (4.13) is of the desired form.

Conversely, it is easy to see that all \((\xi \cup a_1 \cup b_1, \eta \cup a_2 \cup b_2)\) as in the claim arise in (4.13). Notice that they all have multiplicity one in the induction. □

**Corollary 4.39.** Let \(k_2 = mk_1, k_1 \neq 0\) be such that \(m \in \frac{1}{2} \mathbb{Z} \geq 0\), and consider \((\xi, \eta) \in \mathcal{P}_{l,2}\). Let \(a + b + l = n\). Then

\[
\text{tr}_m - \text{Ind}_{S_k \times W_0(B_t)}^{W_0(B_n)}((a) \otimes \text{Ind}_{S_k \times W_0(B_t)}^{W_0(B_n)}((b) \otimes (\xi, \eta))) = \text{tr}_m - \text{Ind}_{S_k \times W_0(B_t)}^{W_0(B_n)}((a) \otimes (b) \otimes (\xi, \eta)).
\]

**Proof.** By the above lemma, both sides are equal to \(\sum_{i,j} \xi \cdot \eta \cup (a_{i,1} + a_{i,2} \cup (b_{j,1} + b_{j,2})\) for certain \(a_{i,1} + a_{i,2} = a, b_{j,1} + b_{j,2} = b\). □

By repeated application of this corollary, we get:

**Proposition 4.40.** Let \(k_2 = mk_1, k_1 \neq 0\) and \(m \in \frac{1}{2} \mathbb{Z} \geq 0\). Let \((\xi, \eta) \in \mathcal{P}_{l,2}\) and \(\lambda \vdash n - l\). Then

\[
\text{tr}_m - \text{Ind}_{S_k \times W_0(B_t)}^{W_0(B_n)}(\text{triv}_{\lambda} \otimes (\xi, \eta)) = \sum_{\mu} (\xi \cup \mu, \eta \cup (\lambda - \mu)),
\]

where \(\mu\) ranges over a set of partitions whose Young diagrams are contained in the one of \(\lambda\), which depends on \((\xi, \eta)\) and \(m\).

### 4.12. Confluence of residual subspaces

In this section we show that \(C_m(W_0cL) \leftrightarrow \Sigma_m(W_0cL)\) and that \(\Sigma_m(W_0cL)\) is a full similarity class.

#### 4.12.1

First we treat the case where a generically residual point is no longer residual at a given special value, but coincides with the center of a higher-dimensional residual subspace:
Proposition 4.41. Let \( k \neq 0, m \in \frac{1}{2} \mathbb{Z} \geq 0 \), and suppose that \( W_0 c(\mu, k, mk) = W_0 c_L \) for \( L \in L_m(n) \) of type \((\kappa, \nu)\). Then the set \( \Sigma_m(W_0 c_L) \) of Springer correspondents of \( W_0 c_L \), i.e., all the irreducible representations of \( W_0 \) which appear in

\[
\sum_{[v'|S_m(v') = mS_m(v)]} \text{tr}_m \text{Ind}_{W_0(B_n)_{S_m \times W_0(B_l)}} \langle \text{triv}_k \otimes S_m(v') \rangle,
\]

satisfies

\[
\Sigma_m(W_0 c_L) = [S_{m+\epsilon}(\mu)]_m = [S_{m-\epsilon}(\mu)]_m.
\]

Moreover, we have a bijection

\[
\Sigma_m(W_0 c_L) \leftrightarrow C_m(W_0 c_L).
\]

Proof. We will prove the Proposition by induction. Therefore we introduce, for \( i = 0, 1, \ldots, l = l(\kappa) \):

\[
\kappa^{(i)} = (\kappa_1, \ldots, \kappa_l); \quad \mu^{(i)} \text{ such that } sp_m(\mu^{(i)}) = (\kappa^{(i)}, \nu),
\]

i.e., \( \mu^{(i)} \) is the partition obtained by removing the \( A \)-strips of length \( \kappa_{i+1}, \ldots, \kappa_l \) from \( T_m(\mu) \). Let \( L^{(i)} \) be the residual coset of type \((\kappa^{(i)}, \nu)\).

We will use induction on \( i \). Suppose that \( i = 0 \). Then the statement reduces to Theorem 4.32.

Therefore we take \( i + 1 \) and suppose that the Proposition holds for \( i \). That is (putting \([\kappa^{(i)}] = n_i \), \( \Sigma_m(W_0 c_L^{(i)}) \), the set of all irreducible characters occurring in

\[
\sum_{[v'|S_m(v') = mS_m(v)]} \text{tr}_m \text{Ind}_{W_0(B_{n_i+1})_{S_{n_i} \times W_0(B_l)}} \langle \text{triv}_{\kappa^{(i)}} \otimes S_m(v') \rangle,
\]

satisfies the bijection

\[
\Sigma_m(W_0 c_L^{(i)}) \leftrightarrow C_m(W_0 c_L^{(i)}),
\]

and

\[
\Sigma_m(W_0 c_L^{(i)}) = [S_{m+\epsilon}(\mu^{(i)})]_m = [S_{m-\epsilon}(\mu^{(i)})]_m.
\]

Suppose that \( \kappa_{i+1} = t \) and pick \((\xi, \eta) \in \Sigma_m(W_0 c_L^{(i)})\). In view of Corollary 4.39, we consider

\[
\text{tr}_m \text{Ind}_{W_0(B_{n_i+1})_{S_{l} \times W_0(B_{n_i+1})}} ((f) \otimes (\xi, \eta)) \tag{4.15}
\]

and we will show that it consists of two elements, which belong to \([S_m(\mu^{(i+1)})]_m\).
In view of the following implication

\[(\xi, \eta) \sim_m (\xi', \eta') \Rightarrow \text{tr}_m - \text{Ind}((t) \otimes (\xi, \eta)) \sim_m \text{tr}_m - \text{Ind}((t) \otimes (\xi', \eta')),\]

we may assume that \((\xi, \eta) \in S_{m \pm \varepsilon}(\mu^{(i)})\). Doing so, we will calculate explicitly both \((4.15)\) and \(S_{m \pm \varepsilon}(\mu^{(i+1)})\) and show that they agree. Recall from 4.15 that an A-strip of length \(t\) is found inside \(T_m(\mu)\) as a hook-shaped strip, whose extremities contain \([t/2]\) (recall also that \(t\) is necessarily odd). For \(m \pm \varepsilon\) the splitting procedure \(S_{m \pm \varepsilon}\) is well-defined; the square in the corner of the hook belongs to a horizontal block in one splitting, and to a vertical one in the other.

We are going to construct the constituents \((\xi', \eta')\) whose \(m\)-symbol has maximal \(a_m\)-value in the induced representation \((4.15)\).

(i) Recall that the Littlewood–Richardson rule implies that the diagrams of \(\xi'\) resp. \(\eta'\) are obtained from those of \(\xi\) resp. \(\eta\) by means of strict expansions. In this case, this means that there are no two new squares in the same column. Suppose for convenience that \(l(\xi) = m + r\), \(l(\eta) = r\). Then we write \(\xi = (0 = \xi_0 \leq \xi_1 \leq \cdots \leq \xi_{r+m})\), \(\eta = (0 = \eta_0 \leq \eta_1 \leq \cdots \leq \eta_r)\), and \(\xi' = (\xi_0' \leq \cdots \leq \xi_{m+r}')\), \(\eta' = (\eta_0' \leq \cdots \leq \eta_{r}')\). We have

\[
\xi_i \leq \xi_i' \leq \xi_{i+1}, \quad \eta_i \leq \eta_i' \leq \eta_{i+1},
\]

and \((\xi, \eta)\) has \(m\)-symbol

\[
\begin{pmatrix}
0 & \xi_1 + 2 & \cdots & \xi_r + 2r & \xi_{r+1} + 2(r + 1) & \cdots & \xi_{r+m} + 2(r + m) \\
0 & \eta_1 + 2 & \cdots & \eta_r + 2r
\end{pmatrix}
\]

which we have to expand in such a way that we maximize the \(a\)-value.

(ii) Now we describe the splittings \(S_{m \pm \varepsilon}(\mu^{(i+1)})\). Suppose for convenience that \((\xi, \eta) = S_{m \pm \varepsilon}(\mu^{(i+1)})\). Recall that by \(j(b)\) we denote the extremity on which the block \(b\) ends. Since the A-strip of length \(t\) fits into \(T_m(\mu^{(i)})\), there exist \(\xi_{a+1}, \xi_a, \eta_{b+1}, \eta_b\) such that \(j(\xi_{a+1}) > [t/2], j(\eta_{b+1}) > [t/2]\), while \(j(\xi_a) < [t/2], j(\eta_a) < [t/2]\). Several of these blocks may be empty, in which case we consider these equalities to hold, since this can only happen for the inner- or outermost blocks.

The A-strip is thus inserted in \(T_m(\mu^{(i)})\) in the place of \((\xi_a, \eta_b)\). Since there are \(m + r\) horizontal blocks and \(r\) vertical ones, this means that the corner of the A-strip is positioned on a square containing \([a - b]\), as seen before. Therefore, \(S_{m \pm \varepsilon}(\mu^{(i+1)}) = S_{m \pm \varepsilon}(\mu^{(i)}) \cup (\xi_a', \eta_b')\), where \((\xi_a', \eta_b')\) is given by

\[
(i) - \begin{cases}
\xi_a' = [t/2] - (a - b) \\
\eta_b' = [t/2] + (a - b)
\end{cases}
\quad \text{for } m + \varepsilon \quad \text{or} \quad (ii) - \begin{cases}
\xi_a' = [t/2] - (a - b) \\
\eta_b' = [t/2] + (a - b)
\end{cases}
\quad \text{for } m - \varepsilon.
\]

(4.17)
(iii) Now we show that we also find (4.17) when we calculate (4.15). By 4.37, a constituent \((\xi', \eta')\) with maximal \(a\)-value is of the form \((\xi', \eta') = (\xi, \eta) \cup (x, y)\) with \(x + y = t\). It remains to show that \(x\) and \(y\) are as in (4.17).

We have to add \(t\) squares to \((\xi, \eta)\), such that \(a_m\) is maximized. Suppose that \((x, \beta)\) occurs in the (full) induction (4.15). Let \(s_m(\xi, \eta)\) have entries \(x'_1 > x'_2 > \cdots \geq x'_{n+m}\) (resp. \(x_1 > x_2 > \cdots \geq x_{n+m}\)). Let \(s_{(\xi, \eta)}\) be such that \(x'_i = x_i\) for \(i = 1, 2, \ldots, s_{(\xi, \eta)}\) and \(x'_{s_{(\xi, \eta)}+1} \neq x_{s_{(\xi, \eta)}+1}\). Then, by 4.33, if \((\xi', \eta')\) occurs in (4.15) then \(s_{(\xi', \eta')}\) is maximal among \(\{s_{(\xi, \eta)} | (x, \beta)\) occurs in Ind\((t) \otimes (\xi, \eta)\)\); i.e., the truncated induction changes the parts of \((\xi, \eta)\) whose entries are “as small as possible”. Since we write the parts of \(\xi\) and \(\eta\) in increasing order, we therefore locate a pair \((\xi_i, \eta_j)\) such that \(i + j\) is maximal among the pairs \(\{(i, j) | (\xi_i, \eta_j) \neq (\xi'_i, \eta'_j), a_m(\xi'_i, \eta'_j)\) is maximal\}.

For example, if \(t > \xi_m + r + \eta_r\), then we cannot perform the induction without enlarging either \(\xi_m + r\) or \(\eta_r\) and we find \((\xi_i, \eta_j) = (\xi_{r+m}, \eta_r)\). In general, there are two analogous situations possible.

(iii) Suppose we have a sequence

\[
e_m(\eta_{j+1}) > e_m(\xi_k) > e_m(\xi_{k-1}) \cdots > e_m(\xi_l) \geq e_m(\eta_f) \tag{4.18}
\]

which is as long as possible, i.e., \(e_m(\xi_{k+1}) > e_m(\eta_{f+1})\) and \(e_m(\xi_{l-1}) < e_m(\eta_f)\), in which

\[\xi_k + \eta_{f+1} \geq t > \xi_l + \eta_f.\]

Some of these parts may not exist, in which case we delete the relation in which they occur. We check if, and how, we need to change parts \((\xi_i, \eta_j)\) into \((\xi'_i, \eta'_j)\).

First we consider \(\eta_f\) and one of the parts \(\xi_i, \ldots, \xi_k\). We do not need to change \(\xi_k\) if \(t - (\xi_k + \eta_f) < e_m(\xi_k) - e_m(\eta_f) = \xi_k - \eta_f + 2(k - f)\). In that case we check if we have to change \(\xi_{k-1}\), etc. This means that we have to look for the smallest \(p \in \{0, 1, \ldots, k - l\}\) such that

\[t \geq 2\xi_{k-p} + 2(k - p - f). \tag{4.19}\]

In fact, since \(t\) is odd, this will be a strict inequality. Suppose for a moment that such \(p\) exists. Then we calculate how the parts of \(\xi_{k-p}\) and \(\eta_f\) should be changed, in order to maximize the \(a\)-value. This means that we distribute the \(t - \xi_{k-p} - \eta_f\) squares of the strip, which need to be added to \(\xi_{k-p}\) and \(\eta_f\), in such a way that we increase the entries \(e_m(\xi_{k-p})\) and \(e_m(\eta_f)\) minimally. This produces the equations

\[
\begin{align*}
(\xi'_{k-p}) &= \frac{t}{2} - (k - p - f) \\
(\eta_f') &= \frac{t}{2} + (k - p - f)
\end{align*}
\]

or

\[
\begin{align*}
(\xi'_{k-p}) &= \lceil \frac{t}{2} \rceil - (k - p - f) \\
(\eta_f') &= \lceil \frac{t}{2} \rceil + (k - p - f)
\end{align*}
\]

(i) or (ii)

One checks that these equations lead to the following:

\[
(\xi'_{k-p}) = \frac{t}{2} - (k - p - f) \quad \text{or} \quad (\xi'_{k-p}) = \lceil \frac{t}{2} \rceil - (k - p - f), \\
(\eta_f') = \frac{t}{2} + (k - p - f) \quad \text{or} \quad (\eta_f') = \lceil \frac{t}{2} \rceil + (k - p - f). \tag{4.20}
\]
It remains to show that \( a = k - p, b = f \), for then we recover (4.17). This can be shown as follows: if \( p > 0 \), then (4.19) and minimality of \( p \) imply that \( \hat{\xi}_{k-p} \leq \lfloor \frac{k}{2} \rfloor - (k-p-f) \) and \( \hat{\xi}_{k-p+1} \geq \lceil \frac{k}{2} \rceil - (k-p-f) \). Also \( e_m(\eta_f) < e_m(\hat{\xi}_{k-p}) \) implies \( \eta_f < \lfloor \frac{k}{2} \rfloor + (k-p-f) \) and \( e_m(\hat{\xi}_{k-p+1}) > e_m(\hat{\xi}_{k-p+1}) \) implies \( \eta_{f+1} > \lceil \frac{k}{2} \rceil + (k-p-f) \) . Thus by (4.16) both (4.20)(i)–(ii) are allowed, unless \( \hat{\xi}_{k-p+1} = \lfloor \frac{k}{2} \rfloor - (k-p-f) \). If \( \hat{\xi}_{k-p+1} \neq 0 \), then it starts on a square containing \( |k-p-f+1| \), hence ends on \( \lfloor \frac{k}{2} \rfloor \), which is impossible. Therefore \( \hat{\xi}_{k-p+1} \) ends on more than \( \lfloor \frac{k}{2} \rfloor \) and \( \hat{\xi}_{k-p} \) on less. On the other hand, if \( \hat{\xi}_{k-p+1} = 0 \), then it follows that the block of length \( \eta_f \) starts on more than \( \lfloor \frac{k}{2} \rfloor \), from which we see that an A-strip of length \( t \) cannot be fitted into the \( m \)-tableau \( T_m(\mu^{(f)}) \), which is a contradiction. Again because of 4.29, we see from (4.18) that if \( \eta_{f+1} \neq 0 \), then \( j(\eta_{f+1}) > \lceil \frac{k}{2} \rceil \) and \( j(\eta_f) < \lfloor \frac{k}{2} \rfloor \). This means that, as we wanted to show, \( a = k - p \) and \( b = f \). The upshot is that we find two 2-partitions with maximal \( a \)-value, because the A-strip of length \( t \) fits into the diagram. The new entries in the \( m \)-symbol are consecutive.

Notice that \( \hat{\xi}_{k-p+1} \geq \lfloor \frac{k}{2} \rfloor - (k-p-f) \) implies \( e_m(\hat{\xi}_{k-p+1}) \geq \lceil \frac{k}{2} \rceil + (k-p-f)+1 \). In equality holds, then the corresponding block of length \( \hat{\xi}_{k-p+1} \) ends on \( \lfloor \frac{k}{2} \rfloor \), and the A-strip of length \( t \) can not be fitted into \( T_m(\mu^{(f)}) \). In that situation only (4.20)(ii) is possible, since (4.20)(ii) would yield two consecutive entries in the top row of the \( m \)-symbol, which is impossible. Then the entries of \( \hat{\xi}_{k-p}^{'} , \eta_f^{'} \) and \( \hat{\xi}_{k-p+1}^{'} \) form three consecutive numbers, i.e., an interval in the sense of Lusztig (see [1]) is formed.

We still have to check the case \( p = 0 \). Then \( t > 2\hat{\xi}_k + 2(k-f) \), so \( \hat{\xi}_k \leq \lfloor \frac{k}{2} \rfloor - (k-f) \), and \( t < \hat{\xi}_k + \eta_{f+1} \) then implies \( \eta_{f+1} \geq \lceil \frac{k}{2} \rceil + (k-f) \). Since \( e_m(\eta_f) \leq e_m(\hat{\xi}_k) \), we also find \( \eta_{f} \leq \lceil \frac{k}{2} \rceil + (k-f) \) and finally \( e_m(\hat{\xi}_{k+1}) \geq e_m(\eta_{f+1}) \) implies \( \hat{\xi}_{k+1} \geq \lfloor \frac{k}{2} \rfloor - (k-f) \). This shows that indeed both (4.20)(i)–(ii) are allowed. As in the case \( p > 0 \), here also \( a = k - p \) and \( b = f \), which can be shown similarly.

The last possibility in situation (4.18) is that there is no \( p \in \{0, 1, \ldots, k-l \} \) satisfying (4.19). Then we have \( t < 2\hat{\xi}_l + 2(l-f) \), so \( \hat{\xi}_l \geq \lceil \frac{l}{2} \rceil - (l-f) \). In this case we do not have to change \( \xi_l \), but we have to change \( \eta_f \), since \( t > \hat{\xi}_l + \eta_f \). From \( e_m(\eta_f) > e_m(\hat{\xi}_{l-1}) \) it follows that \( \hat{\xi}_{l-1} \leq \lceil \frac{l}{2} \rceil - (l-f-1) \). This means (as it did in (4.19)) that we have to change the pair \( (\hat{\xi}_{l-1}, \eta_f) \). Calculating the extension which maximizes the \( a \)-value yields (4.17) with \( a = l-1, b = f \). Since \( \eta_f \) resp. \( \hat{\xi}_l \) start on a square containing \( [l-f-1] \) resp. \( [l-f] \), the blocks of length \( \eta_f \) resp. \( \hat{\xi}_l \) end on at least, resp. at most \( \lfloor \frac{l}{2} \rfloor \). Therefore indeed \( a = l-1, b = f \), and both extensions are allowed since there is no block ending on \( \lfloor \frac{l}{2} \rfloor \).

(iii b) The other possibility is that we have a sequence of the form

\[
\begin{align*}
& e_m(\zeta_{f+1}) > e_m(\eta_k) > \ldots e_m(\eta_l) > e_m(\zeta_f) \\
(4.21)
\end{align*}
\]

such that \( e_m(\eta_{k+1}) > e_m(\zeta_{f+1}) \), \( e_m(\eta_{l-1}) < e_m(\zeta_f) \), and

\[
\hat{\xi}_{f+1} + \eta_k \geq t > \eta_l + \zeta_f.
\]
In this situation, we can do computations analogous to the ones of situation (4.18). We first check if there is \( p \in \{0, 1, \ldots, k - l\} \) such that

\[
t \geq 2\eta_{k-p} + 2(k - p - f).
\] (4.22)

If such \( p \) exists, take the smallest such. Then we find that \( a = f, b = k - p \) and that both (4.17)\(^{(i)}\)–(ii) are allowed, where we need to know that there is no extremity equal to \( \lfloor \frac{x}{2} \rfloor \) if \( i > 0 \). Otherwise, we have to change \((\xi_f, \eta_{l-1})\). Again the \( a \)-value is always maximized by (4.17).

(iv) Suppose that in (iii) we have changed the parts \( \xi_i \) and \( \eta_j \). We check when it is necessary to increase the entries of \( \tilde{\xi}_{i-1} \) and \( \tilde{\eta}_{j-1} \). Since we have placed \( t - \tilde{\xi}_i - \tilde{\eta}_j \) squares, there now remain \( \tilde{\xi}_i + \eta_j \) to be placed. It is easy to see that (4.16) implies that all \( \tilde{\xi}_k = \tilde{\xi}_{k+1} (k = 1, \ldots, i - 1) \) and \( \tilde{\eta}_k = \tilde{\eta}_{k+1} (k = 1, \ldots, j - 1) \), and finally \( \tilde{\xi}_0 = \xi_1, \tilde{\eta}_0 = \eta_1 \). This is also what we found in (ii). We conclude that indeed \((\xi', \eta') = (\xi, \eta) \cup (x, y)\) with \( x, y \) as in (4.17).

(v) Suppose that \((\xi, \eta), (\tilde{\xi}, \tilde{\eta})\) are two different members of \( \Sigma_m(W_0c_{L_{(i)}}) \), that \((\xi', \eta')\) appears in \( tr_m - \text{Ind}((t) \otimes (\xi, \eta)) \), and similarly for \((\xi'', \eta'')\). Then it is easy to see that \((\xi', \eta') \neq (\xi'', \eta'')\). Therefore we find that \(|\Sigma_m(W_0c_{L_{(i)}+1})| = 2|\Sigma_m(W_0c_{L_{(i)}})|\). On the other hand, it is also clear that

\[
|C_m(W_0c_{L_{(i)+1}})| = 2|C_m(W_0c_{L_{(i)}})|,
\]

thus the required bijection between \( C_m(W_0c_{L_{(i)+1}}) \) and \( \Sigma_m(W_0c_{L_{(i)+1}}) \) follows. Moreover, we have already seen that \( \Sigma_m(W_0c_{L_{(i)+1}}) \subset [S_{m \pm (\mu^{(i)}+1)}]_m \), and it is not hard to see that \(|[S_{m \pm (\mu^{(i)}+1)}]_m| = 2|[S_{m \pm (\mu^{(i)})}]_m|\), thus it follows from the induction hypothesis that indeed \( \Sigma_m(W_0c_{L_{(i)+1}}) = [S_{m \pm (\mu^{(i)+1})}]_m \). This proves the induction step, and therefore also the Proposition if \( m \) is integer.

(vi) If \( m \notin \mathbb{Z} \), we can apply the same type of reasoning. We briefly state the results in this case. Now \( t \) must be even. The A-strip contains the entries \((\frac{t-1}{2}, \frac{t-3}{2}, \ldots, \frac{1}{2}, \frac{1}{2}, \ldots, \frac{t-3}{2}, \frac{t-1}{2})\). Again, we consider \((\xi, \eta) \in S_{m \pm (\mu^{(i)})}\). Suppose that blocks \( \xi_{a+1} \) and \( \eta_{b+1} \) end on more than \( \frac{t-1}{2} \), and \( \xi_a \) and \( \eta_b \) end on less. Then the A-strip will account for the entries \((\xi'_{a'}, \eta'_{b'})\), taking the place of \((\xi_a, \eta_b)\). Since the corner square of the strip contains this time the value \(|a - b - 1| \), we find the following analogue of (4.17):

\[
\begin{align*}
\xi'_a &= \frac{t}{2} - (a - b) + 1 & \text{for } m + \varepsilon \text{ or } \\
\eta'_b &= \frac{t}{2} + (a - b) - 1 & \text{for } m - \varepsilon.
\end{align*}
\] (4.23)

This time the entries in the \( m \)-symbol are \( e_m(\xi_i) = \xi_i + 2i \) and \( e_m(\eta_i) = \eta_i + 2i + 1 \), which means that if we are in situation (4.18), this time we need to find the smallest \( p \in \{0, 1, \ldots, k - l\} \) such that

\[
t \geq 2\xi_{k-p} + 2(k - p - f) - 1.
\] (4.24)
In situation (4.21), we have to find the smallest \( p \) for which

\[
t \geq 2\eta_{k-p} + 2(k - p - f) + 1.
\]

(4.25)

Then it turns out, that in the same way as for integer \( m \), we find the same \((\zeta', \eta')\) as obtained in (4.23), where the fact that both 2-partitions occur in the induction is due to the fact that the \( A \)-strip of length \( t \) can be inserted into \( T_m(v) \). □

4.12.2.

Now we can consider the general situation. Let \( L \in \mathcal{L}_m(n) \) be a residual coset. Then we choose \( M \in \mathcal{L}_m(n) \) of minimal dimension among those \( M \) such that \( W_0c_M \in C_m(W_0c_L) \). Suppose \( M \) has type \((\rho, \mu)\), and \( sp_m(\mu) = (\kappa, v) \). Putting \( \alpha = \rho \cup \kappa \), it follows that \( L \) is of type \((\alpha, v)\). It follows from the minimality of \( \dim(M) \) that an \( A \)-strip of length \( \rho_i \) cannot be incorporated into \( T_m(v) \). This means that:

**Lemma 4.42.** Let \( m \in \frac{1}{2}\mathbb{Z} \geq 0 \). An \( A \)-strip of length \( t \) cannot be inserted into \( T_m(\mu) \) if and only if one of the following holds:

(i) \( m \) is integer and \( t \) is even, or \( m \) is not integer and \( t \) is odd;

(ii) One of the extremities of \( T_m(\mu) \) is equal to \( \lfloor \frac{t}{2} \rfloor \);

(iii) \( m > 0 \), \( l(\mu) < m \) and \( \lfloor \frac{t}{2} \rfloor < m - l(\mu) \);

(iii) \( m < 0 \), \( l(\mu') < m \) and \( \lfloor \frac{t}{2} \rfloor < m - l(\mu') \);

**Proof.** This is straightforward. □

Now suppose that we consider a residual subspace of type \((t, \mu)\). Then, as in Proposition 4.41, we look for the constituent in \( \text{Ind}(t) \otimes S_m(\mu) \) with maximal \( a \)-value. Then we observe that such a constituent is unique precisely when there is no confluence:

**Proposition 4.43.** Let \( \mu, \rho \) be partitions, and let \((\xi, \eta) \in \Sigma_m(W_0c_L) \) where \( L \in \mathcal{L}_m(n) \) has type \( sp_m(\mu) = (\kappa, v) \). Suppose that \( A \)-strips of length \( \rho_i \) cannot be inserted into \( T_m(v) \). Then \( \text{tr}_m - \text{Ind}(\text{triv}_\rho \otimes (\xi, \eta)) = (\zeta', \eta') \) for some \((\zeta', \eta') \in \mathcal{P}_{n,2} \) with \( n = |\mu| + |\rho| \).

**Proof.** We adopt the notation of Proposition 4.41 and its proof. We suppose that \( m \in \mathbb{Z} \geq 0 \), the other case being analogous. First we remark that if an \( A \)-strip of length \( t \) cannot be inserted into \( T_m(v) \), then it can also not be inserted into \( T_m(\mu) \). Let \( t \) be odd, such that \( \lfloor \frac{t}{2} \rfloor < m - l(\nu) \). If \( \kappa = \emptyset \) then \( \lfloor \frac{t}{2} \rfloor < m - l(\mu) \), else there is an extremity of \( \mu \) equal to \( \lfloor \frac{t}{2} \rfloor \).

Suppose that \( \kappa = \emptyset \). In this case, it is clear that \( a_m \) is maximal for \((\mu, \rho)\). The reader may verify that this is also the 2-partition obtained in the proof of 4.41. Therefore if \( \kappa = \emptyset \), we are done.

Next, suppose that \( \kappa \neq \emptyset \). We can now go through the proof of 4.41. The only difference is that there may be pairs of equal entries in the \( m \)-symbol of \((\xi, \eta)\). Therefore,
in part (iii), the sequence (4.18) now reads
\[ e_m(\eta_{f+1}) \geq e_m(\xi_k) > e_m(\xi_{k-1}) \cdots > e_m(\xi_l) \geq e_m(\eta_f) \] (4.26)
which is as long as possible, i.e., \( e_m(\xi_{k+1}) > e_m(\eta_{f+1}) \) and \( e_m(\xi_{l-1}) < e_m(\eta_f) \), in which
\[ \xi_k + \eta_{f+1} > t > \xi_l + \eta_f. \] (4.27)
The other possibility is that one finds a sequence
\[ e_m(\xi_{f+1}) \geq e_m(\eta_k) > \cdots e_m(\eta_l) \geq e_m(\xi_f) \] (4.28)
such that \( e_m(\eta_{k+1}) > e_m(\xi_{f+1}) \), \( e_m(\eta_{l-1}) < e_m(\xi_f) \), and
\[ \xi_{f+1} + \eta_k \geq t > \eta_l + \xi_f. \] (4.29)
Notice that there is a unique sequence (4.26) satisfying (4.27) or else a unique sequence (4.28) satisfying (4.29).

In each case, one obtains two potential Springer correspondents given by (4.17) for some \( a, b \). However, if the \( A \)-strip of length \( t \) cannot be fitted into the diagram, then only one of them is a 2-partition, and conversely. For even \( t \), the two possibilities for \( (\xi_i, \eta'_j) \) of 4.41 reduce to one. Therefore in this case as well, we are done. \( \square \)

Now we show that indeed we find the desired result:

**Theorem 4.44.** Let \( m \in \frac{1}{2} \mathbb{Z}_{\geq 0} \) and let \( \mathbb{H} \) have parameters \( k_2 = mk_1, k_1 \neq 0 \). Let \( L \in \mathcal{L}_m(n) \) be a residual subspace for \( \mathbb{H} \). Then \( \Sigma_m(W_0cL) \) forms a similarity class in \( W_0/\sim_m \), and there is a bijection
\[ C_m(W_0cL) \longleftrightarrow \Sigma_m(W_0cL). \]

**Proof.** Let \( L \) have type \((\alpha, \nu)\) and let \( \rho, \kappa, \nu \) be as described in the introduction to this subsection. If \( \rho = \emptyset \) then we have already treated this case in Proposition 4.41. Suppose therefore that we have \( \rho \neq \emptyset \). Let \( (\tilde{\xi}, \eta) \) be a Springer correspondent of the residual subspace of type \((\kappa, \mu)\). By Proposition 4.43 above, \( (\tilde{\xi}, \eta) \) leads (by inducing \( \text{triv}_\rho \otimes (\tilde{\xi}, \eta) \)) to exactly one Springer correspondent \( (\xi', \eta') \) of \( W_0cL \). It remains to check that the thus obtained \( (\xi', \eta') \) are mutually different and in the same similarity class. This is however not difficult to see.

The fact that \( \Sigma_m(W_0cL) \) is a full similarity class in bijection with \( C_m(W_0cL) \) follows from the corresponding fact in Proposition 4.41. Indeed, let \( M \) be the residual subspace of type \((\kappa, \nu)\), then \( \Sigma_m(W_0cM) \) is in bijection with \( C_m(W_0cM) \) by 4.41. But \( |\Sigma_m(W_0cL)| = |\Sigma_m(W_0cM)| \) by Proposition 4.43, and \( |C_m(W_0cL)| = |C_m(W_0cM)| \) as well by definition of \( M \) and \( L \). \( \square \)
4.13. Deformed symbols

Thus far, the \(m\)-symbols only apply to parameters \(k_2 = mk_1\) with \(k_1 \neq 0\) and \(m \in \frac{1}{2}\mathbb{Z}_{\geq 0}\). However, we can use them as well to apply to any choice of parameters with \(k_1 \neq 0\) if we deform them slightly. Choose \(m \in \frac{1}{2}\mathbb{Z}_{\geq 0}\) and let \(\varepsilon > 0\) be very small. Let \((\xi, \eta) \in \mathcal{P}_{n,2}\). Then we define the \((m + \varepsilon)\)-symbol (resp. the \((m - \varepsilon)\)-symbol) of \((\xi, \eta)\) to be the \(m\)-symbol of \((\xi, \eta - \varepsilon)\) (resp. the \(m\)-symbol of \((\xi, \eta + \varepsilon)\)). By \(\eta \pm \varepsilon\) we mean the partition whose parts are \(\eta_i \pm \varepsilon\). Let \(a_{m \pm \varepsilon}(\xi, \eta) = a_m(\xi, \eta \mp \varepsilon)\) be the corresponding \(a\)-value, with corresponding truncated induction \(\text{tr}_{m \pm \varepsilon} - \text{Ind}\). Let \(L\) be the generically residual subspace of type \((\lambda, \mu)\), then it is not hard to show that

\[
\Sigma_{m \pm \varepsilon}(W_0c_L) = \text{tr}_{m \pm \varepsilon} - \text{Ind}_{\lambda \times W_0C_L}(\text{triv}_{\lambda} \otimes S_{m \pm \varepsilon}(\mu))
\]

consists of exactly one element \((\xi, \eta) \in \Sigma_m(W_0c_L)\), and moreover that we obtain bijections

\[
\bigcup_{W_0c_{L'} \in C_m(W_0c_L)} \Sigma_{m \pm \varepsilon}(W_0c_{L'}) \longleftrightarrow \Sigma_m(W_0c_L) \longleftrightarrow \bigcup_{W_0c_{L'} \in C_m(W_0c_L)} \Sigma_{m \pm \varepsilon}(W_0c_{L'}).\]

Finally, one checks that we can now define \(\Sigma_x(W_0c_L)\) for any \(x \in \mathbb{R}\), since the Springer correspondence is constant between two consecutive \(m \in \frac{1}{2}\mathbb{Z}_{\geq 0}\):

**Proposition 4.45.** Let \(m, m' \in \frac{1}{2}\mathbb{Z}_{\geq 0}\) with \(m - m' = \frac{1}{2}\). Let \(L\) be a generically residual subspace. Then

\[
\Sigma_{m - \varepsilon}(W_0c_L) = \Sigma_{m' + \varepsilon}(W_0c_L).
\]

Therefore, for \(m > x > m'\), we put \(\Sigma_x(W_0c_L) = \Sigma_{m - \varepsilon}(W_0c_L) = \Sigma_{m' + \varepsilon}(W_0c_L)\).

**Proof.** This boils down to going through the proof of 4.41 and checking that for both \(m\) and \(m'\) the same of the two possibilities in (4.20) or (4.23) is obtained by \(\text{tr}_{m - \varepsilon} - \text{Ind}\) and \(\text{tr}_{m' + \varepsilon} - \text{Ind}\). For details, see [21]. \(\square\)

4.14. Unipotent classes

In the classical cases, we have the bijection \(C_m(n) \leftrightarrow U_m(n)\), such that the Springer correspondents of \(W_0c_L\) are those of the corresponding unipotent class. We will show that we can define a set \(U_m(n)\) with the analogous properties for every \(m \in \frac{1}{2}\mathbb{Z}_{\geq 0}\). Furthermore, by also defining the generalizations \(f_m^B\) and \(\phi_m\), we will show that our generalized Springer correspondence reduces to the classical one in case \(m \in \{\frac{1}{2}, 1\}\).

Recall that in this case, for a residual point \(c\) with jumps \(j_i\) the partition \(\lambda\) of the corresponding unipotent class has parts \(2j_i + 1\). We therefore calculate what the weight of the corresponding partition for arbitrary \(m\) is:
Lemma 4.46. Let \( k \neq 0, m \in \frac{1}{2} \mathbb{Z}_{\geq 0} \) and let \( c = c(\mu, k, mk) \) be residual with jumps \( \{ j_i \} \). Define \( \lambda \) to be the partition with parts \( 2j_i + 1 \). If \( m \) is integer, then \( |\lambda| = 2n + m^2 \), otherwise \( |\lambda| = 2n + m^2 - \frac{1}{4} \).

Proof. (i) Suppose that \( m \) is integer. First consider \( \mu_1 \leq \mu_2 \leq \cdots \leq \mu_p \) with \( p \leq m - 1 \). Then \( S_m(\mu) = (\mu, -) \), and the jumps of \( c \) are \( \{ m + \mu_p - 1, m + \mu_{p-1} - 2, \ldots, m + \mu_1 - p, m - p - 1, m - p - 2, \ldots, 1, 0 \} \). Therefore, indeed

\[
\sum_i (2j_i + 1) = \left( \sum_{i=1}^l (2(m + \mu_i - i) + 1) + \sum_{i=0}^{m-p-1} (2i + 1) \right)
= 2mp + 2n - 2\frac{p(p + 1)}{2} - 2\frac{(m - p - 1)(m - l)}{2} + (m - p)
= m^2 + 2n.
\]

Now consider the general situation, where we may (and will) assume that \( T_m(\mu) \) is in standard position, i.e., that \( S_m(\mu) = (\xi(W_0c), \eta(W_0c)) \) as in 4.25. Hence, \( S_m(\mu) \) consists of \( m - 1 \) horizontal blocks \( (\xi_{r+2}, \ldots, \xi_{r+m}) = (\mu_{p-m+2}, \ldots, \mu_p) \), followed by a series of alternating horizontal and vertical blocks. Then \( (\xi_r, \eta) \), with \( \xi_r = (\xi_1, \xi_2, \ldots, \xi_{r+1}) \), is the \( m \)-tableau of a residual point for \( m = 1 \). Therefore we know that \( \sum_{i=1}^{2r+1} (2j_i + 1) = 2(\sum_{i=p-m+2}^m \mu_i) + 1 \). Thus

\[
\sum_i (2j_i + 1) = \sum_{i=1}^{2r+1} (2j_i + 1) + \sum_{i=2}^m (2(i + \mu_{p-m+i} - 1) + 1)
= m^2 + 2n.
\]

(ii) If \( m \notin \mathbb{Z} \), we find that if \( l(\mu) = p \leq m - \frac{1}{2} \) that the jumps are \( \{ m + \mu_p - 1, m + \mu_{p-1} - 2, \ldots, m + \mu_1 - p, m - p - 1, m - p - 2, \ldots, \frac{3}{2}, \frac{1}{2} \} \), which after a similar computation yields \( \sum_i 2j_i + 1 = m^2 + 2n - \frac{1}{4} \). The general case is treated analogous to integer \( m \) as well. \( \square \)

Definition 4.47. Let \( m \in \frac{1}{2} \mathbb{Z}_{\geq 0} \). If \( m \in \mathbb{Z} \) we define

\[
U_m(n) := \{ \lambda = (1^r 2^s \ldots) | 2n + m^2 \mid r_i \text{ is even if } i \text{ is even and } \sum_i \text{odd}(r_i \text{ mod } 2) \geq m \}. 
\]

That is, we consider partitions of \( 2n + m^2 \) in which even parts have even multiplicity and which have at least \( m \) odd parts with odd multiplicity.

For \( m \notin \mathbb{Z} \) we define

\[
U_m(n) := \{ \lambda = (1^r 2^s \ldots) | 2n + m^2 - \frac{1}{4} \mid r_i \text{ is even if } i \text{ is odd and } \sum_i \text{even}(r_i \text{ mod } 2) \geq m - \frac{1}{2} \}. 
\]

That is, we consider partitions of \( 2n + m^2 - \frac{1}{4} \) in which odd parts have even multiplicity, and which have at least \( m - \frac{1}{2} \) even parts with odd multiplicity.
Notice that in the equal label cases \( m = \frac{1}{2}, 1 \) we indeed recover the old \( U_m(n) \).

We now define a map \( f_m^{BC} : U_m(n) \to C_m(n) \), associating a central character of \( \hat{H}^t(R) \) to \( \lambda \in U_m(n) \).

**Definition 4.48.** Let \( m \in \frac{1}{2}\mathbb{Z}_{\geq 0} \) and suppose \( \lambda \in U_m(n) \).

(i) If \( m \in \mathbb{Z} \), let \( l_1 < l_2 < \cdots < l_s \) be the parts that occur an odd number of times in \( \lambda \), and put \( n_0 := \sum_j l_j \). All \( l_j \) are odd and \( s \geq m \). Notice that \( n_0 \equiv m(2) \). Let \( l = (n_0 - m^2)/2 \). By removing each \( l_j \) once from \( \lambda \), we obtain a partition where every part occurs an even number of times. If we remove each second part, we find a partition \( \kappa \). The associated residual subspace \( L \) of \( \mathbb{H} \) with parameters \( k_1 \neq 0, k_2 = mk_1 \) has root system \( R_L \) of type \( A_k \times B_l \), whose center is the residual point in \( t_L \) whose \( B_l \)-part is the residual point with jumps \( \frac{l_j - 1}{2} \). Then \( W_0c_L := f_m^{BC}(\lambda) \).

(ii) If \( m \not\in \mathbb{Z} \), let \( l_1 < \cdots < l_s \) be the parts of \( \lambda \) that occur an odd number of times. Now all \( l_i \) are even, and \( s \geq m - \frac{1}{2} \). Let \( n_0 = \sum_j l_j \) and put \( l = (n_0 - m^2 + 1/4)/2 \). By removing each \( l_j \) once from \( \lambda \), we obtain a partition where each part has even multiplicity. Remove each second part to obtain a partition \( \kappa \). Let \( \mathbb{H} \) have parameters \( k_1 \neq 0, k_2 = mk_1 \) and let \( L \) be the residual subspace of \( \mathbb{H} \) which has root system \( R_L \) of type \( A_k \times B_l \), whose center is the residual point in \( t_L \) whose \( B_l \)-part is the residual point with jumps \( \frac{l_j - 1}{2} \). Then \( W_0c_L := f_m^{BC}(\lambda) \).

**Lemma 4.49.** The map \( f_m^{BC} \) is a bijection, i.e., \( U_m(n) \) parametrizes the central characters of \( \hat{H}^t(R) \) with parameters \( k_2 = mk_1, k_1 \neq 0 \).

**Proof.** Consider a residual subspace \( L \) with \( R_L \) of type \( A_k \times B_l \). The residual points in \( B_l \) are characterized by their jumps \( j_i \). From 4.9 we know that the parts \( 2j_i + 1 \) form a partition \( \tilde{\lambda} \) of \( 2l + m^2 \) (resp. \( 2l + m^2 - \frac{1}{2} \)) which has at least \( m \) (resp. at least \( m - \frac{1}{2} \)) distinct odd (resp. even) parts. Since \( |\kappa| = n - l \), by adding two parts \( \kappa_i \) to \( \tilde{\lambda} \) for every \( i \), we obtain \( \lambda \in U_m(n) \). Clearly \( f_m^{BC}(\lambda) = W_0c_L \). Therefore \( f_m^{BC} \) is a bijection between \( U_m(n) \) and \( C_m(n) \). \( \square \)

We now define the generalized \( \phi_m : U_m(n) \to \hat{W}_0/\sim_m \). Later we show that \( \phi_m(\lambda) = \Sigma_m(f_m^{BC}(\lambda)) \).

**Definition 4.50.** Let \( m \in \frac{1}{2}\mathbb{Z}_{\geq 0} \) and consider \( \lambda \in U_m(n) \). Its parts are written in increasing order. If \( m \not\in \mathbb{Z} \) then we ensure that \( l(\lambda) = m - \frac{1}{2} + 2r \) for some \( r \in \mathbb{Z}_{\geq 0} \) by putting \( \lambda_1 = 0 \) if necessary.

(i) For \( m \in \mathbb{Z} \) (resp. \( m \not\in \mathbb{Z} \)), replace in the last \( m \) (resp. \( m + \frac{1}{2} \)) parts of \( \lambda \) each pair of even (resp. odd) consecutive entries \((x, x)\) by \((x + 1, x - 1)\), to obtain an \( n \)-composition \( \mu \).

(ii) Define \( \mu_i^* = \mu_i + (i - 1) \) for \( i = 1, 2, \ldots, 2r \) and \( \mu_{2r+i}^* = \mu_{2n+i} + 2r \) for \( i = 1, 2, \ldots, m \) if \( m \in \mathbb{Z} \); resp. \( \mu_i^* = \mu_i + (i - 1) \) (\( i = 1, 2, \ldots, 2r \)) and \( \mu_{2r+i}^* = \mu_i + (2r - 1) \) for \( i = 1, 2, \ldots, m - \frac{1}{2} \) if \( m \not\in \mathbb{Z} \).
(iii) Form the 2-composition \((\zeta^*, \eta^*)\) where \(\eta^*\) contains the \(\frac{\mu_i}{2}\) for odd \(\mu_i\), and \(\zeta^*\) contains the \(\frac{\mu_i}{2}\) for even \(\mu_i\) (in the order they appear in \(\mu^*\)).

(iv) Define the 2-composition \((\xi, \eta)\) by \(\xi_i = \zeta_i^* + (i - 1)\) and \(\eta_i = \eta_i^* - (i - 1)\). If \(m \not\equiv \mathbb{Z}\), then readjust the lengths of \(\xi\) and \(\eta\) to \(l(\xi) = l(\eta) + m + \frac{1}{2}\).

(v) Finally, \(\phi_m(\lambda) = [(\xi, \eta)]_m = (\langle x, \beta \rangle \in \mathcal{P}_{n, 2} \mid (x, \beta) \sim_m (\xi, \eta))\).

Notice that for \(m \in \{\frac{1}{2}, 1\}\), we recover \(\phi_m\) of Section 4.7. For every \(m\), the \(m\)-symbol of \((\xi, \eta)\) (of part (iv) in Definition 4.50) is increasing. This follows easily from the corresponding, known, fact for \(m \in \{\frac{1}{2}, 1\}\). However, it may happen that \((\xi, \eta) \not\in \mathcal{P}_{n, 2}\) since \(\xi\) need not be a partition.

We want to show that \(\phi_m\) establishes a bijection between \(\mathcal{U}_m(n)\) and \(\hat{W}_0/\sim_m\).

Therefore we define the candidate inverse map \(\psi_m = \phi_m^{-1}\):

\textbf{Definition 4.51.} Let \(m \in \{\frac{1}{2}, 1\}\). For \((x, \beta) \in \mathcal{P}_{n, 2}\), define \(\lambda = \psi_m(x, \beta)\) as follows.

(i) If \(m \in \mathbb{Z}\), let \((\xi, \eta)\) be the 2-composition such that \(\sigma_m(\xi, \eta)\) is the increasing rearrangement of \(\sigma_m(x, \beta)\). If \(m \not\equiv \mathbb{Z}\), let \((\xi, \eta)\) be the 2-composition such that \(\sigma_m(\xi, \eta)\) is the increasing rearrangement of \(\sigma_m(x, \beta)\), and then define \((\xi, \eta)\) by putting \(\xi_i = \eta_i^* - (i - 1)\) and \(\xi_i^* = \xi_i + (i - 1)\) if \(m \not\equiv \mathbb{Z}\).

(ii) Let \(\xi_i^* = \eta_i^* - (i - 1)\) and \(\eta_i^* = \xi_i + (i - 1)\). Note that \(\xi_i^*\) still is in general a composition rather than a partition.

(iii) Let \(\mu^*\) be the following composition. If \(m \in \mathbb{Z}\), let \((\mu_1^*, \mu_2^*, \ldots, \mu_{2n}^*)\) be the increasing rearrangement of \(2\xi^*_1 + 1, 2\xi^*_2 + 1, \ldots, 2\xi^*_{n+1} + 1, 2\eta^*_1, 2\eta^*_2, \ldots, 2\eta^*_m\), and let \(\mu_{2n+i}^* = 2\xi_{n+i}^* + 1\) for \(i = 1, \ldots, m\). If \(m \not\equiv \mathbb{Z}\), let \((\mu_1^*, \mu_2^*, \ldots, \mu_{2n}^*)\) be the increasing rearrangement of \(2\xi^*_1 + 1, \ldots, 2\xi^*_n + 1, 2\eta^*_1, \ldots, 2\eta^*_m\) and let \(\mu_{2n+i}^* = 2\xi_{n+i}^* + 1\) for \(i = 1, \ldots, m - 1\).

(iv) Define the \(n\)-composition \(\mu\). If \(m \equiv \mathbb{Z}\) then \(\mu_i = \mu_i^* - (i - 1)\) for \(i = 1, 2, \ldots, n\) and \(\mu_{2n+i} = \mu_{2n+i}^* - 2n\) for \(i = 1, 2, \ldots, m\). If \(m \not\equiv \mathbb{Z}\) then \(\mu_i = \mu_i^* - (i - 1)\) for \(i = 1, 2, \ldots, 2n\) and \(\mu_{2n+i} = \mu_{2n+i}^* - (2n - 1)\) for \(i = 1, 2, \ldots, m - 1\).

(v) Replace any pair of consecutive entries of the form \((x + 1, x - 1)\) by \((x, x)\) to obtain \(\lambda := \psi_m(x, \beta)\).

\textbf{Remark 4.52.} \(\psi_m\) factors through \(\sim_m\) and thus we can define \(\psi_m(\Sigma) := \psi_m(x, \beta)\) for \((x, \beta) \in \Sigma \in \hat{W}_0/\sim_m\). In the equal label cases where \(m \in \{\frac{1}{2}, 1\}\), we recover the inverse of \(\phi_m\) of Section 4.7.

Both \(\phi_m\) and \(\psi_m\) are easily seen to be injective. Hence, to prove that \(\phi_m\) and \(\psi_m\) realize a bijection \(\mathcal{U}_m(n) \leftrightarrow \hat{W}_0/\sim_m\), it suffices to show that \(\psi_m(\Sigma) \in \mathcal{U}_m(n)\). To do so, we first prove a lemma.

\textbf{Lemma 4.53.} Let \((x, \beta) \in \mathcal{P}_{n, 2}\) and let \(m \in \{\frac{1}{2}, 1\}\). Write \(x = (x_1, \ldots, x_{n+m})\) and \(\beta = (\beta_1, \ldots, \beta_n)\).

(i) Suppose that \(e_m(x_{n+i+1}) > e_m(\beta_i) \geq e_m(x_{n+i})\) for some \(1 \leq i \leq m\). Let \(x_{tr} = (x_1, \ldots, x_{n+i})\), and suppose that \(\psi_m(x_{tr}, \beta) = \lambda_{tr}\). If \(m \equiv \mathbb{Z}\) then \(\psi_1(x_{tr}, \beta) = \lambda_{tr} = (\lambda_1, \lambda_2, \ldots, \lambda_{2n+i})\). If \(m \not\equiv \mathbb{Z}\) then \(\psi_{i-1/2}(x_{tr}, \beta) = \lambda_{tr} = (\lambda_1, \lambda_2, \ldots, \lambda_{2n+i})\).
(iiia) Suppose \( m \in \mathbb{Z} \) and \( e_m(\beta \gamma) \geq e_m(\alpha n) \). Let \( \alpha r = \alpha - \alpha n + m \) and \( \beta r = \beta - \beta n \).

Suppose \( \psi_\gamma(\alpha, \beta) = \lambda \). Then \( \psi_\gamma(\alpha r, \beta r) = \lambda r = (\lambda, \ldots, \lambda) \).

(iiib) Suppose \( m \notin \mathbb{Z} \) and \( e_m(\beta \gamma) \geq e_m(\alpha n + m) \). Let \( \alpha r = \alpha - \alpha n + m + \frac{1}{2} \) and \( \beta r = \beta - \beta n \). If \( \psi_\gamma(\alpha, \beta) = \lambda \), then \( \psi_\gamma(\alpha r, \beta r) = (\lambda, \ldots, \lambda) \).

Proof. (i) This is clear, since if we denote the corresponding 2-compositions with increasing \( \mu \)-symbols by \((\bar{\zeta}, \bar{\eta}) \sim^m (\alpha, \beta) \) and \((\bar{\zeta} \alpha, \bar{\eta} \beta) \sim_i (\alpha r, \beta r) \), then \( \bar{\zeta} = \bar{\zeta} \cup (\bar{\zeta} + i, \bar{\zeta} + n) \).

(ii) Let \( \zeta = (\xi_1, \ldots, \xi_n) \) and \( \eta = (\eta_1, \ldots, \eta_n) \) be such that \((\zeta, \eta) \sim^m (\alpha, \beta) \) and the entries of \( \Lambda_{n, n, n}^m(\zeta, \eta) \) increase when read from left to right. Likewise, let \( \zeta' = (\xi_1, \ldots, \xi_{n+1}) \) and \( \eta' = (\eta_1, \ldots, \eta_{n-1}) \) be such that \((\zeta', \eta') \sim^m (\alpha r, \beta r) \) and \( \Lambda_{n+1, n-1}^m(\zeta', \eta') \) has increasing entries. By Lemma 4.52, we may compute \( \psi_m(\zeta', \eta') \) on \( \Lambda_{n+1, n-1}^m(\zeta', \eta') \).

Then, if we write

\[
\sigma_{n, n}^m(\zeta, \eta) = \begin{pmatrix}
x_1 & \cdots & x_{n-1} & x_n & x_{n+1} & \cdots & x_{n+m}
y_1 & \cdots & y_{n-1} & y_n
\end{pmatrix},
\]

it follows that

\[
\Lambda_{n+1, n-1}^m(\zeta', \eta') = \begin{pmatrix}
x_1 & \cdots & x_{n-1} & x_n & x_{n+1} & \cdots & x_{n+m-2}
y_1 & \cdots & y_{n-1}
\end{pmatrix}.
\]

It follows that \( \zeta_i = \zeta_i \) for \( 1 \leq i \leq n \), \( \xi_{n+1} = \eta_n - 2 \), \( \xi_{n+j} = \xi_{n-2+j} - 2 \) for \( 2 \leq j \leq m - 1 \); and \( \eta_i = (\eta_1, \ldots, \eta_{n-1}) \). Thus,

\[
\zeta_i = (\xi_1, \ldots, \xi_n, \eta_n - 2, \xi_{n+1} - 2, \ldots, \xi_{n-2+m-2}), \eta_i = \eta_i - \eta_n,
\]

\[
\zeta_i^* = (\xi_1, \ldots, \xi_n, \eta_n - 1, \xi_{n+1} - 1, \ldots, \xi_{n-2+m-2}), \eta_i^* = \eta_i - \eta_n^*.
\]

By definition, \( (\mu^*_{r, 1}, \ldots, \mu^*_{r, 2n-1}) \) consists of \( 2 \xi_1 + 1, \ldots, 2 \xi_n + 1, 2 \eta_1^*, \ldots, 2 \eta_n^* \), arranged in increasing order. The first \( 2n - 1 \) parts of \( \mu^* \) are the first \( 2n - 1 \) parts of the increasing rearrangement of \( 2 \xi_1 + 1, \ldots, 2 \xi_n + 1, 2 \eta_1^*, \ldots, 2 \eta_n^* \). Since the symbols (4.30) and (4.31) are increasing, \( \xi_i < \eta_i^* < \xi_i + 1 + 1 \) for all \( i = 1, \ldots, n \), where at least one inequality is strict. Likewise, \( \eta_i^* < \xi_i + 1 + 1 \leq \eta_i^* \) where at least one inequality is strict. This implies that if \( \mu^*_k = 2 \xi_j + 1 \), then either \( \mu^*_k + 1 = 2 \xi_j + 1 + 1 \), \( \mu^*_k + 1 = 2 \eta_j^* \), or \( \mu^*_k + 1 = 2 \eta_j^* + 1 + 1 \) but in the latter case, \( \mu^*_k + 2 = 2 \eta_j^* + 1 \). Analogously, if \( \mu^*_k = 2 \eta_j^* \), then either \( \mu^*_k + 1 = 2 \eta_j^* + 1 \), \( \mu^*_k + 1 = 2 \eta_j^* + 1 + 1 \) or \( \mu^*_k + 1 = 2 \xi_j + 1 + 1 \) but in the latter case, \( \mu^*_k + 2 = 2 \xi_j + 1 + 1 \). In particular, we have \( \mu^*_r, i = \mu^*_i \) for \( i \in \{1, \ldots, 2n - 2 \} \). It also
follows that
\[ \mu_{2n+1}^* = \max((2z_{n+1}^* + 1, 2\eta_n^*)), \] (4.32)
\[ \mu_{2n}^* = \max((2z_{n+1}^* + 1, 2\eta_n^*, 2z_n^* + 1) \setminus \{\mu_{2n+1}^*\}), \] (4.33)
\[ \mu_{2n-1}^* = \max((2z_{n+1}^* + 1, 2\eta_n^*, 2z_n^* + 1, 2\eta_{n-1}^*) \setminus \{\mu_{2n+1}^*, \mu_{2n}^*\}), \] (4.34)
\[ \mu_{tr,2n-1}^* = \max((2z_n^* + 1, 2\eta_n^*) \setminus \{\mu_{2n+1}^*\}) \] (4.35).

From the definition of \( \psi_m \) it is easy to check that \( \mu_{2n+i} = 2z_{n+i} + 2i - 1 \) for \( 2 \leq i \leq m \). Likewise, \( \mu_{tr,2(n-1)+i} = 2z_{n-1+i} + 2i - 1 \) for \( 2 \leq i \leq m \), from which it follows that \( \mu_{tr,2n+i} = \mu_{2n+i} \) for \( 2 \leq i \leq m \), and also that
\[ \mu_{tr,2n+1}^* = 2z_{n+1}^* - 1, \] (4.36)
\[ \mu_{tr,2n}^* = 2\eta_n^* - 1. \] (4.37)

Since \( \mu_{tr,i} = \mu_i \) for \( i \leq 2n - 2 \) and \( i \geq 2n + 2 \), it remains to calculate \( \mu_i \) and \( \mu_{tr,i} \) for \( i \in \{2n - 1, 2n, 2n + 1\} \) and then to compare \( \lambda_{tr} \) to \( \lambda \). We distinguish between various cases.

(A) Suppose that \( e_m(\tilde{\xi}_n) = e_m(\eta_n) \). Then \( (\mu_{2n-1}^*, \mu_{2n}^*, \mu_{2n+1}^*) = (2\eta_n^*, 2z_{n+1}^*, 2z_{n+1}^* + 1) \), hence \( (\mu_{2n-1}, \mu_{2n}, \mu_{2n+1}) = (2\eta_n, 2\tilde{\xi}_n, 2\tilde{\xi}_n + 1) \). Since \( \tilde{\xi}_n = \eta_n \), there is no pair \((x+1, x-1)\) in this triple. Now, \( (\mu_{tr,2n-1}, \mu_{tr,2n}, \mu_{tr,2n+1}) = (2\tilde{\xi}_{n+1}^* + 1, 2\eta_{n-1}^*, 2\tilde{\xi}_{n+1}^* - 1) \) or \( (\mu_{tr,2n-1}, \mu_{tr,2n}, \mu_{tr,2n+1}) = (2\tilde{\xi}_{n+1}^* + 1, 2\eta_{n-1}^*, 2\tilde{\xi}_{n+1}^* - 1) \). Thus \( (\lambda_{2n-1}, \lambda_{2n}, \lambda_{2n}^*) = (\lambda_{tr,2n-1}, \lambda_{tr,2n}, \lambda_{tr,2n+1}) = (2\tilde{\xi}_{n+1}, 2\eta_n, 2\tilde{\xi}_{n+1} + 1) \).

(B) Suppose that \( e_m(\tilde{\xi}_n) < e_m(\eta_n) < e_m(\tilde{\xi}_{n+1}) \). Then \( \tilde{\xi}_n^* < \eta_n^* \leq \tilde{\xi}_{n+1}^* \). Thus, \( (\mu_{2n-1}^*, \mu_{2n}^*, \mu_{2n+1}^*) = (max(2\tilde{\xi}_{n+1}^* + 1, 2\eta_n^*), 2\eta_n^*, 2\tilde{\xi}_{n+1}^* + 1) \), and \( (\mu_{2n-1}, \mu_{2n}, \mu_{2n+1}) = (\mu_{2n-1}^* - (2n - 2), 2\eta_n - 1, 2\tilde{\xi}_{n+1} + 1) \), which is the same as for \( \mu_{tr} \).

(C) Suppose that \( e_m(\eta_n) = e_m(\tilde{\xi}_{n+1}) \). Then \( \eta_n = \tilde{\xi}_{n+1} + 2 \) and \( \eta_n^* = \tilde{\xi}_{n+1}^* + 1 \). It follows that \( (\mu_{2n-1}^*, \mu_{2n}^*, \mu_{2n+1}^*) = (max(2\tilde{\xi}_{n+1}^* + 1, 2\eta_n^*), 2\tilde{\xi}_{n+1}^* + 1, 2\eta_n^*) \) and \( (\mu_{2n-1}, \mu_{2n}, \mu_{2n+1}) = (\mu_{2n-1}^* - (2n - 2), 2\tilde{\xi}_{n+1} + 2, 2\eta_n - 2) \) and these parts are the same in \( \lambda \). On the other hand, \( (\mu_{tr,2n-1}, \mu_{tr,2n}, \mu_{tr,2n+1}) = (\mu_{2n-1}^*, 2\eta_n + 1, 2\tilde{\xi}_{n+1} + 1) \) so \( \lambda_{tr,2n} = \lambda_{2n} = \lambda_{tr,2n+1} = \lambda_{2n+1} = 2\eta_n - 2 = 2\tilde{\xi}_{n+1} + 2 \).

(iib) Let \( (\tilde{\xi}, \eta, \pi) \sim_m (x, \beta, \pi') \) and \( (\tilde{\xi}', \eta', \pi') \sim_m (x, \beta, \pi_{tr}) \) have increasing \( m \)-symbol. Then \( l(\tilde{\xi}) = n + m + \frac{1}{2} \) and \( l(\eta) = n \). By definition 4.5(i), we define \( \tilde{\xi} \) with \( l(\tilde{\xi}) = n + m - \frac{3}{2} \) by \( \tilde{\xi}_i = \tilde{\xi}_{i+1} \), and \( \tilde{\xi}' \) with \( l(\xi') = n + m - \frac{3}{2} \) by \( \tilde{\xi}'_i = \tilde{\xi}'_{i+1} \) for \( i = 1, 2, \ldots, n + m - \frac{3}{2} \) (notice that \( \tilde{\xi}'_1 = \tilde{\xi}_1 = 0 \)). Then one checks, using the obvious analogues of (4.30) and (4.31), that
\[ \tilde{\xi}' = (\tilde{\xi}_1, \ldots, \tilde{\xi}_{n-1}, \eta_n - 1, \tilde{\xi}_n - 2, \ldots, \tilde{\xi}_{n+m-\frac{1}{2}} - 2), \eta_{tr} = (\eta_1, \ldots, \eta_{n-1}), \]
\[ \tilde{\xi}'' = (\tilde{\xi}_1^*, \ldots, \tilde{\xi}_{n-1}^*, \eta_n^* - 1, \tilde{\xi}_n^* - 1, \ldots, \tilde{\xi}_{n+m-\frac{1}{2}}^* - 1), \eta_{tr}^* = (\eta_1^*, \ldots, \eta_{n-1}^*). \]
By definition, \((\mu^*_1, \ldots, \mu^*_{2n-2})\) consists of the increasing rearrangement of \(2\zeta^*_1 + 1, \ldots, 2\zeta^*_n - 1, 1, 2\eta^*_1, \ldots, 2\eta^*_{n-1}\), whereas \((\mu^*_1, \ldots, \mu^*_{2n-2})\) consists of the first \(2n - 2\) terms of the increasing rearrangement of \(2\zeta^*_1 + 1, \ldots, 2\zeta^*_n + 1, 2\eta^*_1, \ldots, 2\eta^*_{n-1}\). Since both \((\zeta, \eta)\) and \((\zeta', \eta')\) have increasing \(m\)-symbols, we have \(\zeta^*_i - 1 \leq \eta^*_i \leq \zeta^*_i + 1\) and \(\eta^*_i \leq \zeta^*_i + 1 \leq \eta^*_{i+1} + 1\), where at least one of either equalities is strict. Thus, if \(\mu^*_k = 2\zeta^*_j + 1\), then \(\mu^*_{k+1} = 2\zeta^*_{j+1} + 1\), \(\mu^*_{j+1} = 2\eta^*_j\), or \(\mu^*_{k+1} = 2\zeta^*_j + 1\) or \(\mu^*_{k+1} = 2\zeta^*_{j-1} + 1\) in the latter case, \(\mu^*_{k+2} = 2\eta^*_j + 1\). Likewise, if \(\mu^*_k = 2\eta^*_j\) then \(\mu^*_{k+1} = 2\eta^*_{j+1}\), \(\mu^*_{k+1} = 2\zeta^*_j + 1\) or \(\mu^*_{k+1} = 2\zeta^*_{j-1} + 1\) in the latter case, \(\mu^*_{k+2} = 2\zeta^*_j + 1\). It follows that \(\mu^*_{t_i} = \mu^*_i\) for \(1 \leq i \leq 2n - 3\).

One checks that \(\mu_{2n+i} = 2\zeta^*_{s+n+i} + 2i\) for \(1 \leq i \leq m - \frac{1}{2}\), and \(\mu_{tr,2(n-1)+i} = 2\zeta^*_{s+n-1+i} - 2i\) for \(i \geq 1\). For \(i \geq 2\), this implies that \(\mu_{tr,2n+i-2} = 2\zeta^*_{s+n-1+i} - 2i = 2(\zeta^*_{s-2} - 2) - 2i = 2\zeta^*_n+(i-2) - 2(i - 2)\). In particular, \(\mu_{2n+j} = \mu_{tr,2n+j}\) for \(j \geq 1\). Thus, it remains to compare \((\mu^*_{tr,2n-2}, \mu^*_{tr,2n-1}, \mu^*_{tr,2n})\) to \((\mu^*_{2n-2}, \mu^*_{2n-1}, \mu^*_{2n})\). Note that \(\mu^*_{tr,2n-2} = \max(2\zeta^*_{s-1} + 1, 2\eta^*_{n-1})\) and \(\mu^*_{2n} = \max(2\zeta^*_n + 1, 2\eta^*_{n})\). We have

\begin{align*}
\mu^*_{2n} &= \max\{(2\zeta^*_n + 1, 2\eta^*_n)\}, \\
\mu^*_{2n-1} &= \max\{(2\zeta^*_{n} + 1, 2\eta^*_n, 2\zeta^*_{n-1} + 1)\} \setminus \{\mu^*_{2n}\}, \\
\mu^*_{2n-2} &= \max\{(2\zeta^*_{n} + 1, 2\eta^*_n, 2\zeta^*_{n-1} + 1, 2\eta^*_{n-1})\} \setminus \{\mu^*_{2n}, \mu^*_{2n-1}\}, \\
\mu^*_{tr,2n} &= 2\zeta^*_n, \\
\mu^*_{tr,2n-1} &= 2\eta^*_n, \\
\mu^*_{tr,2n-2} &= \max\{(2\zeta^*_n + 1, 2\eta^*_{n-1})\}.
\end{align*}

(A) Suppose that \(e_m(\zeta^*_n) = e_m(\eta^*_n)\). Then \(\zeta^*_{n-1} = \eta^*_n + 1, \zeta^*_{n-1} = \eta^*_{n-1}\). Thus \((\mu^*_{2n-2}, \mu^*_{2n-1}, \mu^*_{2n}) = (2\eta^*_n, 2\zeta^*_n - 1 + 1, 2\zeta^*_n + 1)\) and \((\mu^*_{2n-2}, \mu^*_{2n-1}, \mu^*_{2n}) = (2\eta^*_n + 1, 2\zeta^*_n - 1, 2\zeta^*_n)\). On the other hand, \((\mu^*_{tr,2n-2}, \mu^*_{tr,2n-1}, \mu^*_{tr,2n}) = (2\zeta^*_{n-1} + 1, 2\eta^*_{n-1} - 2\zeta^*_n - 1)\), hence \((\mu^*_{tr,2n-2}, \mu^*_{tr,2n-1}, \mu^*_{tr,2n}) = (2\zeta^*_n - 2, 2\eta^*_{n-1}, 2\zeta^*_n)\). Therefore \((\lambda^*_{tr,2n-2}, \lambda^*_{tr,2n-1}) = (\lambda^*_{2n-2}, \lambda^*_{2n-1}) = (2\zeta^*_n - 1, 2\eta^*_n + 1)\) and \(\lambda^*_{2n} = \lambda^*_{tr,2n}\).

(B) Suppose that \(e_m(\zeta^*_n) < e_m(\eta^*_n)\). In this case, \(\zeta^*_{n-1} < \eta^*_n + 1\) and \(\eta^*_n < \zeta^*_{n+1}\). Thus \((\mu^*_{2n-2}, \mu^*_{2n-1}, \mu^*_{2n}) = (\max(2\zeta^*_{n-1} + 1, 2\eta^*_{n-1})), 2\eta^*_{n}, 2\zeta^*_{n+1})\) and it follows that \((\mu^*_{tr,2n-2}, \mu^*_{tr,2n-1}, \mu^*_{tr,2n}) = (\mu^*_{2n-2}, \mu^*_{2n-1}, \mu^*_{2n}) = (\mu^*_{tr,2n-2}, \mu^*_{tr,2n-1}, \mu^*_{tr,2n})\).

(C) Suppose that \(e_m(\zeta^*_n) < e_m(\zeta^*_{n+1})\). Then \(\zeta^*_n = \zeta^*_{n+1} + 1\) and \(\eta^*_{n} = \zeta^*_{n+1} + 1\). Therefore, \((\mu^*_{2n-2}, \mu^*_{2n-1}, \mu^*_{2n}) = (\max(2\zeta^*_{n-1} + 1, 2\eta^*_{n-1}) + 1, 2\zeta^*_{n+1} + 1, 2\eta^*_{n})\) and \((\mu^*_{2n-2}, \mu^*_{2n-1}, \mu^*_{2n}) = (\mu^*_{tr,2n-2}, \mu^*_{tr,2n-1}, \mu^*_{tr,2n}) = (\mu^*_{tr,2n-2}, \mu^*_{tr,2n-1}, \mu^*_{tr,2n})\). On the other hand, \((\mu^*_{tr,2n-2}, \mu^*_{tr,2n-1}, \mu^*_{tr,2n}) = (\lambda^*_{2n-2}, \lambda^*_{2n-1}, \lambda^*_{2n})\).

\[\square\]

**Proposition 4.54.** Suppose \((x, \beta) \in \mathcal{P}_{n,2}\). Then \(\psi_m(x, \beta) \in \mathcal{U}_m(n)\).
Proof. (i) Let $m \in \mathbb{Z}$. Let the 2-composition $(\xi, \eta)$ of $n$ be the one described in part (i) of the definition of $\psi_m$. Following the procedure $\psi_m$, we find that, since we consider $\xi$ as having $n + m$ parts and $\eta$ as having $n$ parts:

$$|\psi_m(\alpha, \beta)| = \sum_{i=1}^{n+m} 2(\xi_i + (i - 1)) + 1 + \sum_{i=1}^{n} 2(\eta_i + (i - 1)) - \sum_{k=1}^{2n-1} k - 2nm = 2n + m^2,$$

so indeed $\lambda := \psi_m(\alpha, \beta) + 2n + m^2$. It remains to be shown that even parts in $\lambda$ occur with even multiplicity and that there are at least $m$ odd parts in $\lambda$ with odd multiplicity.

Notice that although $(\xi, \eta)$ is in general a 2-composition rather than a 2-partition, $\eta$ and $\xi_{tr} = (\xi_1, \ldots, \xi_{n+1})$ are partitions. On $(\xi_{tr}, \eta)$, $\psi_m$ acts as $\psi_1$, so $(\lambda_1, \ldots, \lambda_{2n+1})$ is a partition where even parts have even multiplicity. Since $\mu_{2n+i} = 2\xi_{n+i} + 2i - 1$, the last $m$ parts of $\mu$ are all odd. In the final step $\mu \rightarrow \lambda$, we create pairs of equal (even) parts, so in $\lambda = \psi_m(\alpha, \beta)$ even parts have even multiplicity.

Now let us show that $\psi_m(\alpha, \beta)$ contains at least $m$ odd parts with odd multiplicity. We have seen that $\mu_{2n+i} = 2\xi_{n+i} + 2i - 1$ ($1 \leq i \leq m$) so if $\lambda = \mu$, then we are done. If $\lambda \neq \mu$, let $i$ be maximal such that $(\lambda_{n+i}, \lambda_{n+i+1}) = (\mu_{n+i} - 1, \mu_{n+i+1} + 1)$. Then $\xi_{n+i} = \xi_{n+i+1} + 2$ and $e_m(\xi_{n+i}) = e_m(\xi_{n+i+1})$. Suppose that in $\sigma_m(\xi, \eta)$ the corresponding entries are $e_m(\beta_j) = e_m(\alpha_{n+i})$.

Then there exists $p \in \{1, \ldots, m\}$ such that $e_m(\alpha_{n+p+1}) > e_m(\beta_p) \geq e_m(\alpha_{n+p})$ (where the second inequality is strict if $j \neq n$, in view of our assumption that we consider the biggest pair of equal entries in $\sigma_m(\xi, \eta)$). Then the last $m - p$ entries of $\mu$ are all odd and distinct by Lemma 4.53(i). Therefore, we may consider $\psi_{m-p}$ on $((\alpha_1, \ldots, \alpha_{n+p}), (\beta_1, \ldots, \beta_n))$ instead. Thus, we are reduced to the case where $e_m(\beta_p) \geq e_m(\alpha_{n+p})$. Let $\lambda_{tr} = (\lambda_1, \ldots, \lambda_{2n-2})$, $\alpha_{tr} = (\alpha_1, \ldots, \alpha_{n+m-1})$, $\eta_{tr} = (\beta_1, \ldots, \beta_{n-1})$. By Lemma 4.53(ii), we have $\psi_m(\alpha_{tr}, \beta_{tr}) = \lambda_{tr}$. It is sufficient to show that $\lambda_{tr}$ has $m$ odd parts with odd multiplicity. Combining these arguments, we can reduce ourselves to the case of $(\alpha, \beta)$ without pairs of equal entries $e_m(\beta_j) = e_m(\alpha_{n+i})$. But for such $(\alpha, \beta)$, the statement holds.

(ii) Suppose that $m \notin \mathbb{Z}$. Let $(\xi, \eta)$ be the 2-composition whose $m$-symbol is similar to the one of $(\alpha, \beta)$ and increasing. Then, by definition of $\psi_m$ we have $|\psi_m(\alpha, \beta)| = \sum_{i=1}^{n+m-1/2} 2(\xi_i + (i - 1) + 1) + 2 \sum_{i=1}^{n} 2(\eta_i + (i - 1)) - \sum_{i=1}^{2n} (i - 1) - (2n - 1)(m - \frac{1}{2}) = 2n + m^2 - 1/4$. Again, it remains to be seen that odd parts have even multiplicity and that there are at least $m - \frac{1}{2}$ distinct even parts with odd multiplicity. Analogous to the case $m \in \mathbb{Z}$, this follows from the fact that the statement is true for $m = \frac{1}{2}$, combined with $\mu_{2n+i} = 2\xi_{n+i} + 2i$ for $i \geq 1$ and Lemma 4.53. □

Now we can show that the maps we have defined lead to the desired result:

**Theorem 4.55.** Let $\mathbb{H}$ have parameters $k_1 \neq 0$, $k_2 = mk_1$ and $m \in \frac{1}{2}\mathbb{Z}_{\geq 0}$. We have

(i) bijections

$$C_m(n) \leftrightarrow \{\Sigma_m(W_0L) \mid W_0L \in L_m(n)\} \leftrightarrow \hat{W}_0/ \sim_m;$$

(4.44)
(ii) the identity

\[ (f_m^{\mathcal{B}C} \circ \psi_m \circ \Sigma_m)(W_{0CL}) = W_{0CL} \text{ for all } W_{0CL} \in C(n). \]  \hfill (4.45)

**Proof.** (i) In view of Theorem 4.44, we need to show that if \( W_{0CL} \neq W_{0CL}' \), then \( \Sigma_m(W_{0CL}) \neq \Sigma_m(W_{0CL}') \). However, since \( \mathcal{U}_m(n) \leftrightarrow C(n) \) via \( f_m^{\mathcal{B}C} \), and \( \mathcal{U}_m(n) \leftrightarrow \tilde{W}_0/\sim_m \) via \( \phi_m \), the statement follows from (ii).

(ii) Consider \( L \in \mathcal{L}_m(n) \) of type \((\kappa, \nu)\). We prove the Proposition by induction on \( I(\kappa) \).

(a) First we assume that \( \kappa \in \mathbb{Z} \). If \( l(\kappa) = 0 \), then \( W_0L = W_0c \) is a residual point. Let \( J(W_0c) = \{ j_1, j_2, \ldots, j_{2+m} \} \). Then, by Definition 4.20, we have \( \Sigma_m(W_0c) = \{(\xi(W_0c), \eta(W_0c))\}_{m} \), where \( \xi(W_0c) = (j_1, j_2, \ldots, j_{2+r+2}, \ldots, j_{2+m} - (m - 1)) \) and \( \eta(W_0c) = (j_2 + 1, j_4 + 1, \ldots, j_{2+r} + 1) \). Since \( \sigma_m(\xi(W_0c), \eta(W_0c)) \) is an increasing symbol, we apply the definition of \( \psi_m \) to find \( \xi^* = (j_1, j_3 + 1, j_5 + 2, \ldots, j_{2+r+1} + r, j_{2+r+2} + r, \ldots, j_{2+r+m} + r) \) and \( \eta^* = (j_2 + 1, j_4 + 2, \ldots, j_{2+r} + r) \), hence \( \mu^* = (2j_1 + 1, 2j_2 + 2, 2j_3 + 3, 2j_4 + 4, \ldots, 2j_{2+r+1} + 2r + 1, 2j_{2+r+2} + 2r + 1, \ldots, 2j_{2+r+m} + 2r + 1) \), and \( \lambda = \psi_m(\xi, \eta) = (2j_1 + 1, 2j_2 + 1, \ldots, 2j_{2+r+m} + 1) \). Therefore, \( f_m^{\mathcal{B}C}(\lambda) = W_0c \).

(b) Now suppose that the statement is true for every \( L' \in \mathcal{L}_m(n) \) of type \((\alpha, \beta)\) with \( l(\alpha) < l = l(\kappa) \), i.e., in particular for \( M \) of type \((\kappa - \kappa_l, \nu)\). Put \( t = \kappa_l \), then

\[ \Sigma_m(W_0CL) = \text{tr}_m - \text{Ind}^{W_0(B_n)}_{S_t \times W_0(B_{n-t})}(t \otimes \Sigma_m(W_0CM)). \]

Let \((\alpha, \beta) \in \Sigma_m(W_0CM)\) and \((\alpha', \beta') \in \text{tr}_m - \text{Ind}(t \otimes (\alpha, \beta)) \subset \Sigma_m(W_0CL)\). Recall from (the proof of) 4.41 that \((\alpha', \beta') = (\alpha, \beta) \cup (\alpha'_l, \beta'_l)\) for some decomposition \( t = \alpha'_l + \beta'_l \).

By the induction hypothesis, \( f_m^{\mathcal{B}C}(\psi_m(\alpha, \beta)) = W_0CM \).

To compute \( \psi_m(\alpha, \beta) \), define \((\xi', \eta')\) (resp. \((\xi, \eta)\)) to be the \( 2 \)-composition such that \( \sigma_m(\xi', \eta') \) (resp. \( \sigma_m(\xi, \eta) \)) is the increasing rearrangement of \((\alpha', \beta')\) (resp. \((\alpha, \beta)\)). We want to know to which parts of \((\xi', \eta')\), the parts \((\alpha'_l, \beta'_l)\) correspond. There are three possibilities for the position of the entries \( e_m(\alpha'_l), e_m(\beta'_l) \) in \( \sigma_m(\xi', \eta') \): either \( (e_m(\alpha'_l), e_m(\beta'_l)) = (e_m(\xi'_l), e_m(\eta'_l)) \) for some \( 0 \leq l \leq n \), or \( (e_m(\alpha'_l), e_m(\beta'_l)) = (e_m(\xi'_l), e_m(\eta'_l)) \) for some \( 0 \leq l \leq n \), or \( (e_m(\alpha'_l), e_m(\beta'_l)) = (e_m(\xi'_l), e_m(\eta'_l)) \) for some \( l > n \). This last possibility does not occur in the equal label cases.

Let \( \psi_m(\alpha, \beta) = \lambda \) and \( \psi_m(\alpha', \beta') = \lambda' \). We need to show that \( \lambda' = \lambda \cup (tt) \). We claim that to do so it suffices to compute the parts of \( \lambda' \) which are derived from the parts \((\alpha'_l, \beta'_l)\). Indeed, with \((\xi', \eta')\) as above, write \( \lambda'_b := \psi_m(\xi'_a) \) if \( \xi'_a \) gives rise to part \( \lambda'_b \), and similarly for \( \eta' \). Then one easily checks that \( \psi_m(\xi'_a, \eta'_b) = \psi_m(\xi, \eta) \cup \psi_m(\xi'_a) \cup \psi_m(\eta'_b) \) in each of the above three cases. Therefore, it is sufficient to show that \( \psi_m(\xi'_a) = \psi_m(\eta'_b) = t \).

Let us consider these possibilities one by one. In the first case, notice that by rearranging the m-symbol, we preserve the sum \( \xi'_l + \eta'_l = \alpha'_l + \beta'_l = t \). Thus, if \( t \) is odd, we have \( (\xi'_l, \eta'_l) = (l \frac{1}{2}, \frac{1}{2}) \). Hence, \( \xi'_l^* = \frac{1}{2} + (l - 1), \eta'_l^* = \frac{1}{2} + l - 1, \) and \( \mu^*_l = 2 \frac{1}{2} + 2(l - 1) + 1 = t + 2(l - 1), \mu^*_l = 2 \frac{1}{2} + 2(l - 1) = t + 2l - 1 \). Then
\[ \mu_{2l-1} = \mu_{2l-1}' - (2l - 2) = t \] and \[ \mu_{2l} = \mu_{2l}' - (2l - 1) = t, \] so \( \lambda_{2l-1}' = \lambda_{2l}' = t \) as desired. If \( t \) is even, then \( (\zeta_{i}'^l, \eta_{i}'^l) = (\frac{t}{2}, \frac{t}{2}) \), \( (\zeta_{i}^{2l}, \eta_{i}^{2l}) = (\frac{t}{2} + l - 1, \frac{t}{2} + l - 1) \), and so \( \mu_{2l-1}' = t + 2l - 2, \mu_{2l}' = t + 2l - 1 \) which again yields \( \lambda_{2l-1}' = \lambda_{2l}' = t \).

In the second case, one has \( (\zeta_{i+1}'^l, \eta_{i}'^l) = (\frac{t}{2}, \frac{t}{2}) \) or \( (\zeta_{i+1}'^l, \eta_{i}'^l) = (\frac{t}{2} - 1, \frac{t}{2} + 1) \) and the claim follows analogously.

Finally consider the third possibility where the new entries in \( (\zeta_i', \eta_i') \) are two entries of \( \zeta_i' \). Then \( (\alpha'_i, \beta'_i) = (\alpha, \beta) \cup (\alpha_{n+i}'^l, \beta_{n+i}'^l) \) for some \( i \geq 1 \). One has \( \alpha_{n+i}'^l = \lfloor \frac{t}{2} \rfloor - i \) and \( \beta_{n+i}' = \lfloor \frac{t}{2} \rfloor + i, \) so \( e_m(\alpha_{n+i}'^l) = \alpha_{n+i}' + 2(n + i - 1) = \lfloor \frac{t}{2} \rfloor - i + 2(n + i - 1) = \lfloor \frac{t}{2} \rfloor + 2n - 2 + 2i \). If \( m = 0 \), then \( (\alpha_{n+i}'^l, \beta_{n+i}'^l) = (\frac{t}{2} - i, \frac{t}{2} + i) \), \( e_m(\alpha_{n+i}'^l) \) is rearrangement \( e_m(\alpha_{n+i}'^l) \), these entries are \( e_m(\zeta_{n+i}'^l), e_m(\zeta_{n+i}'^l) \). Thus, \( \zeta_{n+i}'^l = e_m(\zeta_{n+i}'^l) - 2(n + i - 2) = \lfloor \frac{t}{2} \rfloor - i + 2 \) and \( \zeta_{n+i}'^l = \lfloor \frac{t}{2} \rfloor - i \). It follows that \( \mu_{2n+i+1}' = 2\zeta_{n+i}'^l + 2(i - 1) - 1 = 2\lfloor \frac{t}{2} \rfloor + 1, \) and likewise \( \mu_{2n+i+2}' = 2\lceil \frac{t}{2} \rceil - 1. \) If \( t \) is even then \( (\mu_{2n+i+1}'^l, \mu_{2n+i+2}'^l) = (t + 1, t - 1) \) and \( (\lambda_{2n+i+1}'^l, \lambda_{2n+i+2}'^l) = (t, t) \) as well.

(c) If \( m \notin \mathbb{Z} \), the proof is analogous. \( \square \)

**Corollary 4.56.** If \( m \in \{ \frac{1}{4}, 1 \} \), and the central character \( W_0C_L \) for \( L \in \mathcal{L}_m(n) \) corresponds to the unipotent class \( C_J \), then \( \Sigma_m(W_0C_L) = \Sigma_m(C_J) \). Thus, we retrieve the classical Springer correspondence.

**Proof.** For residual points, this has been remarked already. The Corollary follows since the maps \( f_B^{BC}, \varphi_m, \psi_m \) which we have defined for general \( m \), reduce to the classical ones in the equal label cases. \( \square \)

### 4.15. Generalized Green functions

Shoji has defined in [20] a matrix of functions \( P = (P_{A,B})_{A,B \in \mathcal{P}_{n,2}} \) as the solution matrix of the following equation. Recall the order \( \succ_m \) of Section 4.3 on \( \mathcal{P}_{n,2} \) which is obtained from the \( m \)-symbols. Let \( t \) be a formal variable. Then we consider matrices \( P^m, \Lambda^m \) of unknowns such that,

\[
\begin{align*}
A_{A,B}^m &= 0 \text{ unless } A \sim_m B, \\
P_{A,B}^m &= 0 \text{ unless either } A \succ_m B \text{ and } A \prec_m B, \text{ or } A = B, \\
P_{A,A}^m &= t_{eq}(A), \\
P_{m,A}^m &= p_{m,A}^m = \Omega.
\end{align*}
\]

(4.46)

Here, \( \Omega \) is as in (3.7) (i.e., \( \Omega = (\omega_{AB}) \) and \( \omega_{AB} = \omega_{ij} \) if \( A \leftrightarrow i, B \leftrightarrow j \)). Shoji shows that (4.46) has unique solution matrices \( P^m, \Lambda^m \) with entries in \( \mathcal{Q}(t) \), and conjectures that moreover \( \Lambda_{A,B} \in \mathbb{Z}[t], P_{A,B} \in \mathbb{Z}_{\geq 0}[t] \). For \( m = \frac{1}{2} \) resp. \( m = 1 \), it is known that \( a_m(\chi) = \text{dim}(B_\chi) \) if \( \chi = \chi_{t,\rho} \) and that if \( C \subset \bar{C}' \) then \( a_m(\chi) \geq a_m(\chi') \) for all \( \chi \in \Sigma_m(C), \chi' \in \Sigma_m(C') \). Thus, in these cases one recovers the (indeed polynomial) Green functions of \( \text{Sp}_{2n}(\mathbb{F}_q) \) resp. \( \text{SO}_{2n+1}(\mathbb{F}_q) \) for \( \text{char}(\mathbb{F}_q) \neq 2 \); one has

\[
P_{A,B}^m = \pi_{ji} \text{ if } A \leftrightarrow i, B \leftrightarrow j.
\]

(4.47)
Therefore the $P_{A,B}^m$ are in general also called Green functions. On the other hand, they are a generalization of Kostka functions $K_{\lambda,\mu}(t)$ in the sense that they form the transition matrix between (appropriately defined, cf. [20]) Schur functions and Hall–Littlewood polynomials.

Our objective is to use these Green functions to describe the $\hat{W}_0$-structure of $\hat{H}_t^f(\mathbb{R})$ for all parameters $k_1, k_2$ with $k_1 \neq 0$. By Remarks 2.4 and 2.5, it is sufficient to determine $\hat{H}_t^f(\mathbb{R})$ with $k_1 \geq 0, k_2 = mk_1 \geq 0$, for every $m \geq 0$.

5. Conjectures

5.1. The graded Hecke algebra

Recall that in the equal label cases, $\hat{H}_t^f(\mathbb{R})$ is parametrized by $\hat{W}_0$. We transfer this parametrization to $P_{n,2}$ by writing $M_A^m$ for $M_A^{\chi_A}$ if $\chi = \chi_A$, where $m = 1$ if $k_1 = k_2$ (the $B_n$-case) and $m = \frac{1}{2}$ if $k_2 = \frac{1}{2} k_1$ (the $C_n$-case). As a $W_0$-module, $M_A^m$ is naturally graded, and we write $M_A^{m;l}$ for its degree-$l$ part. Fix $j = (u, \rho) \in I_0$ and let (with notation as in paragraph 3.2)

$$H^{2l}(B_u) \rho \simeq \sum_{\chi \in \hat{W}_0} n^{l}_{j,\chi} V_{\chi},$$

for $l = 0, 1, \ldots, d_u$ (it is known that all odd cohomology vanishes).

Suppose that we write the Green polynomial $\pi_{ji}$ as

$$\pi_{ji}(q) = \sum_{l \geq 0} \pi^{l}_{ji} q^l. \quad (5.1)$$

Then clearly, if $\chi \leftrightarrow i$, it follows that

$$\pi^{l}_{ji} = n^{l}_{j,\chi}.$$

If we write $P_{B,A}^m$ as $P_{B,A}^m(q) = \sum_{l \geq 0} P_{B,A}^{m;l} q^l$, then by (3.2), (3.6), (4.47),

$$M_A^{m;l} = \sum_B P_{B,A}^{m;l} V_B \otimes \varepsilon.$$

We can now state our conjecture.

**Conjecture 5.1.** Let $k_2 = mk_1$ with $k_1 > 0$ and $m \in \frac{1}{2} \mathbb{Z} \geq 0$. Let $\mathbb{H}$ be the graded Hecke algebra associated to a root system of type $B_n$, whose root labels are $k_1, k_2$. Then we conjecture that

(i) the Green functions $P_{A,B}^m$ are polynomials with non-negative coefficients of degree $\leq a_m(B)$, i.e., can be written as $P_{A,B}^m(q) = \sum_{l=0}^{a_m(B)} P_{A,B}^{m;l} q^l$ with $P_{A,B}^{m;l} \in \mathbb{Z} \geq 0$;
(ii) We have a bijection

\[ \hat{\mathbb{H}}^f (\mathbb{R}) \leftrightarrow \hat{W}_0, \]

written as \( M^m_A \leftrightarrow A \in \mathcal{P}_{n,2} \), which is uniquely determined by requiring that \( \chi_A \otimes \varepsilon \) occurs in \( M^m_A \) (viewed as a \( W_0 \)-module), for all \( A \in \mathcal{P}_{n,2} \).

(iii) The \( W_0 \)-modules \( M^m_A \) can be viewed as graded modules, via a grading inherited from the Green functions \( P^m_{BA} \), as follows. We write its degree-\( l \) part as \( M^{m;l}_A \). It is given by the formula

\[ M^{m;l}_A \simeq \sum_B P^{m;l}_{B,A} V_B \otimes \varepsilon \]

(iv) \( M^m_A \) has central character \( W_0 c \) if and only if \( A \in \Sigma_m(W_0 c) \); that is, \( M^m_A \) has central character \( f^\mathcal{BC}_m(\psi_m(A)) \).

**Corollary 5.2.** The bijection in (ii) is obtained by taking the top degrees of the \( M^m_A \), i.e., we have \( M_{m;}^{m,\text{max}} = \chi_A \otimes \varepsilon \).

**Corollary 5.3.** Every irreducible tempered representation of \( \mathbb{H} \) with real central character has a multiplicity free \( W_0 \)-type.

**Proof.** Granted the Conjecture, we have

\[ (V_A \otimes \varepsilon, M^m_A|W_0) = \left( V_A \otimes \varepsilon, \sum_{l \geq 0} P^{m;l}_{BA} V_B \otimes \varepsilon \right) = \sum_{l \geq 0} P^{m;l}_{AA} = 1. \]

hence the \( W_0 \)-module \( V_A \otimes \varepsilon \) occurs exactly once in \( M^m_A \). \( \square \)

Although there may be several \( W_0 \)-characters occurring with multiplicity one in a given \( V \in \hat{\mathbb{H}}^f (\mathbb{R}) \), there is a unique such character \( \chi_V \otimes \varepsilon \in \hat{W}_0 \) for every \( V \in \hat{\mathbb{H}}^f (\mathbb{R}) \), if we demand that \( \hat{\mathbb{H}}^f (\mathbb{R}) \to \hat{W}_0 : V \mapsto \chi_V \otimes \varepsilon \) is injective. Then \( \chi_V \otimes \varepsilon \) forms the top degree of \( V|W_0 \), and \( \chi_V \) is a Springer correspondent of the central character of \( V \). If \( V \) runs through the modules in \( \hat{\mathbb{H}}^f (\mathbb{R}) \) with central character \( W_0 c \) then \( \chi_V \) runs through \( \Sigma_m(W_0 c) \).

**Remark 5.4.** Conjecture 5.1 together with Remark 2.5 gives a description of \( \hat{\mathbb{H}}^f (\mathbb{R}) \) for all parameters \( k_1, k_2 \) with \( k_1 \neq 0 \). On the other hand, we can define Green functions \( P^{-m} \) and Springer maps \( \Sigma_{m, \cdot}, \psi_{-, \cdot}, f^\mathcal{BC}_m \) for \( \mathbb{H} \) with parameters \( k_2/k_1 = -m \in \frac{1}{2} \mathbb{Z} \leq 0 \) such that Conjecture 5.1 can be extended to these parameters. This is done as follows. We fix \( m \in \frac{1}{2} \mathbb{Z} \geq 0 \). Let, for \( (\xi, \eta) \in \mathcal{P}_{n,2} \), \( \Phi(\xi, \eta) = (\eta, \xi) = (\xi, \eta) \otimes (-, n) \) (viewed as Weyl group characters). We define the \((-m)\)-symbol as

\[
\sigma_{-m}(\xi, \eta) = \begin{cases} 
\Lambda_{n,n+m}^2(\xi, \eta) & \text{if } m \in \mathbb{Z} \\
\Lambda_{n,n+m}^{2,1}(\xi, \eta) & \text{if } m \notin \mathbb{Z}
\end{cases}
\quad \text{and} \quad
\tilde{\sigma}_{-m}(\xi, \eta) = \begin{cases} 
\tilde{\Lambda}_{n}^{2,0}(\xi, \eta) & \text{if } m \in \mathbb{Z}, \\
\tilde{\Lambda}_{m,n}^{2,1}(\xi, \eta) & \text{if } m \notin \mathbb{Z}.
\end{cases}
\]
where, for \( m \notin \mathbb{Z} \), we define \( m' = m + \frac{1}{2} \). We write the entry of \( \eta_i \) to the left of the entry of \( \xi_j \), e.g.

\[
\sigma_{-2}(23, 3) = \begin{pmatrix} 2 & 5 \\ 2 & 4 & 9 \end{pmatrix}.
\]

If \( m = 0 \), then we obtain two symbols, denoted by \( \sigma_{+0}(A) \) and \( \sigma_{-0}(A) \).

It follows directly that \( a_{-m}(A) = a_m(\Phi(A)) \) for all \( A \in \mathcal{P}_{n,2} \). Let \( c \) be a residual point for \( \mathbb{H}(\pm k_1, \pm k_2) \). Then there exists a unique residual point \( c' \) for \( \mathbb{H}(\pm k_1, \mp k_2) \) with \( J(c) = J(c') \). One checks that this induces a bijection between the residual subspaces of \( \mathbb{H}(\pm k_1, \pm k_2) \) and those of \( \mathbb{H}(\pm k_1, \mp k_2) \), such that \( L \) of type \((\kappa, \nu)\) corresponds to \( L' \) of type \((\kappa, \nu')\), where \( \nu' \) is the conjugate partition of \( \nu \). We denote this bijection by \( L \leftrightarrow \Phi(L) \), and we write \( \Phi(W_0c_L) = W_0c_{\Phi(L)} \). This enables us to define, for \( L \) of type \((\kappa, \nu)\) with \(|\nu| = l\),

\[
\Sigma_{-m}(L) = [\text{tr}_{-m} - \text{Ind}_{S_0 \times W_0(B)}^{W_0}(\text{triv}_K \otimes S_{-m}(\nu))]_{-m}.
\]

Since \( S_{-m}(\nu) = \Phi(S_{m}(\nu')) \), it is easy to see that we have

\[
\Sigma_{-m}(\Phi(L)) = \Phi(\Sigma_{m}(L)). \tag{5.3}
\]

Putting \( U_{-m}(n) = U_m(n) \), we define \( f_{-m}^{BC}(\lambda) = \Phi(f_{m}^{BC}(\lambda)) \) and \( \psi_{-m}(A) = \psi_{m}(\Phi(A)) \). One can check that with these definitions, (4.45) also holds for \( m \in \frac{1}{2} \mathbb{Z} \leq 0 \) and that the definitions for \( m = 0 \) are unambiguous.

We use (4.46) to define Green functions \( P_{A,B}^{-m} \). Using the defining equation, it is easy to see that these satisfy, for all \( A, B \in \mathcal{P}_{n,2} \),

\[
P_{A,B}^{-m} = P_{\Phi(A), \Phi(B)}^{-m}. \tag{5.4}
\]

This identity means that 5.1(iii) and (iv), applied to \(-m\), are compatible. Let \( k_1 < 0, k_2 = mk_1, m \in \frac{1}{2} \mathbb{Z} \geq 0 \). Choose \( M_A^m \in \mathbb{H}_{1}^f(k_1, k_2; \mathbb{R}) \) and let its central character be \( W_0c_L = f_{m}^{BC}(\psi_{m}(A)) \). By Remark 2.5, \( M_A^m \otimes (-, n) \) is the \( W_0 \)-restriction of \( M_C^{-m} \) in \( \mathbb{H}_{1}^f(k_1, -k_2; \mathbb{R}) \) with central character \( \Phi(W_0c_L) \). By (5.4),

\[
\sum_B P_{B,A}^m V_B \otimes V_{(-,n)} = \sum_B P_{B,A}^m V_{\Phi(B)} = \sum_B P_{\Phi(B), A}^m V_B = \sum_B P_{B,\Phi(A)}^{-m} V_B,
\]

so by 5.1(iii), \( C = \Phi(A) \). This is compatible with 5.1(iv) since \( f_{-m}^{BC}(\psi_{-m}(\Phi(A))) = \Phi(f_{m}^{BC}(\psi_{m}(A))) = \Phi(W_0c_L) \), and \( \Phi(A) \in \Sigma_{-m}(\Phi(W_0c_L)) \) by (5.3). Thus, we can extend Conjecture 5.1 to apply to all \( m \in \frac{1}{2} \mathbb{Z} \). According to this conjecture we can
label the modules in \( \hat{\mathbb{H}}^t(\mathbb{R}) \) with \( P_{n,2} \) such that for any \( m \in \frac{1}{2}\mathbb{Z} \), \( M_A^m \) has central character \( f_m^{BC}(\psi_m(A)) \) and \( W_0 \)-decomposition

\[
M_A^{m;l} = \begin{cases} 
\sum_B P_{BA}^m v_B & \text{if } k_1 < 0, \\
\sum_B P_{BA}^m v_B \otimes \varepsilon & \text{if } k_1 > 0.
\end{cases}
\] (5.5)

If we replace \( k_2 \) by \(-k_2\) then we have \( \Phi(M_A^m) = M_{\Phi(A)}^{-m} \), in every degree. In particular, \( \Phi(M_A^{0;l}) = M_{\Phi(A)}^{0;l} \) for all \( l \), hence, if \( A = \Phi(A) \), the \( W_0 \)-module \( M_A^0 \) is invariant for \( \Phi \) in every graded part. In this situation, \( \mathbb{H} \) contains the graded Hecke algebra \( \mathbb{H}(D_n) \) associated to the sub-root system of \( R_0 \) consisting of the long roots, which is of type \( D_n \), as a sub-algebra. One can restrict a module of \( \hat{\mathbb{H}}^t(\mathbb{R}) \) to \( \hat{\mathbb{H}}(D_n) \) to obtain a tempered \( \hat{\mathbb{H}}(D_n) \)-module. The above formulas suggest that \( M_A^0 \simeq_{W_0(D_n)} M_{\Phi(A)}^0 \) remains irreducible unless \( A = \Phi(A) \), in which case it splits into two irreducible components.

We do not necessarily assume in Conjecture 5.1 that the parameters are special. Indeed, the equivalence classes in \( \hat{W}_0 \) under \( \sim_m \) are singletons if \( m > n - 1 \). This means that for such parameters, the members of \( \hat{\mathbb{H}}^t(\mathbb{R}) \) are separated by their central character. Thus, we recover the special parameters (4.1) computed in [4].

**Remark 5.5.** For \( B_3 \) and \( B_4 \), and special parameters, Conjecture 5.1 has been shown to be true, thanks to the explicit calculations of the appropriate Green functions by Gunter Malle, and the determination of \( \hat{\mathbb{H}}^t(\mathbb{R}) \). See Section 6.4 for an example.

Notice that we may also formulate Conjecture 5.1 as follows, if we transfer the Springer correspondents of \( C_m(n) \) to \( U_m(n) \) by putting \( \Sigma_m(\lambda) = \Sigma_m(W_0cL) \) if \( W_0cL = f_m^{BC}(\lambda) \):

**Conjecture 5.6.** Let \( \hat{\mathbb{H}} \) be the graded Hecke algebra associated to the root system of type \( B_n \) with parameters \( k_1, k_2 \) such that \( k_1 > 0 \) and \( k_2 = mk_1 \) for \( m \in \frac{1}{2}\mathbb{Z}_{\geq 0} \). Then \( \hat{\mathbb{H}}^t(\mathbb{R}) \) has the following description. For \( \lambda \in U_m(n) \), let \( \Sigma_m(\lambda) = [\phi_m(\lambda)]_m \) be the set of Springer correspondents of \( \lambda \in U_m(n) \).

Then \( \hat{\mathbb{H}}^t(\mathbb{R}) \) has a set of representing modules

\[
\{ M_{\lambda, (\xi, \eta)}^m \mid \lambda \in U_m(n), \ (\xi, \eta) \in \Sigma_m(\lambda) \},
\]

where this indexation is uniquely characterized by the requirements

- **The central character of** \( M_{\lambda, (\xi, \eta)}^m \) **is** \( W_0cL = f_m^{BC}(\lambda) \).
- **The** \( W_0 \)-module \( M_{\lambda, (\xi, \eta)}^m \) **contains the irreducible** \( W_0 \)-character \( \chi(\xi, \eta) \otimes \varepsilon \).

Moreover, given this parametrization, the multiplicity of \( (\xi, \eta) \otimes \varepsilon \) in the \( W_0 \)-module \( M_{\lambda, (\xi, \eta)}^m \) is equal to one.
In the notation of Conjecture 5.1, we have $M^m_A = M^m_{\hat{\xi}, A}$. In the above formulation, the modules $H^m_{A, (\hat{\xi}, \eta)}$ are presented as the analogue of the Springer modules $H(B_u)\rho$, where $\lambda$ replaces $u$ and $(\hat{\xi}, \eta)$ replaces $\rho$. Indeed, in the classical case, the character $\rho$ can be read off from the symbol of $(\hat{\xi}, \eta)$.

5.1.1. Special case: $A = (-, 1^n)$

In the equal label cases, the $W_0$-module $H(B_u)$ is for $u = 1$ isomorphic to the coinvariant algebra (with degrees doubled). The only Springer correspondent of $C = \{1\}$ is the sign representation, indexed by $(-, 1^n)$. For arbitrary $m$, the conjecture shows that the module $M^m_{(-, 1^n)}$ is isomorphic to the coinvariant algebra as well, and the $W_0$-grading has undergone a shift.

Lemma 5.7. Let $0 \leq m \in \frac{1}{2} \mathbb{Z}_{\geq 0}$. Then, with $\varepsilon = (-, 1^n)$ the sign representation, we have $P^m_{A, \varepsilon}(t) = r^{n(m-1)} P^1_{A, \varepsilon}(t)$ if $m \in \mathbb{Z}$, and $P^m_{A, \varepsilon}(t) = r^{n(m-1/2)} P^{1/2}_{A, \varepsilon}(t)$ if $m \notin \mathbb{Z}$.

Proof. Suppose that $m \in \mathbb{Z}$. One checks that the $a_m$-value is maximal among $\mathcal{P}_{n,2}$. Therefore one can show, analogous to Lemma 7.2 in [20] that $r^{-a_m(\varepsilon)} R(\varepsilon) P^m_{A, \varepsilon} = R(A)$ (where we identify $\hat{W}_0$ with $\mathcal{P}_{n,2}$ and $R$ is as in Section 3. On the other hand one checks with an easy computation that $a_m(\varepsilon) = n^2 + n(m - 1)$. The proof for $m \notin \mathbb{Z}$ is analogous. □

5.1.2. Generic parameters

We also conjecture the structure of $\hat{H}^I(\mathbb{R})$ at generic parameters:

Conjecture 5.8. Suppose that the parameters are generic, say $k_2 = tk_1$ where $t = m + \varepsilon$ or $t = m - \varepsilon$, $k_2 = mk_1$ with $k_1 \neq 0$ are special parameters and $\varepsilon > 0$ is such that there are no special parameter values between $k_2 = mk_1$ and $k_2 = tk_1$. Recall from Theorem 4.44 that we have a bijection $\Sigma_m(W_{0CL}) \leftrightarrow C_m(W_{0CL})$. Let $L'$ be a generically residual subspace of type $(\kappa, v)$ such that $W_{0CL'} \subset C_m(W_{0CL})$. Suppose it has Springer correspondent $(\hat{\xi}, \eta) = \Sigma_r(W_{0CL'}) \in \Sigma_m(W_{0CL})$. Then, for $\hat{H}$ with labels $k_2 = tk_1$, the irreducible tempered module $M^I_{(\xi, \eta)}$ with central character $W_{0CL'}$ has $W_0$-structure given by

$$M^I_{(\xi, \eta)} |_{W_0} = \text{Ind}_{W_0(R_{L'})}^{W_0} (\text{triv}_K \otimes M^m_{S_r(v)}).$$

(5.6)

In other words, the Springer correspondent for $k_2 = tk_1$ of $W_{0CL'}$ is obtained by truncatedly inducing $\text{triv}_K \otimes S_r(v)$ (the Springer correspondent of $W_{LCL'}$), whereas the entire module $M^I_{(\hat{\xi}, \eta)}$ is obtained as the induction of the whole corresponding module. We give an example in Section 6.3.

5.2. The affine Hecke algebra

We note that Conjecture 5.1 applies to the affine Hecke algebra as well:
Remark 5.9. Let $\mathcal{R} = (R_0, X, \bar{R}_0, Y, \Pi_0)$ be a root datum such that $R_0$ has type $B_n$. Let the simple roots be $\alpha_1, \ldots, \alpha_n$. Let $k_1, k_2$ be the root labels of the long, resp. short roots and consider $\mathcal{H} = \mathcal{H}(\mathcal{R}^{\deg}, k)$. In view of Remark 2.4 and Eq. (2.11), we can transfer the conjecture on $\mathcal{H}'(\mathbb{R})$ to $\mathcal{H}'(\mathbb{R})$, where $\mathcal{H} = \mathcal{H}(\mathcal{R}', q)$ is the affine Hecke algebra associated to $\mathcal{R}' = (R'_0, X', \bar{R}_0', Y', \Pi_0')$ and root label function $q$, in the following two cases:

- $R_0 = R'_0$, $\Pi_0 = \Pi'_0$, $\text{rank}(X) = \text{rank}(X')$ and (2.8) holds, i.e., $k_1 = \log(q_{\alpha_1})$, $k_2 = \log(q_{\alpha_2}q_{\alpha_3}^{1/2})$;
- $R'_0$ has type $C_n$, $R'_0$ (resp. $\Pi'_0$) is obtained from $R_0$ (resp. $\Pi_0$) by multiplying the short roots by 2, $\text{rank}(X) = \text{rank}(X')$ and (2.8) holds, that is, $k_1 = \log(q_{\alpha_1})$, $k_2 = \log(q_{\alpha_2}^{1/2})$.

In these cases, $\mathcal{H}'(\mathbb{R})$ is in bijection with $\mathcal{H}'(\mathbb{R})$, and under the correspondence of $\mathcal{H}_0 \to \mathcal{W}_0$ obtained in the limit $q_{\lambda}^{1/2} \to 1$, the statements on the $W_0$-structure of the modules in $\mathcal{H}'(\mathbb{R})$ carry over to the $\mathcal{H}_0$-structure of the corresponding modules in $\mathcal{H}'(\mathbb{R})$.

6. Examples

We end by giving some examples of the conjectures and the constructions in the previous sections.

6.1. Splitting and joining

Let $n = 22, m = 2, k \neq 0$ and let $c$ be the residual point with coordinates $(7, 6, 6, 5, 5, 4, 4, 4, 3, 3, 3, 3, 2, 2, 2, 2, 1, 1, 1, 0, 0)k$, then $J(W_0c) = (0, 1, 3, 4, 6, 7)$. Therefore $\Sigma_m(W_0c)$ is the $m$-equivalence class of

$$
\begin{pmatrix}
0 & 3 & 10 & 12 \\
2 & 7 & & \\
& & &
\end{pmatrix}
$$

(6.1)

The 2-symbol (6.1) is the symbol of $(366, 25)$. There are 6 possible 2-symbols with these entries, i.e., there are 15 generically residual points $c(\lambda, k_1, k_2)$ $(i = 0, 1, \ldots, 14)$ which, when $k_2 = 2k_1 \neq 0$, become equal to $c$. The partitions $\lambda_i$ can be determined by applying $\mathcal{F}_2$ to all double partitions whose 2-symbols is similar to (6.1). We construct first $\lambda(W_0c)$, obtained from (6.1). It is obtained as $\lambda(W_0c) = \mathcal{F}_2((366, 25))$ and shown in Fig. 7. Now let us demonstrate how we can, given an arbitrary $\lambda \in P_2(c)$, reach $\lambda(W_0c)$ through a series of flips. We take $\lambda = (2^83^2)$. We need to rearrange the 2-symbol of $\mathcal{F}_2(\lambda)$ into (6.1), by interchanging the first entry which is bigger than its successor with the unique entry in the other row with which it can be interchanged (the entries in each row must remain increasing). In the following figure, we carry this out.
Observe that each permutation in the symbol corresponds to a flip of one part of \( \xi \) to a part of \( \eta \), until we reach the initial partition \( \lambda(W_{0C}) \). Notice also that we have at every step 4 horizontal and 2 vertical extremities.

6.2. Calculation of Springer correspondents

Let \( n = 35, m = 2 \) and consider the residual subspace \( L \) of type \(((3, 6, 9), \nu)\) where \( \nu = (1^42^2) \). We will describe the sets \( C_m(W_{0C_L}) \) and \( \Sigma_m(W_{0C_L}) \), as well as the partition \( \lambda \in \mathcal{U}_m(n) \) such that \( f_m^{BC}(\lambda) = W_{0C_L} \). We start with constructing \( \Sigma_m(W_{0C_L}) \) by using the definition of truncated induction. We observe by drawing \( T_2(\nu) \) that one can insert \( A \)-strips of length 3 and 9 into \( T_2(\nu) \) to obtain a partition \( \mu \). Then \( T_2(\mu) \) is as in Fig. 8.

We calculate \( \Sigma_m(W_{0C_L}) \). Put \( \kappa = (39), \alpha = (7), \rho = \alpha \cup \kappa = (379) \) as in 4.12.2. We adopt the notation of Proposition 4.41. We start with

\[
\text{tr}_m - \text{Ind}_{S_9 \times W_6(B_{16})}^{W_6(B_{25})} ((9) \otimes (66, 4)).
\]
The 2-symbol of \((\zeta, \eta) = (66, 4)\), written with \(l(\zeta) = 4, l(\eta) = 2\) is
\[
\begin{pmatrix}
0 & 2 & 10 & 12 \\
0 & 6 &  & \\
\end{pmatrix}.
\]

Recall that \(\zeta = (\zeta_0, \ldots, \zeta_3)\) and \(\eta = (\eta_0, \eta_1)\). With \(t = 9\), we have
\[
e_2(\zeta_3) > e_2(\zeta_2) > e_2(\eta_1) > e_2(\zeta_1),
\]
and \(\zeta_2 + \eta_1 = 10 > t > \zeta_1 + \eta_1 = 4\). Thus, we are in situation (4.21) with \(k = 1, l = 1, f = 1\). Since \(t > 2\eta_1\), it follows that (4.20) applies with \(p = 0\) and we obtain \((\zeta'_1, \eta'_1) = (4, 5)\) or \((5, 4)\). The resulting 2-partitions \((0466, 45)\) and \((0566, 44)\) have 2-symbols
\[
\begin{pmatrix}
0 & 6 & 10 & 12 \\
4 & 7 &  & \\
\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}
0 & 7 & 10 & 12 \\
4 & 6 &  & \\
\end{pmatrix}.
\]

Notice that they are indeed similar. We proceed with the second induction, choosing \((0466, 45)\) (the outcome does not depend on this choice), i.e., we calculate
\[
\text{tr}_m - \text{Ind}_{W_0(B_{32})/S_7 \times W_0(B_{22})}^W ((7) \otimes (466, 45)).
\]

We put \(t = 7\) and consider the 2-symbol of \((\zeta, \eta)\), where we have put \(l(\zeta) = 5, l(\eta) = 3\) and both indexations starting with zero:
\[
\begin{pmatrix}
0 & 2 & 8 & 12 & 14 \\
0 & 6 & 9 &  & \\
\end{pmatrix}.
\]

Then we have
\[
e_2(\zeta_4) > e_2(\zeta_3) > e_2(\eta_2) > e_2(\zeta_2) > e_2(\eta_1) > e_1(\zeta_1) > e_2(\eta_0) \geq e_2(\zeta_0),
\]
while \(\zeta_2 + \eta_1 = 8 > t > \zeta_1 + \eta_1 = 4\). Thus, we are in situation (4.21) with \(f = 1, k = l = 1\). We have \(t < 2\eta_1\), hence we find \((\zeta'_1, \eta'_0) = (3, 4)\) or \((2, 5)\). However, the latter is impossible since \(\eta_1 = 4\). This is an illustration of how the impossibility of fitting the \(A\)-strip of length seven into \(T_2(v)\), implies that this induction does not change the number of Springer correspondents. Thus we find the 2-partition \((03466, 445)\). Finally, we consider the induction
\[
\text{tr}_m - \text{Ind}_{W_0(B_{35})/S_3 \times W_0(B_{32})}^W ((3) \otimes (3466, 445)).
\]
We start from \((\zeta, \eta) = (003466, 0445)\) with 2-symbol
\[
\begin{pmatrix}
0 & 2 & 7 & 10 & 14 & 16 \\
0 & 6 & 8 & 11
\end{pmatrix}.
\]

Observe also the interval 6,7,8 which has been formed due to the induction of the factor \(A_6\) which could not be fitted into \(T_2(v)\). We have \(t = 3\),
\[
e_2(\xi_5) > e_2(\xi_4) > e_2(\eta_3) > e_2(\xi_3) > e_2(\xi_2) > e_1(\xi_2) > e_2(\eta_1) > e_2(\xi_1) > e_2(\eta_0) \geq e_2(\xi_0),
\]
and \(\xi_1 + \eta_1 = 4 > t > \xi_1 + \eta_0\). Thus we are in situation (4.18) with \(f = 0, k = l = 1\).
We have \(2\xi_1 + 2 < t\) and hence \((\xi_1^2, \eta_0^2) = (0, 3)\) or \((1, 2)\). Indeed, both are possible. Thus we obtain the 2-partitions \((3466, 3445)\) and \((13466, 2445)\) with 2-symbols
\[
\begin{pmatrix}
0 & 2 & 7 & 10 & 14 & 16 \\
3 & 6 & 8 & 11
\end{pmatrix}
\]
and
\[
\begin{pmatrix}
0 & 3 & 7 & 10 & 14 & 16 \\
2 & 6 & 8 & 11
\end{pmatrix}.
\]

We conclude that \(\Sigma_m(W_0cL) = [(13466, 2445)]_m\). Let us check that this is also what we find by applying \(\phi_m\) to \(\lambda\), the partition corresponding to \(L\). This partition consists of the parts \(2j_i + 1\) where the \(j_i\) are the jumps of \(c(v, k, mk)\), and the parts \((t, t)\) for every \(A_{t-1}\), hence \(\lambda = (1, 3^2, 7^3, 9^2, 13, 15)\). Using the definition of \(\phi_m\), one easily checks that \(\phi_m(\lambda) = (13466, 2445)\).

Finally we check the bijection between \(\Sigma_m(W_0cL)\) and \(C_m(W_0cL)\). Since \(l(\kappa) = 2, m = 2, r = 1\), we have \(|\Sigma_m(W_0cL)| = 2^2 \cdot \binom{2+2-1}{1} = 16\).

The corresponding 2-symbols are found as follows. It is not hard to see that there are exactly three other partitions \(v_1, v_2, v_3\) such that \(S_2(v_i) \sim_2 S_2(v)\). We display them in Fig. 9 where we insert also the blocks of length 3 and 9 to obtain \(\mu_1, \mu_2, \mu_3\).

One checks that \(\{S_2(v_i); i = 1, 2, 3\} = \{(1236, 268), (1234, 26, 10), (34566, 4)\}\). Applying
\[
\text{tr}_m - \text{Ind}_{S_7 \times W_0(B_28)}^{W_0(B_{55})}((7) \otimes (\alpha, \beta)),
\]
where \((\alpha, \beta) = S_{2 \pm \varepsilon}(\mu_i)\) yields the other 2-partitions in \(\Sigma_2(W_0cL)\). One easily checks that to \(\mu_1\) the following 2-partitions are associated: we have
\[
\begin{pmatrix}
0 & 3 & 6 & 8 & 11 & 16 \\
2 & 7 & 10 & 14
\end{pmatrix}.
\]
and those resulting by interchanging 2 and 3, and/or 10 and 11; similarly from \( \mu_2 \) we find

\[
\begin{pmatrix}
0 & 3 & 6 & 8 & 11 & 14 \\
2 & 7 & 10 & 16 \\
\end{pmatrix},
\]

and those resulting by interchanging 2 and 3, and/or 10 and 11; and finally from \( \mu_2 \) we find

\[
\begin{pmatrix}
3 & 6 & 8 & 11 & 14 & 16 \\
0 & 2 & 7 & 10 \\
\end{pmatrix},
\]

and those resulting by interchanging 2 and 3, and/or 10 and 11. One checks that these 16 2-partitions exhaust a similarity class.

We give another example for non-integer \( m \). Let \( m = \frac{7}{2} \), \( n = 35 \) and consider \( U_m(n) \ni \lambda = (1 \ 1 \ 4 \ 7 \ 7 \ 10 \ 12 \ 13 \ 13 \ 14) \). An application of \( f_m^{BC} \) yields that \( \lambda \) corresponds to \( L \in \mathcal{L}_m(n) \) of type \(((7, 13), \nu)\) where \( c = c(v, k, mk) \) is a residual point for \( B_{14} \) with jumps \( J(W_0c) = \{\frac{3}{2}, \frac{9}{2}, \frac{11}{2}, \frac{13}{2}\} \), as in Fig. 10.
We apply $\phi_m$. According to the definition, first we need to add a zero to $\lambda$ and to replace the parts $(13 13)$ by $(14 12)$. Then we have $\mu = (0 1 1 4 7 7 10 12 14 12 14)$, $\mu^* = (0 2 3 7 11 12 16 19 21 19 21)$, $(\xi^*, \eta^*) = (1 3 5 9 10 9 10, 0 1 6 8)$, and $(\xi, \eta) = (1236644, 0045)$ whose lengths are readjusted to form $(\xi, \eta) = (123644, 045)$ in the $7\#$-symbol, which reads:

$$\left(\begin{array}{ccccccc}
1 & 4 & 7 & 12 & 14 & 14 & 16 \\
1 & 7 & 10 \\
\end{array}\right).$$

We replace it by

$$\left(\begin{array}{ccccccc}
1 & 4 & 7 & 10 & 12 & 14 & 16 \\
1 & 7 & 14 \\
\end{array}\right),$$

which is the symbol of $(1234444, 49) = \phi_m(\lambda)$.

We now check that we also find $\phi_m(\lambda)$ by computing $\Sigma_m(W_0cL)$ using truncated induction. Thus, using again the notation of Section 4.12.2 and Proposition 4.41, we have $\nu = \mu$, $\kappa = \emptyset$, $\alpha = \rho = (1 7 13)$. We start with computing

$$\text{tr}_m - \text{Ind}_{W_0(B_{27})}^{W_0(B_{34})}((13) \otimes (2444, -)).$$

The symbol of $(02444, 0)$ is

$$\left(\begin{array}{cccc}
0 & 4 & 8 & 10 \\
1 \\
\end{array}\right),$$

such that with $t = 13$ we have

$$e_m(\zeta_4) > e_m(\zeta_3) > e_m(\zeta_2) > e_m(\zeta_1) > e_m(\eta_0) > e_m(\zeta_0),$$

and $t > \zeta_1 + \eta_0$. That is, we are in situation (4.18) with $f = 0$, $k = 4$, $l = 3$. In (4.24), we get $p = 3$ since $2\zeta_3 + 6 - 1 = 13$. According to (4.23) we thus find $(\zeta'_3, \eta'_0) = (4, 9)$. We thus obtain the 2-partition $(24444, 9)$. We proceed to calculate

$$\text{tr}_m - \text{Ind}_{W_0(B_{27})}^{W_0(B_{34})}((7) \otimes (24444, 9)).$$

The symbol of $(02444, 09)$ is

$$\left(\begin{array}{ccccccc}
0 & 4 & 8 & 10 & 12 & 14 \\
1 & 12 \\
\end{array}\right),$$

such that

$$e_m(\zeta_5) > e_m(\eta_1) \geq e_m(\zeta_4) > e_m(\zeta_3) > e_m(\zeta_2) > e_m(\eta_0) > e_m(\zeta_0).$$
and $\eta_1 + \xi_4 = 13 > 7 = t > \xi_1 + \eta_0$. Analogous to the first induction we find this time that the $a$-value is maximal for $(234444, 49)$. The last step, the induction

$$\text{tr}_m - \text{Ind}_W^W(35) \otimes (1) \otimes (234444, 49)$$

yields analogously that indeed

$$\Sigma_m(W_0cL) = [(123444, 49)]_m = \phi_m(\lambda)$$

as expected.

### 6.3. Tempered modules at generic parameters

Let $n = 6$, $m = 1$ and consider $L$ of type $A_0 \times A_2 \times B_2(2)$. Then one checks that $\Sigma_1(W_0cL) = \{W_0cL_1, \ldots, W_0cL_4\}$ where $L_1, L_2, L_3, L_4$ are of type, respectively, $B_6(222), A_0 \times B_5(122), A_2 \times B_3(12), A_0 \times A_2 \times B_2(2)$ (see the figure below). From

$$\begin{array}{ccc}
1 & 2 \\
0 & i & 0 \\
\vdots & \vdots & \vdots \\
\end{array}$$

Definition 4.20 it follows that $\Sigma_1(W_0cL) = \{(122, 1), (112, 2), (22, 11), (12, 12)\}$. Using the deformed symbols of Section 4.13, we find the following $a$-values around $m = 1$:

$$\begin{array}{ccc}
(\xi, \eta) & (1 \pm \varepsilon) - \text{symbol} & a_{1-\varepsilon}(\xi, \eta) & a_{1+\varepsilon}(\xi, \eta) \\
(122, 1) & \begin{pmatrix}
1 & 4 \\
\mp \varepsilon & 3 \mp \varepsilon \\
6 \\
\end{pmatrix} & 7 + 2\varepsilon & 7 + \varepsilon \\
(22, 11) & \begin{pmatrix}
0 & 4 \\
1 \mp \varepsilon & 3 \mp \varepsilon \\
6 \\
\end{pmatrix} & 7 + \varepsilon & 7 + 2\varepsilon \\
(112, 2) & \begin{pmatrix}
1 & 3 \\
\mp \varepsilon & 4 \mp \varepsilon \\
6 \\
\end{pmatrix} & 7 + \varepsilon & 7 + 2\varepsilon \\
(12, 12) & \begin{pmatrix}
0 & 3 \\
1 \mp \varepsilon & 4 \mp \varepsilon \\
6 \\
\end{pmatrix} & 7 \pm 3\varepsilon \\
\end{array}$$

The modules in $\hat{H}_k(\mathbb{R})$ with $k_2 = k_1 \neq 0$ with central character $W_0cL$ are denoted $\{M^1_{(\xi, \eta)} | (\xi, \eta) \in \Sigma_1(W_0cL)\}$. Let $V^0_1, \ldots, V^0_4$ denote the $W_0$-types of $M^1_{(122, 1)}, \ldots, M^1_{(12, 12)}$ (in the order above) respectively. We have calculated these $W_0$-modules via Lusztig’s theorem (3.4) and the explicit knowledge of the Green functions, computed by G. Malle.
Thus, in this example, generically (around character $W^{106}$), Slooten / Advances in Mathematics 203 (2006) 34 – 108

components, as $W_m$-modules for $m$.

6.4. Calculation of $\hat{\mathbb{H}}$ parameters

At generic parameters around $m(W_{0L_2}) = (122, 1)$. Then, according to (5.8),

$$V_1^− = M^{1−ε}_{(122,1)} \mid w_0 = \text{Ind}_{W_0(R_{L_2})}^W((1) \otimes (3) \otimes M^1_{(2,−)}),$$

and it turns out that $V_1^− \simeq V_1^0 \oplus \cdots \oplus V_4^0$.

If $L' = L_3$ and $m = 1 + ε$ then $\Sigma_{1+ε}(W_{0L_3}) = (22, 11)$. According to (5.6),

$$V_2^+ = M^{1+ε}_{(22,11)} \mid w_0 = \text{Ind}_{W_0(R_{L_3})}^W((3) \otimes M^1_{(12,−)}),$$

and it turns out that $V_2^+ \simeq V_2^0 \oplus V_4^0$.

Continuing, one checks that according to Conjecture 5.6 we find the following modules for $m = 1 − ε$, $m = 1$, $m = 1 + ε$.

<table>
<thead>
<tr>
<th>$L$</th>
<th>$M^{1−ε}<em>{\Sigma</em>{m−ε}(W_{0L_2})} \mid w_0$</th>
<th>$m = 1$</th>
<th>$M^{1+ε}<em>{\Sigma</em>{m+ε}(W_{0L_2})} \mid w_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L_1$</td>
<td>$V_4^0$</td>
<td>$−$</td>
<td>$V_1^0$</td>
</tr>
<tr>
<td>$L_2$</td>
<td>$V_4^0 \oplus V_3^0$</td>
<td>$−$</td>
<td>$V_1^0 \oplus V_2^0$</td>
</tr>
<tr>
<td>$L_3$</td>
<td>$V_4^0 \oplus V_3^0$</td>
<td>$−$</td>
<td>$V_3^0 \oplus V_4^0$</td>
</tr>
<tr>
<td>$L_4$</td>
<td>$V_1^0 \oplus V_2^0 \oplus V_3^0 \oplus V_4^0 {V_1^0, V_2^0, V_3^0, V_4^0}$</td>
<td>$V_1^0 \oplus V_2^0 \oplus V_3^0 \oplus V_4^0$</td>
<td></td>
</tr>
</tbody>
</table>

Thus, in this example, generically (around $m = 1$), the representation with central character $W_{0L'}$ for $L'$ of type $A_0 \times A_2 \times B_2(2)$ is irreducible. At $m = 1$, it splits into four irreducible components. At generic parameters around $m = 1$, the other modules in $\hat{\mathbb{H}}(\mathbb{R})$ with central character $W_{0L''}$ for $L'' \in \Sigma_m(W_{0L})$ are direct sums of these components, as $W_0$-modules.

6.4. Calculation of $\hat{\mathbb{H}}(\mathbb{R})$

To illustrate Conjecture 5.1 we consider a root system of type $B_3$. We consider the special parameters $k_2 = mk_1$ with $m = 2$ and $k_1, k_2 > 0$. Then we order $P_{3,2}$, refining the order given by the $a$-value, and such that similarity classes form intervals. In this case, we have

<table>
<thead>
<tr>
<th>$(\xi, \eta)$</th>
<th>$a_2(\xi, \eta)$</th>
<th>$(\xi, \eta)$</th>
<th>$a_2(\xi, \eta)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(-, 111)$</td>
<td>12</td>
<td>$(11, 1)$</td>
<td>3</td>
</tr>
<tr>
<td>$(-, 12)$</td>
<td>8</td>
<td>$(111, −)$</td>
<td>3</td>
</tr>
<tr>
<td>$(1, 11)$</td>
<td>6</td>
<td>$(2, 1)$</td>
<td>2</td>
</tr>
<tr>
<td>$(-, 3)$</td>
<td>5</td>
<td>$(12, −)$</td>
<td>1</td>
</tr>
<tr>
<td>$(1, 2)$</td>
<td>4</td>
<td>$(3, −)$</td>
<td>0</td>
</tr>
</tbody>
</table>
and also \((11, 1) \sim_2 (111, -)\); we therefore keep this ordering. The matrix \((P^2_{A,B})\) has been computed by G. Malle. The results are as follows. For all \(A = (\xi, \eta) \in \mathcal{P}_{n,2}\), we tabulate the partition \(\lambda = \psi_2(\xi, \eta) \in \mathcal{U}_2(3)\) of which it is a Springer correspondent, the residual subspace \(L\) such that \(W_{0CL} = f_{2}^{BC}(\lambda)\) and for all \(l\) the degree-\(l\) component \(M^{2;l}_{A}\) of \(M^{2}_{A}\) viewed as \(W_0\)-module.

<table>
<thead>
<tr>
<th>((\xi, \eta))</th>
<th>(\psi_2(\xi, \eta)) type of (L)</th>
<th>(l)</th>
<th>(M^{2;l}_{(\xi, \eta)})</th>
</tr>
</thead>
<tbody>
<tr>
<td>((-1, 111))</td>
<td>((1^7 3))</td>
<td>(\emptyset)</td>
<td>3 (3, -)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>(2, 1)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>4</td>
<td>(1, 2) + (12, -)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>5</td>
<td>(1, 1) + (11, 1) (2, 1)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>6</td>
<td>(1, 1) + (1, 2) (12, -)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>7</td>
<td>(1, 12) + (11, 1) (2, 1)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>8</td>
<td>(1, 11) + (1, 2)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>9</td>
<td>(1, -) + (12, 1)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>10</td>
<td>(1, 11)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>11</td>
<td>(1, 11)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>12</td>
<td>(-, 111)</td>
</tr>
<tr>
<td>((-1, 12))</td>
<td>((1^{3} 2^2 3))</td>
<td>(A_1)</td>
<td>3 (3, -)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>(2, 1)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>4</td>
<td>(1, 2) + (12, -)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>5</td>
<td>(1, -) + (11, 1) (2, 1)</td>
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<td></td>
<td></td>
<td>6</td>
<td>(1, 1) + (1, 2)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>7</td>
<td>(1, 12)</td>
</tr>
<tr>
<td>((1, 11))</td>
<td>((1^5 5))</td>
<td>(B_1)</td>
<td>2 (12, -) + (3, -)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>(2, 1)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>3</td>
<td>(11, 1) + (2, 1)</td>
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<tr>
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<td>4</td>
<td>(1, 2) + (111, -) (12, -)</td>
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<tr>
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<td>6</td>
<td>(1, 11)</td>
</tr>
<tr>
<td>((-1, 3))</td>
<td>((13^3))</td>
<td>(A_2)</td>
<td>2 (3, -)</td>
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<td></td>
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<td>3</td>
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<td></td>
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<td>4</td>
<td>(-, 3)</td>
</tr>
<tr>
<td>((1, 2))</td>
<td>((12^{2} 5))</td>
<td>(A_1 \times B_1)</td>
<td>2 (12, -) + (3, -)</td>
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<tr>
<td></td>
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<td></td>
<td>(2, 1)</td>
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<tr>
<td></td>
<td></td>
<td>3</td>
<td>(11, 1) + (2, 1)</td>
</tr>
<tr>
<td>((11, 1))</td>
<td>((1^{2} 35))</td>
<td>(B_2, (11))</td>
<td>2 (12, -)</td>
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<td></td>
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<td></td>
<td>(2, 1)</td>
</tr>
<tr>
<td>((111, -))</td>
<td>((1^{2} 35))</td>
<td>(B_2, (11))</td>
<td>3 (111, -)</td>
</tr>
<tr>
<td>((2, 1))</td>
<td>((1^{3} 7))</td>
<td>(B_2, (2))</td>
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</tr>
<tr>
<td></td>
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</tr>
<tr>
<td>((12, -))</td>
<td>((37))</td>
<td>(B_3, (12))</td>
<td>1 (12, -)</td>
</tr>
<tr>
<td>((3, -))</td>
<td>((19))</td>
<td>(B_3, (3))</td>
<td>0 (3, -)</td>
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</table>
Notice that the decomposition into graded parts of the regular representation (which we obtain for the center $0$ of the subspace $L = \alpha$ which has $R_L = \emptyset$), is the one of the coinvariant algebra with a degree shift, as seen in Section 5.1.1.

We have verified by explicit calculation that the rightmost column indeed contains the $W_0$-types of the modules in $\hat{\mathcal{H}}^f(R)$. For details, see [21].

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References