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Equivalence of the Gelfand–Shilov Spaces

Jaeyoung Chung

Department of Mathematics, Kunsan National University, Kunsan 573-360, Korea

Soon-Yeong Chung

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and

Dohan Kim

Department of Mathematics, Seoul National University, Seoul 151-742, Korea

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We prove that there is a one to one correspondence between the Gelfand–Shilov space W_M^{Ω} of type W and the space $S_{M_p}^{N_p}$ of generalized type S. As an application we prove the equality $W_M \cap W^{\Omega} = W_M^{\Omega}$, which is a generalization of the equality $S_r \cap S^s = S_r^s$ found by I. M. Gelfand and G. E. Shilov (''Generalized Functions, II, III,'' Academic Press, New York/London, 1967). © 1996 Academic Press, Inc.

INTRODUCTION

The purpose of this paper is to investigate the relations between the Gelfand–Shilov spaces of generalized type S and of type W which were introduced by I. M. Gelfand and G. E. Shilov in [GS]. They used these spaces to investigate the uniqueness of the solutions of the Cauchy problems of partial differential equations. Although the Gelfand–Shilov spaces of generalized type S are defined by means of the positive sequences M_p and N_p and the spaces of type W are defined by means of the weight functions $M(x)$ and $\Omega(y)$, we show that the class of the spaces of type W is exactly the same as a class of the spaces of generalized type S.

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In Section 1 we state definitions of the Gelfand–Shilov spaces of generalized type S and type W, some relations between sequences and their associated functions, and basic concepts of Young conjugates and their properties. In Section 2 we prove that the spaces W_M, W^Ω, and W_m^Ω of type W are equal to some spaces S_{M_p}, S^{N_q}, and $S_{M_p}^{N_q}$ of generalized type S, respectively, under the natural condition $M \subset \Omega$ and vice versa. In Section 3 as an application of the above result we show the equality

$$W_M \cap W^\Omega = W_M^\Omega \tag{W}$$

under the natural condition $M \subset \Omega$ (see Section 1 for definition). For this we first show the equality

$$S_{M_p} \cap S^{N_p} = S_{M_p}^{N_p}. \tag{S}$$

The above two equalities are generalizations of the equality

$$S_r \cap S^s = S_r^s \tag{S0}$$

which was suggested by Gelfand and Shilov in [GS] and proved affirmatively by Kashpirovsky in [Ka] in 1979.

An equality of type (S) was also proved in Pathak [P] under the condition (P). But, this result cannot be applied to prove the equality (W) as the defining sequence N_p of $\Omega^*(y)$ satisfies (M.1)* (see Lemma 2.5) and (M.1)* implies the reverse inequality (P)* of (P). See Section 3 for definitions of the conditions (P) and (P)* and for an example. So we replace (P) by a natural condition

$$M_p N_p \supset p!$$

which is always satisfied if $M(x) \subset \Omega(y)$, and we modify the proof in [P] in order to make use of the condition $M_p N_p \supset p!$ instead of (P). Finally we also prove the triviality of the spaces $S_{M_p} \cap S^{N_p}$ and $S_{M_p}^{N_p}$ under the condition $M_p N_p \subset p!^s$, $0 < s < 1$, which will complete the generalization of the equality (S0).

1. PRELIMINARIES

Let M_p, $p = 0, 1, 2, \ldots,$ be a sequence of positive numbers. We impose the conditions denoted (M.1), (M.2), and (M.2)′ which denote logarithmic convexity, stability under ultradifferential operators, and stability under differential operators. We assume that $M_0 = 1$ for simplicity and refer to [K, p. 26] for details. We also use the multi-index notations $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $\alpha! = \alpha_1! \cdots \alpha_n!$, and $\partial^\alpha = (\partial/\partial x_1)^{\alpha_1} \cdots (\partial/\partial x_n)^{\alpha_n}$ for $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$.

DEFINITION 1.1. For each sequence M_p of positive numbers we define the *associated function* $M(\rho)$ on $[0, \infty)$ by

$$M(\rho) = \sup_p \log \frac{\rho^p}{M_p}. \tag{1.1}$$

DEFINITION 1.2. Let M_p and N_p be sequences of positive numbers satisfying (M.1). Then we write $M_p \subset N_p$ ($M_p \prec N_p$, respectively) if there are constants L and C (for any $L > 0$ there is a constant $C > 0$, respectively) such that $M_p \leq CL^p N_p$, $p = 0, 1, 2, \ldots$. M_p and N_p are *equivalent* if $M_p \subset N_p$ and $M_p \supset N_p$ hold.

For example, $M_p = p!^s$ and $N_p = p^{ps}$ are equivalent if $s > 0$, by Stirling's formula.

DEFINITION 1.3. Let $M(x)$ and $N(x)$ be real functions. Then we write $M(x) \subset N(x)$ ($M(x) \prec N(x)$, respectively) if there are constants L and C (for every $L > 0$ there is a constant $C > 0$, respectively) such that $M(x) \leq N(Lx) + C$. $M(x)$ and $N(x)$ are *equivalent* if $M(x) \subset N(x)$ and $M(x) \supset N(x)$.

DEFINITION 1.4. If $M(\rho)$ is an increasing convex function in $\log \rho$ and increases more rapidly than $\log \rho^p$ for any p as ρ tends to infinity, we define its *defining sequence* by

$$M_\rho = \sup_{\rho > 0} \frac{\rho^p}{\exp M(\rho)}, \qquad p = 0, 1, 2, \ldots.$$

DEFINITION 1.5. Let $M : [0, \infty) \to [0, \infty)$ be a convex and increasing function with $M(0) = 0$ and $\lim_{x \to \infty} x / M(x) = 0$. Then we define its *Young conjugate* M^* by

$$M^*(y) = \sup_x (xy - M(x)).$$

We now refer to [GS, Vol. II, Chap. IV] for the definitions of the Gelfand–Shilov spaces of type S and to [GS, Vol. II, Appendix 1, Chap. IV] for the Gelfand–Shilov spaces of generalized type S.

Let $M(x)$ and $\Omega(y)$ be differentiable functions on $[0, \infty)$ satisfying the condition

(K) $M(0) = \Omega(0) = M'(0) = \Omega'(0) = 0$ and their derivatives continuous, increasing, and tending to infinity.

We refer to [GS, Vol. II, Appendix 2, Chap. IV] for the Gelfand–Shilov spaces of type W.

DEFINITION 1.6. A space $\tilde{\Phi}$ of Fourier transforms $\hat{\varphi}(\xi) = \int_{\mathbb{R}^n} \varphi(x) e^{-ix\cdot\xi}\, dx$ of the functions $\varphi \in \Phi$ is called the *Fourier-dual* of Φ.

2. EQUIVALENCE OF THE GELFAND–SHILOV SPACES

In this section we prove the equivalence of the Gelfand–Shilov spaces of type W and the spaces of generalized type S.

Let M_p, $p = 0, 1, 2, \ldots,$ be a sequence of positive numbers. We impose the following conditions on M_p;

(M.1)′ (Strong Logarithmic Convexity) $m_p = M_p/M_{p-1}$ is increasing and tending to infinity as $p \to \infty$.

(M.1)* (Duality) $p!/M_p$ satisfies (M.1)′.

PROPOSITION 2.1 [K, p. 49]. *If M_p satisfies* (M.1), *then M_p is the defining sequence of the associated function of itself, that is,*

$$M_p = \sup_{\rho \geq 0} \frac{\rho^p}{\exp M(\rho)},$$

where $M(\rho)$ is the associated function of M_p.

We can easily obtain the following Lemma 2.2 from the Lemma 2.3 in [CCK, p. 370].

LEMMA 2.2. *Let $M : [0, \infty) \to [0, \infty)$ be the function satisfying the condition* (K). *Then $M(x)$ is equivalent to the associated function of the defining sequence of itself, i.e.,*

$$M(x) \cong \sup_p \log \frac{x^p}{M_p},$$

where M_p is the defining sequence of $M(x)$.

LEMMA 2.3 [CCK]. *Let $M(\rho)$ be a function satisfying the condition* (K). *Then the defining sequence M_p^* of the Young conjugate $M^*(\rho)$ of $M(\rho)$ is equivalent to $p!/M_p$, where M_p is the defining sequence of $M(\rho)$. In fact, $M_p^* = (p/e)^p/M_p$.*

Conversely, if M_p satisfies (M.1)′ *and* (M.1)*, *then the associated function $M^\#(\rho)$ of $p!/M_p$ and the Young conjugate $M^*(\rho)$ of the associated function $M(\rho)$ of M_p are equivalent.*

LEMMA 2.4. *Let M_p be a sequence of positive numbers satisfying the conditions* (M.1)′ *and* (M.1)*. *Then the associated function $M(\rho)$ of M_p is equivalent to a function $M_0(\rho)$ which satisfies the condition* (K).

Conversely, let $M(\rho)$ *be a function satisfying the condition* (K). *Then the defining sequence* M_p *of* $M(\rho)$ *satisfies the conditions* (M.1)′ *and* (M.1)* *up to equivalence.*

Proof. We may assume that $m_p = M_p/M_{p-1}$ and p/m_p are strictly increasing. Let $w(t)$ be the line segments connecting $(m_p, p/m_p)$ and $(m_{p+1}, (p+1)/m_{p+1})$, $p = 1, 2, \ldots,$ where $(m_p, p/m_p) = (0,0)$ for $p = 0$ and let $M_0(\rho) = \int_0^\rho w(t)\,dt$. Then the function $M_0(\rho)$ satisfies the condition (K). We claim that the associated function $M(\rho)$ of M_p is equivalent to $M_0(\rho)$. Denote by $m(\lambda)$ the number of $m_p \le \lambda$. We obtain the following inequalities for $m_p \le \lambda < m_{p+1}$

$$p/m_p \le w(\lambda) < (p+1)/m_{p+1} \tag{2.1}$$

$$p/m_{p+1} \le m(\lambda)/\lambda < p/m_p. \tag{2.2}$$

Combining (2.1) and (2.2) we have

$$\tfrac{1}{2} w(\lambda) \le m(\lambda)/\lambda \le w(\lambda) \tag{2.3}$$

for all $\lambda > 0$.

Integrating (2.3) and applying the formula $M(\rho) = \int_0^\rho m(\lambda)/\lambda\,d\lambda$ in [K, p. 50] and the convexity of M_0, we conclude that $M_0(\frac{1}{2}\lambda) \le M(\lambda) \le M_0(\lambda)$ for all $\lambda \ge 0$.

To prove the converse, let $M(\rho)$ be a function satisfying the condition (K). Then the defining sequence M_p of $M(\rho)$ satisfies (M.1)′. Since the Young conjugate $M^*(\rho)$ of $M(\rho)$ also satisfies the condition (K) the defining sequence $(p/e)^p/M_p$ of $M^*(\rho)$ also satisfies (M.1)′. We complete the proof by Stirling's formula.

LEMMA 2.5 [GS, p. 245]. *Let* M_p *and* N_p *satisfy* (M.1) *and* (M.1)*. *Then*

$$\widetilde{S_{M_p}} = S^{M_p} \qquad \text{and} \qquad \widetilde{S^{N_p}} S_{N_p}.$$

DEFINITION 2.6. We call $S_{M_p}^{N_p}$ a space of type S_0 if the sequences M_p and N_p satisfying the condition (M.1)′, (M.1)*, and $M_p N_p \supset p!$. Also, we call W_M^Ω a space of type W_0 if the functions M and Ω satisfy the condition (K) and the relation $M \subset \Omega$.

Remark 2.7. (i) Let $M_p = p!^r$, $N_p = p^{ps}$, $0 < r, s < 1$, $r + s \ge 1$. Then $S_{M_p}^{N_p}$ is a space of type S_0.

(ii) Let $M(x) = e^x - x - 1$, $\Omega(y) = ye^y - y$. Then $M(x)$ and $\Omega(y)$ satisfy (K) and $M(x) \subset \Omega(y)$ and, W_M^Ω is a space of type W_0. In fact, $M(x) \supset \Omega(y)$ also holds and W_Ω^M is a space of type W_0.

(iii) Let M_p and N_p be the defining sequences of the above functions $M(x)$ and $\Omega^*(y)$, respectively. Then $S_{M_p}^{N_p}$ is a space of type S_0. Also, let M_p and N_p be the defining sequences of the above functions $M(x)$ and $\Omega(y)$. Then $S_{M_p}^{p!/N_p}$ is a space of type S_0 by Lemma 2.3 and Lemma 2.4.

We now prove the main theorem in this section.

THEOREM 2.8. *There is a one to one correspondence between the spaces of type S_0 and type W_0. In other words, for any given space $S_{M_p}^{N_p}$ of type S_0, there is W_M^Ω of type W_0 such that $S_{M_p}^{N_p} = W_M^\Omega$ and vice versa.*

Proof. For any given spaces $S_{M_p}^{N_p}$ of type S_0, we claim that $S_{M_p}^{N_p} = W_M^{N^*}$ where M is the associated function of M_p and N^* is the Young conjugate of the associated function N of N_p. Let $\varphi \in S_{M_p}^{N_p}$. Then for every $\alpha, \beta \in \mathbb{N}_0^n$ we obtain

$$\left|\xi^\alpha \partial^\beta \varphi(\xi)\right| \le CA^{|\alpha|}B^{|\beta|}M_{|\alpha|}N_{|\beta|} \tag{2.4}$$

for some $A, B > 0$. Since N_p satisfies (M.1)* or $p!/N_p$ satisfies (M.1)′, it is easy to see that $N_p \prec p!$. Hence the function $\varphi(\xi)$ can be continued analytically into the complex domain as an entire analytic function. Applying the Taylor expansion and the inequality (2.4) we have

$$\begin{aligned} |\xi^\alpha \varphi(\xi + i\eta)| &\le \sum_{\gamma \in \mathbb{N}_0^n} \frac{|\xi^\alpha \partial^\gamma \varphi(\xi)|}{\gamma!} |\eta|^{|\gamma|} \\ &\le C \sum_{\gamma \in \mathbb{N}_0^n} A^{|\alpha|} M_{|\alpha|} N_{|\gamma|} |B\eta|^{|\gamma|}/\gamma! \\ &\le 2^n C A^{|\alpha|} M_{|\alpha|} \exp N^\#(2B|\eta|). \end{aligned} \tag{2.5}$$

Dividing $|\xi|^{|\alpha|}$ in both sides of the inequality (2.5) and taking infimum for $|\alpha|$ in the right hand side of (2.5), we have

$$|\varphi(\xi + i\eta)| \le 2^n C \exp\left[-M(|\xi|/A) + N^\#(2B|\eta|)\right].$$

Note that we may use $|\xi|^{|\alpha|}\,\partial^\beta\varphi(\xi)|$ instead of $|\xi^\alpha \partial^\beta \varphi(\xi)|$ in (2.4). Also Lemma 2.3 implies $N^\#(2B|\eta|) \le N^*(B'|\eta|)$ for some $B' > 0$. Thus, we have

$$|\varphi(\xi + i\eta)| \le C_1 \exp\left[-M(|\xi|/A) + N^*(B'|\eta|)\right]. \tag{2.6}$$

It follows that $S_{M_p}^{N_p} \subset W_M^{N^*}$ where M is the associated function of M_p, $N^\#$ is the associated function of $p!/N_p$, and N^* is the Young conjugate of the associated function N of N_p.

By Lemma 2.4, $W_M^{N^*}$ is a space of type W and also the fact $M_p N_p \supset p!$ implies $M \subset N^\# \cong N^*$. Hence $W_M^{N^*}$ is a space of type W_0. Conversely, let $\varphi \in W_M^{N^*}$. Then the inequality (2.6) is satisfied. Hence by the Cauchy integral formula together with the inequality (2.6) we have

$$\begin{aligned}
|\partial^\beta \varphi(x)| &= \left| \frac{\beta!}{(2\pi i)^n} \int_{|\zeta_j - x_j| = R} \frac{\varphi(\zeta)\, d\zeta_1 \ldots d\zeta_n}{(\zeta_1 - x_1)^{\beta_1+1} \cdots (\zeta_n - x_n)^{\beta_n+1}} \right| \\
&\le C \frac{\beta!}{R^{|\beta|}} \sup_{|\zeta_j - x_j| \le R} \exp[-M(a|\xi|) + N^*(b|\eta|)] \\
&\le C \frac{\beta!}{R^{|\beta|}} \exp N^*(b'R) \sup_{|\zeta_j - x_j| \le R} \exp[-M(a|\xi|)] \\
&= C \frac{\beta!}{R^{|\beta|}} \exp N^*(b'R) \exp\left[-M\left(a|\xi^0|\right)\right], \qquad (2.7)
\end{aligned}$$

where the quantity $M(a|\xi|)$ attains its minimum at $\xi = \xi^0$. In fact, we can write $\xi^0 = x + \theta R$ where $\theta = (\theta_1, \ldots, \theta_n)$, $\theta_i = 0$ or ± 1, $i = 1, 2, \ldots, n$. By the convexity of M and the relation $M \subset N^*$, we have

$$\begin{aligned}
\exp\left[-M\left(a|\xi^0|\right)\right] &= \exp[-M(a|x + \theta R|)] \\
&\le \exp[-M(a(|x| - |\theta R|))] \\
&\le \exp\left[-M\left(\frac{a}{2}|x|\right)\right] \exp M(a_1 R) \\
&\le \exp\left[-M\left(\frac{a}{2}|x|\right)\right] \exp N^*(a_2 R).
\end{aligned}$$

Hence the inequality (2.7) is reduced to

$$\begin{aligned}
|\partial^\beta \varphi(x)| &\le C\beta! \frac{\exp[2N^*(a_2 + b')R]}{R^{|\beta|}} \exp\left[-M\left(\frac{a}{2}|x|\right)\right] \\
&\le C\beta! \frac{\exp N^*[2(a_2 + b')R]}{R^{|\beta|}} \exp\left[-M\left(\frac{a}{2}|x|\right)\right]. \qquad (2.8)
\end{aligned}$$

Multiplying $|x^\alpha|$ in both sides of (2.8), taking the supremum for $|x|$, and taking the infimum for R in the right hand side of (2.8) we have

$$|x^\alpha \partial^\beta \varphi(x)| \le C_1 \beta! \inf_R \frac{\exp N^*(cR)}{R^{|\beta|}} \sup_x \frac{|x|^{|\alpha|}}{\exp M((a/2)|x|)}$$

$$\leq C_1(2/a)^{|\alpha|}M_{|\alpha|}\,\beta!\left(\sup_R \frac{R^{|\beta|}}{\exp N^*(cR)}\right)^{-1}$$

$$\leq C_1(c/a)^{|\alpha|}c^{|\beta|}M_{|\alpha|}\,\beta!\left(\frac{(|\beta|/e)^{|\beta|}}{N_{|\beta|}}\right)^{-1}$$

$$\leq C_1 A^{|\alpha|}B^{|\beta|}M_{|\alpha|}N_{|\beta|},$$

where $A = 2/a, B = ce$.

It follows that $W_M^{N^*} \subset S_{M_p}^{N_p}$. Now, for a given space W_M^{Ω} of type W_0 let M_p and N_p be the defining sequences of M and Ω^*. Then M_p and N_p satisfy (M.1)′ and (M.1)* by Lemma 2.4. Furthermore, the relation $M \subset \Omega$ implies $M_p \supset p!/N_p$ or $M_p N_p \supset p!$. Thus $S_{M_p}^{N_p}$ is a space of type S_0. But by the first part of the proof, the space $S_{M_p}^{N_p}$ is equal to $W_{M_1}^{N_1^*}$ where M_1 is the associated function of M_p and N_1^* is the Young conjugate of the associated function N_1 of N_p. Also $W_{M_1}^{N_1^*}$ is equal to W_M^{Ω} since M_1 and M are equivalent and N_1 and Ω^* are equivalent and so are N_1^* and $(\Omega^*)^* = \Omega$. Therefore we have $W_M^{\Omega} = S_{M_p}^{N_p}$ which completes the proof. ∎

Using the similar method as in Theorem 2.8 and Lemma 2.5 we obtain the following theorem.

THEOREM 2.9. *Let W_M and W^{Ω} be spaces of type W. Then there exist spaces S_{M_p} and S^{N_p} of generalized type S such that $W_M = S_{M_p}$ and $W^{\Omega} = S^{N_p}$. In this case, the sequences M_p and N_p satisfy the conditions* (M.1)′ *and* (M.1)*. *Conversely, if M_p and N_p satisfy the conditions* (M.1)′ *and* (M.1)*, *then the spaces S_{M_p} and S^{N_p} are equal to some spaces W_M and W^{Ω} of type W, respectively.*

3. EQUALITY FOR THE SPACES OF GENERALIZED TYPE S AND TYPE W

Applying the results of the above section we prove, in this section, the equality (W) under the non-triviality condition $M(x) \subset \Omega(y)$. For this equality we first prove the equality (S) under the conditions which are satisfied by the defining sequences M_p and N_p of $M(x)$ and $\Omega^*(y)$, respectively, where $M(x) \subset \Omega(y)$.

First we state Pathak's result on the equality (S).

THEOREM 3.1 [P]. *Suppose that there exists a positive constant C such that*

$$N_{p+q} \geq C\binom{p+q}{p}N_pN_q, \qquad p, q = 0, 1, \ldots \tag{P}$$

and that (M.2) *holds for* N_p *and* (M.1) *and* (M.2) *hold for* M_p. *Then the equality* (S) *holds*.

But, we cannot apply this result to prove the equality (W) as the defining sequence N_p of $\Omega^*(y)$ satisfies (M.1)* (see Lemma 2.5) and (M.1)* implies the following reverse inequality

$$N_{p+q} \leq C\binom{p+q}{p} N_p N_q, \qquad p, q = 0, 1, \dots \tag{P*}$$

for some constant C.

For example, the condition (P) is not satisfied by $N_p = p!^s$, $0 < s < 1$, which is the defining sequence of some $\Omega^*(y)$ where $\Omega(y)$ satisfies the condition (K). So we replace (P) by a natural condition

$$M_p N_p \supset p!$$

which is always satisfied if $M(x) \subset \Omega(y)$, and we modify the proof in [P] in order to make use of the condition $M_p N_p \supset p!$ instead of (P).

Let M_p, $p = 0, 1, 2, \dots,$ be a sequence of positive numbers. We impose one of the following conditions on M_p:

(M.0) (Nontriviality) $M_p \supset p!$;

(M.0$'$ (Triviality) $M_p \subset p!^s$, $0 < s < 1$.

We first prove the equality $S_{M_p} \cap S^{N_p} = S_{M_p}^{N_p}$ for the case $M_p N_p \supset p!$, which is a generalization of the equality $S_r \cap S^s = S_r^s$ for the case $r + s \geq 1$.

Making use of integration by parts, the Leibniz formula, and the Schwarz inequality we can obtain the following:

LEMMA 3.2. *If* M_p *and* N_p *satisfy the condition* (M.2)$'$, *then the supremum norm* $\|\cdot\|_\infty$ *and the* L^2*-norm* $\|\cdot\|_2$ *are equivalent for the spaces of type* S.

THEOREM 3.3. *If* M_p *and* N_p *satisfy the conditions* (M.1) *and* (M.2), *and if* $M_p N_p$ *satisfies* (M.0), *then the equality*

$$S_{M_p} \cap S^{N_p} = S_{M_p}^{N_p}$$

holds.

Proof. Using integration by parts, the Leibniz formula, and the Schwarz inequality as in [Ka] we obtain

$$\begin{aligned}\|x^{\alpha}\,\partial^{\beta}\varphi(x)\|_2^2 &= \int_{\mathbb{R}^n}\big[x^{2\alpha}\,\partial^{\beta}\varphi(x)\big]\,\partial^{\beta}\varphi(x)\,dx\\ &\le \sum_{\substack{k\le 2\alpha\\ k\le\beta}}\binom{\beta}{k}\binom{2\alpha}{k}k!\|\,\partial^{2\beta-k}\varphi(x)\|_2\|x^{2\alpha-k}\varphi(x)\|_2\\ &\le C^2\sum_k\binom{\beta}{k}\binom{2\alpha}{k}k!\,A^{2(|\alpha|+|\beta|-|k|)}M_{|2\alpha|-|k|}N_{|2\beta|-|k|}\\ &\le C^2A^{2(|\alpha|+|\beta|)}M_{|2\alpha|}N_{|2\beta|}\sum_k\binom{\beta}{k}\binom{2\alpha}{k}k!\big(M_{|k|}N_{|k|}\big)^{-1}\\ &\le C^2(2AH)^{2(|\alpha|+|\beta|)}M_{|\alpha|}^2N_{|\beta|}^2. \qquad (3.1)\end{aligned}$$

This implies that $\varphi(x)$ belongs to $S_{M_p}^{N_p}$ in view of Lemma 3.2.

The reverse inclusion is obvious, which completes the proof.

Remark 3.4. Let $S_{M_p}^{N_p}$ be a space of type S_0. Then M_p and N_p satisfy (M.1) and (M.1)*. Since (M.1)* implies (P)* and (P)* implies (M.2), the equality (S) holds by Theorem 3.3.

THEOREM 3.5. *Let W_M^{Ω} be a space of type W_0. Then the equality*

$$W_M\cap W^{\Omega}=W_M^{\Omega}$$

holds.

Proof. For given spaces W_M, W^{Ω}, and W_M^{Ω} of type W_0 there exist S_{M_p}, S^{N_p}, and $S_{M_p}^{N_p}$ of type S_0 such that

$$W_M=S_{M_p},\qquad W^{\Omega}=S^{N_p},\qquad\text{and}\qquad W_M^{\Omega}=S_{M_p}^{N_p}$$

by Theorem 2.8 and Theorem 2.9.

Since $S_{M_p}^{N_p}$ is a space of type S_0, the equality $S_{M_p}\cap S^{N_p}=S_{M_p}^{N_p}$ holds by Remark 3.4. Consequently, we have the equality $W_M\cap W^{\Omega}=W_M^{\Omega}$.

We now prove the triviality of the spaces $S_{M_p}^{N_p}$ and $S_{M_p}\cap S^{N_p}$ under the condition $M_pN_p\subset p!^s$, $0<s<1$, which generalizes the equality $S_r\cap S^s=S_r^s$ for the other case $r+s<1$, which will complete the generalization of the quality (S0).

THEOREM 3.6. *If M_p and N_p satisfy the condition* (M.2)′ *and $M_p N_p$ satisfies* (M.0)′, *then both the spaces $S_{M_p}^{N_p}$ and $S_{M_p} \cap S^{N_p}$ are trivial.*

Proof. If $\varphi(x)$ belongs to $S_{M_p}^{N_p}$ or $S_{M_p} \cap S^{N_p}$, then by the condition (M.0)′, $\varphi(x)$ is continued analytically into the complex plane as an entire analytic function. For $\varphi(x) \in S_{M_p}^{N_p}$ we have $\|x^\alpha \partial^\alpha \varphi(x)\|_\infty \le CA^{|\alpha|}|\alpha|!^s$, $0 < s < 1$. Also, applying integration by parts, the Leibniz formula, and the Schwarz inequality we have for $\varphi \in S_{M_p} \cap S^{N_p}$

$$\begin{aligned} &\|x^\alpha \partial^\alpha \varphi(x)\|_\infty \\ &\quad \le CA^{|\alpha|}\Bigg(\sum_{\substack{k \le \alpha \\ k \le (1,\dots,1)}} \|x^{\alpha-k+1} \partial^{\alpha-k+1}\varphi(x)\|_2 \\ &\qquad + \sum_{\substack{k \le \alpha \\ k \le (1,\dots,1)}} \|x^{\alpha-k}\, \partial^{\alpha-k+1}\varphi(x)\|_2 \Bigg) \end{aligned}$$

for some constants C and A.

Replacing α, β by $\alpha - k + 1$ in (3.1), we have

$$\begin{aligned} \|x^{\alpha-k+1}\, \partial^{\alpha-k+1}\varphi(x)\|_2^2 &\le C^2(2A)^{4|\alpha|} \sum_{j \le \alpha+1} j! M_{|2\alpha+2-j|} N_{|2\alpha+2-j|} \\ &\le C^2(2A)^{4|\alpha|} \sum_{j \le \alpha+1} |j|!^{1-s}(|j|!|2\alpha + 2 - j|!)^s \\ &\le C_1^2 A_1^{2|\alpha|}|\alpha|!^{1+s}. \end{aligned}$$

Similarly, we have $\|x^{\alpha-k}\, \partial^{\alpha-k+1}\varphi(x)\|_2^2 \le C_2^2 A_2^{2|\alpha|}|\alpha|!^{1+s}$. Therefore, in view of Lemma 3.2 we obtain that

$$\|x^\alpha \partial^\alpha \varphi(x)\|_\infty \le C_3 A_3^{|\alpha|}|\alpha|!^{(1+s)/2} \le C_3 A_4^{|\alpha|}\alpha!^{(1+s)}/2$$

for $\varphi(x) \in S_{M_p} \cap S^{N_p}$.

Now it is easy to show that if an entire analytic function $\varphi(\zeta)$ on $\mathbb{C}^n$ satisfies the inequality

$$\|\xi^\alpha \partial^\alpha \varphi(\xi)\|_\infty \le CA^{|\alpha|}\alpha!^s, \qquad 0 < s < 1, \tag{3.2}$$

then $\varphi(\zeta)$ degenerates to a constant function.

In fact, by the Taylor expansion we obtain

$$\partial^\beta \varphi(0) = \sum_\alpha \partial^{\alpha+\beta}(\xi)(-\xi)^\alpha/\alpha!.$$

By letting $|\xi| \to \infty$, (3.2) implies that $\partial^{\beta}\varphi(0) = 0$ for $\beta \neq 0$, hence φ is constant. Therefore the spaces $S_{M_p}^{N_p}$ and $S_{M_p} \cap S^{N_p}$ are trivial.

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