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Time-Periodic Solutions of Quasilinear Parabolic Differential Equations

I. Dirichlet Boundary Conditions

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We study boundary value problems for quasilinear parabolic equations when the initial condition is replaced by periodicity in the time variable. Our approach is to relate the theory of such problems to the classical theory for initial-boundary value problems. In the process, we generalize many previously known results. © 2001 Elsevier Science

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1. INTRODUCTION

The usual theory of quasilinear parabolic equations (see [16, 26] and the references therein) is concerned with initial-boundary value problems for operators P defined by

$$Pu = -u_t + a^{ij}(X, u, Du)D_{ij}u + a(X, u, Du), \tag{1.1}$$

for a positive definite matrix-valued function a^{ij} and a scalar-valued function a. A typical such problem is $Pu = 0$ in Ω, $u = \varphi$ on $S\Omega$, $u = u_0$ on ω for some domain $\Omega \subset \mathbb{R}^{n+1}$ with lateral boundary $S\Omega$ and initial surface ω. (If $\Omega = \omega \times (0, T)$, then $S\Omega = \partial\omega \times (0, T)$.) Under appropriate conditions on Ω and the functions in the equations (in particular, sufficient smoothness), this problem is known to have a solution, with known smoothness.

In this work, we replace the initial condition $u = u_0$ on ω with the periodic condition $u(\cdot, 0) = u(\cdot, T)$ on ω. This periodic problem has been studied by a number of authors [1, 3, 4, 9, 11–13, 29–31, 34, 37], etc. (see [36]

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for a more complete discussion), but only under severe assumptions. All of these authors assume that $\Omega = \omega \times (0, T)$ and that the functions a^{ij}, a, and φ are periodic with period T. We shall study problems in which Ω may be noncylindrical (although it must be periodic in the sense described in Section 2) and only φ is periodic.

These authors make further hypotheses on the problem. Ŝmulev [34] assumes that a^{ij} is independent of Du, and Nakao and Ohara [30] and Seidman [31] assume that $Pu = -u_t + \operatorname{div} \psi(|Du|)Du - G(X, u)$ for suitable functions ψ and G. We shall show that, when specialized to this particular structure, our results hold under weaker hypotheses on ψ and G than theirs. Further comparisons to known results will be made in later sections of this paper. We also study equations with $a^{ij}D_{ij}u$ a suitable multiple of the mean curvature operator $\mathscr{M}u = \operatorname{div}(Du/(1 + |Du|^2)^{1/2}))$ under suitable additional hypotheses. These mean curvature equations seem to fall outside all previous work on periodic problems.

We also improve previous results by weakening the hypotheses on Ω. For example, Nakao and Ohara [30] assumed that $\partial\omega$ has nonnegative mean curvature with respect to the unit inner normal. We shall show that this curvature condition is not needed for their class of equations, and we consider weaker smoothness hypotheses on the domains than in other works.

An important element of our work here is the use of estimates already proved for initial-boundary value problems. This approach is not new (see, for example, [1, 11–13, 37]), but we use sharper estimates than in the cited papers. Whenever possible, we shall simply quote such estimates; however, in order to examine a broad class of problems, we shall also present proofs of some estimates not already in the literature but the proofs are usually very simple modifications of previous proofs.

For the reader's convenience, we begin with some notation and a reduction of the solvability of quasilinear problems in Section 2 to establishing certain *a priori* estimates on the solutions. Sections 3 and 4 are devoted to these estimates. Examples are discussed in Section 5, and some remarks directly related to other, uniformly parabolic examples are given in Section 6. Additional remarks appear in Section 7.

2. NOTATION AND REDUCTION TO *A PRIORI* ESTIMATES

For the most part, our notation follows that in [26]. For example, points in $\mathbb{R}^{n+1}$ ($n \geq 2$) are denoted by $X = (x, t)$ and the summation convention is observed throughout. We use Ω for a domain (nonempty, connected open set) in $\mathbb{R}^{n+1}$, and we define diam Ω to be the infimum of those numbers R such that $|x - x_0| < R$ for all $X \in \Omega$ and some $x_0 \in \mathbb{R}^n$. We refer to [26] for the definitions of $\Omega(t)$, $Q(X_0, R)$, $\mathscr{P}\Omega$, $B\Omega$, $C\Omega$, and $S\Omega$. We also use the weighted parabolic Hölder space $H_a^{*(b)}$ from [22], and we write Γ for $\Omega \times \mathbb{R} \times \mathbb{R}^n$.

We say that $\Omega \subset \mathbb{R}^n \times (0, T)$ is *periodic* if the set

$$\Omega^* = \{X : 0 < t < 2T : X \in \Omega \text{ or } (x, t - T) \in \Omega\}$$

is a domain, and if any (topological) boundary point X of Ω^* with $0 < t < 2T$ is in $S\Omega$. For simplicity, we write ω for the set of all $x \in \mathbb{R}^n$ such that $(x, 0) \in B\Omega$. When Ω is a cylinder $\omega \times (0, T)$, then Ω is automatically periodic with $S\Omega = \partial\omega \times (0, T]$. In addition, if Ω is a periodic domain, we write $S\Omega \in H_{1+\alpha}$ if $\mathscr{P}\Omega^* \in H_{1+\alpha}$ (as defined on p. 76 of [26]). We write our problem as

$$Pu = 0 \text{ in } \Omega, \qquad u = \varphi \text{ on } S\omega, \qquad u(\cdot, 0) = u(\cdot, T) \text{ in } \Omega, \tag{2.1}$$

with P defined by (1.1). We always assume that φ is T-periodic. When $Pu = -u_t + \operatorname{div} A(X, u, Du) + B(X, u, Du)$, we say that P is in *divergence form*. In addition, by virtue of the linear theory in [22] (see [27] for the application to periodic problems), the coefficients of P need not be continuous with respect to t.

The basis for our existence program is the following reduction to *a priori* estimates via the Leray–Schauder fixed point theorem. To state this theorem more simply, for any subset K of Γ and $\beta \in (0, 1]$, we use $H_\beta^*(K)$ to denote the space of all functions f defined on K such that

$$|f(x, t, z, p) - f(y, t, w, q)| \le F(|x - y| + |z - w| + |p - q|)^\beta$$

for some constant F and all (x, t, z, p) and (y, t, w, q) in K.

LEMMA 2.1. *Suppose that $S\Omega \in H_{1+\alpha}$ and $\varphi \in H_{1+\alpha}(S\Omega)$ is T-periodic for some $\alpha \in (0, 1)$ and $T > 0$. Suppose that a^{ij} and a are in $H_\beta^*(K)$ for any bounded subset K of Γ. If there are constants $M \ge 0$ and $\eta \in (0, \alpha)$ (independent of $\tau \in (0, 1)$) such that any solution of*

$$u_t = a^{ij}(X, u, Du)D_{ij}u + \tau a(X, u, Du) \qquad \text{in } \Omega, \tag{2.2a}$$

$$u = \tau\varphi \qquad \text{on } S\Omega, \tag{2.2b}$$

$$u(\cdot, 0) = u(\cdot, T) \qquad \text{in } \omega \tag{2.2c}$$

obeys the estimate $|u|_0 + |Du|_\eta < M$, then (2.1) *has a solution in $H_{2+\beta}^{*(-1-\alpha)}$.*

Proof. We follow the proof of [20, Lemma 5.1] with two changes: We replace the initial condition in that work with the periodic condition, and we use the linear theory from [27]. ∎

Problem (2.1) can be embedded in a more general family of problems as in [20, Lemma 5.1], but this additional generality is not relevant to the problems considered here. On the other hand, it is useful to note that the coefficients a^{ij} and a need only be measurable with respect to t in Lemma 2.1, assuming that the Hölder continuity with respect to the other variables is uniform in t.

3. POINTWISE ESTIMATES

We now start to prove the estimates required by Lemma 2.1. In this section, we estimate the L^∞ norm of the solution. Our first estimates deal with equations in divergence form. First we set $R = \operatorname{diam}\Omega$, and we assume that there are positive constants a_1, b_0, b_1, and M along with an increasing convex function H such that $H(0) = 0$ and

$$p \cdot A(X, z, p) \geq H(|p|) - a_1 H\left(\frac{|z|}{R}\right), \tag{3.1a}$$

$$zB(X, z, p) \leq b_0 (p \cdot A(X, z, p))^+ + b_1 H\left(\frac{|z|}{R}\right) \tag{3.1b}$$

for all $(X, z, p) \in \Gamma$ with $|z| \geq M$. If we assume that a_1, b_0, and b_1 are small enough, then we first obtain an integral estimate on solutions.

LEMMA 3.1. *Let H, A, and B satisfy conditions* (3.1a), (3.1b), *let P be in divergence form, and let u be a solution of* (2.1) *with $|\varphi| \leq M$. Suppose also that there is a positive constant $\varepsilon < 1$ such that $(a_1 + b_1)/(1 - b_0) \leq \varepsilon$. Then*

$$\int_\Omega H\left(\frac{|u|}{R}\right) dX \leq C(\varepsilon)|\Omega| H\left(\frac{2M}{R}\right), \tag{3.2a}$$

$$\int_{\{|u| \geq M\}} |Du \cdot A(X, u, Du)| dX \leq C(\varepsilon)|\Omega| H\left(\frac{2M}{R}\right). \tag{3.2b}$$

If also there is a constant $m \geq 1$ such that

$$\sigma H'(\sigma) \leq m H(\sigma) \tag{3.3}$$

for all $\sigma > M$, then, for any $q \geq 1$, there is a constant C determined only by ε, m, $M_0 = H(M/R)T/M$, n, and q such that

$$\int_\Omega H\left(\frac{|u|}{R}\right)^q dX \leq C|\omega| H\left(\frac{M}{R}\right)^q. \tag{3.4}$$

Proof. Inequalities (3.2a), (3.2b) follow from the proof of [21, Lemma 2.3] after noting that $0 \le (Du \cdot A)^- \le a_1 H(|u|/R)$ and observing, as in [30], the terms obtained by integrating the u_t term are zero.

For (3.4), we combine the proofs of [25, Theorem 2.2; 26, Theorem 9.11]. First, we define $J(\tau) = \min\{M, (\tau - M)^+\}$ and $\chi(\tau) = \int_M^\tau J(\sigma)\, d\sigma$. If we use the test function $J(|u|) \operatorname{sgn} u$ along with (3.1) and (3.2) and note that $0 \le J'(|u|) \le 1$ and $J(|u|) \le |u|$ wherever $|u| > M$, we find that

$$\begin{aligned}\int_{\Omega(t_1)} \chi(|u|)dx - \int_{\Omega(t_0)} \chi(|u|)dx \\ = \int_{t_0}^{t_1} \int_{\Omega(t)} [-J'(|u|)Du \cdot A + J(|u|) \operatorname{sgn} uB] dX \\ \le C(\varepsilon)|\Omega| H\left(\frac{2M}{R}\right)\end{aligned}$$

for $t_0 \in (0, T)$ and $t_1 \in (t_0, t_0 + T)$. From (3.2a), it follows that there is $t_0 \in (0, T)$ such that

$$\int_{\Omega(t_0)} H\left(\frac{|u|}{R}\right) dx \le C(n, \varepsilon) R^n H\left(\frac{2M}{R}\right).$$

Because H is convex and increasing, it follows that $H(\tau)/\tau$ is an increasing function of τ. We therefore see that $|u| \le MH(|u|/R)/H(M/R)$, and hence (recalling (3.3))

$$\chi(|u|) \le M|u| \le 3M^2 H(|u|/R)/H(M/R) \le C(m) M^2 H(|u|/R)/H(2M/R)$$

wherever $|u| \ge M$. Combining these estimates, we find that

$$\int_{\Omega(t_0)} \chi(|u|)dx \le C(m, n, \varepsilon) R^n M^2.$$

By explicit evaluation of χ, it is easily seen that $\chi(|u|) \ge M|u|/4$ wherever $|u| \ge 2M$, so

$$\int_{\{x \in \Omega(t_1): |u(x,t_1)| \ge 2M\}} |u(x, t_1)| dx \le C(m, n, \varepsilon)(1 + M_0) M R^n$$

for any $t_1 \in (0, T)$. For $\tau \ge 0$ and $t \in (0, T)$, we write $\Omega_\tau(t)$ for the subset of $\Omega(t)$ on which $|u| \ge \tau M$ to infer from this result that

$$\sup_{0<t<T} \int_{\Omega_\tau(t)} 1\, dx \le C_0(m, M_0, n, \varepsilon) R^n / \tau.$$

Next, for $\tau \ge 1$ to be chosen, we define

$$w_0(\sigma) = \left(H(\sigma/R)^{q-1}(\sigma/R) - H(\tau M/R)^{q-1}(\tau M/R)\right)^+$$

and use $w_0(|u|)$ sgn u as a test function to infer that

$$\int_{\Omega^+} H\left(\frac{|u|}{R}\right)^{q-1} H(|Du|)dX \le C(m)q \int_{\Omega^+} H\left(\frac{|u|}{R}\right)^q dX, \tag{3.5}$$

where Ω^+ denotes the set on which $|u| \ge \tau M$. Next, set

$$w = \left(H(|u|/R)^q - H(\tau M/R)^q\right)^+$$

and use Hölder's inequality along with the Sobolev imbedding theorem (for each fixed t) to infer that

$$\begin{aligned}\int_\Omega w\,dX &\le \int_0^T \left(\int_{\Omega_\tau(t)} 1dx\right)^{1/n} \left(\int_{\Omega_\tau(t)} w^{n/(n-1)}\,dx\right)^{(n-1)/n} dt \\ &\le C(n)\left[\frac{C_0}{\tau}\right]^{1/n} R\int_\Omega |Dw|\,dX.\end{aligned}$$

Now (3.3) and Young's inequality in the form $aH(b)/b \le H(a) + H(b)$ [21, Lemma 1.1(e)] with $a = |Du|$ and $b = |u|/R$ imply that

$$R\int_\Omega |Dw|\,dx \le mq\left[\int_{\Omega^+} H\left(\frac{|u|}{R}\right)^q dX + \int_{\Omega^+} H\left(\frac{|u|}{R}\right)^{q-1} H(|Du|)\,dX\right].$$

We now combine these last two inequalities with (3.5) to see that

$$\int_\Omega w\,dX \le C(m,n,q)\left[\frac{C_0}{\tau}\right]^{1/n} \int_\Omega H\left(\frac{|u|}{R}\right)^q dX.$$

For $\tau = \max\{1, (2C(m,n,q))^n C_0\}$, we conclude from this inequality via simple rearrangement that

$$\int_{\Omega^+} H\left(\frac{|u|}{R}\right)^q dX \le 4\int_{\Omega^+} H\left(\frac{\tau M}{R}\right)^q dX \le C(m,q,\tau)|\Omega| H\left(\frac{M}{R}\right)^q.$$

The proof is completed by combining this estimate with the obvious one for the integral over $\Omega \setminus \Omega^+$. ∎

Note that the smallness hypothesis on a_1, b_0, and b_1 is equivalent to the single assumption $a_1 + b_0 + b_1 < 1$, which is only needed to prove (3.2a), (3.2b). Although the condition $b_0 < 1$ is necessary for our method, the smallness condition on a_1 and b_1 can be relaxed since it is only used to apply the Poincaré inequality [21, Lemma 2.2]. In particular, for $H(\sigma) = \sigma^m$, a sharp upper bound is known for $a_1 + b_1$. When $H(\sigma) = \sigma^2$, it is just the first eigenvalue of the Laplacian with zero Dirichlet data. With this observation, we can reproduce the results in [3, 4].

Examples of functions H which satisfy (3.3) are $H(\sigma) = \sigma^m$, or $H(\sigma) = \sigma^k(\ln(1+\sigma))^K$ for $1 \le k < m$ and $K > 0$. In addition (see [21]), H may satisfy the growth conditions

$$\limsup_{\sigma\to\infty} \sigma^{-k} H(\sigma) = \infty, \qquad \liminf_{\sigma\to\infty} \sigma^{-k} H(\sigma) = 0$$

for all k in an arbitrary interval $[k_1, k_2] \subset (1, \infty)$.

Once we have these integral estimates, the L^∞ bound is proved via Moser iteration. For this iteration, we no longer need the close connection between the p and z behavior.

LEMMA 3.2. *Suppose that there are constants* a_0, a_1, b_0, b_1, m, *and* M *with* $m \ge 2$ *such that*

$$p \cdot A \le a_0|p| - a_1|z|^m, \qquad zB \le b_0(p \cdot A)^+ + b_1|z|^m \tag{3.6}$$

for $|z| \ge M$. *If* $|\varphi| \le M$, *then any solution of* (2.1) *obeys the estimate*

$$\sup_\Omega |u| \le 2M + C(a_0, a_1, b_0, b_1, m, M, n) \int_\Omega |u|^{(n+1)(m-1)}\, dX. \tag{3.7}$$

Proof. We imitate the proof of [26, Theorem 9.8]. For $q > \max\{1, 2b_0 - 2n\}$, we use the test function

$$\eta = \left[t\left(1 - \frac{M}{|u|}\right)^+\right]^{(n+1)q-n} |u|^{q+nm-n-1} \operatorname{sgn} u.$$

Then for $\zeta = (1 - M/|u|)^+ t$ and

$$v = (q + nm - n - 1)\left(1 - \frac{M}{|u|}\right) + [(n+1)q - n]\frac{M}{|u|},$$

we see that

$$\sup_{0<s<2T} \int_{\Omega(s)} |u|^{q+nm-n} \zeta^{(n+1)(q-1)}\, dx + \int_\Omega |Du||u|^{q+nm-n-2} \zeta^{(n+1)(q-1)} v\, dX$$
$$\le c_1 q^2 \int_\Omega |u|^{q+nm-n-2+m} \zeta^{(n+1)(q-1)}\, dX,$$

with c_1 determined by a_0, a_1, a_2, m, and T. The iteration described in [26, Theorem 9.8] then gives

$$\sup_{\Omega^*\setminus\Omega} |u| \le 2M + C \int_{\Omega^*} |u|^{(n+1)(m-1)}\, dX = 2M + 2C \int_\Omega |u|^{(n+1)(m-1)}\, dX.$$

The desired estimate now follows by noting that $\sup_{\Omega^*\setminus\Omega} |u| = \sup_\Omega |u|$. ∎

Note that (3.1) and (3.3) imply (3.6) with the same m although the constants a_1, b_0, b_1, and M must be suitably modified. When $H(\sigma) = \sigma^m$, our result is the periodic analog of a result of Aronson and Serrin [2] for solutions of the initial-boundary value problem.

For nondivergence equations, the L^∞ estimate follows by easy modifications of the corresponding estimates for elliptic equations. In particular, many of the estimates in [10, Chap. 10] have periodic-parabolic analogs. Rather than provide a complete description of the possible structure conditions, we give two simple ones. The first one, based on [10, Theorem 10.3], uses the important function $\mathscr{E}$ defined by

$$\mathscr{E}(X, z, p) = a^{ij}(X, z, p)p_i p_j. \tag{3.8}$$

LEMMA 3.3. *If there are nonnegative constants μ_1 and μ_2 such that*

$$\frac{a(X, z, p)\operatorname{sgn} z}{\mathscr{E}(X, z, p)} \leq \frac{\mu_1|p| + \mu_2}{|p|^2} \tag{3.9}$$

for all $(X, z, p) \in \Gamma$ with $|p| \neq 0$ and if u is a solution of (2.1), *then*

$$\sup_\Omega |u| \leq \sup |\varphi| + C(\mu_1, R)\mu_2. \tag{3.10}$$

Proof. As in the proof of [10, Theorem 10.3], we assume without loss of generality that $0 \leq x^1 \leq 2R$ in Ω and that $\mu_2 > 0$. Then the function v defined by

$$v(x) = \sup_{S\Omega} \varphi^+ + \mu_2(\exp(2(\mu_1 + 1)R) - \exp((\mu_1 + 1)x^1))$$

satisfies the inequalities $v \geq u$ on $S\Omega$ and

$$-v_t + a^{ij}(X, u, Dv)D_{ij}v + a(X, u, Dv) < 0$$

in Ω. It's easy to see that there is a linear operator L given by $Lw = -w_t + \bar{a}^{ij}D_{ij}w + b^iD_iw$ such that $L(u - v) > 0$ in Ω and $u - v \leq 0$ on $S\Omega$. The strong maximum principle implies that $u - v$ can attain a positive maximum only on $B\Omega$, and the periodic condition implies that a positive maximum would also occur on $\omega \times \{T\}$, and hence $u - v \leq 0$ in Ω. Because of the explicit form of v, this inequality gives the upper bound for u and a lower bound follows by similar reasoning, and the case $\mu_2 = 0$ follows by sending $\mu_2 \to 0$. ∎

Our second L^∞ bound for equations in nondivergence form is approximately analogous to the bound for equations in divergence form. It uses another important structure function $\mathscr{T}$, the trace of (a^{ij}).

LEMMA 3.4. *Suppose that there are positive constants M and L such that*

$$\operatorname{sgn} z a(X, z, p) \leq \frac{|p|}{R}\mathcal{T}(X, z, p) \tag{3.11}$$

for all $(X, z, p) \in \Gamma$ *with* $|z| \geq M$ *and* $|p| \geq L$. *Then*

$$\sup_{\Omega} |u| \leq \max\{M, \sup|\varphi|\} + 2LR. \tag{3.12}$$

Proof. Modify the proof of [32, Theorem 3] in the same way that we modified the proof of [10, Theorem 10.3] to prove Lemma 3.3. ∎

4. GRADIENT ESTIMATES

Now we turn to the crux of the results for periodic solutions: estimates on the gradient of the solution. Although it is possible to obtain global gradient estimates directly in some cases (see, for example, [30]), we shall follow the path used for the Cauchy–Dirichlet problem of first deriving a boundary gradient estimate and then using this estimate to obtain a global one.

It is well known from the theory for the Cauchy–Dirichlet problem that the functions $\mathcal{E}$ and $\mathcal{T}$, defined above, play a crucial role in the boundary gradient estimate. The proof of [26, Theorem 10.4] and the strong maximum principle give our first boundary gradient estimate, in which $d(X)$ denotes the parabolic distance from X to $S\Omega$.

LEMMA 4.1. *Suppose* $S\Omega \in H_{1+\alpha}$. *If there are positive constants* μ *and* p_0 *such that*

$$|p|^{2-\alpha}\mathcal{T} + |a| \leq \mu\mathcal{E}, \qquad |p|^{2-\alpha} \leq \mu\mathcal{E} \tag{4.1}$$

for $|p| \geq p_0$, *then there is a constant C determined only by* G, n, p_0, R, T, α, μ, *and* $|\varphi|_{1+\alpha}$ *such that* $|u - \varphi| \leq Cd$ *in* Ω *and* $\sup_{S\Omega}|Du| \leq C$.

In general, the theorems in [26, Chap. 10] extend to the periodic case provided the hypotheses of those theorems are suitably modified. We include a particular version of this observation related to curvature equations.

LEMMA 4.2. *Let* $\Omega = \omega \times (0, T)$ *with* $\partial\omega \in C^2$, *write* H_0 *for the mean curvature of* $\partial\omega$, *and suppose there are functions* $g(X, z, p)$ *and* $G(X, z)$ *with* $g > 0$ *and G decreasing with respect to z for fixed X such that*

$$a^{ij}(X, z, p) = g(X, z, p)\left[\delta^{ij} - \frac{p_i p_j}{1 + |p|^2}\right] \tag{4.2a}$$

$$a(X, z, p) = g(X, z, p)G(X, z), \tag{4.2b}$$

and such that there are constants K_1 and R with

$$|G(x,t,z) - G(y,t,z)| \leq K_1|x-y| \tag{4.3}$$

for all x and y with $d(x), d(y) \leq R$, all $t \in [0,T]$ and $z \in \mathbb{R}$. Suppose also that

$$(n-1)H_0(x) \geq |G(X, \varphi(X))| \tag{4.4}$$

for all $X \in S\Omega$. If either $1 \leq \mu\mathscr{E}$ for $|p| \geq p_0$ or

$$\sup|\varphi_t| \leq g(X,z,p)[(n-1)H_0(x) - |G(X,\varphi(X))|] + \mu\mathscr{E}, \tag{4.5}$$

then there is a constant C determined by K_1, p_0, R, T, α, and $|\varphi|_{1+\alpha}$ such that $|u-\varphi| \leq Cd$ in Ω and $\sup_{S\Omega}|Du| \leq C$.

From a boundary gradient estimate, a global one follows by known gradient estimates which are local in time, but not in space. For equations in divergence form and u a (sufficiently smooth) solution of (2.1), we define

$$v = (1+|Du|^2)^{1/2}, \qquad \nu = Du/v, \qquad g^{ij} = \delta^{ij} - \nu^i\nu^j, \qquad d\mu = v\,dx.$$

We also assume that A^i is differentiable with respect to (x,z,p) and that there are two matrix-valued functions (C^i_k) and (D^i_k) such that (D^i_k) is differentiable with respect to (x,z,p) and

$$C^i_k + D^i_k = \frac{\partial A^i}{\partial z}p_k + \frac{\partial A^i}{\partial x^k} + B\delta^i_k.$$

Next, we define

$$A^{ij} = v\frac{\partial A^i}{\partial p_j}, \qquad D^{ij}_k = \frac{\partial D^i_k}{\partial p_j}, \qquad \mathscr{F} = p_i\nu^k\frac{\partial D^i_k}{\partial z} + \nu^k\frac{\partial D^i_k}{\partial x^i}.$$

To simplify our structure conditions, we use $\beta, \beta_1, \ldots, \beta_7$ to denote nonnegative constants and we suppose that there is a nonnegative constant τ_0 along with a bounded function Λ_1 such that

$$C^i_k g^{jk}\zeta_{ij} \leq \beta_1\Lambda_1^{1/2}(A^{ij}\zeta_{ik}\zeta_{jk})^{1/2}, \tag{4.6a}$$

$$C^i_k\nu^k\xi_i \leq \beta_1\Lambda_1^{1/2}(A^{ij}\xi_i\xi_j)^{1/2}, \tag{4.6b}$$

$$vD^{ij}_k\nu^k\zeta_{ij} \leq \beta_1\Lambda_1^{1/2}(A^{ij}\zeta_{ik}\zeta_{jk})^{1/2}, \tag{4.6c}$$

$$\mathscr{F} \leq \beta_1^2\Lambda_1 \tag{4.6d}$$

for all $n \times n$ matrices ζ, all n-vectors ξ and η, and all $(X,z,p) \in \Gamma$ such that $z = u(X)$ and $v > \tau_0$.

We also write Ω_τ for the subset of Ω on which $v > \tau$. The first step in proving our gradient bound is to reduce the L^∞ estimate of $|Du|$ to an integral one by applying [26, (11.49)] with $\rho = (2T)^{1/2}$.

LEMMA 4.3. *Suppose conditions* (4.6a)–(4.6d) *hold and that there are* $C^1([\tau_0,\infty))$ *functions* w, λ, *and* Λ *such that*

$$w \text{ is increasing}, \tag{4.7a}$$

$$\xi^{-\beta} w(\xi) \text{ is a decreasing function of } \xi, \tag{4.7b}$$

$$w(\xi)^{\beta}(\Lambda(\xi)/\lambda(\xi))^{N/2}/\xi \text{ is an increasing function of } \xi, \tag{4.7c}$$

$$\xi^{-\beta}(\Lambda(\xi)/\lambda(\xi))^{N/2} \text{ is a decreasing function of } \xi \tag{4.7d}$$

for $N = n$ *if* $n \geq 3$, $N > 2$ *if* $n = 2$. *Suppose also that*

$$v\lambda(v)\left(1 + \left(\frac{v\lambda'(v)}{\lambda(v)}\right)^2\right) g^{ij}\xi_i\xi_j \leq A^{ij}\xi_i\xi_j, \tag{4.8}$$

$$\Lambda_1 \leq v\Lambda, \qquad 1 \leq \Lambda \tag{4.9}$$

on Ω_{τ_0}. *If also* $v \leq \tau$ *on* $S\Omega$ *for some* $\tau \geq \tau_0$, *then*

$$\begin{aligned} \sup_{\Omega_\tau}\left(1 - \frac{\tau}{v}\right)^{N+2} w &\leq C(n, \beta, \beta_1 T^{1/2}) T^{-(N+2)/2} \\ &\quad \times \int_{\Omega_\tau} w(\Lambda/\lambda)^{N/2}\Lambda \, d\mu \, dt. \end{aligned} \tag{4.10}$$

Next, we assume that there is a differentiable vector-valued function $\bar{A}$ such that

$$\left(\frac{\Lambda}{\lambda}\right)^{N/2} \Lambda v \leq \beta_2 w^{\beta_3} Du \cdot \overline{A}. \tag{4.11}$$

Then we reduce our estimate to one on $\int w^q Du \cdot \overline{A}\, dX$. In many cases, we can take $\overline{A} = A$, but our examples will show the utility of considering other choices. To estimate this integral, we need some additional structure conditions on $\overline{A}$, some of which are related to A.

LEMMA 4.4. *Suppose conditions* (4.6), (4.7a), (4.7b), *and* (4.11) *hold and that* $v < \tau$ *on* $S\Omega$. *Suppose also that there is a positive, decreasing function* ε *such that*

$$w'(v)\overline{A} \cdot \xi \leq \beta_4 (A^{ij}\xi_i\xi_j)^{1/2} (\nu \cdot \overline{A})^{1/2}, \tag{4.12a}$$

$$v|\overline{A}_z| + |\overline{A}_x| \leq \beta_5 Du \cdot \overline{A}, \tag{4.12b}$$

$$w\overline{A}^{ij}\zeta_{ij} \leq \beta_4 (vA^{ij}\zeta_{ik}\zeta_{jk})^{1/2} (Du \cdot \overline{A})^{1/2}, \tag{4.12c}$$

$$\Lambda_1 v \leq \varepsilon(v) w^2 Du \cdot \overline{A}, \tag{4.12d}$$

$$Du \cdot \overline{A} \leq Du \cdot A, \tag{4.12e}$$

on Ω_{τ_0}. *Set* $\sigma = \sup_t \operatorname{osc}_{\Omega(t)} u$ *and* $E = \exp(\beta_5\sigma)$. *If there is a constant* $\tau_1 \geq \tau$ *such that*

$$400(\beta_1\sigma)^2E^2q^2\beta_4^2\varepsilon(\tau_1)(1+\beta q)^3 \leq 1 \tag{4.13}$$

for some $q \geq 2$, *then there is a constant* C *determined only by* n, q, β, β_1, *and* β_4 *such that*

$$\int_{\Omega_\tau} w^q Du \cdot \overline{A}\, dX \leq C(w(\tau_1)+E\sigma)^q \int_{\Omega_\tau} Du \cdot A\, dX. \tag{4.14}$$

Proof. We follow the proof of [26, Lemma 11.12] except that we keep track of the presence of $\overline{A}$. ∎

Our final estimate for equations in divergence form is an immediate consequence of [26, Lemma 11.13] and is related to Lemma 3.1.

LEMMA 4.5. *Suppose that*

$$|Du||A| \leq \beta_6 Du \cdot A, \qquad |B| \leq \beta_7 Du \cdot A \tag{4.15}$$

for $v \geq \tau_0$. *Let* $\sigma_1 = \operatorname{osc}_\Omega u$, *and suppose that* $u = \varphi$ *on* $S\Omega$ *and that there is a constant* Φ *such that* $|D\varphi| \leq \Phi$ *and* $|u-\varphi| \leq \Phi d$ *in* Ω. *Set* $\tau_2 = \max\{\tau_0, 4\beta_6\sigma_1/R)$ *and*

$$\Delta = \sup_{v<\tau_2}\left([B-\beta_7 Du\cdot A]^+ + [Du\cdot A]^+ + \frac{\sigma_1}{R}|A|\right). \tag{4.16}$$

If k_0 *is the smallest positive integer such that* $T \leq k_0R^2$, *then*

$$\int_{\Omega_{\tau_2}} Du \cdot A\, dX \leq k_0C(n)\exp(\beta_7\sigma_1)R^n[\sigma_1^2 + (\Delta + C(\Omega)\Phi^2)R^2]. \tag{4.17}$$

To discuss equations in nondivergence form, we define the operators δ and $\bar\delta$ by

$$\delta f(X,z,p) = \frac{\partial f}{\partial z} + \frac{p}{|p|^2}\cdot\frac{\partial f}{\partial x}, \qquad \bar\delta f(X,z,p) = p\cdot\frac{\partial f}{\partial p},$$

we suppose that a^{ij} can be decomposed as

$$a^{ij}(X,z,p) = a_*^{ij}(X,z,p) + \frac{1}{2}[p_if_j(X,z,p) + p_jf_i(X,z,p)] \tag{4.18}$$

for some differentiable functions a_*^{ij} and f_i with (a_*^{ij}) positive-definite with minimum eigenvalue λ_*, and we define

$$A_\infty = \limsup_{|p|\to\infty}\ \sup_{\{X\in\Omega,\,|z|\leq\sup|u|\}} \frac{1}{\mathscr{E}}\left(\frac{v}{2\lambda_*}\sum_{i,j}(\bar\delta a_*^{ij})^2 + (\bar\delta-1)\mathscr{E}\right), \tag{4.19a}$$

$$B_\infty = \limsup_{|p|\to\infty}\ \sup_{\{X\in\Omega,\,|z|\leq\sup|u|\}} \frac{1}{\mathscr{E}}\left(\delta\mathscr{E} + (\bar\delta-1)a\right), \tag{4.19b}$$

$$C_\infty = \limsup_{|p|\to\infty}\ \sup_{\{X\in\Omega,\,|z|\leq\sup|u|\}} \frac{1}{\mathscr{E}}\left(\frac{v}{2\lambda_*}\sum_{i,j}(\delta a_*^{ij})^2 + \delta a\right). \tag{4.19c}$$

Then we have the following global gradient bound, which is just a consequence of [26, Theorem 11.1] (which, in turn, is based on [33, Sect. 6]) and the strong maximum principle.

THEOREM 4.6. *Let* $u \in C^{2,1}(\Omega)$ *be a solution of* (2.1) *with* $Du \in C^0(\overline{\Omega})$. *If* A_∞, B_∞, *and* C_∞ *are all finite and if* $A_\infty \le 0$ *or* $C_\infty \le 0$, *then there is a constant* c_1 *determined only by* $\sup_{S\Omega} |Du|$, A_∞, B_∞, C_∞, p_0, μ, *and the limit behavior in* (4.19) *such that* $\sup_\Omega |Du| \le c_1$.

Finally, if A^i is Lipschitz with respect to (x, z, p) and B is bounded or if a^{ij} is Lipschitz with respect to (x, z, p) and a is bounded, a Hölder gradient estimate holds. We refer the reader to [26, Chap. 12] for precise versions of these estimates.

5. EXAMPLES

We now present some examples to apply the estimates of Sections 3 and 4. In all examples, we assume that $S\Omega \in H_{1+\alpha}$ and that $\varphi \in H_{1+\alpha}(S\Omega)$ for some $\alpha \in (0, 1)$. When $\Omega = \omega \times (0, T)$, we write H_0 for the mean curvature of $\partial\omega$. Additional regularity assumptions may be made as needed.

EXAMPLE 1. *Nakao–Ohara–Seidman type equations*. We start with a structure based on [30], [31]. Let P have the form

$$Pu = -u_t + \operatorname{div}(\psi(|Du|)Du) + G(X, u) \tag{5.1}$$

for a positive, continuous function ψ defined on $(0, \infty)$ and a Caratheodory function G defined on $\Omega \times \mathbb{R}$ such that there are constants $\delta \in (0, 1]$, $m \ge 0$, and $M > 0$ along with an increasing, convex function H such that

$$\sigma^{1-\delta}\psi(\sigma) \text{ is an increasing function of } \sigma \tag{5.2}$$

for $\sigma \ge M$ and

$$\sigma H'(\sigma) \le mH(\sigma), \qquad \sigma^2\psi(\sigma) \ge H(\sigma) \tag{5.3}$$

for all $\sigma > M$. Although $\psi(0)$ need not be defined, we adopt the convention that $\psi(|p|)p = 0$ when $p = 0$. (Note that (5.2) implies $\lim_{\sigma\to 0^+} \sigma\psi(\sigma) = 0$.) We also assume that

$$zG(X, z) \le \varepsilon H(|z|/R) + G_0 \tag{5.4}$$

for some nonnegative constants ε and G_0 with $\varepsilon < 1$, and that $G \in L^\infty(\Omega \times K)$ for any bounded subset K of $\mathbb{R}$. Nakao and Ohara [30] assume (among other things) that

$$\psi(\sigma) \ge k_0\sigma^{m-2}, \qquad \psi'(\sigma) \ge k_0\sigma^{m-3} \tag{5.2$'$}$$

for constants $k_0 > 0$ and $m \geq 2$ and that $G(X, z) = g(x, z) - f(X)$ with

$$zg(x, z) \leq k_2|z| + k_3|z|^\beta \tag{5.4$'$}$$

for some $\beta \in (1, m)$ and f and g are locally Lipschitz; these conditions easily imply (5.2) and (5.3) with $\delta = 1$ and $H(\sigma) = k_0\sigma^m$ and (5.4). Moreover, when $m = 2$, the conditions in (5.2$'$) imply that $\psi(\sigma) \geq k_0[\ln \sigma + 1]$ for $\sigma \geq 1$, so they cannot allow the special case $\psi \equiv 1$ (although it appears that the proofs in [30] allow this case). Also, when $m = 2$, we can allow $zG \leq k_2|z| + \varepsilon_1|z|^2 \ln(1 + |z|)$ for sufficiently small ε_1, which is strictly stronger than (5.4$'$). Seidman [31] assumes that $G \equiv 0$, that (5.2) and (5.3) hold with $\delta = m$ and $H(\sigma) = \sigma^m$, and that $\psi(\sigma) \leq C\sigma^m$.

Now, (3.1) and (3.3) hold with $a_1 = b_0 = 0$, $b_1 = (1 + \varepsilon)/2$, and M sufficiently large, so we have an L^∞ bound for solutions of (2.2) which is uniform in $\tau \in [0, 1]$. Next, $\mathcal{T} = n\psi(|p|) + |p|\psi'(|p|)$ and $\mathcal{E} = \psi(|p|)|p|^2 + \psi'(|p|)|p|^3$, so (4.1) holds with $\mu = 1$ and p_0 sufficiently large. Therefore Lemma 4.1 gives a boundary gradient estimate; Nakao and Ohara also assume that $\Omega = \omega \times (0, T)$ with $H_0 \geq 0$ and that $\varphi \equiv 0$. Seidman allows nonzero C^∞ boundary data.

For the global gradient bound, we have (4.6)–(4.9) with $D_k^i = 0$, $\Lambda_1 = v^\delta$, $\tau_0 = 2 + \sup_{S\Omega} v$, β_1 suitably large, $\beta = 1$, $w = v$, Λ, $\Lambda_1 = v$, and $\lambda = v^{\delta-1}$, so Lemma 4.3 gives a bound on $\sup v$ in terms of

$$\int_{\Omega_\tau} w(\Lambda/\lambda)^{N/2}\Lambda \, d\mu \, dt = \int_{\Omega_\tau} v^{2+(1-\delta)N/2} \, dX$$

for $\tau \geq \tau_0$. Next, (4.11), (4.12), and (4.13) hold with $w = v$, $\beta_3 = (1 - \delta)(N + 2)/2$, $\beta_2 = 1$, $\bar{A}(X, z, p) = v^{\delta-1}Du$, β_4 sufficiently large, $\beta_5 = 0$, $\varepsilon(v) = v^{-2}$ and τ_1 sufficiently large, and Lemma 4.4 gives an estimate on $\int v^{2+(1-\delta)N/2} \, dX$ in terms of $\int Du \cdot A \, dX$, which is estimated via Lemma 4.5 with $\beta_6 = \beta_7 = 1$. If $\psi \in C^2[0, \infty)$ and $G \in C^1(\overline{\Omega} \times \mathbb{R})$, then we have a Hölder gradient estimate, and then Lemma 2.1 provides the existence of solutions under this additional smoothness.

To remove the additional smoothness assumptions, we first note that there are sequences of smooth functions (ψ_j) and (G_j) which satisfy (5.2), (5.3), and (5.4) with uniform constants such that $A_j \to A$ uniformly on bounded subsets of $\mathbb{R}^n$, where $A_j(p) = p\psi_j(p|)$, and $G_j(\cdot, z) \to G(\cdot, z)$ a.e. for any $z \in \mathbb{R}$. We write u_j for the solution of (2.1) when P is given by (5.1) with ψ and G replaced by ψ_j and G_j, respectively. From our estimates, there is a constant U (independent of j) such that $\sup|u_j| + \sup|Du_j| \leq U$. To continue, we prove a uniform continuity estimate for u_j with respect to t. For $X_0 \in \Omega$ and $r > 0$ sufficiently small, set $t_1 = t_0 - r^2$ and write $B = B(x_0, r)$ and $Q = B \times (t_1, t_0)$. There is a constant k determined only by the geometry of $S\Omega$ such that $Q(X_1, kr) \subset Q \cap \Omega$ for some $X_1 \in \Omega$. Let $\zeta = C(n)r^{-n}[(1 - |x - x_1|^2)^+]^2$ with $C(n)$ chosen so that $\int \zeta \, dx = 1$

and set $U_j(t) = \int \zeta(x)u_j(x,t)\,dx$. From the weak form of the equation, we have

$$|U_j(t_0) - U_j(t_1)| = \left| \int_Q [D\zeta \cdot Du_j \psi_j(|Du_j|) + \zeta G_j(X, u_j)] dX \right|$$
$$\leq C|Q|r^{-n-1} + Cr^2 \leq Cr$$

because $|D\zeta| \leq Cr^{-n-1}$. Next, we note that $U(t) = u(x^*, t)$ for some $x^* \in B$, so $|u_j(x_0, t_i) - U_j(t_i)| \leq Ur$ for $i = 0, 1$. Hence $|u_j(x_0, t_0) - u_j(x_0, t_1)| \leq Cr$, and the sequence (u_j) is equicontinuous, so there is a function u such that $u_j \to u$ uniformly. Minty's Lemma can be used to show that u is a weak solution of (2.1) (see [30]), but we shall show directly that (Du_j) converges strongly to Du.

To prove this convergence, let j and k be positive integers and set

$$I = \int_\Omega [A(Du_j) - A(Du_k)] \cdot [Du_j - Du_k]\,dX.$$

The proof of [21, (5.8)] shows that

$$\int_\Omega H(|Du_j - Du_k|)\,dX \leq C(n, m)\left[I + I^{1/2} \left(\int_\Omega H(|Du_j|)\,dX \right)^{1/2} \right],$$

and the differential equations for these functions yield

$$I = \int_\Omega [A(Du_j) - A_j(Du_j)] \cdot [Du_j - Du_k]\,dX$$
$$+ \int_\Omega [A_k(Du_k) - A(Du_k)] \cdot [Du_j - Du_k]\,dX$$
$$+ \int_\Omega [G_j(X, u_j) - G_k(X, u_k)][u_j - u_k]\,dX.$$

Because of the uniform convergence of (A_j) and (u_j) to their respective limits, we can make I arbitrarily small by choosing j and k large enough, and hence

$$\int_\Omega H(|Du_j - Du_k|)\,dX \to 0$$

as $j, k \to \infty$, and then Jensen's inequality implies that (Du_j) is convergent in L^1. By uniqueness of limits, we must have $Du_j \to Du$, and then Hölder's inequality implies that $Du_j \to Du$ in L^q for any $q \in (1, \infty)$. It then follows from the dominated convergence theorem that u is a weak solution of (5.1).

EXAMPLE 2. *The time-dependent prescribed mean curvature equation*. Now let $\psi(\sigma) = (1 + \sigma^2)^{-1/2}$ in (5.1) and suppose that there are constants $M > 0$ and $\varepsilon \in (0, 1)$ such that $(\operatorname{sgn} z)G(X, z) \leq \varepsilon(n-1)/R$ for $|z| \geq M$. Then Lemma 3.4 implies that $|u|_0 \leq \max\{|\varphi|_0, M\} + C(\varepsilon)R$.

For the gradient estimates, we also assume that $\Omega = \omega \times (0, T)$ with $\partial\omega \in C^2$ and $\sup|\varphi_t| + |G(X, \varphi(X))| \le (n-1)H_0(x)$ at each point $X \in S\Omega$. We also assume that $G_z \le 0$ and that $|G_x|$ is bounded on bounded subsets of $\Omega \times \mathbb{R}$, and then a boundary gradient estimate follows from Lemma 4.2 with $g(X, z, p) = 1/(1+|p|^2)$.

The global gradient estimate is more delicate. First, conditions (4.6)–(4.9) hold with $C_k^i = 0$, $\Lambda = \Lambda_1 = 1$,$w = v$, $\lambda = 1/v$, and $\beta_1 = \sup|G_x|^{1/2}$, so Lemma 4.3 implies that

$$\sup v \le C \int_{\Omega_\tau} v^{2+(N/2)} Du \cdot A\, dX \tag{5.5}$$

provided $\tau \ge \tau_0 = 2 + \sup_{S\Omega} v$. Since (4.12a) only holds for $w' \le 1/v$, we can't estimate the right hand side of (5.5) directly. Instead, we use Lemma 4.4 with $\overline{A} = A$, $w = \ln v$, $q = 2$, $\beta_4 = 1$, $\beta_5 = 0$, and $\varepsilon(v) = (\ln v)^{-2}$ and then Lemma 4.5 (with $\beta_6 = \beta_7 = 1$) to infer that

$$\int_{\Omega_\tau} (\ln v)^2 Du \cdot A\, dX \le C \int_{\Omega_\tau} Du \cdot A\, dX \le C \tag{5.6}$$

for $\tau \ge \tau_0$.

Now we use a slight variation of Lemma 4.4 similar to the proof of (3.4). Let θ be a differentiable function such that $\theta(v) = 0$ on $S\Omega$ and set $\Theta(v) = \int_\tau^v \theta(s)\, ds$. In addition, we define

$$\mathscr{C}^2 = v^{-2} g^{ij} g^{km} D_{ik} u D_{jm} u, \qquad \mathscr{E}_1 = v^{-2} g^{ij} D_i v D_j v.$$

By multiplying the equation for u by $\operatorname{div}(\theta(v)\nu)$ and integrating by parts, we see that

$$\begin{aligned} &\int_{\omega\times\{t_2\}} \Theta(v)\, dx + \int_{t_1}^{t_2} \int_\omega [\mathscr{C}^2 \theta(v) + \mathscr{E}_1 v \theta'(v)]\, dx\, dt \\ &\quad = \int_{t_1}^{t_2} \int_\omega \mathscr{F} \theta(v)\, dx\, dt + \int_{\omega\times\{t_1\}} \Theta(v)\, dx \end{aligned} \tag{5.7}$$

for any t_1 and t_2 such that $0 \le t_1 < t_2 \le 2T$. In particular, we take $\theta(v) = (\ln v)^2 (1 - \tau/v)^+$, so θ' and θ are nonnegative, and

$$\frac{1}{4}\theta(v)(v-\tau) \le \Theta(v) \le \theta(v)(v-\tau).$$

Further, $\mathscr{F}\theta(v) \le Cv$ on Ω_τ, and $\mathscr{C}^2$ and $\mathscr{E}_1$ are nonnegative. Combining these estimates with (5.6) then yields

$$\int_{\omega\times\{t_2\}} (\ln v)^2 [(1-\tau/v)^+]^2\, d\mu \le C + 4\int_{\omega\times\{t_1\}} (\ln v)^2 [(1-\tau/v)^+]^2\, d\mu.$$

Using (5.6) again and arguing as in Lemma 3.1, we conclude that

$$\sup_{t\in(0,T)} \int_{\{x: v(x,t)\ge\tau\}} 1\, d\mu \le C(\ln\tau)^{-2}$$

for $\tau \geq \tau_0$. To finish the proof, we take $\theta(v) = v^q(v-\tau)^+$, with $q = 2 + N/2$, along with $t_1 = 0$ and $t_2 = T$ in (5.7) to infer that

$$\int_{\Omega_\tau} [v^q(1-\tau/v)\mathscr{C}^2 + v^q(1-\tau/v)\mathscr{E}_1]\, d\mu\, dt \leq C(q) \int_{\Omega_\tau} v^q(1-\tau/v)\, d\mu\, dt.$$

Then Hölder's inequality and the Michael–Simon version [28] of the Sobolev inequality (in the form [19, (1.4)]) with $h = v^q[(1-\tau/v)^+]^3$ imply as before that

$$\int_{\Omega_\tau} v^q(1-\tau/v)^3\, d\mu\, dt \leq C(q)(\ln\tau)^{-2} \int_{\Omega_\tau} v^q(1-\tau/v)\, d\mu\, dt.$$

From this inequality, we conclude that

$$\int_{\Omega_\tau} v^q\, d\mu\, dt \leq C(q),$$

so we infer a gradient bound from (5.5). A Hölder gradient estimate follows directly from the smoothness of G.

EXAMPLE 3. *Motion by mean curvature.* Now we consider the equation

$$u_t = (1+|Du|^2)^{1/2}[\operatorname{div}(Du/(1+|Du|^2)^{1/2}) + G(X,u)]$$

with G and Ω as in Example 2 but with the estimate on H_0 relaxed to $(n-1)H_0(x) \geq |G(X,\varphi(X))|$. As before, we obtain an L^∞ estimate for u from Lemma 3.4 and a boundary gradient estimate from Lemma 4.2 with $g \equiv 1$. If we choose $a_*^{ij} = \delta^{ij}$ and $f_i = p_i/v^2$, then $A_\infty = -1$, $B_\infty = 0$, and $C_\infty = \sup|G_x|$, so Theorem 4.6 gives a global gradient bound.

EXAMPLE 4. *Uniformly parabolic equations.* First, we suppose that the equation is in divergence form and let H be a convex function with $H(0) = 0$ such that

$$1 + \delta \leq \frac{sH'(s)}{H(s)} \leq m \tag{5.8}$$

for some positive constants m and δ. If A and B satisfy conditions (3.1a), (3.1b) with $a_1 + b_0 + b_1 < 1$, then Lemma 3.1 and 3.2 give an L^∞ bound for u. For the gradient bound, we also assume that there are positive constants θ_1, θ_2, and θ_3 such that

$$\theta_1 H(|p|)|p|^{-2}|\xi|^2 \leq \frac{\partial A^i}{\partial p_j}\xi_i\xi_j \leq \theta_2 H(|p|)|p|^{-2}|\xi|^2, \tag{5.9a}$$

$$|\frac{\partial A}{\partial x}| + |p||\frac{\partial A}{\partial z}| + |B| \leq \theta_3 H(|p|) \tag{5.9b}$$

for $|p| \geq 1$. Lemma 4.1 with $p_0 = 1$ and $\mu = C(\theta_1, \theta_2, \theta_3)$ gives a boundary gradient estimate. In addition, conditions (4.6)–(4.9) are satisfied with

$D_k^i = 0$, $\Lambda_1 = H|p|$, $w = v$, $\lambda = H(v)/v$, $\Lambda = H(v)v$, and β_1 determined by θ_1, θ_2, and n, so Lemma 4.3 gives a gradient bound in terms of $\int v^q H(v)\,dX$ for some $q \geq 1$ determined only by n. To estimate this integral, we first observe that conditions (4.11) and (4.12) hold with $\overline{A}(p) = H(|p|)|p|^{-2}p$, $w = v$, $\theta = 2$, $\beta_5 = 0$, and suitable β_4; if also θ_3 is sufficiently small (which is true if the left-hand side of (5.9b) is $o(H)$), then (4.13) holds. Finally, if $|A| = O(H(|p|)/|p|)$, then (4.15) holds with some β_6 and β_7 so the gradient estimate now follows from Lemmata 4.4 and 4.5. Assuming finally that A and B are continuous, we infer the existence of solutions via the approximation argument in Example 1.

For equations in nondivergence form, we suppose first that there are positive constants μ_1 and μ_2 along with a positive function λ (defined on Γ) such that

$$a^{ij}\xi_i\xi_j \geq \lambda|\xi|^2, \tag{5.10a}$$

$$\operatorname{sgn} za \leq \mu_1\lambda|p| + \mu_2 \tag{5.10b}$$

for all $(X, z, p) \in \Gamma$ and all $\xi \in \mathbb{R}^n$. Then Lemma 3.3 gives an L^∞ bound for u. Next, we suppose that the operator is uniformly parabolic, in the sense that there is a positive constant μ such that $|a^{ij}| \leq \mu\lambda$. A boundary gradient estimate then follows from Lemma 4.1 if there are positive constants p_0, θ_1, and θ_2 such that $|a| \leq \theta_1\lambda|p|^2$ and $|p|^{-\alpha} \leq \theta_2\lambda$ for $|p| \geq p_0$. Next, we assume that there are positive constants L and θ_3 and a decreasing function ε with $\varepsilon(\sigma) \to 0$ as $\sigma \to \infty$ such that

$$|\bar{\delta}a^{ij}| \leq \theta_3\lambda,\ (\bar{\delta} - 1)a \leq \theta_3\lambda|p|^2, \tag{5.11a}$$

$$|\delta a^{ij}| \leq \varepsilon(|p|)\lambda,\ \delta a \leq \varepsilon(|p|)\lambda|p|^2 \tag{5.11b}$$

for $|p| \geq L$. The global gradient estimate is a consequence of Theorem 4.6 (with $C_\infty \leq 0$), and the Hölder gradient estimate then follows.

6. REFINEMENTS FOR UNIFORMLY PARABOLIC OPERATORS

By invoking additional local estimates, we can prove existence of solutions to (2.1) under even weaker hypotheses. Rather than attempting to give a complete description of these results, we mention some applications to uniformly parabolic operators which improve the results discussed in Example 4.

In the divergence structure case, we suppose that H satisfies (5.8), that A and B satisfy conditions(3.1a), (3.1b) with $a_1 + b_0 + b_1 < 1$, and that there are positive constants θ_1, θ_2, and θ_3 such that (5.9a), (5.9b) hold for $|p| \geq 1$. If we also assume either $m \leq 2$ or $\delta \geq 1$ (in particular if $H(s) = s^m$

for some $m > 1$), then the arguments in [5, 7] (see [6]) along with (5.8) and (5.9) imply a Hölder estimate for u. Because of this continuity estimate, a local version of our interior gradient bound is true (see, e.g. [26, p. 280]), and a Hölder gradient estimate holds if we assume that (5.9a) holds for all p by the arguments in [23].

For equations in nondivergence form, we recall first that conditions (5.10a), (5.10b) imply an L^∞ bound and that a Hölder gradient estimate follows from a global gradient estimate. (We also recall that $|a^{ij}| \leq \mu\lambda$.) If there are constants $m \geq -\alpha$, $\theta_2, \ldots, \theta_5$ with $\theta_3 \geq \theta_2 > 0$ and a decreasing function ε with $\varepsilon(\sigma) \to 0$ as $\sigma \to \infty$ such that

$$\theta_2 |p|^m \leq \lambda \leq \theta_3 |p|^m,$$
$$\left| \frac{\partial a^{ij}}{\partial p_k} - \frac{\partial a^{ik}}{\partial p_j} \right| \leq \theta_3 \lambda / |p|,$$
$$|p||a_x^{ij}| + |p|^2 |a_z^{ij}| + |a| \leq \varepsilon(|p|)\lambda |p|^2$$

for $|p| \geq 1$, then a boundary gradient estimate follows from Lemma 4.1 and we can invoke the results of [26, Sect. 11.7] (see also [17]) in place of Theorem 4.6 to infer a global gradient bound. In particular, no conditions are made on the derivatives of a. Moreover, we can relax the condition on m to $m \geq -2$ if Ω is a cylinder ([26, Theorem 10.4] with (10.13b) holding) and we can remove the lower bound on m altogether if Ω is cylindrical and φ is time-independent [26, Corollary 10.5]. In particular, we can remove the hypotheses on the derivatives of a in [13]. Finally, if $m = 0$, then (5.10a) and the estimate $|a| = O(|p|^2)$ imply a Hölder estimate for u by virtue of the straightforward extension [18, Theorem 3.2; 26, Lemma 11.4] of the corresponding result for linear equations by Krylov and Safonov [15], so we can take ε to be a positive constant in this case.

When a^{ij} does not depend on p, then we can weaken our hypotheses still further. If $|a^{ij}|$, $1/\lambda$, and $|a|/(\lambda(|p|^2 + 1))$ are uniformly bounded for bounded z, and if a^{ij} is continuous with respect to (x, z) uniformly on bounded subsets of $\Omega \times \mathbb{R}$, then the interpolation technique of Trudinger [35] gives the required gradient estimate via the last inequality on page 173 of [26]. Again, Ω need not be cylindrical. This result improves [34, Theorem 6] by removing the conditions on derivatives of a^{ij} and a.

7. FURTHER RESULTS

All of the results given above can be further refined in several ways. Here, we indicate some extensions which relate directly to previous results for periodic solutions of parabolic equations.

First, if $\partial\omega$ satisfies a uniform exterior $H_{1+\alpha}$ condition (as defined in [26, p. 228]), then ω can be approximated (from inside) by smooth domains satisfying a uniform exterior $H_{1+\alpha}$ condition with uniform constants for the approximating domains. By using our existence results and uniform estimates in the approximating domains, we see that a solution exists provided P satisfies the hypotheses of Lemma 4.1 along with the hypotheses giving L^∞ bounds on u and on Du. In particular, Examples 1 and 4 fall into in this category.

Next, we can often allow continuous boundary values by imitating the arguments in [10, Sect. 14.5; 26, Sect. 12.5(v)]. In particular, continuous boundary values are allowed in Examples 2, 3, and 4 (but not in Example 1 without further structure conditions), and also for the uniformly parabolic equations from Section 6. (Suitable interior gradient bounds are given in [26, Sect. 11.5] for Examples 2 and 4 and for Section 6, and in [8] for Example 3.) The analysis of semilinear equations from Section 6 in this situation gives an appropriate extension of the results in [34].

If P is in divergence form with $A^i(X, z, p) = |p_i|^{m_i-2}p_i$ for $m_i > 2$ and $|B(X, z, p)| = O(\sum |p_i|^{m_i})$ uniformly as $|p| \to \infty$, a gradient bound follows from [24, Example 2]. Further generalizations of this example are given in [24]. Similar problems were studied by Nakao [29] and Yamada [37], but they were unable to prove an L^∞ bound for the gradient. The hypotheses of [37] do not, in fact, imply such a bound because the equation is replaced by a differential inclusion with inhomogeneous terms in L^2. We can recover Yamada's results (under our weaker regularity hypotheses on Ω) by first approximating the differential inclusion by an equation with smooth coefficients and then proving appropriate uniform bounds. Yamada also assumed Ω to be not necessarily cylindrical but smoother than we consider here.

In many cases, the hypotheses of Section 3 can be replaced by the existence of an ordered subsolution-supersolution pair, that is, a pair of functions $(\underline{v}, \bar{v})$ with $\underline{v} \le \bar{v}$ in Ω such that

$$P\underline{v} \ge 0 \text{ in } \Omega, \qquad \underline{v} \le \varphi \text{ on } S\Omega, \qquad \underline{v}(\cdot, 0) \le \underline{v}(\cdot, T) \text{ in } \omega$$

and

$$P\bar{v} \le 0 \text{ in } \Omega, \qquad \bar{v} \ge \varphi \text{ on } S\Omega, \qquad \bar{v}(\cdot, 0) \ge \bar{v}(\cdot, T) \text{ in } \omega.$$

The argument used in [13] applies to all of our examples except the uniformly parabolic equations of nondivergence form in Example 4, and hence the hypotheses of Section 3 can be replaced by the existence of an ordered subsolution-supersolution pair in all these cases. The key new idea here is that the coefficients of P do not need to be continuous with respect to t, so the nonperiodicity of $\underline{v}$ and $\bar{v}$ are irrelevant to our argument.

We close with some observations on the regularity of solutions. We have assumed that a^{ij} and a are Hölder with respect to (x, z, p) but not t

to obtain an $H_{1+\alpha}$ solution with locally Hölder continuous second spatial derivatives. The time derivative is locally Hölder continuous with respect to x but not necessarily t. Further smoothness follows via the linear theory. For example, if $S\Omega \in H_{2+\alpha}$, $\varphi \in H_{2+\alpha}$, and a^{ij} and a in H_α, then $u \in H_{2+\alpha}(\Omega)$. If, additionally, a^{ij} and a are time-periodic with period T, then it follows that there is a periodic solution with u_t also time-periodic.

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