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On Some New Integral Inequalities in N Independent Variables

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Integral inequalities of the Gollwitzer type in n independent variables are established which generalize some known results obtained by Gollwitzer, Bondge, Pachpatte, Shih, and Yeh.

1. INTRODUCTION

One reason for much of the successful mathematical development in the theory of differential and integral equations is the availability of various kinds of inequalities. During the past few years, the discovery and the application of the new generalizations of the Gronwall–Bellman inequality in more than one independent variables have attracted the interest of many authors. The aim of this paper is to establish some new n-independent-variable integral inequalities which unify and extend some known results due to Gollwitzer [2], Bondge and Pachpatte [1], Pachpatte [3, 4], and Shih and Yeh [5]. The inequalities obtained here are useful for some problems in the theory of partial differential and integral equations in several variables. Throughout this paper the following notations will be used.

Let $R_+ = [0, \infty)$ and $I = [0, h)$, where $0 < h \leqslant \infty$. A point $(x_1^i, x_2^i, ..., x_n^i)$ in the n-dimensional Euclidean space R^n will be denoted by x^i, and the origin of R^n is denoted by 0. For any two points x, y in R^n, $x < y$ (that is, $x_i < y_i$, $i = 1, 2, ..., n$), we denote

$$\int_{x_1}^{y_1} \cdots \int_{x_n}^{y_n} \cdot\, ds_n\, ds_{n-1} \cdots ds_1 \text{ by } \int_x^y \cdot\, ds; \qquad D_i = \frac{\partial}{\partial x_i}, \quad i = 1, 2, ..., n,$$

and $D = D_1 D_2 \cdots D_n$. The natural partial ordering on R^n is defined by $x \leqslant y$ if and only if $x_i \leqslant y_i$ for $i = 1, 2, ..., n$. Let $C(I^n, R_+)$ be the class of all continuous functions on I^n with range in R_+.

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In what follows, we define the functionals $E_k^{(j)}(s, x; v)$ on $C(\Gamma^n, R_+)$ by

$$E_k^{(j)}(s, x; v) = \int_x^s a_k^{(j)}(s, t^k) \int_{t^k}^s a_{k+1}^{(j)}(s, t^{k+1}) \cdots \int_{t^{j-1}}^s a_j^{(j)}(s, t^j) \times v(t^j)\, dt^j dt^{j-1} \cdots dt^k,$$

$$j = 1, 2, \ldots, m;\ k = 1, 2, \ldots, j;\ 0 \leqslant x \leqslant s,\ s \in \Gamma^n,$$

where $v \in C(\Gamma^n, R_+)$, $a_k^{(j)}(s, x)\colon \Gamma^n \times \Gamma^n \to R_+$ are continuous functions.

2. Main Result

THEOREM 1. *Let the functions u and w be in the class $C(\Gamma^n, R_+)$, and let $a_k^{(j)}(s, x)\colon \Gamma^n \times \Gamma^n \to R_+$ be continuous functions. Suppose that the inequality*

$$u(s) \geqslant w(x) - \sum_{j=1}^m E_1^{(j)}(s, x; w) \tag{2.1}$$

is satisfied for $0 \leqslant x \leqslant s$, where $s \in \Gamma^n$. Then the following two inequalities are also valid for $0 \leqslant x \leqslant s$:

(I)
$$u(s) \geqslant w(x) \exp\left(-\int_x^s \sum_{j=1}^m A_j(s, t)\, dt\right), \tag{2.2}$$

where

$$A_j(s, x) = \max[a_j^{(j)}(s, x), a_j^{(j+1)}(s, x), \ldots, a_j^{(m)}(s, x)], \tag{2.3}$$

for each $s \in \Gamma^n$ fixed, $j = 1, 2, \ldots, m$;

(II)
$$u(s) \geqslant w(x)/q_m(s, x), \tag{2.4}$$

where the function $q_m(s, x)$ is defined by

$$\begin{aligned} q_1(s, x) &= \exp \int_x^s \sum_{j=1}^m A_j(s, t)\, dt, \\ q_r(s, x) &= 1 + \int_x^s \sum_{j=1}^{m-r+1} A_j(s, t)\, q_{r-1}(s, t)\, dt, \qquad r = 2, 3, \ldots, m. \end{aligned} \tag{2.5}$$

Proof. We first prove the validity of (2.2). Fixing $s \in \Gamma^n$, the inequality (2.1) can be rewritten as

$$w(x) \leqslant r_1(x) \qquad \text{for } 0 \leqslant x \leqslant s, \tag{2.6}$$

where

$$r_1(x) = u(s) + \sum_{j=1}^{m} E_1^{(j)}(s, x; w).$$

Therefore,

$$r_1(x) = u(s) \qquad \text{on } x_i = s_i,\ i = 1, 2, ..., n,$$

and by using (2.6) we obtain

$$\begin{aligned}(-)^n Dr_1(x) &= a_1^{(1)}(s, x)\, w(x) + \sum_{j=2}^{m} a_1^{(j)}(s, x)\, E_2^{(j)}(s, x; w) \\ &\leqslant A_1(s, x) \left\{ r_1(x) + \sum_{j=2}^{m} E_2^{(j)}(s, x; r_1) \right\}, \end{aligned} \tag{2.7}$$

here $A_1(s, x)$ is given by (2.3). We define

$$r_2(x) = r_1(x) + \sum_{j=2}^{m} E_2^{(j)}(s, x; r_1),$$

then $r_1(x) \leqslant r_2(x)$ when $0 \leqslant x \leqslant s$, and $r_2(x) = u(s)$ on $x_i = s_i$, $i = 1, 2, ..., n$.
By applying (2.7) we derive

$$\begin{aligned}(-1)^n Dr_2(x) &= (-1)^n Dr_1(x) + \sum_{j=3}^{m} a_2^{(j)}(s, x)\, E_3^{(j)}(s, x; r_1) \\ &\quad + a_2^{(2)}(s, x)\, r_1(x) \\ &\leqslant A_1(s, x)\, r_2(x) + A_2(s, x)\, r_3(x), \quad 0 \leqslant x \leqslant s, \end{aligned} \tag{2.8}$$

where $A_2(s, x)$ is given by (2.3) and $r_3(x)$ is defined by

$$r_3(x) = r_2(x) + \sum_{j=3}^{m} E_3^{(j)}(s, x; r_2).$$

Continuing in this way then we obtain

$$(-1)^n Dr_k(x) \leqslant r_{k+1}(x) \sum_{j=1}^{k} A_j(s, x), \qquad 0 \leqslant x \leqslant s,\ k = 1, 2, ..., m-1, \tag{2.9}$$

$$w(x) \leqslant r_1(x) \leqslant r_2(x) \leqslant \cdots \leqslant r_m(x), \qquad 0 \leqslant x \leqslant s, \tag{2.10}$$

and

$$r_1(x) = r_2(x) = \cdots = r_m(x) = u(s) \quad \text{on } x_i = s_i,\ i = 1, ..., n, \tag{2.11}$$

where

$$r_1(x) = u(s) + \sum_{j=1}^{m} E_1^{(j)}(s, x; w),$$
$$r_{k+1}(x) = r_k(x) + \sum_{j=k+1}^{m} E_{k+1}^{(j)}(s, x; r_k), \quad k = 1, 2, ..., m-1. \tag{2.12}$$

From the last equality in (2.12) with $k = m-1$, we derive

$$(-1)^n D r_m(x) = (-1)^n D r_{m-1}(x) + a_m^{(m)}(s, x)\, r_{m-1}(x)$$
$$\leqslant r_m(x) \sum_{j=1}^{m-1} A_j(s, x) + a_m^{(m)}(s, x)\, r_{m-1}(x)$$
$$\leqslant r_m(x) \sum_{j=1}^{m} A_j(s, x), \qquad 0 \leqslant x \leqslant s, \tag{2.13}$$

since (2.9) and (2.10) hold. We see from (2.12) that $r_j(x) \geqslant u(s) > 0$ is valid for $j = 1, 2, ..., m$ and $0 \leqslant x \leqslant s$. Hence, we obtain from (2.13) that

$$\frac{(-1)^n D r_m(x)}{r_m(x)} \leqslant \sum_{j=1}^{m} A_j(s, x), \qquad 0 \leqslant x \leqslant s,\ s \in \Gamma^n \text{ fixed.} \tag{2.14}$$

The above inequality (2.14) can be rewritten as

$$\frac{(-1)^n r_m(x) \cdot D_1 D_2 \cdots D_n r_m(x)}{r_m^2(x)} \leqslant \sum_{j=1}^{m} A_j(s, x)$$
$$+ \frac{(-1)^n D_n r_m(x) \cdot D_1 D_2 \cdots D_{n-1} r_m(x)}{r_m^2(x)},$$

i.e.,

$$(-1)^n D_n \left[\frac{D_1 \cdots D_{n-1} r_m(x)}{r_m(x)} \right] \leqslant \sum_{j=1}^{m} A_j(s, x),$$

since $(-1)^n D_n r_m(x) \cdot D_1 \cdots D_{n-1} r_m(x) \geqslant 0$ holds. Keeping $x_1, x_2, ..., x_{n-1}$ fixed in the above inequality, setting $x_n = t_n$, and integrating the both sides with respect to t_n from x_n to s_n, we obtain

$$\frac{(-1)^{n-1} D_1 \cdots D_{n-1} r_m(x)}{r_m(x)} \leqslant \int_{x_n}^{s_n} \sum_{j=1}^{m} A_j(s, x_1, ..., x_{n-1}, t_n)\, dt_n,$$

since $D_1 \cdots D_{n-1} r_m(x_1, ..., x_{n-1}, s_n) = D_1 \cdots D_{n-1} u(s) = 0$ for $n \geqslant 2$.

The above inequality implies

$$(-1)^{n-1} D_{n-1}\left[\frac{D_1 \cdots D_{n-2} r_m(x)}{r_m(x)}\right]$$

$$\leqslant \int_{x_n}^{s_n} \sum_{j=1}^{m} A_j(s, x_1, \ldots, x_{n-1}, t_n)\, dt_n,$$

since here we have $(-1)^{n-1} D_{n-1} r_m(x) \cdot D_1 \cdots D_{n-2} r_m(x) \geqslant 0$.

Keeping $x_1, \ldots, x_{n-2}, x_n$ fixed in the above inequality, setting $x_{n-1} = t_{n-1}$, and then integrating with respect to t_{n-1} from x_{n-1} to s_{n-1}, we derive

$$\frac{(-1)^{n-2} D_1 \cdots D_{n-2} r_m(x)}{r_m(x)}$$

$$\leqslant \int_{x_{n-1}}^{s_{n-1}} \int_{x_n}^{s_n} \sum_{j=1}^{m} A_j(s, x_1, \ldots, x_{n-2}, t_{n-1}, t_n)\, dt_n dt_{n-1},$$

since $D_1 \cdots D_{n-2} r_m(x_1, \ldots, x_{n-2}, s_{n-1}, x_n) = D_1 \cdots D_{n-2} u(s) = 0$ for $n \geqslant 3$.

Proceeding in this way we easily obtain

$$D_2\left[\frac{D_1 r_m(x)}{r_m(x)}\right] \leqslant \int_{x_3}^{s_3} \cdots \int_{x_n}^{s_n} \sum_{j=1}^{m} A_j(s, x_1, x_2, t_3, \ldots, t_n)\, dt_n \cdots dt_3.$$

Keeping $x_1, x_3, \ldots, x_n$ fixed in the above inequality, setting $x_2 = t_2$, and integrating with respect to t_2 from x_2 to s_2, and in view of $D_1 r_m(x_1, s_2, x_3, \ldots, x_n) = 0$, then we obtain

$$\frac{-D_1 r_m(x)}{r_m(x)} \leqslant \int_{x_2}^{s_2} \cdots \int_{x_n}^{s_n} \sum_{j=1}^{m} A_j(s, x_1, t_2, \ldots, t_n)\, dt_n\, dt_{n-1} \cdots dt_2.$$

Now keeping $x_2, \ldots, x_n$ fixed in the above inequality, setting $x_1 = t_1$, and integrating with respect to t_1 from x_1 to s_1, and using $r_m(s_1, x_2, \ldots, x_n) = u(s)$, we obtain

$$-\ln(u(s)/r_m(x)) \leqslant \int_{x}^{s} \sum_{j=1}^{m} A_j(s, t)\, dt,$$

or

$$r_m(x) \leqslant u(s) \exp \int_{x}^{s} \sum_{j=1}^{m} A_j(s, t)\, dt \qquad (\equiv u(s)\, q_1(s, x)). \tag{2.15}$$

Hence the desired inequality (2.2) follows from (2.10) and (2.15).

We now prove the inequality (2.4). Substituting the bound for $r_m(x)$ in (2.15) in the inequality (2.9) with $k=m-1$, we have

$$(-1)^n D r_{m-1}(x) \leqslant u(s)\, q_1(s, x) \sum_{j=1}^{m-1} A_j(s, x), \quad 0 \leqslant x \leqslant s.$$

Integrating the above inequality with respect to x_n from x_n to s_n, and using $D_1 \cdots D_{n-1} r_{m-1}(x_1, ..., x_{n-1}, s_n) = 0$, we get

$$(-1)^{n-1} D_1 \cdots D_{n-1} r_{m-1}(x) \leqslant \int_{x_n}^{s_n} u(s) \sum_{j=1}^{m-1} A_j(s, x_1, ..., x_{n-1}, t_n) \times q_1(s, x_1, ..., x_{n-1}, t_n)\, dt_n.$$

Integrating the above inequality with respect to x_{n-1} from x_{n-1} to s_{n-1}, and using $D_1 \cdots D_{n-2} r_{m-1}(x_1, ..., x_{n-2}, s_{n-1}, x_n) = 0$, we have

$$(-1)^{n-2} D_1 \cdots D_{n-2} r_{m-1}(x) \leqslant \int_{x_{n-1}}^{s_{n-1}} \int_{x_n}^{s_n} u(s) \sum_{j=1}^{m-1} A_j(s, x_1, ..., x_{n-2}, t_{n-1}, t_n) \times q_1(s, x_1, ..., x_{n-2}, t_{n-1}, t_n)\, dt_n\, dt_{n-1}.$$

Continuing in this way then we obtain

$$-D_1 r_{m-1}(x) \leqslant \int_{x_2}^{s_2} \cdots \int_{x_n}^{s_n} u(s) \sum_{j=1}^{m-1} A_j(s, x_1, t_2, ..., t_n) \times q_1(s, x_1, t_2, ..., t_n)\, dt_n \cdots dt_2.$$

Now, integrating the above inequality with respect to x_1 from x_1 to s_1 and using $r_{m-1}(s_1, x_2, ..., x_n) = u(s)$ we obtain

$$r_{m-1}(x) \leqslant u(s)\, q_2(s, x) \qquad \text{when } 0 \leqslant x \leqslant s,$$

where $q_2(s, x)$ is given by (2.5). Similarly, substituting the above bound for $r_{m-1}(x)$ in the inequality (2.9) with $k=m-2$, after integration we get the bound on $r_{m-2}(x)$ such that

$$r_{m-2}(x) \leqslant u(s)\, q_3(s, x) \qquad \text{when } 0 \leqslant x \leqslant s.$$

Continuing in this way, we can easily derive the desired inequality (2.4). The proof of the Theorem 1 is now completed.

COROLLARY 1. *In the above Theorem* 1, *if* $m=1$ *and* $a_1^{(1)}(s,x)=a(s)\,b(x)$, *where* $a(x)$ *and* $b(x)$ *are nonnegative continuous functions defined on* I^n, *then we derive Theorem* 1 *in Shih and Yeh* [5] *which in turn is an extension of Gollwitzer* [2, *Theorem* 2] *and Bondge and Pachpatte* [1, *Theorem* 1].

COROLLARY 2. *In the above Theorem* 1, *if* $m=2$, $a_1^{(i)}(s,x)=f(s,x)$ $(i=1,2)$, *and* $a_2^{(2)}(s,x)=g(s,x)$, *here* f *and* $g\colon I^n\times I^n\to R_+$ *are continuous functions, then we derive the lower bound for* $u(s)$ *such that*

$$u(s)\geqslant w(x)\left\{1+\int_x^s f(s,r)\left(\exp\int_r^s (f(s,t)+g(s,t))\,dt\right)dr\right\}^{-1},\qquad 0\leqslant x\leqslant s.$$

We note that Theorem 4 in [5] is a special case of the above Corollary 2 in which $f(s,x)=a(s)\,b(x)$ and $g(s,x)=c(x)$.

3. FURTHER EXTENSIONS

In this section we shall give some further extensions of Theorem 1, which unify and extend several known inequalities in [1–5].

THEOREM 2. *Let all of the hypotheses in Theorem* 1 *be satisfied, and let* $H(r)$ *be a positive, strictly increasing, convex, submultiplicative, and continuous function defined for* $r>0$, $H(0)=0$, *and* $\operatorname{Lim} H(r)=+\infty$ *as* $r\to\infty$. *Let* $p(x)$, $q(x)$ *be positive continuous functions on* I^n *with* $p(x)+q(x)=1$. *Suppose that the inequality*

$$u(s)\geqslant w(x)-b(s)\,H^{-1}\left\{\sum_{j=1}^{m} E_1^{(j)}(s,x;H(w))\right\}\tag{3.1}$$

is satisfied for $0\leqslant x\leqslant s$, $s\in I^n$, *where* H^{-1} *denotes the inverse of* H, *and* $b(x)$ *is a nonnegative continuous function on* I^n. *Then for* $0\leqslant x\leqslant s$ *we have*

$$u(s)\geqslant p(s)\,H^{-1}\left\{\frac{H(w(x))}{p(s)}\exp\left(-\int_x^s [q(s)\,H(b(s)/q(s))\,A_1(s,t)\right.\right.$$
$$\left.\left.+A_2(s,t)+\cdots+A_m(s,t)]\,dt\right)\right\},\tag{3.2}$$

and

$$u(s)\geqslant p(s)\,H^{-1}\left[\frac{H(w(x))}{p(s)\,v_m(s,x)}\right],\tag{3.3}$$

where

$$v_1(s, x) = \exp \int_x^s [q(s)\, H(b(s)/q(s))\, A_1(s, t) + A_2(s, t) + \cdots + A_m(s, t)]\, dt,$$

$$v_k(s, x) = 1 + \int_x^s [q(s)\, H(b(s)/q(s))\, A_1(s, t) + A_2(s, t) + \cdots$$
$$+ A_{m-k+1}(s, t)]\, v_{k-1}(s, t)\, dt, \qquad k = 2, 3, ..., m-1, \tag{3.4}$$
$$v_m(s, x) = 1 + \int_x^s q(s)\, H(b(s)/q(s))\, A_1(s, t)\, v_{m-1}(s, t)\, dt.$$

Proof. Rewrite the inequality (3.1) as

$$w(x) \leqslant p(s)(u(s)/p(s)) + q(s)(b(s)/q(s))\, H^{-1}\left\{\sum_{j=1}^{m} E_1^{(j)}(s, x; H(w))\right\}.$$

Since H is increasing, convex, and submultiplicative, from the above inequality we observe

$$H(w(x)) \leqslant p(s)\, H(u(s)/p(s)) + q(s)\, H(b(s)/q(s)) \sum_{j=1}^{m} E_1^{(j)}(s, x; H(w)),$$

i.e.,

$$p(s)\, H(u(s)/p(s)) \geqslant H(w(x)) - q(s)\, H(b(s)/q(s)) \sum_{j=1}^{m} E_1^{(j)}(s, x; H(w)).$$

Now, a suitable application of Theorem 1 to the above inequality yields the desired inequalities (3.2) and (3.3).

The above Theorem 2 generalizes the known results due to Bondge and Pachpatte [1, Theorem 2], Gollwitzer [2, Theorem 1], Pachpatte [4, Theorem 2], and Shih and Yeh [5, Theorems 2 and 5].

THEOREM 3. *Let all of the hypotheses in Theorem* 1 *be satisfied, and let the function* $b(x)$ *be the same as defined in Theorem* 2. *Let* $G(r)$ *be a positive, continuous, strictly increasing, subadditive, and submultiplicative function for* $r > 0$, $G(0) = 0$, *and* $G(r) \to +\infty$ *as* $r \to +\infty$. *Suppose that the inequality*

$$u(s) \geqslant w(x) - b(s)\, G^{-1}\left\{\sum_{j=1}^{m} E_1^{(j)}(s, x; G(w))\right\} \tag{3.5}$$

is satisfied for $0 \leqslant x \leqslant s$, $s \in \Gamma$, *where* G^{-1} *denotes the inverse function of* G. *Then when* $0 \leqslant x \leqslant s$ *we also have the following inequalities*:

$$u(s) \geqslant G^{-1}\left\{G(w(x)) \exp\left(-\int_x^s [G(b(s))\, A_1(s, t) + A_2(s, t) + \cdots + A_m(s, t)]\, dt\right)\right\}, \tag{3.6}$$

and

$$u(s) \geqslant G^{-1}[G(w(x))/z_m(s, x)], \tag{3.7}$$

where the function $z_m(s, x)$ is defined by

$$\begin{aligned} z_1(s, x) &= \exp \int_x^s [G(b(s))\, A_1(s, t) + A_2(s, t) + \cdots + A_m(s, t)]\, dt, \\ z_k(s, x) &= 1 + \int_x^s [G(b(s))\, A_1(s, t) + A_2(s, t) + \cdots \\ &\quad + A_{m-k+1}(s, t)]\, z_{k-1}(s, t)\, dt, \qquad k = 2, 3, \ldots, m-1, \\ z_m(s, x) &= 1 + \int_x^s G(b(s))\, A_1(s, t)\, z_{m-1}(s, t)\, dt. \end{aligned} \tag{3.8}$$

Proof. Rewrite the given inequality (3.5) as

$$w(x) \leqslant u(s) + b(s)\, G^{-1}\left\{\sum_{j=1}^{m} E_1^{(j)}(s, x; G(w))\right\}.$$

Since G is increasing, subadditive, and submultiplicative, so we obtain from- the above inequality

$$G(w(x)) \leqslant G(u(s)) + G(b(s)) \sum_{j=1}^{m} E_1^{(j)}(s, x; G(w)),$$

i.e.,

$$G(u(s)) \geqslant G(w(x)) - G(b(s)) \sum_{j=1}^{m} E_1^{(j)}(s, x; G(w)).$$

A suitable application of Theorem 1 to the above inequality yields the desired inequalities (3.6) and (3.7).

The above Theorem 3 generalizes the known results obtained in Bondge and Pachpatte [1, Theorem 6], Pachpatte [3, Theorem 3], and Shih and Yeh [5, Theorem 6]. By the way, we note here that the additional condition $G(r) \to +\infty$ as $r \to +\infty$ should be added to the Theorem 6 in [5] to ensure the desired lower bound for $u(s)$.

In concluding this paper, we remark that there are different ways to applying the inequalities in Theorems 1–3, according to the choice of the unknown function from u and w. If u is unknown and we set $w(x) = F(u(x))$, here $F: R_+ \to R_+$ is any known continuous function, then as discussed in [1], [2], and [5], we may use these inequalities to obtain the lower bounds for $u(s)$ from the corresponding integral inequalities for u. If the function w is chosen as unknown function and u is a known continuous function, then as we will show in the following example, the upper bound on $w(x)$ can be obtained by using our inequalities.

EXAMPLE. Suppose the following integral equation

$$v(x) = k(s) + \int_x^s A(s, t; v(t))\, dt + \int_x^s g(s, t) \left(\int_t^s B(s, r; v(r))\, dr \right) dt \qquad (*)$$

is satisfied for $0 \leqslant x \leqslant s$, where $s \in I^n$ is a vector-valued parameter; and $k: I^n \to R$, $g: I^n \times I^n \to R$, and A and $B: I^n \times I^n \times R \to R$ are known continuous functions. We assume further that the inequalities

$$\begin{aligned} |A(s, t; p)| &\leqslant f(s, t)\,|p|, \\ |B(s, t; q)| &\leqslant h(s, t)\,|q|, \end{aligned} \qquad \text{for } s, t \in I^n,\ t \leqslant s;\ p, q \in R,$$

are satisfied, where f and $h: I^n \times I^n \to R_+$ are known continuous functions. Then, if $v(x)$ is a continuous solution of $(*)$ on I^n we easily obtain from Eq. $(*)$ that

$$\begin{aligned} |k(s)| \geqslant |v(x)| &- \int_x^s f(s, t)\, |v(t)|\, dt \\ &- \int_x^s |g(s, t)| \left(\int_t^s h(s, r)\, |v(r)|\, dr \right) dt, \qquad 0 \leqslant x \leqslant t \leqslant s,\ s \in I^n. \end{aligned} \qquad (**)$$

Setting $u(s) = |k(s)|$ and $w(x) = |v(x)|$ in $(**)$, and applying Theorem 1 to the above inequality, then we obtain the upper bound on $|v(x)|$ such that

$$|v(x)| \leqslant |k(s)| \exp \int_x^s [f(s, t) + |g(s, t)| + h(s, t)]\, dt, \qquad 0 \leqslant x \leqslant s,\ s \in I^n,$$

and

$$\begin{aligned} |v(x)| \leqslant |k(s)| \Big\{ 1 &+ \int_x^s (f(s, t) + |g(s, t)|) \\ &\times \left(\exp \int_t^s (f(s, r) + |g(s, r)| + h(s, r))\, dr \right) dt \Big\}, \\ &\qquad 0 \leqslant x \leqslant t \leqslant s,\ s \in I^n, \end{aligned}$$

since here we have $A_1(s, x) \leqslant f(s, x) + |g(s, x)|$, where $A_1(s, x)$ is defined by $A_1 = \max(f(s, x), |g(s, x)|)$ for each $s \in I^n$ fixed.

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