Equivalence of the Gelfand–Shilov Spaces

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We prove that there is a one to one correspondence between the Gelfand–Shilov space $W^\omega_\alpha$ of type $W$ and the space $S^\omega_\alpha$ of generalized type $S$. As an application we prove the equality $W^\omega_\alpha \cap W^\omega_\beta = W^\omega_\gamma$, which is a generalization of the equality $S^\omega_\alpha \cap S^\omega_\beta = S^\omega_\gamma$ found by I. M. Gelfand and G. E. Shilov ("Generalized Functions, II, III," Academic Press, New York/London, 1967).

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INTRODUCTION

The purpose of this paper is to investigate the relations between the Gelfand–Shilov spaces of generalized type $S$ and of type $W$ which were introduced by I. M. Gelfand and G. E. Shilov in [GS]. They used these spaces to investigate the uniqueness of the solutions of the Cauchy problems of partial differential equations. Although the Gelfand–Shilov spaces of generalized type $S$ are defined by means of the positive sequences $M(\omega)$ and $N(\eta)$, and the spaces of type $W$ are defined by means of the weight functions $M(x)$ and $\Omega(y)$, we show that the class of the spaces of type $W$ is exactly the same as a class of the spaces of generalized type $S$. 

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In Section 1 we state definitions of the Gelfand–Shilov spaces of generalized type $S$ and type $W$, some relations between sequences and their associated functions, and basic concepts of Young conjugates and their properties. In Section 2 we prove that the spaces $W_{M}$, $W^\Omega$, and $W^\Omega_{M}$ of type $W$ are equal to some spaces $S_{M_{\tau}}$, $S^N_{r}$, and $S^N_{M_{\tau}}$ of generalized type $S$, respectively, under the natural condition $M \subset \Omega$ and vice versa. In Section 3 as an application of the above result we show the equality
\[ W_{M} \cap W^\Omega = W^\Omega_{M} \] (W)
under the natural condition $M \subset \Omega$ (see Section 1 for definition). For this we first show the equality
\[ S_{M_{\tau}} \cap S^N_{r} = S^N_{M_{\tau}}. \] (S)
The above two equalities are generalizations of the equality
\[ S_{r} \cap S^{s} = S^{s}, \] (S0)
which was suggested by Gelfand and Shilov in [GS] and proved affirmatively by Kashpirovsky in [Ka] in 1979.

An equality of type (S) was also proved in Pathak [P] under the condition (P). But, this result cannot be applied to prove the equality (W) as the defining sequence $N_{p}$ of $\Omega^{*}(y)$ satisfies (M.1)$^{*}$ (see Lemma 2.5) and (M.1)$^{*}$ implies the reverse inequality (P)$^{*}$ of (P). See Section 3 for definitions of the conditions (P) and (P)$^{*}$ and for an example. So we replace (P) by a natural condition
\[ M_{p}N_{p} \triangleright p! \]
which is always satisfied if $M(x) \subset \Omega(y)$, and we modify the proof in [P] in order to make use of the condition $M_{p}N_{p} \triangleright p!$ instead of (P). Finally we also prove the triviality of the spaces $S_{M_{\tau}} \cap S^N_{r}$ and $S^N_{M_{\tau}}$ under the condition $M_{p}N_{p} \subset p^r$, $0 < s < 1$, which will complete the generalization of the equality (S0).

1. Preliminaries

Let $M_{p}$, $p = 0, 1, 2, \ldots$, be a sequence of positive numbers. We impose the conditions denoted (M.1), (M.2), and (M.2)$^{*}$ which denote logarithmic convexity, stability under ultradifferential operators, and stability under differential operators. We assume that $M_{0} = 1$ for simplicity and refer to [K, p. 26] for details. We also use the multi-index notations $|\alpha| = \alpha_{1} + \cdots + \alpha_{n}$, $\alpha! = \alpha_{1}! \cdots \alpha_{n}!$, and $\partial^{\alpha} = (\partial/\partial x_{1})^{\alpha_{1}} \cdots (\partial/\partial x_{n})^{\alpha_{n}}$ for $\alpha = (\alpha_{1}, \ldots, \alpha_{n}) \in \mathbb{N}^{n}_{0}.$
DEFINITION 1.1. For each sequence $M_\rho$ of positive numbers we define the associated function $M(\rho)$ on $[0,\infty)$ by

$$M(\rho) = \sup_{\rho \geq 0} \frac{\rho^p}{\exp M(\rho)}.$$  \hfill (1.1)

DEFINITION 1.2. Let $M_\rho$ and $N_\rho$ be sequences of positive numbers satisfying (1.1). Then we write $M_\rho \subset N_\rho$ ($M_\rho < N_\rho$, respectively) if there are constants $L$ and $C$ for any $L > 0$ there is a constant $C > 0$, respectively) such that $M_\rho \leq CL^p N_\rho$, $p = 0, 1, 2, \ldots$. $M_\rho$ and $N_\rho$ are equivalent if $M_\rho \subset N_\rho$ and $M_\rho \supset N_\rho$ hold.

For example, $M_\rho = \rho!^p$ and $N_\rho = \rho^{p^p}$ are equivalent if $s > 0$, by Stirling's formula.

DEFINITION 1.3. Let $M(x)$ and $N(x)$ be real functions. Then we write $M(x) \subset N(x)$ ($M(x) < N(x)$, respectively) if there are constants $L$ and $C$ (for every $L > 0$ there is a constant $C > 0$, respectively) such that $M(x) \leq N(Lx) + C$. $M(x)$ and $N(x)$ are equivalent if $M(x) \subset N(x)$ and $M(x) \supset N(x)$.

DEFINITION 1.4. If $M(\rho)$ is an increasing convex function in $\log \rho$ and increases more rapidly than $\log \rho^p$ for any $p$ as $\rho$ tends to infinity, we define its defining sequence by

$$M_\rho = \sup_{\rho > 0} \frac{\rho^p}{\exp M(\rho)}, \quad p = 0, 1, 2, \ldots.$$ 

DEFINITION 1.5. Let $M : [0, \infty) \to [0, \infty)$ be a convex and increasing function with $M(0) = 0$ and $\lim_{x \to \infty} x/M(x) = 0$. Then we define its Young conjugate $M^*$ by

$$M^*(y) = \sup_x (xy - M(x)).$$ 


Let $M(x)$ and $\Omega(y)$ be differentiable functions on $[0, \infty)$ satisfying the condition

(K) \hspace{1cm} M(0) = \Omega(0) = M'(0) = \Omega'(0) = 0 \hspace{1cm} \text{and their derivatives continuous, increasing, and tending to infinity.} \hspace{1cm} \text{We refer to [GS, Vol. II, Appendix 2, Chap. IV] for the Gelfand–Shilov spaces of type $W$.}
**Definition 1.6.** A space $\tilde{\Phi}$ of Fourier transforms $\hat{\varphi}(\xi) = \int_{\mathbb{R}^n} \varphi(x)e^{-i\xi x} \, dx$ of the functions $\varphi \in \Phi$ is called the *Fourier-dual* of $\Phi$.

2. EQUIVALENCE OF THE GELFAND–SHILOV SPACES

In this section we prove the equivalence of the Gelfand–Shilov spaces of type $W$ and the spaces of generalized type $S$.

Let $M_p, p = 0, 1, 2, \ldots$, be a sequence of positive numbers. We impose the following conditions on $M_p$:

(M.1) (Strong Logarithmic Convexity) $m_p = M_p/M_{p-1}$ is increasing and tending to infinity as $p \to \infty$.

(M.1)* (Duality) $p!/M_p$ satisfies (M.1).

**Proposition 2.1** [K, p. 49]. If $M_p$ satisfies (M.1), then $M_p$ is the defining sequence of the associated function of itself, that is,

$$M_p = \sup_{p \geq 0} \frac{\rho^p}{\exp M(\rho)},$$

where $M(\rho)$ is the associated function of $M_p$.

We can easily obtain the following Lemma 2.2 from the Lemma 2.3 in [CCK, p. 370].

**Lemma 2.2.** Let $M : [0, \infty) \to [0, \infty)$ be the function satisfying the condition (K). Then $M(x)$ is equivalent to the associated function of the defining sequence of itself, i.e.,

$$M(x) \equiv \sup_p x^p \frac{\log x}{M_p},$$

where $M_p$ is the defining sequence of $M(x)$.

**Lemma 2.3** [CCK]. Let $M(\rho)$ be a function satisfying the condition (K). Then the defining sequence $M^*_p$ of the Young conjugate $M^*(\rho)$ of $M(\rho)$ is equivalent to $p!/M_p$, where $M_p$ is the defining sequence of $M(\rho)$. In fact, $M^*_p = (p/e)^p/M_p$.

Conversely, if $M_p$ satisfies (M.1) and (M.1)*, then the associated function $M^*(\rho)$ of $p!/M_p$ and the Young conjugate $M^*(\rho)$ of the associated function $M(\rho)$ of $M_p$ are equivalent.

**Lemma 2.4.** Let $M_p$ be a sequence of positive numbers satisfying the conditions (M.1) and (M.1)*. Then the associated function $M(\rho)$ of $M_p$ is equivalent to a function $M_0(\rho)$ which satisfies the condition (K).
Conversely, let $M(\rho)$ be a function satisfying the condition (K). Then the defining sequence $M_p$ of $M(\rho)$ satisfies the conditions (M.1) and (M.1)* up to equivalence.

Proof. We may assume that $m_p = M_p/M_{p-1}$ and $p/m_p$ are strictly increasing. Let $w(t)$ be the line segments connecting $(m_p, p/m_p)$ and $(m_{p+1}, (p + 1)/m_{p+1})$, $p = 1, 2, \ldots$, where $(m_p, p/m_p) = (0, 0)$ for $p = 0$ and let $M_\rho(\rho) = \int_0^\rho w(t) \, dt$. Then the function $M_\rho(\rho)$ satisfies the condition (K). We claim that the associated function $M(\rho)$ of $M$ is equivalent to $M_\rho(\rho)$. Denote by $m(\lambda)$ the number of $m_p \leq \lambda < m_{p+1}$

\begin{align}
p/m_p \leq w(\lambda) < (p + 1)/m_{p+1} \\
p/m_{p+1} \leq m(\lambda)/\lambda < p/m_p.
\end{align}

Combining (2.1) and (2.2) we have

\begin{equation}
\frac{1}{2}w(\lambda) \leq m(\lambda)/\lambda \leq w(\lambda)
\end{equation}

for all $\lambda > 0$.

Integrating (2.3) and applying the formula $M(\rho) = \int_0^\rho m(\lambda)/\lambda \, d\lambda$ in [K, p. 50] and the convexity of $M_\rho$, we conclude that $M_\rho(\lambda) \leq M(\lambda) \leq M_\rho(\lambda)$ for all $\lambda \geq 0$.

To prove the converse, let $M(\rho)$ be a function satisfying the condition (K). Then the defining sequence $M_p$ of $M(\rho)$ satisfies (M.1). Since the Young conjugate $M^*(\rho)$ of $M(\rho)$ also satisfies the condition (K) the defining sequence $(p/e)^p/M_p$ of $M^*(\rho)$ also satisfies (M.1). We complete the proof by Stirling’s formula.

**Lemma 2.5** [GS, p. 245]. Let $M_p$ and $N_p$ satisfy (M.1) and (M.1)*. Then

\[ S_{M_p} = S_{M_p} \quad \text{and} \quad S_{N_p} = S_{N_p}. \]

**Definition 2.6.** We call $S_{M_p}$ a space of type $S_0$ if the sequences $M_p$ and $N_p$ satisfying the condition (M.1), (M.1)*, and $M_p, N_p \succ \rho$. Also, we call $W_{M_p}$ a space of type $W_0$ if the functions $M$ and $\Omega$ satisfy the condition (K) and the relation $M \subset \Omega$.

**Remark 2.7.** (i) Let $M_p = p^r, N_p = p^s, 0 < r, s < 1, r + s \geq 1$. Then $S_{M_p}$ is a space of type $S_0$.

(ii) Let $M(x) = e^x - x - 1, \Omega(y) = ye^y - y$. Then $M(x)$ and $\Omega(y)$ satisfy (K) and $M(x) \subset \Omega(y)$ and, $W_{M_p}^\Omega$ is a space of type $W_0$. In fact, $M(x) \subset \Omega(y)$ also holds and $W_{M_p}^\Omega$ is a space of type $W_0$.\]
Let $M_p$ and $N_p$ be the defining sequences of the above functions $M(x)$ and $\Omega^*(y)$, respectively. Then $S_{M_p}^{N_p}$ is a space of type $S_0$. Also, let $M_p$ and $N_p$ be the defining sequences of the above functions $M(x)$ and $\Omega(y)$. Then $S_{M_p}^{N_p/N_p}$ is a space of type $S_0$ by Lemma 2.3 and Lemma 2.4.

We now prove the main theorem in this section.

**Theorem 2.8.** There is a one to one correspondence between the spaces of type $S$ and type $W$. In other words, for any given space $S_{M_p}^{N_p}$ of type $S$, there is $W_{M_p}^{N_p}$ of type $W$ such that $S_{M_p}^{N_p} = W_{M_p}^{N_p}$ and vice versa.

**Proof.** For any given spaces $S_{M_p}^{N_p}$ of type $S$, we claim that $S_{M_p}^{N_p}$ is $W$ of type $W$ such that $S_{M_p}^{N_p}$ is $W$ $V$ of type $W$. Let $\varphi \in S_{M_p}^{N_p}$. Then for every $\alpha, \beta \in \mathbb{N}_0^n$ we obtain

$$|\xi^\alpha \partial^\beta \varphi(\xi)| \leq CA|\alpha|B|\varphi|A|N_p|\beta|$$

for some $A, B > 0$. Since $N_p$ satisfies (M.1)$^*$ or $p!/N_p$ satisfies (M.1), it is easy to see that $N_p < p!$. Hence the function $\varphi(\xi)$ can be continued analytically into the complex domain as an entire analytic function. Applying the Taylor expansion and the inequality (2.4) we have

$$|\xi^\alpha \varphi(\xi + i\eta)| \leq \sum_{\gamma \in \mathbb{N}_0^n} |\xi^\alpha \partial^\gamma \varphi(\xi)| |\eta|^{\gamma}$$

$$\leq C \sum_{\gamma \in \mathbb{N}_0^n} A|\alpha|M_p|N_p|B|\eta|^{\gamma}/\gamma!$$

$$\leq 2^n CA|\alpha|M_p \exp N_p^*(2B|\eta|).$$

(2.5)

Dividing $|\xi|^{\alpha_1}$ in both sides of the inequality (2.5) and taking infimum for $|\alpha|$ in the right hand side of (2.5), we have

$$|\varphi(\xi + i\eta)| \leq 2^n C \exp[-M(|\xi|/A) + N_p^*(2B|\eta|)].$$

Note that we may use $|\xi|^{\alpha_1} \partial^\beta \varphi(\xi)$ instead of $|\xi^\alpha \partial^\beta \varphi(\xi)$ in (2.4). Also Lemma 2.3 implies $N_p^*(2B|\eta|) \leq N_p^*(B'|\eta|)$ for some $B' > 0$. Thus, we have

$$|\varphi(\xi + i\eta)| \leq C \exp[-M(|\xi|/A) + N_p^*(B'|\eta|)].$$

(2.6)

It follows that $S_{M_p}^{N_p} \subset W_{M_p}^{N_p}$, where $M$ is the associated function of $M_p$, $N^*$ is the associated function of $p!/N_p$, and $N^*$ is the Young conjugate of the associated function $N$ of $N_p$. 

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By Lemma 2.4, \( W_M^{N^*} \) is a space of type \( W \) and also the fact \( M_p N_p \supset p! \)
implies \( M \subset N^* \equiv N^* \). Hence \( W_M^{N^*} \) is a space of type \( W_0 \). Conversely, let
\( \varphi \in W_M^{N^*} \). Then the inequality (2.6) is satisfied. Hence by the Cauchy
integral formula together with the inequality (2.6) we have

\[
|\partial^\beta \varphi(x)| = \left| \frac{\beta!}{(2\pi i)^n} \int_{|\xi - x|^\beta = R} \frac{\varphi(\xi) d\xi_1 \ldots d\xi_n}{(\xi_1 - x_1)^{\beta_1+1} \ldots (\xi_n - x_n)^{\beta_n+1}} \right|
\leq C \frac{\beta!}{R^{\beta}|} \sup_{|\xi - x| \leq R} \exp \left[ -M(a|\xi|) + N^*(b|\eta|) \right]
\leq C \frac{\beta!}{R^{\beta}|} \exp N^*(b'R) \sup_{|\xi - x| \leq R} \exp \left[ -M(a|\xi|) \right]
= C \frac{\beta!}{R^{\beta}|} \exp N^*(b'R) \exp \left[ -M(a|\xi^0|) \right],
\tag{2.7}
\]

where the quantity \( M(a|\xi|) \) attains its minimum at \( \xi = \xi^0 \). In fact, we can
write \( \xi^0 = x + \theta R \) where \( \theta = (\theta_1, \ldots, \theta_n), \theta_i = 0 \) or \( \pm 1, i = 1, 2, \ldots, n \).
By the convexity of \( M \) and the relation \( M \subset N^* \), we have

\[
\exp \left[ -M(a|\xi^0|) \right] = \exp \left[ -M(a|x + \theta R|) \right]
\leq \exp \left[ -M(a(|x| - |\theta R|)) \right]
\leq \exp \left[ -M \left( \frac{a}{2} |x| \right) \right] \exp M(a_1 R)
\leq \exp \left[ -M \left( \frac{a}{2} |x| \right) \right] \exp N^*(a_2 R).
\]

Hence the inequality (2.7) is reduced to

\[
|\partial^\beta \varphi(x)| \leq C \beta! \frac{\exp \left[ 2N^*(a_2 + b') R \right]}{R^{\beta|}} \exp \left[ -M \left( \frac{a}{2} |x| \right) \right]
\leq C \beta! \frac{\exp N^* \left[ 2(a_2 + b') R \right]}{R^{\beta|}} \exp \left[ -M \left( \frac{a}{2} |x| \right) \right]. \tag{2.8}
\]

Multiplying \( |x|^{a|} \) in both sides of (2.8), taking the supremum for \( |x| \), and
taking the infimum for \( R \) in the right hand side of (2.8) we have

\[
|x|^{a|} |\partial^\beta \varphi(x)| \leq C_1 \beta! \inf_R \frac{\exp N^* \left( c R \right)}{R^{\beta|}} \sup_x \frac{|x|^{a|}}{\exp M \left( (a/2) |x| \right)}
\]
\[
\begin{align*}
\leq C_1 (2/a)^{|\alpha|} M_{|\alpha|} \beta \left( \sup_R \frac{R^{|\beta|}}{\exp N^* (cR)} \right)^{-1} \\
\leq C_1 (c/a)^{|\alpha|} |\beta| M_{|\alpha|} \beta \left( \frac{(|\beta|/c)^{|\beta|}}{N_{|\beta|}} \right)^{-1} \\
\leq C_1 A^{|\alpha|} B^{|\beta|} M_{|\alpha|} N_{|\beta|},
\end{align*}
\]
where \( A = 2/a, B = ce \).

It follows that \( W_M^{N^*} \subset S_V^{N_p} \). Now, for a given space \( W_M^{\Omega} \) of type \( W_\Omega \) let \( M_p \) and \( N_p \) be the defining sequences of \( M \) and \( \Omega^* \). Then \( M_p \) and \( N_p \) satisfy \((M.\Omega)^* \) by Lemma 2.4. Furthermore, the relation \( M \subset \Omega \) implies \( M_p \supset p! N_p \) or \( M_p N_p \supset p! \). Thus \( S_p^{N_p} \) is a space of type \( S_\Omega \). But by the first part of the proof, the space \( S_p^{N_p} \) is equal to \( W_M^{N_p} \) where \( M_p \) is the associated function of \( M_p \) and \( N_p^* \) is the Young conjugate of the associated function \( N_p \) of \( N_p \). Also \( W_M^{N^*} \) is equal to \( W_M^{\Omega} \) since \( M_1 \) and \( M \) are equivalent and \( N_1 \) and \( \Omega^* \) are equivalent and so are \( N_p^* \) and \( (\Omega^*)^* = \Omega \). Therefore we have \( W_M^{\Omega} = S_p^{N_p} \) which completes the proof.

Using the similar method as in Theorem 2.8 and Lemma 2.5 we obtain the following theorem.

**Theorem 2.9.** Let \( W_M \) and \( W_\Omega \) be spaces of type \( W \). Then there exist spaces \( S_M \) and \( S_\Omega \) of generalized type \( S \) such that \( W_M = S_M \) and \( W_\Omega = S_\Omega \).

In this case, the sequences \( M_p \) and \( N_p \) satisfy the conditions \((M.\Omega)^* \) and \((M.\Omega)^p \). Conversely, if \( M_p \) and \( N_p \) satisfy the conditions \((M.\Omega)^* \) and \((M.\Omega)^p \), then the spaces \( S_M \) and \( S_\Omega \) are equal to some spaces \( W_M \) and \( W_\Omega \) of type \( W \), respectively.

### 3. Equality for the Spaces of Generalized Type \( S \) and Type \( W \)

Applying the results of the above section we prove, in this section, the equality \((W)\) under the non-triviality condition \( M(x) \subset \Omega(y) \). For this equality we first prove the equality \((S)\) under the conditions which are satisfied by the defining sequences \( M_p \) and \( N_p \) of \( M(x) \) and \( \Omega^*(y) \), respectively, where \( M(x) \subset \Omega(y) \).

First we state Pathak's result on the equality \((S)\).

**Theorem 3.1 (P).** Suppose that there exists a positive constant \( C \) such that

\[
N_{p+q} \geq C \left( \frac{p}{p+q} \right) N_{p} N_{q}, \quad p, q = 0, 1, \ldots
\]
and that (M.2) holds for \( N_p \) and (M.1) and (M.2) hold for \( M_p \). Then the equality (S) holds.

But, we cannot apply this result to prove the equality (W) as the defining sequence \( N_p \) of \( \Omega^s(y) \) satisfies (M.1)* (see Lemma 2.5) and (M.1)* implies the following reverse inequality

\[
N_{p+q} \leq C \left( \frac{p+q}{p} \right) N_p N_q, \quad p, q = 0, 1, \ldots
\]

for some constant \( C \).

For example, the condition (P) is not satisfied by \( N_p = p^s, 0 < s < 1 \), which is the defining sequence of some \( \Omega^s(y) \) where \( \Omega(y) \) satisfies the condition (K). So we replace (P) by a natural condition

\[
M_p N_p \ni p!
\]

which is always satisfied if \( M(x) \subset \Omega(y) \), and we modify the proof in [P] in order to make use of the condition \( M_p N_p \ni p! \) instead of (P).

Let \( M_p, p = 0, 1, 2, \ldots \), be a sequence of positive numbers. We impose one of the following conditions on \( M_p \):

- **(M.0)** (Nontriviality) \( M_p \ni p! \);
- **(M.0')** (Triviality) \( M_p \subset p^s, 0 < s < 1 \).

We first prove the equality \( S_{M_p} \cap S^{N_p} = S^{N_p}_{M_p} \) for the case \( M_p N_p \ni p! \), which is a generalization of the equality \( S_r \cap S^s = S^s_r \) for the case \( r + s \geq 1 \).

Making use of integration by parts, the Leibniz formula, and the Schwarz inequality we can obtain the following:

**Lemma 3.2.** If \( M_p \) and \( N_p \) satisfy the condition (M.2), then the supremum norm \( \| \cdot \|_{\infty} \) and the \( L^2 \)-norm \( \| \cdot \|_2 \) are equivalent for the spaces of type \( S \).

**Theorem 3.3.** If \( M_p \) and \( N_p \) satisfy the conditions (M.1) and (M.2), and if \( M_p N_p \) satisfies (M.0), then the equality

\[
S_{M_p} \cap S^{N_p} = S^{N_p}_{M_p}
\]

holds.
Proof. Using integration by parts, the Leibniz formula, and the Schwarz inequality as in Ka we obtain
\[ \| x^n \partial^\beta \varphi(x) \|_2^2 = \int_{\mathbb{R}^n} \left[ x^{2\alpha} \partial^\beta \varphi(x) \right] \partial^\beta \varphi(x) \, dx \]
\[ \leq \sum_{k \leq 2\alpha} \left( \frac{\beta}{k} \right) \left( \frac{2\alpha}{k} \right)^k \| \partial^{2\beta-k} \varphi(x) \|_2 \| x^{2\alpha-k} \varphi(x) \|_2 \]
\[ \leq C^2 \sum_{k} \left( \frac{\beta}{k} \right) \left( \frac{2\alpha}{k} \right)^k A^{2(\alpha)+|\beta|-|k|} M_{2\alpha-|k|} N_{2\beta-|k|} \]
\[ \leq C^2 A^{2(\alpha)+|\beta|} M_{2\alpha} N_{2\beta} \sum_{k} \left( \frac{\beta}{k} \right) \left( \frac{2\alpha}{k} \right)^k (M_{|k|} N_{|k|})^{-1} \]
\[ \leq C^2 (2AH)^{2(\alpha)+|\beta|} M_{2\alpha}^2 N_{2\beta}^2. \] (3.1)

This implies that \( \varphi(x) \) belongs to \( S_{M_p} \) in view of Lemma 3.2.

The reverse inclusion is obvious, which completes the proof.

Remark 3.4. Let \( S_{M_p}^N \) be a space of type \( S_0 \). Then \( M_p \) and \( N_p \) satisfy (M.1) and (M.1)**. Since (M.1)** implies (P)** and (P)** implies (M.2), the equality (S) holds by Theorem 3.3.

Theorem 3.5. Let \( W_\Omega^M \) be a space of type \( W_0 \). Then the equality
\[ W_\Omega \cap W^\Omega = W_\Omega^M \]
holds.

Proof. For given spaces \( W_\Omega, W^\Omega, \) and \( W_\Omega^M \) of type \( W_0 \) there exist \( S_{M_p}^N \), \( S_{N_p}^N \), and \( S_{M_p}^N \) of type \( S_0 \) such that
\[ W_\Omega = S_{M_p}, \quad W^\Omega = S_{N_p}, \quad \text{and} \quad W_\Omega^M = S_{M_p}^N \]
by Theorem 2.8 and Theorem 2.9.

Since \( S_{M_p}^N \) is a space of type \( S_0 \), the equality \( S_{M_p}^N \cap S_{N_p}^N = S_{M_p}^N \) holds by Remark 3.4. Consequently, we have the equality \( W_\Omega \cap W^\Omega = W_\Omega^M \).

We now prove the triviality of the spaces \( S_{M_p}^N \) and \( S_{N_p}^N \) under the condition \( M_p N_p < p^s \), \( 0 < s < 1 \), which generalizes the equality \( S_p \cap S = S \) or the other case \( r + s < 1 \), which will complete the generalization of the quality (S0).
THEOREM 3.6. If $M_p$ and $N_p$ satisfy the condition (M.2) and $M_p N_p$ satisfies (M.0), then both the spaces $S_{M_p}^N$ and $S_{M_p} \cap S_{N_p}$ are tri-
ial.

Proof. If $\varphi(x)$ belongs to $S_{M_p}^N$ or $S_{M_p} \cap S_{N_p}$, then by the condition (M.0), $\varphi(x)$ is continued analytically into the complex plane as an entire 
analytic function. For $\varphi(x) \in S_{M_p}^N$, we have $\|x^a \partial^a \varphi(x)\| \leq CA^a \|\alpha\|^s$, $0 < s < 1$. Also, applying integration by parts, the Leibniz formula, and the Schwarz inequality we have for $\varphi \in S_{M_p} \cap S_{N_p}$

$$\|x^a \partial^a \varphi(x)\| \leq CA^a \left( \sum_{k \leq a} \|x^{a-k+1} \partial^{a-k+1} \varphi(x)\|_2 \right)$$

for some constants $C$ and $A$.

Replacing $\alpha, \beta$ by $\alpha - k + 1$ in (3.1), we have

$$\|x^{a-k+1} \partial^{a-k+1} \varphi(x)\|_2 \leq C^2 (2A)^{4|a|} \sum_{j \leq a+1} j! |M_{2\alpha+2-j}| N_{2\alpha+2-j}$$

$$\leq C^2 (2A)^{4|a|} \sum_{j \leq a+1} |j|! (2\alpha + 2 - j)!$$

$$\leq C_1 A_1^2 \|\alpha\|^{1+s}.$$  

Similarly, we have $\|x^{a-k} \partial^{a-k+1} \varphi(x)\|_2 \leq C_2 A_2^2 \|\alpha\|^{1+s}$. Therefore, in view of Lemma 3.2 we obtain that

$$\|x^a \partial^a \varphi(x)\| \leq C_3 A_3^{a|a|} \|\alpha\|^{(1+s)/2} \leq C_3 A_4^{a|a|} \|\alpha\|^{(1+s)/2}$$

for $\varphi(x) \in S_{M_p} \cap S_{N_p}$.

Now it is easy to show that if an entire analytic function $\varphi(\xi)$ on $\mathbb{C}^n$ satisfies the inequality

$$\|\xi^a \partial^a \varphi(\xi)\| \leq CA^a \|\alpha\| s$$  

then $\varphi(\xi)$ degenerates to a constant function.

In fact, by the Taylor expansion we obtain

$$\partial^\beta \varphi(0) = \sum_\alpha \partial^{a+\beta} \varphi(0) (-\xi)^a / \alpha!.$$
By letting $|ξ| \to \infty$, (3.2) implies that $\partial^β ϕ(0) = 0$ for $β \neq 0$, hence $ϕ$ is constant. Therefore the spaces $S_{M_p}^{N_p}$ and $S_{M_p}^{N_p} \cap S_{N_p}$ are trivial.

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