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Decomposing the essential spectrum

E.B. Davies

Department of Mathematics, King's College London, Strand, London, WC2R 2LS, United Kingdom

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Abstract

We use C^*-algebra theory to provide a new method of decomposing the essential spectra of self-adjoint and non-self-adjoint Schrödinger operators in one or more space dimensions.

Keywords: Essential spectrum; C^*-algebra; Non-self-adjoint; Schrödinger operator

1. Introduction

In a recent study by Hinchcliffe [14] of the spectrum of a periodic, discrete, non-self-adjoint Schrödinger operator acting on $\mathbf{Z}^2$ with a dislocation along $\{0\} \times \mathbf{Z}$, we were struck by the fact that the essential spectrum of the operator, defined by means of the Calkin algebra, divides into two parts, one of which occupies a region in the complex plane, the other being one or more simple curves; the curves are associated with surface states confined to a neighbourhood of the dislocation. The same phenomenon occurs in the self-adjoint case, but here the distinction is between parts of the (real) spectrum that have infinite spectral multiplicity and other parts with finite multiplicity, at least in two dimensions.

In this paper we describe a method of decomposing the essential spectrum of a self-adjoint or non-self-adjoint Schrödinger operator into parts by using the two-sided ideals of a certain standard C^*-algebra. Our conclusion is that one can define different types of essential spectrum, provided one is given this extra structure; we warn the reader that the spectral classification that we obtain is not a unitary invariant of the operators concerned. However, the C^*-algebra used

E-mail address: e.brian.davies@kcl.ac.uk.

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is the same for all the applications considered so the results obtained have a high degree of model-independence.

At a broad conceptual level, the C^*-algebras that we consider are closely related to algebras and modules that were introduced by Georgescu, Măntoiu, Roe and others [1–3,15,16,19,20]; many further references and useful comments may be found in [12,13]. However, our treatment works in very general metric space setting and does not depend on the presence of any group action, whether abelian or nonabelian. As a result it is applicable to more or less arbitrary waveguides, discrete and continuous graphs and Riemannian manifolds, as well as to $\mathbf{R}^d$ and $\mathbf{Z}^d$. The idea of studying the essential spectrum of an operator by constructing a C^*-algebra with a large class of closed two-sided ideals is also not new. It first appeared in [2,3], but all previous treatments apply in much more restricted contexts than that here.

Some of our spectral results can be proved by methods that are geometric in the sense that they involve Hilbert space methods rather than C^*-algebras. An advantage of the approach described here is that instead of dealing with new applications by invoking analogy and experience, the use of C^*-algebras enables one to formulate simple general theorems that cover applications directly. The method accommodates many of the technical hypotheses that have been used in the field within a single formalism.

In Sections 2 and 4 we investigate the relevant C^*-algebra theory without reference to its application. Section 3 is devoted to showing how to apply the results to discrete Schrödinger operators. Theorems 10 and 11 describe the spectrum when a periodic potential has a dislocation on one or both of the two axes in $\mathbf{Z}^2$; the second possibility has not previously been considered. After a substantial amount of preparatory work, we turn in Section 7 to the study of Schrödinger and more general differential operators acting in $L^2(\mathbf{R}^d)$, and show that the abstract methods developed earlier can be applied to their resolvent operators under suitable hypotheses. The spectral mapping theorem then allows one to pull the results back to the original operators. Example 42 explains the application of the methods to multi-body Schrödinger operators. Finally, in Section 8 we show that our methods are not only relevant in a Euclidean context. We prove that the C^*-algebraic assumptions are satisfied when considering the Laplace–Beltrami operator on three-dimensional hyperbolic space by writing down the explicit formulae available in this case; the same applies to a wide variety of other Riemannian manifolds but general heat kernel bounds are needed for the proofs.

2. Some C^*-algebra theory

Throughout this section $\mathcal{A}$ will denote a (usually non-commutative) C^*-algebra with identity, and $\mathcal{J}$ will denote a (closed, two-sided) ideal in $\mathcal{A}$. It is well-known that such an ideal is necessarily closed under adjoints and that $\mathcal{A}/\mathcal{J}$ is again a C^*-algebra with respect to the quotient norm. See [10, Chapter 1] or [17, Chapter 1] for various standard facts about C^*-algebras that we will use without further comment.

If $x \in \mathcal{A}$ then we denote the spectrum of x by $\sigma(x)$; it is known that if $\mathcal{A}$ is replaced by a larger C^*-algebra, $\sigma(x)$ does not change. If $\mathcal{J}$ is an ideal in $\mathcal{A}$ we denote the natural map of $\mathcal{A}$ onto the quotient algebra $\mathcal{A}/\mathcal{J}$ by $\pi_{\mathcal{J}}$. If several ideals $\mathcal{J}_r$ are labelled by a parameter r, we write π_r instead of $\pi_{\mathcal{J}_r}$ for brevity, and also put $\sigma_r(x) = \sigma(\pi_{\mathcal{J}_r}(x))$

Lemma 1. *If the ideals $\mathcal{J}_1, \mathcal{J}_2$ in $\mathcal{A}$ satisfy $\mathcal{J}_2 \subseteq \mathcal{J}_1 \subseteq \mathcal{A}$ then*

$$\sigma_1(x) \subseteq \sigma_2(x) \subseteq \sigma(x).$$

Proof. We first put $\mathcal{J}_3 = \{0\}$, so that $\mathcal{A}/\mathcal{J}_3 = \mathcal{A}$ and $\sigma_3(x) = \sigma(x)$. Suppose that $1 \leqslant r < s \leqslant 3$ and that $\lambda \notin \sigma_s(x)$. Then there exists $y \in \mathcal{A}$ such that

$$\big(\pi_s(x) - \lambda 1\big)\pi_s(y) = \pi_s(y)\big(\lambda 1 - \pi_s(x)\big) = 1$$

in $\mathcal{A}/\mathcal{J}_s$. Hence there exist $u, v \in \mathcal{J}_s$ such that

$$(x - \lambda 1)y = 1 + u, \qquad y(\lambda 1 - x) = 1 + v.$$

Applying π_r to both equations and using the fact that $\mathcal{J}_s \subseteq \mathcal{J}_r$ we obtain

$$\big(\pi_r(x) - \lambda 1\big)\pi_r(y) = \pi_r(y)\big(\lambda 1 - \pi_r(x)\big) = 1.$$

Hence $\lambda \notin \sigma_r(x)$ and $\sigma_r(x) \subseteq \sigma_s(x)$. $\square$

Note. If $\mathcal{A} = \mathcal{L}(\mathcal{H})$ and $\mathcal{J}$ is the ideal $\mathcal{K}(\mathcal{H})$ of all compact operators on the Hilbert space $\mathcal{H}$, then $\sigma(\pi_{\mathcal{J}}(x))$ is (one of several inequivalent definitions of) the essential spectrum of x by [5, Theorem 4.3.7]. Needless to say we are interested in more general examples.

There are several ways of constructing $\mathcal{A}$ and the relevant ideals $\mathcal{J}_r$. Given $\mathcal{J}$, the largest choice of $\mathcal{A}$ is described in (2) and more concretely in Lemma 3. If one wishes make another choice, call it $\widetilde{\mathcal{A}}$, one has to confirm that $\mathcal{J} \subseteq \widetilde{\mathcal{A}} \subseteq \mathcal{A}$.

Theorem 2. *Let $\mathcal{B}$ be a C^*-algebra with identity and let $\{p_n\}_{n=1}^{\infty}$ be an increasing sequence of orthogonal projections in $\mathcal{B}$ with $p_n \neq 1$ for every n. Then the norm closure $\mathcal{J}$ of*

$$\mathcal{J}_0 = \{x \in \mathcal{B}\colon\ \exists n \geqslant 1.\ p_n x = x p_n = x\}$$

is a C^*-subalgebra that does not contain the identity of $\mathcal{B}$. We have

$$\mathcal{J} = \Big\{x \in \mathcal{B}\colon\ \lim_{n\to\infty} \|x - p_n x p_n\| = 0\Big\}. \tag{1}$$

Moreover $\mathcal{J}$ is an ideal in the C^*-algebra with identity $\mathcal{A}$ defined by

$$\mathcal{A} = \{a \in \mathcal{B}\colon\ a\mathcal{J} \subseteq \mathcal{J} \text{ and } \mathcal{J}a \subseteq \mathcal{J}\}. \tag{2}$$

If $\mathcal{B} = \mathcal{L}(\mathcal{H})$ and p_n converge strongly to I as $n \to \infty$ then

$$\mathcal{K}(\mathcal{H}) \subseteq \mathcal{J} \subseteq \mathcal{A},$$

so

$$\sigma\big(\pi_{\mathcal{J}}(x)\big) \subseteq \sigma_{\mathrm{ess}}(x) \subseteq \sigma(x)$$

for all $x \in \mathcal{A}$.

Proof. First note that if $p_n x = x p_n = x$ then $p_m x = x p_m = x$ for all $m \geqslant n$. It follows by elementary algebra that $\mathcal{J}_0$ is a *-subalgebra of $\mathcal{B}$, and this implies the same for $\mathcal{J}$. If $x \in \mathcal{J}_0$ then there exists n for which $1 - p_n = (1 - p_n)(1 - x)$. Therefore

$$1 = \|1 - p_n\| = \big\|(1 - p_n)(1 - x)\big\| \leqslant \|1 - p_n\| \|1 - x\| = \|1 - x\|$$

because $p_n \neq 1$ for every n. Hence $\|1 - x\| \geqslant 1$ for all $x \in \mathcal{J}$ and we can deduce that $1 \notin \mathcal{J}$.

If $x \in \mathcal{B}$ and $\lim_{n\to\infty} \|x - p_n x p_n\| = 0$ then the fact that $p_n x p_n \in \mathcal{J}_0$ implies that $x \in \mathcal{J}$. Conversely if $x \in \mathcal{J}$ and $\varepsilon > 0$ then there exists $y \in \mathcal{J}_0$ such that $\|x - y\| < \varepsilon$. There now exists $N \geqslant 1$ such that $y = p_n y p_n$ for all $n \geqslant N$. For all such n we have

$$\begin{aligned} \|x - p_n x p_n\| &\leqslant \|x - y\| + \|y - p_n x p_n\| \\ &= \|x - y\| + \|p_n y p_n - p_n x p_n\| \\ &\leqslant 2\|x - y\| \\ &< 2\varepsilon. \end{aligned}$$

Hence $\lim_{n\to\infty} \|x - p_n x p_n\| = 0$.

The proofs that $\mathcal{A}$ is a C^*-algebra with identity and that $\mathcal{J}$ is an ideal in $\mathcal{A}$ are both elementary algebra.

If $\mathcal{B} = \mathcal{L}(\mathcal{H})$ then in order to prove that $\mathcal{K}(\mathcal{H}) \subseteq \mathcal{J}$ it is sufficient by (1) and a density argument to observe that if x is a finite rank operator then $\lim_{n\to\infty} \|x - p_n x p_n\| \to 0$. The final inclusion of the theorem follows from Lemma 1. $\square$

The following provides an alternative description of $\mathcal{A}$.

Lemma 3. *Let $\mathcal{B}$, $\{p_n\}_{n=1}^\infty$, $\mathcal{J}$ and $\mathcal{A}$ be defined as in Theorem* 2. *Let*

$$\mathcal{D}_0 = \{a \in \mathcal{B}\colon\ \forall n \geqslant 1.\ \exists m \geqslant n.\ p_m a p_n = a p_n.\} \tag{3}$$

and

$$\mathcal{D} = \{a \in \mathcal{B}\colon\ \forall n \geqslant 1.\ a p_n \in \mathcal{J}\}. \tag{4}$$

Then $\mathcal{D}_0 \subseteq \mathcal{D}$ and $\mathcal{A} = \mathcal{D} \cap \mathcal{D}^*$.

Proof. The inclusions $\mathcal{D}_0 \subseteq \mathcal{D}$ and $\mathcal{A} \subseteq \mathcal{D} \cap \mathcal{D}^*$ are elementary. If $a \in \mathcal{D}$ and $x \in \mathcal{J}_0$ then for some $n \geqslant 1$ we have

$$ax = a(p_n x) = (a p_n)x \in \mathcal{J}. \quad \mathcal{J}_0 \subseteq \mathcal{J}.$$

A density argument now implies that $a\mathcal{J} \subseteq \mathcal{J}$. By taking adjoints we conclude that $\mathcal{D} \cap \mathcal{D}^* \subseteq \mathcal{A}$. $\square$

Note. In spite of the notation we do not claim that $\mathcal{D}$ is the norm closure of $\mathcal{D}_0$.

Lemma 4. *Let $\{p_n\}_{n=1}^{\infty}$ be an increasing sequence of projections on $\mathcal{H}$ that converge strongly to 1 and let $\mathcal{J}$ and $\mathcal{A}$ be constructed as described in Theorem 2. If $\{\phi_r\}_{r=1}^{\infty}$ is a sequence of unit vectors in $\mathcal{H}$ and $\lim_{r\to\infty}\|p_n\phi_r\|=0$ for every $n\geqslant 1$ then $\lim_{r\to\infty}\|a\phi_r\|=0$ for every $a\in\mathcal{J}$.*

Proof. This is elementary if $a\in\mathcal{J}_0$ and follows for all $a\in\mathcal{J}$ by approximation. □

We say that a sequence $\{\phi_r\}_{r=1}^{\infty}$ of unit vectors in $\mathcal{H}$ is localized (with respect to $\mathcal{J}$) if there exists $n\geqslant 1$ and $c>0$ such that $\|p_n\phi_r\|\geqslant c$ for all $r\geqslant 1$.

Theorem 5. *If $x\in\mathcal{A}$ and $\lambda\in\sigma(x)\setminus\sigma(\pi_{\mathcal{J}}(x))$ then there exists a sequence $\{\phi_r\}_{r=1}^{\infty}$ that is localized with respect to $\mathcal{J}$ and satisfies either*

$$\lim_{r\to\infty}\|x\phi_r-\lambda\phi_r\|=0 \tag{5}$$

or

$$\lim_{r\to\infty}\|x^*\phi_r-\bar{\lambda}\phi_r\|=0. \tag{6}$$

Proof. If $\lambda\in\sigma(x)$ then there exists a sequence $\{\phi_r\}_{r=1}^{\infty}$ of unit vectors such that either (5) or (6) holds; see [5, Lemma 1.2.13]. Both cases are similar and we only consider the first.

If $\lambda\in\sigma(x)\setminus\sigma(\pi_{\mathcal{J}}(x))$ and (5) holds and $\lim_{r\to\infty}\|p_n\phi_r\|=0$ for all $n\geqslant 1$ then $\pi_{\mathcal{J}}(\lambda 1-x)$ is invertible in $\mathcal{A}/\mathcal{J}$, so there exist $y\in\mathcal{A}$ and $a\in\mathcal{J}$ such that

$$y(\lambda 1-x)=1+a.$$

Lemma 4 now yields

$$\begin{aligned}1&=\lim_{r\to\infty}\big\|(1+a)\phi_r\big\|\\&\leqslant\lim_{r\to\infty}\big(\|y\|\big\|(\lambda 1-x)\phi_r\big\|\big)\\&=0.\end{aligned}$$

The contradiction establishes that if $\lambda\in\sigma(x)\setminus\sigma(\pi_{\mathcal{J}}(x))$ then $\|p_n\phi_r\|$ does not converge to 0 as $r\to\infty$ for some $n\geqslant 1$. It follows that there exists a subsequence $\{\psi_r\}_{r=1}^{\infty}$ and $c>0$ such that $\|p_n\psi_r\|\geqslant c$ for all $r\geqslant 1$. □

Note. Theorem 5 has no converse. If $a\in\mathcal{A}$ is a self-adjoint operator then any eigenvalue λ of a that is embedded in the continuous spectrum satisfies the conclusion of the theorem for the choice $\mathcal{J}=\mathcal{K}(\mathcal{H})$. One simply defines ϕ_n to be the normalized eigenvector of a corresponding to the eigenvalue λ for all n.

Sometimes one has several ideals in $\mathcal{A}$ but neither is contained in the other.

Theorem 6. *Let $\mathcal{J}_1$ and $\mathcal{J}_2$ be two ideals in the C^*-algebra $\mathcal{A}$ with identity, and put $\mathcal{J}_3=\mathcal{J}_1\cap\mathcal{J}_2$. Then*

$$\sigma_3(x)=\sigma_1(x)\cup\sigma_2(x)$$

for all $x\in\mathcal{A}$.

Proof. It is elementary that $\mathcal{J}_3$ is an ideal. Let $\mathcal{B} = \mathcal{A}/\mathcal{J}_1 \oplus \mathcal{A}/\mathcal{J}_2$ and define the C^*-homomorphism $\pi : \mathcal{A} \to \mathcal{B}$ by $\pi = \pi_1 \oplus \pi_2$. Then the image $\mathcal{C} = \pi(\mathcal{A})$ is a C^*-subalgebra of $\mathcal{B}$ and the kernel of π is $\mathcal{J}_3$. If $x \in \mathcal{A}$ then the spectrum of $\pi(x)$ is the same whether regarded as an element of $\mathcal{B}$ or $\mathcal{C}$. In the former case the spectrum is $\sigma_1(x) \cup \sigma_2(x)$ and in the latter case it is $\sigma_3(x)$. $\square$

We next describe one of the C^*-algebras that we shall be using in the next section. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be infinite-dimensional Hilbert spaces and let $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ be their Hilbert space tensor product. Let I_i denote the identity operator on $\mathcal{H}_i$ for $i = 1, 2$.

Theorem 7. *Let $\{P_n\}_{n=1}^{\infty}$ be an increasing sequence of finite rank projections in $\mathcal{H}_1$ which converges strongly to I_1 as $n \to \infty$ and put $p_n = P_n \otimes I_2$. Then $\mathcal{J}$ defined as in Theorem 2 is the closed linear span of all operators $A_1 \otimes A_2$ where $A_1 \in \mathcal{K}(\mathcal{H}_1)$ and $A_2 \in \mathcal{L}(\mathcal{H}_2)$. Also $\mathcal{A}$, defined as in Theorem 2, contains the closed linear span of all operators $A_1 \otimes A_2$ where $A_i \in \mathcal{L}(\mathcal{H}_i)$ for $i = 1, 2$.*

Proof. Let $\mathcal{J}'$ denote the closed linear span of all operators $a = A_1 \otimes A_2$ where $A_1 \in \mathcal{K}(\mathcal{H}_1)$ and $A_2 \in \mathcal{L}(\mathcal{H}_2)$. The formula

$$\lim_{n\to\infty} \|A_1 - P_n A_1 P_n\| = 0$$

implies

$$\lim_{n\to\infty} \|a - p_n a p_n\| = 0.$$

We deduce that $a \in \mathcal{J}$ and hence that $\mathcal{J}' \subseteq \mathcal{J}$. Conversely if $x \in \mathcal{J}_0$ then there exists $n \geqslant 1$ such that $x = p_n x p_n$. If P_n has rank k then $p_n x p_n$ can be written as the sum of k^2 terms of the form $A_1 \otimes A_2$ where each A_1 has rank 1. Hence $p_n x p_n \in \mathcal{J}'$. The inclusion $\mathcal{J}_0 \subseteq \mathcal{J}'$ implies $\mathcal{J} \subseteq \mathcal{J}'$. The final statement of the theorem follows directly from the inclusions

$$(A_1 \otimes A_2)\mathcal{J}' \subseteq \mathcal{J}', \qquad \mathcal{J}'(A_1 \otimes A_2) \subseteq \mathcal{J}'. \qquad \square$$

3. Application to discrete Schrödinger operators

In this section we construct a C^*-subalgebra $\mathcal{A}$ of $\mathcal{L}(\mathcal{H})$ where $\mathcal{H} = l^2(\mathbf{Z}^d)$ by an ad hoc procedure. A more systematic approach that uses a standard C^*-algebra is described in Section 4.

We put $\mathcal{H}_1 = l^2(\mathbf{Z})$ and $\mathcal{H}_2 = l^2(\mathbf{Z}^{d-1})$, so that

$$\mathcal{H} \simeq \mathcal{H}_1 \otimes \mathcal{H}_2 \simeq l^2(\mathbf{Z}, \mathcal{H}_2) \tag{7}$$

by means of canonical unitary isomorphisms. We define the projections p_n by

$$(p_n\phi)(x) = \begin{cases} \phi(x) & \text{if } -n \leqslant x_1 \leqslant n, \\ 0 & \text{otherwise,} \end{cases}$$

for all $\phi \in \mathcal{H}$ and $x \in \mathbf{Z}^d$. We also define the C^*-algebra $\mathcal{A}$ and the ideal $\mathcal{J}$ as in Theorems 2 and 7. The ideal $\mathcal{J}$ contains all bounded operators on $\mathcal{H}$ that are ‘concentrated’ in some neighbourhood of the dislocation set $S = \{0\} \times \mathbf{Z}^{d-1}$. In Section 4 we explain how to generalize the ideas in this section by allowing the dislocation set to have a completely general shape.

Lemma 8. *The C^*-algebra $\mathcal{A}$ contains all 'Schrödinger operators' of the form*

$$(A\phi)(x) = \sum_{r=1}^{m} a_r(x)\phi(x + b_r) \tag{8}$$

where $\phi \in l^2(\mathbf{Z}^d)$, $x \in \mathbf{Z}^d$, $m \in \mathbf{Z}_+$, $b_r \in \mathbf{Z}^d$ and $a_r \in l^\infty(\mathbf{Z}^d)$ for all $r \in \{1, 2, \dots, m\}$.

Proof. An elementary calculation implies that $p_{n+k}Ap_n = Ap_n$ for all $n \geqslant 1$ where $k = \max\{|b_r|\colon 1 \leqslant r \leqslant m\}$, so $A \in \mathcal{D}_0$. The same applies to A^*, so we may apply Lemma 3. □

We say that the Schrödinger operator A on $\mathcal{H}$ is periodic in the $\mathbf{Z}$ direction with period k if $T_kA = AT_k$ where $(T_k\phi)(m) = \phi(m + k)$ for all $\phi \in l^2(\mathbf{Z}, \mathcal{H}_2)$. This holds if and only if the coefficients a_r are all periodic in the $\mathbf{Z}$ direction with period k.

Theorem 9. *If the Schrödinger operator A is periodic in the $\mathbf{Z}$ direction with period k then*

$$\sigma\big(\pi_{\mathcal{J}}(A)\big) = \sigma_{\mathrm{ess}}(A) = \sigma(A). \tag{9}$$

If in addition $H = A + B + C$ where $B \in \mathcal{J}$ and $C \in \mathcal{K}(\mathcal{H})$, then

$$\sigma_{\mathrm{ess}}(A) \subseteq \sigma_{\mathrm{ess}}(A + B) = \sigma_{\mathrm{ess}}(H) \subseteq \sigma(H). \tag{10}$$

Proof. The identities in (9) follow directly from Lemma 1 provided we can prove that $\sigma(A) \subseteq \sigma(\pi_{\mathcal{J}}(A))$. If $\lambda \in \sigma(A)$ then there exists a sequence $\{\phi_r\}_{r=1}^\infty$ of unit vectors such that either $\lim_{r\to\infty} \|A\phi_r - \lambda\phi_r\| = 0$ or $\lim_{r\to\infty} \|A^*\phi_r - \bar{\lambda}\phi_r\| = 0$; see [5, Lemma 1.2.13]. Both cases are similar, so we only consider the first.

By translating the ϕ_r sufficiently and using the translation invariance of A, we see that there exists a sequence $\{\psi_r\}_{r=1}^\infty$ of unit vectors such that $\lim_{r\to\infty} \|A\psi_r - \lambda\psi_r\| = 0$ and $\lim_{r\to\infty} \|p_n\psi_r\| = 0$ for every n. The argument of Theorem 5 establishes that $\lambda \in \sigma(\pi_{\mathcal{J}}(A))$ and hence that $\sigma(A) \subseteq \sigma(\pi_{\mathcal{J}}(A))$.

The statements in (10) now follow from Lemma 1 as soon as one observes that $\sigma(\pi_{\mathcal{J}}(H)) = \sigma(\pi_{\mathcal{J}}(A))$ and $\sigma(\pi_{\mathcal{K}(\mathcal{H})}(H)) = \sigma(\pi_{\mathcal{K}(\mathcal{H})}(A + B))$. □

The following theorem identifies the asymptotic part of the spectrum of certain Schrödinger operators H as $x_1 \to -\infty$. The operators concerned have much in common with those of [7], but we allow them to be non-self-adjoint and require the underlying space to be discrete.

Theorem 10. *Let $S = \{x \in \mathbf{Z}^d\colon x_1 \geqslant 0\}$ and put*

$$(p_n\phi)(x) = \begin{cases} \phi(x) & \text{if } x_1 \geqslant -n, \\ 0 & \text{otherwise,} \end{cases}$$

for all $\phi \in l^2(\mathbf{Z}^d)$ and $n \geqslant 0$. Let A be of the form (8) *and suppose that it is periodic in the x_1 direction. Also let $H = A + B$ where B is any bounded operator confined to S in the sense that $p_0 B = B p_0 = B$. If $\mathcal{J}$ is defined as in Theorem* 2 *then*

$$\sigma(A) = \sigma_{\mathrm{ess}}(A) = \sigma\big(\pi_{\mathcal{J}}(H)\big) \subseteq \sigma_{\mathrm{ess}}(H) \subseteq \sigma(H).$$

We omit the proof, which is similar to that of Theorem 9 and uses the fact that $B \in \mathcal{J}$.

We finally come to an application that involves two different closed ideals. Let $H = A + V_1 + V_2$ where A acts on $\mathcal{H} = l^2(\mathbf{Z}^2)$, is of the form (8) and is periodic in both horizontal and vertical directions. We assume that the bounded potential V_1 has support in $\mathbf{Z} \times [-a_2, a_2]$ while the bounded potential V_2 has support in $[-a_1, a_1] \times \mathbf{Z}$ for some finite a_1, a_2. Let $\mathcal{J}_1$ be the ideal associated with the sequence of projections

$$(p_n\phi)(i,j) = \begin{cases} \phi(i,j) & \text{if } -n \leqslant i \leqslant n, \\ 0 & \text{otherwise,} \end{cases}$$

and let $\mathcal{J}_2$ be the ideal associated with the sequence of projections

$$(q_n\phi)(i,j) = \begin{cases} \phi(i,j) & \text{if } -n \leqslant j \leqslant n, \\ 0 & \text{otherwise.} \end{cases}$$

The appropriate C^*-algebra $\mathcal{A}$ is defined by

$$\mathcal{A} = \big\{x \in \mathcal{L}(\mathcal{H}):\ x\mathcal{J}_1 \subseteq \mathcal{J}_1,\ \mathcal{J}_1 x \subseteq \mathcal{J}_1,\ x\mathcal{J}_2 \subseteq \mathcal{J}_2,\ \mathcal{J}_2 x \subseteq \mathcal{J}_2\big\}.$$

Theorem 11. *Under the above assumptions $H \in \mathcal{A}$ and*

$$\sigma_{\mathrm{ess}}(H) = \sigma_1(A + V_1) \cup \sigma_2(A + V_2).$$

If V_1 is periodic in the x_1 direction and V_2 is periodic in the x_2 direction then

$$\sigma_{\mathrm{ess}}(H) = \sigma(A + V_1) \cup \sigma(A + V_2).$$

Proof. Since $V_2 \in \mathcal{J}_1$, we have $\sigma_1(H) = \sigma_1(A + V_1)$. Since $V_1 \in \mathcal{J}_2$, we have $\sigma_2(H) = \sigma_2(A + V_2)$. In order to apply Theorem 6 we need to prove that $\sigma_3(H) = \sigma_{\mathrm{ess}}(H)$. This follows if $\mathcal{J}_1 \cap \mathcal{J}_2 = \mathcal{K}(\mathcal{H})$. The only non-trivial part is to prove that if $x \in \mathcal{J}_1 \cap \mathcal{J}_2$ then $x \in \mathcal{K}(\mathcal{H})$.

Given such an x put $x_{m,n} = p_m q_n x q_n p_m$ for all $m, n \geqslant 1$. Noting that p_m and q_n commute and that their product is of finite rank we see that $x_{m,n} \in \mathcal{K}(\mathcal{H})$ for all m, n. Since $x \in \mathcal{J}_2$ we have

$$\lim_{n\to\infty} x_{m,n} = p_m x p_m$$

and since $x \in \mathcal{J}_1$ we have

$$\lim_{m\to\infty} p_m x p_m = x.$$

Therefore $x \in \mathcal{K}(\mathcal{H})$.

The final statement of the theorem involves an application of Theorem 9. □

4. The standard C^*-algebra

If $\mathcal{A}=\mathcal{L}(\mathcal{H})$ for some infinite-dimensional, separable Hilbert space $\mathcal{H}$ then $\mathcal{A}$ contains only one non-trivial ideal, namely $\mathcal{K}(\mathcal{H})$. In this section we construct a 'slightly smaller' C^*-algebra which has a rich ideal structure. We formulate the theory at a very general metric space level, so that it is applicable not only to $\mathbf{Z}^d$ and $\mathbf{R}^d$, but to unbounded discrete and continuous graphs and waveguides, in which X is an unbounded region in $\mathbf{R}^d$. In Section 8 we show that it may also be applied to Schrödinger operators on Riemannian manifolds, writing out the details in the case of three-dimensional hyperbolic space.

If $\mathcal{H}=L^2(\mathbf{R}^d)$ or $\mathcal{H}=l^2(\mathbf{Z}^d)$ then the C^*-algebra $\mathcal{A}$ constructed below coincides with the algebra $C^u(Q)$ of [11] by virtue of [11, Propositions 4.11, 4.12]. However, this fact depends on the use of Fourier transforms on $\mathbf{R}^d$ or $\mathbf{Z}^d$, which have no analogue in our more general context.

Let (X,d,μ) denote a space X provided with a metric d and a measure μ; we require X to be a complete separable metric space with infinite diameter in which every closed ball is compact; all balls in this paper are taken to have positive and finite radius. We also require that the measure of every open ball $B(a,r)=\{x\in X\colon d(x,a)<r\}$ is positive and finite. Let $\mathcal{U}$ denote the class of all non-empty, open subsets of X.

If $S,T\subseteq X$ we put

$$d(S,T)=\inf\bigl\{d(s,t)\colon s\in S \text{ and } t\in T\bigr\}.$$

The function $x\to d(x,S)$ is continuous on X; indeed

$$\bigl|d(x,S)-d(y,S)\bigr|\leqslant d(x,y)$$

for all $x,y\in X$ and $S\subseteq X$. If (X,d) is a length space in the sense of Gromov then

$$\overline{B(a,r)}=\bigl\{x\in X\colon d(x,a)\leqslant r\bigr\}$$

and

$$d\bigl(B(a,r),B(b,s)\bigr)=\max\bigl\{d(a,b)-r-s,0\bigr\}$$

for all $a,b\in X$ and $r,s>0$. However, if $X=\mathbf{Z}^d$ with the Euclidean metric, neither of these identities need hold.

Now put $\mathcal{H}=L^2(X,\mu)$. For any $S\in\mathcal{U}$ we define the projection P_S on $\mathcal{H}$ by

$$(P_S\phi)(x)=\begin{cases}\phi(x) & \text{if } x\in S,\\ 0 & \text{otherwise.}\end{cases}$$

We abbreviate $P_{B(a,r)}$ to $P_{a,r}$.

Lemma 12. *If $A\in\mathcal{L}(\mathcal{H})$ then there exists a largest open set U such that $AP_U=0$. There also exists a largest open set V such that $P_VA=0$.*

Proof. If $\mathcal{V}$ is the class of all open sets V such that $AP_V = 0$ then the only candidate for U is $U = \bigcup_{V \in \mathcal{V}} V$ and by Lindelöf's theorem we may also write $U = \bigcup_{n=1}^{\infty} V_n$ where V_n is a sequence of sets in $\mathcal{V}$. Put $W_1 = V_1$ and $W_{n+1} = W_n \cup V_{n+1}$. If $W_n \in \mathcal{V}$ then

$$AP_{W_{n+1}} = AP_{W_n} + AP_{V_{n+1}}(1 - P_{W_n}) = 0,$$

so $W_{n+1} \in \mathcal{V}$. It follows by induction that $AP_{W_n} = 0$ for all $n \geqslant 1$. Now P_{W_n} is an increasing sequence of projections that converges weakly to P_U so $AP_U = 0$. The second statement of the lemma has a similar proof. □

Lemma 13. *If $A, B \in \mathcal{L}(\mathcal{H})$ and $AB \neq 0$ then for every $\varepsilon > 0$ there exists $a \in X$ such that $AP_{a,\varepsilon} \neq 0$ and $P_{a,\varepsilon} B \neq 0$.*

Proof. Let $\{a_n\}_{n=1}^{\infty}$ be a countable dense set in X and define the sets E_N inductively by $E_1 = B(a_1, \varepsilon)$ and

$$E_{n+1} = B(a_{n+1}, \varepsilon) \setminus (E_1 \cup \cdots \cup E_n).$$

It follows directly that the sets E_n are disjoint and that their union is X. Therefore

$$\lim_{n \to \infty} \sum_{r=1}^{n} P_{E_r} = I,$$

the limit being in the weak operator topology. Therefore

$$\lim_{n \to \infty} \sum_{r=1}^{n} AP_{E_r} B = AB \neq 0$$

in the same sense and there must exist n such that $AP_{E_n} B \neq 0$. We conclude first that $AP_{E_n} \neq 0$ and $P_{E_n} B \neq 0$ and then that $AP_{a_n,\varepsilon} \neq 0$ and $P_{a_n,\varepsilon} B \neq 0$. □

We say that $A \in \mathcal{L}(\mathcal{H})$ lies in $\mathcal{A}_m$ (or that A has range m) if $P_{a,r} A P_{b,s} \neq 0$ implies $d(a,b) \leqslant r + s + m$. If A has an integral kernel K this amounts to requiring that $K(x,y) \neq 0$ implies $d(x,y) \leqslant m$, but we do not require that A has such a kernel.

Lemma 14. *If $A \in \mathcal{A}_m$ and $B \in \mathcal{A}_n$ then $A^* \in \mathcal{A}_m$, $A + B \in \mathcal{A}_{\max(m,n)}$ and $AB \in \mathcal{A}_{m+n}$.*

Proof. The invariance of $\mathcal{A}_m$ under adjoints follows immediately from its definition.

If $P_{a,r}(A+B)P_{b,s} \neq 0$ then $P_{a,r} A P_{b,s} \neq 0$ or $P_{a,r} B P_{b,s} \neq 0$. Therefore $d(a,b) \leqslant r + s + m$ or $d(a,b) \leqslant r + s + n$. In both cases we deduce that $d(a,b) \leqslant r + s + \max(m,n)$.

If $P_{a,r} AB P_{b,s} \neq 0$ then Lemma 13 implies that for every $\varepsilon > 0$ there exists $c \in X$ such that $P_{a,r} A P_{c,\varepsilon} \neq 0$ and $P_{c,\varepsilon} B P_{b,s} \neq 0$. Therefore $d(a,c) \leqslant r + \varepsilon + m$ and $d(c,b) \leqslant \varepsilon + s + n$. These imply that $d(a,b) \leqslant r + s + m + n + 2\varepsilon$. Letting $\varepsilon \to 0$ we finally deduce that $AB \in \mathcal{A}_{m+n}$. □

We will frequently refer to the standard C^*-algebra $\mathcal{A}$ below; this is defined in the next theorem. The algebra $\widetilde{\mathcal{A}}$ below is called the set of all finite range operators in [11, Section 4].

Theorem 15. *If $\mathcal{A}$ is the norm closure of $\widetilde{\mathcal{A}} = \bigcup_{n=0}^{\infty} \mathcal{A}_n$ then $\mathcal{A}$ is a C^*-subalgebra of $\mathcal{L}(\mathcal{H})$. If $V \in L^\infty(X, \mu)$ and V also denotes the operator of multiplication by the function V, then $V \in \mathcal{A}$. Moreover $\mathcal{K}(\mathcal{H}) \subseteq \mathcal{A}$.*

Proof. The first statement follows directly from Lemma 14. If $P_{a,r} V P_{b,s} \neq 0$ then $P_{a,r} P_{b,s} V \neq 0$ and hence $P_{a,r} P_{b,s} \neq 0$. Therefore the open set $U = B(a,r) \cap B(b,s)$ is not empty, and there exists $c \in X$ with $d(a,c) < r$ and $d(b,c) < s$. Therefore $d(a,b) < r + s \leqslant r + s + 0$ and $V \in \mathcal{A}_0$.

If A is compact and $A = AP_U = P_U A$ for some open set U with diameter n then $P_{a,r} A P_{b,s} \neq 0$ implies $P_{a,r} P_U A P_U P_{b,s} \neq 0$ and hence $P_{a,r} P_U \neq 0$ and $P_U P_{b,s} \neq 0$. Hence there exist $u, v \in U$ such that $d(a,u) < r$ and $d(v,b) < s$. We deduce that $d(a,b) < r + s + n$ so $A \in \mathcal{A}_n$. Since the set of all such A is norm dense in $\mathcal{K}(\mathcal{H})$, we conclude that $\mathcal{K}(\mathcal{H}) \subseteq \mathcal{A}$. □

If $S \in \mathcal{U}$ and $r > 0$ we put

$$S(r) = \{x \in X\colon\ d(x, S) < r\} = \bigcup \{B(x,r)\colon\ x \in S\}.$$

The following alternative definition of $\mathcal{A}$ is slightly more transparent in spite of the fact that it quantifies over a much larger class of sets.

Theorem 16. *Given $m \geqslant 1$, let $\mathcal{Y}_m$ denote the set of all $A \in \mathcal{L}(\mathcal{H})$ such that for every $S \in \mathcal{U}$ one has $AP_S = P_{S(m)} A P_S$. Then $\mathcal{A}$ is the norm closure of $\bigcup_{m=0}^{\infty} \mathcal{Y}_m$.*

Proof. If we put $T(m) = X \setminus S(m)$ then $A \in \mathcal{Y}_m$ if and only if for every $S \in \mathcal{U}$ one has $P_{T(m)} A P_S = 0$.

Let $A \in \mathcal{A}_m$, $0 < r < 1/3$ and $s = 1/3$. If $S \in \mathcal{U}$ and $b \in T(m+1)$ then $B(a,r) \subseteq S$ implies $d(a,b) \geqslant m + 1 > r + s + m$ and then $P_{b,s} A P_{a,r} = 0$. Since S may be written as the union of a countable number of balls $B(a,r)$ with $0 < r < 1/3$, Lemma 12 implies that $P_{b,s} A P_S = 0$. Since $T(m+1)$ may be covered by a countable number of balls $B(b,s)$, all with $s = 1/3$, we deduce that $P_{T(m+1)} A P_S = 0$. Therefore $A \in \mathcal{Y}_{m+1}$.

Conversely let $A \in \mathcal{Y}_m$, $r,\ s > 0$ and $d(a,b) > r + s + m$. If we put $S = B(a,r)$ then $B(b,s) \subseteq T(m)$, so $P_{T(m)} A P_S = 0$ implies $P_{b,s} A P_{a,r} = 0$. Therefore $A \in \mathcal{A}_m$.

The two inclusions together imply

$$\bigcup_{m=1}^{\infty} \mathcal{A}_m = \bigcup_{m=1}^{\infty} \mathcal{Y}_m$$

and hence the statement of the theorem. □

We wish to associate an ideal $\mathcal{J}_S$ with every non-empty open subset S of X. This may be done in two ways and we will prove that they yield the same result. The idea is to identify operators that 'decrease in size' as one moves away from S. It will become clear that $\mathcal{J}_S$ depends only on the asymptotic form of S at infinity, and that two sets S_1 and S_2 that move away from each other as one goes to infinity give rise to different ideals, however slowly this separation occurs.

If $S \in \mathcal{U}$ and $r > 0$, we put

$$\begin{aligned}\mathcal{J}_{S,n} &= \{A \in \mathcal{A}\colon\ A = P_{S(n)} A P_{S(n)}\}\\ &= \{A \in \mathcal{A}\colon\ A = A P_{S(n)} = P_{S(n)} A\}\\ &= \{A \in \mathcal{A}\colon\ 0 = A P_{T(n)} = P_{T(n)} A\}\end{aligned}$$

where $T(n) = X \setminus S(n) = \{x \in X\colon\ d(x,S) \geqslant n\}$. We also define

$$\begin{aligned}\mathcal{K}_{S,n} &= \big\{A \in \mathcal{A}\colon\ A P_{a,r} \neq 0 \ \Rightarrow\ d(a,S) \leqslant n + r\big\}\\ &\quad \cap \big\{A \in \mathcal{A}\colon\ P_{a,r} A \neq 0 \ \Rightarrow\ d(a,S) \leqslant n + r\big\}\\ &= \big\{A \in \mathcal{A}\colon\ d(a,S) > n + r \ \Rightarrow\ A P_{a,r} = P_{a,r} A = 0\big\}.\end{aligned}$$

Lemma 17. *If $n \geqslant 1$ then*

$$\begin{aligned}&\bigcup \big\{B(x,r)\colon\ d(x,S) > r + n\big\}\\ &\quad \subseteq T(n) \subseteq \bigcup \big\{B(x,r)\colon\ d(x,S) > r + n - 1\big\}.\end{aligned} \tag{11}$$

Hence

$$\mathcal{K}_{S,n-1} \subseteq \mathcal{J}_{S,n} \subseteq \mathcal{K}_{S,n}. \tag{12}$$

Proof. If $y \in B(x,r)$ and $d(x,S) > r + n$ then $d(y,S) > n$. Hence $B(x,r) \subseteq T(n)$. This proves the first inclusion of (11).

If $x \in T(n)$ then $d(x,S) \geqslant n$. Putting $r = 1/2$ we deduce that $x \in B(x,r)$ and $d(x,S) > r + n - 1$. This proves the second inclusion of (11).

If $A \in \mathcal{K}_{S,n-1}$ then $A P_{x,r} = P_{x,r} A = 0$ for all x, r such that $d(x,S) > r + n - 1$, so Lemma 12 and the second inclusion of (11) together imply that $A P_{T(n)} = P_{T(n)} A = 0$. Therefore $A \in \mathcal{J}_{S,n}$. On the other hand if $A \in \mathcal{J}_{S,n}$ then $A P_{T(n)} = P_{T(n)} A = 0$. The first inclusion (11) of now implies that $A P_{x,r} = P_{x,r} A = 0$ whenever $d(x,S) > r + n$. Therefore $A \in \mathcal{K}_{S,n}$. This completes the proof of (12). $\square$

Let $\mathcal{F}$ denote the family of all non-empty open sets S such that $S(n) \neq X$ for every $n \geqslant 1$. We say that $S, T \in \mathcal{F}$ are asymptotically equivalent if for all $n \geqslant 1$ there exists $m \geqslant 1$ such that $S(n) \subseteq T(m)$ and $T(n) \subseteq S(m)$. In particular all non-empty, open, bounded sets are asymptotically equivalent to each other.

Theorem 18. *If $S \in \mathcal{U}$ then*

$$\overline{\bigcup_{n=1}^{\infty} \mathcal{J}_{S,n}} = \overline{\bigcup_{n=1}^{\infty} \mathcal{K}_{S,n}}.$$

If $S \in \mathcal{F}$ then this set, denoted by $\mathcal{J}_S$, is a proper, closed, two-sided ideal in $\mathcal{A}$ and it contains $\mathcal{K}(\mathcal{H})$. If S, T are asymptotically equivalent then $\mathcal{J}_S = \mathcal{J}_T$.

Proof. Lemma 17 implies that

$$\bigcup_{n=1}^{\infty} \mathcal{J}_{S,n} = \bigcup_{n=1}^{\infty} \mathcal{K}_{S,n}.$$

We denote this linear subspace of $\mathcal{A}$ by $\mathcal{J}_S^\circ$.

Let $A \in \mathcal{A}_m$ and $B \in \mathcal{K}_{S,n}$. If $ABP_{a,r} \neq 0$ then $BP_{a,r} \neq 0$ so $d(a,S) \leqslant n+r$. If $P_{a,r}AB \neq 0$ then Lemma 13 implies that for every $\varepsilon > 0$ there exists $b \in X$ such that $P_{a,r}AP_{b,\varepsilon} \neq 0$ and $P_{b,\varepsilon}B \neq 0$. Therefore $d(a,b) \leqslant m+r+\varepsilon$ and $d(b,S) \leqslant n+\varepsilon$. We conclude that $d(a,S) \leqslant m+n+r+2\varepsilon$. Since $\varepsilon > 0$ is arbitrary we deduce that $d(a,S) \leqslant m+n+r$. Therefore $AB \in \mathcal{K}_{S,m+n}$. A similar argument can be applied to BA. These calculations imply that $\widetilde{\mathcal{A}}\mathcal{J}_S^\circ \subseteq \mathcal{J}_S^\circ$ and $\mathcal{J}_S^\circ \widetilde{\mathcal{A}} \subseteq \mathcal{J}_S^\circ$. The statement of the theorem now follows by a density argument.

In order to prove that $\mathcal{J}_S$ is proper we need to establish that $\|I - A\| \geqslant 1$ for all $A \in \mathcal{J}_S$, or equivalently that this holds for all $A \in \mathcal{J}_{S,n}$ and all $n \geqslant 1$. If $A = AP_{S(n)} = P_{S(n)}A$ then this follows from

$$\|I - P_{S(n)}\| = \left\|(I - P_{S(n)})(I - A)\right\| \leqslant \|I - P_{S(n)}\|\|I - A\| \leqslant \|I - A\|$$

provided $\|I - P_{S(n)}\| = 1$. Since $S \in \mathcal{F}$ there exists $a \in X \setminus S(n+2)$. This implies that $B(a,1) \cap S(n) = \emptyset$. Since $B(a,1)$ has positive measure there exists a non-zero $\phi \in \mathcal{H}$ whose support is contained in $B(a,1)$ and for which $(I - P_{S(n)})\phi = \phi$.

If A is a finite rank operator then $\lim_{n\to\infty} \|A - P_{S(n)}AP_{S(n)}\| = 0$, so $A \in \mathcal{J}_S$. The same applies to all $A \in \mathcal{K}(\mathcal{H})$ by a density argument.

If S, T are asymptotically equivalent then routine algebra shows that $\mathcal{J}_S^\circ = \mathcal{J}_T^\circ$. This implies immediately that $\mathcal{J}_S = \mathcal{J}_T$. $\square$

If $A \in \mathcal{A}$ and $S \in \mathcal{F}$ we put $\sigma_S(A) = \sigma(\pi_{\mathcal{J}_S}(A))$.

Theorem 19. *Let $S, T \in \mathcal{F}$. If $S \subseteq T$ then $\mathcal{J}_S \subseteq \mathcal{J}_T$ and $\sigma_S(A) \supseteq \sigma_T(A)$ for every $A \in \mathcal{A}$. If $S, T \in \mathcal{F}$ are asymptotically independent in the sense that*

$$\forall n \geqslant 1.\ \exists m \geqslant 1. \quad S(n) \cap T(n) \subseteq (S \cap T)(m), \tag{13}$$

then

$$\mathcal{J}_{S\cap T} = \mathcal{J}_S \cap \mathcal{J}_T \tag{14}$$

and

$$\sigma_{S\cap T}(A) = \sigma_S(A) \cup \sigma_T(A) \tag{15}$$

for every $A \in \mathcal{A}$.

Proof. If $A \in \mathcal{J}_S^\circ$ then there exists $n \geqslant 1$ such that $A = AP_{S(n)} = P_{S(n)}A$. If $S \subseteq T$, this implies $A = AP_{T(n)} = P_{T(n)}A$ and hence $\mathcal{J}_S^\circ \subseteq \mathcal{J}_T^\circ$. Therefore $\mathcal{J}_S \subseteq \mathcal{J}_T$ and $\sigma_S(A) \supseteq \sigma_T(A)$ for every $A \in \mathcal{A}$ by Lemma 1.

If $S, T \in \mathcal{F}$ we deduce that $\mathcal{J}_{S\cap T} \subseteq \mathcal{J}_S \cap \mathcal{J}_T$. Now suppose that S, T are asymptotically independent and that $A \in \mathcal{J}_S \cap \mathcal{J}_T$. Eq. (1) implies

$$\lim_{n\to\infty} \|A - P_{S(n)} A P_{S(n)}\| = 0, \qquad \lim_{n\to\infty} \|A - P_{T(n)} A P_{T(n)}\| = 0.$$

If we put

$$\begin{aligned} A_n &= P_{S(n)} P_{T(n)} A P_{T(n)} P_{S(n)} \\ &= P_{S(n)\cap T(n)} A P_{T(n)\cap S(n)}, \end{aligned}$$

then asymptotic independence implies

$$A_n = P_{\{S\cap T\}(m)} A_n = A_n P_{\{S\cap T\}(m)},$$

so $A_n \in \mathcal{J}^{\circ}_{S\cap T}$. Finally

$$\begin{aligned} \lim_{n\to\infty} \|A - A_n\| &\leqslant \lim_{n\to\infty} \|A - P_{S(n)} A P_{S(n)}\| \\ &\quad + \lim_{n\to\infty} \left\| P_{S(n)} (A - P_{T(n)} A P_{T(n)}) P_{S(n)} \right\| \\ &\leqslant \lim_{n\to\infty} \|A - P_{S(n)} A P_{S(n)}\| + \lim_{n\to\infty} \|A - P_{T(n)} A P_{T(n)}\| \\ &= 0. \end{aligned}$$

Therefore $A \in \mathcal{J}_{S\cap T}$. Eq. (15) finally follows from Theorem 6. □

The C^*-algebra $\mathcal{A}$ contains $L^\infty(X)$ and is therefore not separable. It is unlikely that one can obtain a useful classification of its irreducible representations, but a *partial* classification of its ideals can be obtained as follows.

Let $\overline{X}$ be some compactification of X and let $\partial X = \overline{X} \setminus X$ denote the 'points at infinity'. The restriction of any $f \in C(\overline{X})$ to X lies in $L^\infty(X, \mu)$. Since every non-empty open subset of X has positive measure we see that

$$\|f\|_{C(\overline{X})} = \|f\|_{L^\infty} = \|f\|_{\mathcal{L}(\mathcal{H})} = \|f\|_{\mathcal{A}}. \tag{16}$$

It follows that $\mathcal{B} = C(\overline{X})$ is a commutative C^*-subalgebra of $\mathcal{A}$. Note that there is a order-preserving one-one correspondence between the ideals $\mathcal{I}$ in $\mathcal{B}$ and the open subsets V of $\overline{X}$. It is given by

$$V_{\mathcal{I}} = \left\{x \in \overline{X}:\ f(x) \neq 0 \text{ for some } f \in \mathcal{I}\right\}$$

and

$$\mathcal{I}_V = \left\{f \in C(\overline{X}):\ f|_{\overline{X}\setminus V} = 0\right\}.$$

We will write $\overline{E}$ to denote the (compact) closure of a set $E \subseteq \overline{X}$ in $\overline{X}$, even if $E \subseteq X$. If U is an open subset of X then we define its set of asymptotic directions $\widetilde{U} \subseteq \partial X$ to be the set of all

$a \in \partial X$ that possess a neighbourhood $V \subseteq \overline{X}$ for which $V \cap X \subseteq U$. It is immediate that $\widetilde{U}$ is an open subset of ∂X and that $U \cup \widetilde{U}$ is an open subset of $\overline{X}$ with complement $\overline{X \setminus U}$.

If $S \in \mathcal{U}$ then $S(n)$ is an increasing sequence of open sets in X, so $\widetilde{S(n)}$ is an increasing sequence of open subsets of ∂X. We put

$$\widehat{S} = \bigcup_{n \geqslant 1} \widetilde{S(n)}$$

and observe that $\widehat{S}$ is also an open subset of ∂X.

Example 20. Let $X = \mathbf{R}^d$ with the usual Euclidean metric and let Σ be the 'sphere at infinity' parametrized by unit vectors e, called directions.

(a) If

$$S = \left\{x \in X\colon \forall i \in \{1, 2, \ldots, d\}.\ x_i > 0\right\},$$

then

$$\widetilde{S(n)} = \widehat{S} = \left\{e \in \Sigma\colon \forall i \in \{1, 2, \ldots, d\}.\ e_i > 0\right\}$$

for all $n \geqslant 1$. Therefore $\mathcal{B}/(\mathcal{J}_S \cap \mathcal{B}) \simeq C(K)$ where

$$K = \left\{e \in \Sigma\colon \exists i \in \{1, 2, \ldots, d\}.\ e_i \leqslant 0\right\}.$$

(b) If we are only interested in asymptotics in a particular direction $e \in \Sigma$ then we may define

$$S = \mathbf{R}^d \setminus \bigcup_{r>0} \overline{B\left(re, r^{1/2}\right)}.$$

One sees that $S \in \mathcal{F}$ and

$$\widetilde{S(n)} = \widehat{S} = \Sigma \setminus \{e\}$$

for all $n \geqslant 1$. The quotient map π from $\mathcal{B}$ to $\mathcal{B}/(\mathcal{J}_S \cap \mathcal{B}) \simeq \mathbf{C}$ is given by $\pi(f) = f(e)$.

Lemma 21. *If $S \in \mathcal{U}$ then $\mathcal{J}_{S,0} \cap L^\infty(X, \mu)$ is dense in $\mathcal{J}_S \cap L^\infty(X, \mu)$.*

Proof. Let $f \in \mathcal{J}_S \cap L^\infty(X, \mu)$. If p_n is the multiplication operator associated with the characteristic function of $S(n)$ then $p_n f \in \mathcal{J}_{S,0} \cap L^\infty$ for all $n \geqslant 1$ and $\lim_{n\to\infty} \|f - p_n f\| = 0$ by (1). $\square$

Theorem 22. *The map $\mathcal{J} \to V_{\mathcal{J}\cap\mathcal{B}}$ defines an order-preserving map from ideals in $\mathcal{A}$ to open subsets of $\overline{X}$. If $S \in \mathcal{U}$ then*

$$V_{\mathcal{J}_S \cap \mathcal{B}} = \widehat{S} \cup X.$$

If $S \in \mathcal{F}$ then $\widehat{S} \cup X \neq \overline{X}$.

Proof. The first statement of the theorem depends on the observation that if $\mathcal{J}$ is an ideal in $\mathcal{A}$ then $\mathcal{J}\cap\mathcal{B}$ is an ideal in $\mathcal{B}$.

Given $S\in\mathcal{U}$, we put $V=V_{\mathcal{J}_S\cap\mathcal{B}}$. It follows directly from the definitions that

$$C_c\big(S(n)\cup\widetilde{S(n)}\big)\subseteq\mathcal{J}_{S,0}\cap\mathcal{B}\subseteq\mathcal{J}_S\cap\mathcal{B}$$

where C_c denotes the space of continuous functions with compact support in the stated set. Therefore $S(n)\cup\widetilde{S(n)}\subseteq V$ for all $n\geqslant 1$. Since S is non-empty, letting $n\to\infty$ we obtain $X\cup\widehat{S}\subseteq V$.

If $a\notin X\cup\widehat{S}$ then there exists $f\in C(\overline{X})$ such that $f(a)=1$. Given $g\in\mathcal{J}_{S,0}\cap L^\infty(X)$ there exists $n\geqslant 1$ such that $g=gp_n=p_ng$, where p_n is the characteristic function of $S(n)$. Since $a\in\overline{X\setminus S(n+2)}$, given $\varepsilon>0$, there exists $b\in X\setminus S(n+2)$ such that $|f(b)-1|<\varepsilon$. Putting $\varepsilon=1/2$ there exists $\delta\in(0,1)$ such that $x\in B(b,\delta)$ implies $|f(x)|>1/2$ and $x\notin S(n)$. The set $B(b,\delta)$ has positive measure so $\|f-g\|_\infty>1/2$. Lemma 21 now implies that $\|f-h\|_\infty\geqslant 1/2$ for all $h\in\mathcal{J}_S\cap L^\infty(X)$ so $f\notin\mathcal{J}_S\cap\mathcal{B}$. Since this holds for all $f\in C(\overline{X})$ such that $f(a)=1$ we conclude that $a\notin V$ and $V\subseteq X\cup\widehat{S}$.

The final statement of the theorem follows from the fact that $S\in\mathcal{F}$ implies $1\notin\mathcal{J}_S$. □

Corollary 23. *If $S\in\mathcal{F}$, then*

$$\mathcal{B}/(\mathcal{J}_S\cap\mathcal{B})\simeq C(\partial X\setminus\widehat{S}).$$

5. Pseudo-resolvents

If one has a family of resolvent operators $R(z,A)$ all lying in a C^*-algebra $\mathcal{A}$ and $\pi:\mathcal{A}\to\mathcal{B}$ is an algebra homomorphism with a non-trivial kernel $\mathcal{J}$, then $\pi(R(z))$ satisfy the resolvent equations in $\mathcal{B}$. In this section we show how to define the spectrum of this new family, which is not the resolvent family of any obvious operator. This will be a crucial ingredient of our general theory.

Let $\mathcal{A}^\circ$ denote the set of invertible elements of an associative algebra $\mathcal{A}$ with identity. If $a\in\mathcal{A}$ the spectrum of a is defined by

$$\sigma(a)=\big\{\alpha\in\mathbf{C}\colon\ \alpha 1-a\notin\mathcal{A}^\circ\big\}.$$

If we put $U=\mathbf{C}\setminus\sigma(a)$ and define $r:U\to\mathcal{A}$ by $r_z=(z1-a)^{-1}$ then r satisfies the resolvent equations

$$r_\alpha-r_\gamma=(\gamma-\alpha)r_\alpha r_\gamma \tag{17}$$

for all $\alpha,\gamma\in U$. Moreover

$$1+(\gamma-\alpha)r_\alpha=(\gamma 1-a)r_\alpha$$

so $\sigma(a)=\{z\colon\ 1+(z-\alpha)r_\alpha\notin\mathcal{A}^\circ\}$.

Our goal in this section is to define the spectrum of a pseudo-resolvent, defined as a function $r:U\to\mathcal{A}$ that satisfies (17) even though it is not generated by any $a\in\mathcal{A}$.

If A is a closed, unbounded operator on a Banach space $\mathcal{B}$ and $R(z,A)$ denotes its family of resolvent operators, defined for all $z\notin\sigma(A)$, then

$$\sigma\big(R(z,A)\big)=\{0\}\cup\big\{(z-s)^{-1}\colon\ s\in\sigma(A)\big\} \tag{18}$$

by [5, Lemma 8.1.9]. This motivates our analysis, which is, however, purely algebraic, making no reference to Banach spaces or to unbounded operators. The advantage of this is that the results are immediately applicable to quotient algebras $\mathcal{A}/\mathcal{J}$, for which no geometric interpretation exists.

Theorem 24. *If $U\subseteq\mathbf{C}$ and $r:U\to\mathcal{A}$ is a pseudo-resolvent and $\alpha\in U$ then $1+(z-\alpha)r_\alpha\in\mathcal{A}^\circ$ for all $z\in U$. If*

$$\widetilde{U}=\big\{z\colon\ 1+(z-\alpha)r_\alpha\in\mathcal{A}^\circ\big\}$$

then $U\subseteq\widetilde{U}$ and the formula

$$\tilde{r}_z=r_\alpha\big(1+(z-\alpha)r_\alpha\big)^{-1} \tag{19}$$

defines an extension of the pseudo-resolvent from U to $\widetilde{U}$. Moreover $\tilde{r}:\widetilde{U}\to\mathcal{A}$ is a maximal pseudo-resolvent. The set $\sigma(r)=\mathbf{C}\setminus\widetilde{U}$ is called the spectrum of the pseudo-resolvent r and satisfies

$$\sigma(r)=\big\{z\colon\ 1+(z-\alpha)r_\alpha\notin\mathcal{A}^\circ\big\} \tag{20}$$

for every choice of $\alpha\in U$.

Proof. By interchanging the labels α, γ in (17) we see that r_α and r_γ commute. Moreover

$$\begin{aligned}\big(1+(\gamma-\alpha)r_\alpha\big)\big(1+(\alpha-\gamma)r_\gamma\big)&=1+(\gamma-\alpha)\big\{r_\alpha-r_\gamma-(\gamma-a)r_\alpha r_\gamma\big\}\\&=1,\end{aligned}$$

so both terms on the left-hand side are invertible. This proves that $U\subseteq\widetilde{U}$. If $\alpha,z\in U$ then (17) implies that

$$r_\alpha=r_z\big(1+(z-\alpha)r_\alpha\big)$$

so $\tilde{r}_z=r_z$ for all $z\in U$ and $\tilde{r}$ is an extension of r to $\widetilde{U}$.

If $\beta,\gamma\in\widetilde{U}$, then starting from (19) we obtain

$$\begin{aligned}(\gamma-\beta)\tilde{r}_\beta\tilde{r}_\gamma&=(\gamma r_\alpha-\beta r_\alpha)r_\alpha\big(1+(\beta-\alpha)r_\alpha\big)^{-1}\big(1+(\gamma-\alpha)r_\alpha\big)^{-1}\\&=\big\{\big(1+(\gamma-\alpha)r_\alpha\big)-\big(1+(\beta-\alpha)r_\alpha\big)\big\}\\&\quad\times r_\alpha\big(1+(\beta-\alpha)r_\alpha\big)^{-1}\big(1+(\gamma-\alpha)r_\alpha\big)^{-1}\\&=r_\alpha\big\{\big(1+(\beta-\alpha)r_\alpha\big)^{-1}-\big(1+(\gamma-\alpha)r_\alpha\big)^{-1}\big\}\\&=\tilde{r}_\beta-\tilde{r}_\gamma.\end{aligned}$$

Therefore $\tilde{r}$ is a pseudo-resolvent on $\widetilde{U}$.

Now let $\hat{r}$ be a further extension of $\tilde{r}$ to a pseudo-resolvent on $\widehat{U} \supseteq \widetilde{U}$. If $z \in \widehat{U}$ then by the first half of this proof $1 + (z - \alpha)r_\alpha \in \mathcal{A}^\circ$, so $z \in \widetilde{U}$. Therefore $\widehat{U} = \widetilde{U}$ and $\tilde{r} : \widetilde{U} \to \mathcal{A}$ is a maximal pseudo-resolvent.

We have proved that $\widetilde{U} = \{z\colon\ 1 + (z - \alpha)r_\alpha \in \mathcal{A}^\circ\}$ for all $\alpha \in \widetilde{U}$, and this proves (20). □

Corollary 25. *Let $\mathcal{J}$ be a two-sided ideal in the associative algebra $\mathcal{A}$ with identity and let $\pi : \mathcal{A} \to \mathcal{A}/\mathcal{J}$ be the quotient map. If $U \subseteq \mathbf{C}$ and $r : U \to \mathcal{A}$ is a maximal pseudo-resolvent then*

$$\sigma\big(\pi(r)\big) \subseteq \sigma(r).$$

Proof. We need only observe that $z \to \pi(r_z)$ is a pseudo-resolvent in $\mathcal{A}/\mathcal{J}$ but its domain U need not be maximal. If its maximal extension has domain $V \supseteq U$ then

$$\sigma\big(\pi(r)\big) = \mathbf{C} \setminus V \subseteq \mathbf{C} \setminus U = \sigma(r). \qquad \square$$

6. Perturbation theory

When extending the theory of Section 3 to differential operators, one has to be careful not to refer to strong operator convergence, because the standard C^*-algebra $\mathcal{A}$ is only closed under norm convergence. In this section we collect some of the technical results that will be needed. These are formulated at the natural level of generality, but the reader should keep in mind that they will be applied to a resolvent operator A acting in $L^2(\mathbf{R}^d, \mathrm{d}x)$.

Let X be a set with a countably generated σ-field and a σ-finite measure μ, and put $L^2 = L^2(X, \mu)$.

Lemma 26. *Let A be a linear operator on L^2 that is positive in the sense that if $0 \leqslant \phi \in L^2$ then $0 \leqslant A\phi \in L^2$. Then A is bounded and*

$$\|A\| = \sup\big\{\|A\phi\|\colon\ 0 \leqslant \phi \in L^2 \text{ and } \|\phi\| \leqslant 1\big\} < \infty.$$

Moreover $|A(\phi)| \leqslant A(|\phi|)$ for all $\phi \in L^2$.

Proof. See [5, Lemma 13.1.1 and Theorem 13.1.2]. □

In the following discussion V will always denote a (possibly unbounded) measurable function $V : X \to \mathbf{C}$, which we call a potential, and also its associated multiplication operator. Given a positive operator A, let $\widetilde{\mathcal{V}}_A$ denote the set of potentials V that are relatively bounded with respect to A in the sense that

$$\|V\|_A = \sup\big\{\big\|V(A\phi)\big\|\colon\ \|\phi\| \leqslant 1\big\}$$

is finite.

Lemma 27. *We have $\|V\|_A \leqslant \|A\|\|V\|_\infty$ for all $V \in L^\infty$. Therefore $L^\infty(X) \subseteq \widetilde{\mathcal{V}}_A$. If $|W| \leqslant |V|$ and $V \in \widetilde{\mathcal{V}}_A$ then $W \in \widetilde{\mathcal{V}}_A$ and $\|W\|_A \leqslant \|V\|_A$. The space $\widetilde{\mathcal{V}}_A$ is a Banach space with respect to the norm $\|\cdot\|_A$.*

Proof. The last statement is the only one that is not elementary. Let $\xi \in L^2$ satisfy $\|\xi\|_2 = 1$ and $\xi(x) > 0$ almost everywhere in X and let $\psi = A\xi$, so that $\psi \geqslant 0$. The exists a measurable set E such that $\psi(x) > 0$ almost everywhere in E and $\psi(x) = 0$ almost everywhere in $X \setminus E$. In many cases $E = X$ but we do not assume this. If $\phi \in L^2$ and

$$\phi_n(x) = \begin{cases} \phi(x) & \text{if } |\phi(x)| \leqslant n\xi(x), \\ \frac{n\xi(x)\phi(x)}{|\phi(x)|} & \text{otherwise,} \end{cases}$$

then $|\phi_n| \leqslant |\phi|$ and $|\phi_n| \leqslant n\xi$. The dominated convergence theorem implies that $\|\phi_n - \phi\|_2 \to 0$ as $n \to \infty$. Moreover

$$\left|A(\phi_n)\right| \leqslant A\left(|\phi_n|\right) \leqslant A(n\xi) = n\psi$$

so $A(\phi_n)$ has support in E. Letting $n \to \infty$ we conclude that the same holds for $A(\phi)$. We conclude that if V has support in $X \setminus E$ then $VA = 0$, so we focus attention henceforth on the restriction of all the potentials involved to E.

We next observe that $\|V\psi\|_2 \leqslant \|V\|_A$ so if V_n is a Cauchy sequence in $\widetilde{\mathcal{V}}_A$ then $V_n\psi$ is a Cauchy sequence in $L^2(E,\mu)$. Therefore $V_n\psi$ converges in L^2 norm to a limit $V\psi$ in $L^2(E,\mu)$. There exists a subsequence $n(r)$ such that $V_{n(r)}$ converges almost everywhere in E to V.

Given $\varepsilon > 0$ there exists N_ε such that for all $m, n \geqslant N_\varepsilon$ we have

$$\left\|(V_m - V_n)(A\phi)\right\|_2 \leqslant \varepsilon\|\phi\|_2$$

for all $\phi \in L^2$. Replacing n by $n(r)$, letting $r \to \infty$ and using Fatou's lemma we obtain

$$\left\|(V_m - V)(A\phi)\right\|_2 \leqslant \varepsilon\|\phi\|_2$$

for all $m \geqslant N_\varepsilon$ and all $\phi \in L^2$. Hence $V \in \widetilde{\mathcal{V}}_A$ and $\|V_m - V\|_A \to 0$ as $m \to \infty$. □

Now let $\mathcal{V}_A$ denote the closure of L^∞ in $\widetilde{\mathcal{V}}_A$.

Lemma 28. *If $V \in \widetilde{\mathcal{V}}_A$ then $V \in \mathcal{V}_A$ if and only if $\lim_{n\to\infty} \|V^{(n)} - V\|_A = 0$ where*

$$V^{(n)}(x) = \begin{cases} V(x) & \textit{if } |V(x)| \leqslant n, \\ \frac{nV(x)}{|V(x)|} & \textit{otherwise.} \end{cases}$$

If $|W| \leqslant |V|$ and $V \in \mathcal{V}_A$ then $W \in \mathcal{V}_A$.

Proof. If $V \in \mathcal{V}_A$ then there exist $X_n \in L^\infty$ such that $\|X_n\|_\infty \leqslant n$ and $\|V - X_n\|_A \to 0$ as $n \to \infty$. By carrying out a separate calculation at every $x \in X$ we see that

$$\left|V - V^{(n)}\right| \leqslant |V - X_n|.$$

Lemma 27 now implies that

$$\lim_{n\to\infty} \left\|V - V^{(n)}\right\|_A \leqslant \lim_{n\to\infty} \|V - X_n\|_A = 0.$$

The converse statement, that $\lim_{n\to\infty} \|V^{(n)} - V\|_A = 0$ implies $V \in \mathcal{V}_A$, is elementary. The second statement of the lemma follows in a similar way from the inequality

$$\left|W - W^{(n)}\right| \leqslant \left|V - V^{(n)}\right|. \qquad \square$$

Lemma 29. *If* $0 \leqslant A \leqslant B$ *as operators on* L^2, *in the sense that* $0 \leqslant A\phi \leqslant B\phi$ *for all* ϕ *such that* $0 \leqslant \phi \in L^2$, *then* $\mathcal{V}_B \subseteq \mathcal{V}_A$.

Proof. If $\phi \in L^2$ and $V \in \widetilde{\mathcal{V}}_B$ then

$$\left|V(A\phi)\right| = |V|\left|A(\phi)\right| \leqslant |V|A\left(|\phi|\right) \leqslant |V|B\left(|\phi|\right)$$

so

$$\left\|V(A\phi)\right\|_2 \leqslant \left\||V|\left(B|\phi|\right)\right\|_2 \leqslant \left\||V|\right\|_B \left\||\phi|\right\|_2 = \|V\|_B \|\phi\|_2$$

for all $\phi \in L^2$. This implies $\|V\|_A \leqslant \|V\|_B < \infty$ and hence $V \in \widetilde{\mathcal{V}}_A$. The proof of the lemma is completed as in Lemma 28. $\square$

We now specialize to the case in which $\mathcal{H} = L^2(\mathbf{R}^d, \mu)$. Our goal is to describe certain classes of potential in $\mathcal{V}_A$, particularly when A is a positive convolution operator. Such operators arise as the resolvents of constant coefficient, second order partial differential operators and in certain other contexts; the reader primarily interested in Schrödinger operators should keep Example 34 in mind. We will use the classical L^p inequalities due to Hölder, Young, Hausdorff–Young and Riesz–Thorin without further mention.

Lemma 30. *If* $a \in L^1(\mathbf{R}^d)$ *then the operator* A *on* $L^2(\mathbf{R}^d)$ *defined by* $A\phi = a * \phi$ *lies in the standard* C^**-algebra* $\mathcal{A}$.

Proof. If

$$a_n(x) = \begin{cases} a(x) & \text{if } |x| \leqslant n, \\ 0 & \text{otherwise} \end{cases}$$

and $A_n\phi = a_n * \phi$ then

$$\lim_{n\to\infty} \|A_n - A\| \leqslant \|a_n - a\|_1 = 0$$

by Lemma 43. We combine this with the observation that $A_n \in \mathcal{A}_n$, because the support of $A_n\phi$ must lie within a distance n of the support of ϕ. $\square$

Let $\mathcal{C}_d$ denote the set of operators A on $L^2(\mathbf{R}^d, \mathrm{d}x)$ given by $A\phi = a * \phi$, where $0 \leqslant a \in L^1(\mathbf{R}^d, \mathrm{d}x)$.

Lemma 31. *If* $A \in \mathcal{C}_d$ *and* $a \in L^p$ *for some* $1 < p \leqslant 2$ *then* $L^q \subseteq \mathcal{V}_A$, *where* $1/p + 1/q = 1$.

Proof. If $V \in L^q$ then

$$\left\| V(a * \phi) \right\|_2 \leqslant \|V\|_q \|a\|_p \|\phi\|_2,$$

so

$$\|V\|_A \leqslant \|V\|_q \|a\|_p. \qquad \square$$

Lemma 32. *If $A \in \mathcal{C}_d$ and $\hat{a} \in L^p$ where $\hat{a}$ denotes the Fourier transform of a and $2 \leqslant p < \infty$, then $L^p \subseteq \mathcal{V}_A$.*

Proof. This uses the bound

$$\|VA\| \leqslant c_{d,p} \|V\|_p \|\hat{a}\|_p. \tag{21}$$

See, for example, [5, Theorem 5.7.3]. $\square$

There are many other results of a similar type in which both of the L^p norms in (21) are replaced by other choices. See [21, Chapter 4] for details.

The following type of bound is used when analyzing multi-body Schrödinger operators. The decomposition of $\mathbf{R}^d$ used below may be combined with a Euclidean rotation of $\mathbf{R}^d$, since this amounts to a change of coordinate system.

Theorem 33. *Let $x = (x_1, x_2) \in \mathbf{R}^{d_1} \times \mathbf{R}^{d_2}$ where $d = d_1 + d_2$ and suppose that $|V(x_1, x_2)| \leqslant W(x_1)$ for all $x \in \mathbf{R}^d$ where $W \in L^p(\mathbf{R}^{d_1})$ and $2 \leqslant p < \infty$. Suppose also that $A \in \mathcal{C}_d$, $B \in \mathcal{C}_{d_1}$ and*

$$\left|\hat{a}(\xi_1, \xi_2)\right| \leqslant \left|\hat{b}(\xi_1)\right|$$

for all $\xi \in \mathbf{R}^d$, where $0 \leqslant b \in L^1(\mathbf{R}^{d_1})$ and $\hat{b} \in L^p(\mathbf{R}^{d_1})$. Then $V \in \mathcal{V}_A$.

Proof. We may write V=XW where $|X| \leqslant 1$. We may also write $A = BC$ where $\|C\| \leqslant 1$; in fact $C = \mathcal{F}^{-1} M \mathcal{F}$ where $\mathcal{F}$ is the Fourier transform and M is the operator of multiplication by a function m with $|m| \leqslant 1$. Therefore

$$\|VA\| = \|XWBC\| \leqslant \|WB\| \leqslant c\|W\|_p$$

by applying Lemma 32 in $\mathbf{R}^{d_1}$. $\square$

Example 34. If $H = -\overline{\Delta}$ acting in $L^2(\mathbf{R}^d)$ with the usual domain then $A = (I + H)^{-1}$ is of the form $A\phi = a * \phi$ where $0 \leqslant a \in L^1(\mathbf{R}^d)$ and $\hat{a}(\xi) = (1 + |\xi|^2)^{-1}$ for all $\xi \in \mathbf{R}^d$. Theorem 33 is applicable in this context because

$$\left(1 + |\xi|^2\right)^{-1} \leqslant \left(1 + |\xi_1|^2\right)^{-1}$$

whenever $\xi = (\xi_1, \xi_2)$. One needs to assume that $p \geqslant 2$ and $p > d_1/2$.

7. Applications to differential operators

In this section we show that the C^*-algebra methods developed above can be used to study the spectra of certain differential operators. Instead of trying to study $\sigma(A)$ directly we may redirect our attention to the spectrum of one of its resolvent operators by virtue of the results in Section 5. We say that the closed, unbounded operator A is *affiliated to* the C^*-subalgebra $\mathcal{A}$ of $\mathcal{L}(\mathcal{H})$ if the conditions of the following lemma are satisfied.

Lemma 35. *Let $\mathcal{A}$ be a C^*-subalgebra of $\mathcal{L}(\mathcal{H})$ and let $R(z,A)\in\mathcal{A}$ for some $z\notin\sigma(A)$. Then $R(w,A)\in\mathcal{A}$ for all $w\notin\sigma(A)$.*

Proof. If $X=I+(w-z)R(z,A)$ then $X\in\mathcal{A}$ and

$$\sigma(X)=\{1\}\cup\left\{1+\frac{w-z}{z-s}\colon s\in\sigma(A)\right\}$$
$$=\{1\}\cup\left\{\frac{w-s}{z-s}\colon s\in\sigma(A)\right\}.$$

Since this does not contain 0 we deduce that X is invertible in $\mathcal{L}(\mathcal{H})$, and hence also invertible in $\mathcal{A}$. Since $R(w,A)=R(z,A)X^{-1}$ as in [5, Theorem 1.2.10], we deduce that $R(w,A)\in\mathcal{A}$. □

We say that a one-parameter group or semigroup T_t is affiliated to $\mathcal{A}$ if its generator is affiliated in the above sense, i.e. if the associated resolvent family lies in $\mathcal{A}$. If H is a typical Schrödinger operator acting in $L^2(\mathbf{R}^d)$, then the unitary operators e^{-iHt} do not lie in the standard C^*-algebra $\mathcal{A}$, but we will see they are affiliated to it.

Let $\mathcal{H}=L^2(\mathbf{R}^d)$ and let H_0 be a constant coefficient differential operator whose symbol is the polynomial p, so that $H_0\phi=\mathcal{F}^{-1}p\mathcal{F}\phi$ where $\mathcal{F}$ is the Fourier transform operator and p is regarded as an unbounded multiplication operator. It is immediate that H_0 is a closed operator on

$$\mathrm{Dom}(H_0)=\{\phi\in\mathcal{H}\colon\ p\mathcal{F}\phi\in\mathcal{H}\}.$$

Theorem 36. *Suppose that $\lim_{|\xi|\to\infty}|p(\xi)|=+\infty$ and that there exists a real constant b such that $\mathrm{Re}(p(\xi))\leqslant b$ for all $\xi\in\mathbf{R}^d$. Then $\sigma(H_0)\subseteq\{z\colon\ \mathrm{Re}(z)\leqslant b\}$. If $\mathrm{Re}(z)>b$ then $R(z,H_0)$ lies in the standard C^*-algebra $\mathcal{A}$.*

Proof. We have $R(z,H_0)=\mathcal{F}^{-1}\rho\mathcal{F}$ where $\rho\in C_0(\mathbf{R}^d)$ is defined by

$$\rho(\xi)=\big(z-p(\xi)\big)^{-1}.$$

If $n\geqslant 1$ we define

$$\rho_n(\xi)=\mathrm{e}^{-|\xi|^2/n}\big(z-p(\xi)\big)^{-1}.$$

Putting $R=R(z,H_0)$ and $R_n=\mathcal{F}^{-1}\rho_n\mathcal{F}$ we see that

$$\lim_{n\to\infty}\|R_n-R\|=\lim_{n\to\infty}\|\rho_n-\rho\|_\infty=0$$

so it is enough to prove that $R_n \in \mathcal{A}$ for all $n \geqslant 1$. Since ρ_n lies in the Schwartz space $\mathcal{S}$ it is enough to observe that $R_n\phi = k_n * \phi$ for all $\phi \in L^2$ where $k_n \in \mathcal{S} \subseteq L^1$; we may then apply Lemma 30. □

Before starting applications we change conventions so as to conform to the standard practice in quantum theory, writing $-H$ where one might expect to see H.

Example 37. The differential operator

$$(H_0\phi)(x, y) = -\frac{\partial^2\phi}{\partial x^2} - \frac{\partial^3\phi}{\partial y^3}$$

acting in $L^2(\mathbf{R}^2)$ has symbol $p(\xi, \eta) = \xi^2 + i\eta^3$ and is highly non-elliptic. Nevertheless the conditions of Theorem 36 are satisfied. The same applies to the non-negative, self-adjoint, differential operator acting in $L^2(\mathbf{R}^2)$ with real symbol

$$p(\xi, \eta) = \xi^2 + \left(\eta - \xi^n\right)^2$$

where $n \geqslant 2$.

The following hypothesis is valid for a variety of second order elliptic differential operators with variable coefficients; see [4].

Hypothesis 1. The operator $-H_0$ is the generator of a strongly continuous one-parameter semigroup $\mathrm{e}^{-H_0 t}$ on $L^2(\mathbf{R}^d)$. Moreover there exist positive constants c, α and an integral kernel $K(t, x, y)$ such that

$$0 \leqslant K(t, x, y) \leqslant ct^{-d/2}\mathrm{e}^{-\alpha|x-y|^2/t} \tag{22}$$

for all $t > 0$ and $x, y \in \mathbf{R}^d$ and

$$\left(\mathrm{e}^{-H_0 t}\phi\right)(x) = \int_{\mathbf{R}^d} K(t, x, y)\phi(y)\,\mathrm{d}y \tag{23}$$

for all $\phi \in L^2(\mathbf{R}^d)$ and $x \in \mathbf{R}^d$.

Lemma 38. *Under Hypothesis* 1

$$\sigma(H_0) \subseteq \left\{z\colon \operatorname{Re}(z) \geqslant 0\right\}$$

and $(\lambda I + H_0)^{-1}$ *has an integral kernel* $G(\lambda, x, y)$ *for every* $\lambda > 0$. *There exists a function* $g_\lambda \in L^1(\mathbf{R}^d)$ *and a constant* $c_1 > 0$ *such that*

$$0 \leqslant G(\lambda, x, y) \leqslant g_\lambda(x - y)$$

and

$$\left\|(\lambda I + H_0)^{-1}\right\| \leqslant \|g_\lambda\|_1 = c_1\lambda^{-1} < \infty.$$

Proof. If we put

$$k_t(x) = ct^{-d/2}\mathrm{e}^{-\alpha|x|^2/t}$$

then there exists $c_1 > 0$ such that $\|k_t\|_1 = c_1$ for all $t > 0$. Therefore $\|\mathrm{e}^{-H_0 t}\| \leqslant c_1$ for all $t > 0$ and $\sigma(H_0) \subseteq \{z\colon \operatorname{Re}(z) \geqslant 0\}$. If $\lambda > 0$ the kernel G satisfies

$$\begin{aligned} 0 \leqslant G(\lambda, x, y) &= \int_0^\infty K(t, x, y)\mathrm{e}^{-\lambda t}\,\mathrm{d}t \\ &\leqslant \int_0^\infty k_t(x-y)\mathrm{e}^{-\lambda t}\,\mathrm{d}t \\ &= g_\lambda(x-y), \end{aligned}$$

where the positivity of the functions involved implies that

$$\|g_\lambda\|_1 = \int_0^\infty \|k_t\|_1 \mathrm{e}^{-\lambda t}\,\mathrm{d}t = c_1/\lambda.$$

Note finally that

$$\hat{g}_\lambda(\xi) = \frac{c_1}{\lambda + c_2|\xi|^2}$$

for some $c_2 > 0$, all $\lambda > 0$ and all $\xi \in \mathbf{R}^d$. □

Example 39. A bound of the type (22) is not valid for fractional powers of the Laplacian, i.e. $H_0 = (-\overline{\Delta})^\alpha$ where $0 < \alpha < 1$. However, in this case the one-parameter semigroup $\mathrm{e}^{-H_0 t}$ has the kernel

$$K(t, x, y) = k_t(x-y) > 0$$

for all $t > 0$, where $\|k_t\|_1 = 1$ and $\hat{k}_t(\xi) = \mathrm{e}^{-t|\xi|^{2\alpha}}$ for all $t > 0$ and $\xi \in \mathbf{R}^d$. The construction of k_t uses the theory of fractional powers of generators of one-parameter semigroups; see [22, Chapter 9.11]. The resolvent operator $(\lambda I + H_0)^{-1}$ has the kernel

$$G(\lambda, x, y) = g_\lambda(x-y) > 0$$

for all $\lambda > 0$, where

$$g_\lambda(x) = \int_0^\infty k_t(x)\mathrm{e}^{-\lambda t}\,\mathrm{d}t > 0.$$

One deduces that $\|g_\lambda\|_1 = \lambda^{-1} < \infty$ and

$$\hat{g}_\lambda(\xi) = \left(\lambda + |\xi|^{2\alpha}\right)^{-1}$$

for all $\lambda > 0$ and $\xi \in \mathbf{R}^d$. The methods developed in this paper still apply.

The above results allow us to reformulate our problem.

Hypothesis 2. Let $K : \mathbf{R}^d \times \mathbf{R}^d \to \mathbf{R}$ and $k \in L^1(\mathbf{R}^d)$ satisfy

$$0 \leqslant K(x, y) \leqslant k(x - y)$$

for all $x, y \in \mathbf{R}^d$. Let R_0 be the positive operator associated with $K(x, y)$ and let B be the positive operator associated with $k(x - y)$, so that $0 \leqslant R_0 \leqslant B$. Lemma 29 now implies that $\mathcal{V}_B \subseteq \mathcal{V}_{R_0}$.

If $R_0 = (\lambda I + H_0)^{-1}$ in the following theorem then $R = (\lambda I + H_0 + V)^{-1}$ and the assumption $\|V\|_{R_0} < 1$ states that V has relative bound less than 1 with respect to H_0 in the conventional language of perturbation theory.

Lemma 40. *Given Hypothesis* 2*, let the potential* $V \in \mathcal{V}_{R_0}$ *satisfy* $\|V\|_{R_0} < 1$ *and put*

$$R = R_0(I + V R_0)^{-1} = R_0 \sum_{n=0}^{\infty} (-V R_0)^n. \tag{24}$$

Then the operators R_0 *and* R *both lie in the standard* C^**-algebra* $\mathcal{A}$.

Proof. Given $\varepsilon > 0$ there exists $c \in \mathbf{Z}_+$ such that $\int_{|x_1|>c} |k(x)| \, \mathrm{d}x < \varepsilon$. If we put

$$K_c(x, y) = \begin{cases} K(x, y) & \text{if } |x - y| \leqslant c, \\ 0 & \text{otherwise,} \end{cases}$$

and define the operator T on $\mathcal{H}$ by

$$(T_c \phi)(x) = \int_{\mathbf{R}^d} K_c(x, y)\phi(y) \, \mathrm{d}y$$

then $\|R_0 - T_c\| < \varepsilon$ and $T_c P_{S(n)} = P_{S(n+c)} T_c P_{S(n)}$ for every $S \in \mathcal{F}$ and $n \geqslant 1$, hence $T_c \in \mathcal{D}_0$ and $R_0 \in \mathcal{D}$. Applying the same argument to R_0^* yields $R_0 \in \mathcal{A}$ by virtue of Lemma 3. Defining $V^{(r)}$ as in Lemma 28, the identities $V^{(r)} P_{S(n)} = P_{S(n)} V^{(r)}$ for all S, n and r imply that $V^{(r)} R_0 \in \mathcal{A}$. Hence $V R_0 \in \mathcal{A}$. The norm convergence of the series in (24) now implies that $R \in \mathcal{A}$. $\square$

We conclude with two applications to quantum theory. In the first we consider with the Schrödinger operator $H = H_0 + V$ acting in $L^2(\mathbf{R}^d)$, where $H_0 = -\overline{\Delta}$ and $V = W + X$ is a sum of possibly complex-valued potentials satisfying the conditions specified below. Passing to the resolvent operators we actually consider $R_0 = (aI + H_0)^{-1}$, $R_1 = (aI + H_0 + W)^{-1}$ and $R = (aI + H)^{-1}$, where $a > 0$ is large enough to ensure that all the inverses exist.

Theorem 41. *Suppose that V and W lie in the space $\mathcal{V}_{R_0}$ defined just before Lemma* 28 *and that $\|V\|_{R_0} < 1$, $\|W\|_{R_0} < 1$. Suppose that W is periodic in the x_1 direction. Let $S = \{x \in \mathbf{R}^d\colon |x_1| < 1\}$, so that $S(n) = \{x \in \mathbf{R}^d\colon |x_1| < n+1\}$ for all $n \geqslant 1$. Suppose that X has support in $S(c)$ for some $c \geqslant 1$. Finally define the ideal $\mathcal{J}_S \subseteq \mathcal{A}$ as in Theorem* 18. *Then*

$$\begin{aligned}\sigma(H_0+W) &= \sigma_{\mathrm{ess}}(H_0+W) = \sigma_S(H_0+W)\\ &= \sigma_S(H) \subseteq \sigma_{\mathrm{ess}}(H) \subseteq \sigma(H)\end{aligned} \tag{25}$$

where $\sigma_S(A) = \sigma(\pi_{\mathcal{J}_S}(A))$ for every $A \in \mathcal{A}$.

Proof. The operators R_0, R_1 and R all lie in $\mathcal{A}$ for large enough $a > 0$ by Lemma 40. Eq. (25) is equivalent, by definition, to

$$\sigma(R_1) = \sigma_{\mathrm{ess}}(R_1) = \sigma_S(R_1) = \sigma_S(R) \subseteq \sigma_{\mathrm{ess}}(R) \subseteq \sigma(R). \tag{26}$$

The proof of the first two equalities in (26) uses the periodicity of R_1 in the x_1 direction as in the proof of Theorem 9. We next observe that

$$\begin{aligned} I + VR_0 &= I + WR_0 + XR_0\\ &= \bigl\{I + XR_0(1+WR_0)^{-1}\bigr\}(I+WR_0)\\ &= (I+XR_1)(I+WR_0).\end{aligned}$$

Since $I + VR_0$ and $I + WR_0$ are invertible, it follows that $I + XR_1$ is invertible. Therefore

$$\begin{aligned} R &= R_0(I+VR_0)^{-1}\\ &= R_0(I+WR_0)^{-1}(I+XR_1)^{-1}\\ &= R_1(I+XR_1)^{-1}.\end{aligned}$$

Since $X \in \mathcal{V}_{R_0}$ has support in $S(c)$ and $\mathcal{J}_S$ is an ideal we can use Lemma 28 to deduce that $XR_1 \in \mathcal{J}_S$. Therefore

$$\pi_{\mathcal{J}_S}(R) = \pi_{\mathcal{J}_S}(R_1)\bigl\{I + \pi_{\mathcal{J}_S}(XR_1)\bigr\}^{-1} = \pi_{\mathcal{J}_S}(R_1).$$

The inclusions in (26) now follow by applying Lemma 1. $\square$

Example 42. We next point out the relevance of the above results to multi-body Schrödinger operators. Let $\mathcal{H} = L^2(\mathbf{R}^3 \times \mathbf{R}^3)$ and put $x = (x_1, x_2)$ where $x_i \in \mathbf{R}^3$. Let $H_0 = -\overline{\Delta}$ and define

$$H = H_0 + V_1(x_1) + V_2(x_2) + V_3(x_1 - x_2)$$

where all three potentials lie in $L^2(\mathbf{R}^3) + C_0(\mathbf{R}^3)$. By allowing V_1, V_2 and V_3 to be complex-valued we include in our analysis the non-self-adjoint Schrödinger operators that arise in when discussing resonances via complex scaling. For suitable choices of V_i this operator might be regarded as describing two (spinless) electrons orbiting around a fixed nucleus (a simplified Helium atom). Standard estimates imply that V_i all have relative bound 0 with respect to H_0 and that they all lie in $\mathcal{V}_{R_0}$ with $\|V_i\|_{R_0} < 1/3$ provided $R_0 = (aI + H_0)^{-1}$ and $a > 0$ is large enough. Lemma 40 implies that all of the relevant resolvent operators lie in the standard C^*-algebra $\mathcal{A}$.

One can produce several asymptotic sets from $\{x\colon |x_1| < 1\}$, $\{x\colon |x_2| < 1\}$, $\{x\colon |x_1 - x_2| < 1\}$, and we will concentrate on two of these. If one puts

$$S = \bigl\{x\colon |x_1| < 1\bigr\} \cup \bigl\{x\colon |x_2| < 1\bigr\} \cup \bigl\{x\colon |x_1 - x_2| < 1\bigr\},$$

it is evident that $S \in \mathcal{F}$ and that $V_1 + V_2 + V_3 \in \mathcal{J}_S$. Hence

$$\sigma_S(H) = \sigma_S(H_0) = [0, \infty).$$

This set relates to the states in which both particles move away to infinity and they also separate from each other. On the other hand if one puts

$$T = \bigl\{x\colon |x_2| < 1\bigr\} \cup \bigl\{x\colon |x_1 - x_2| < 1\bigr\},$$

it is evident that $T \in \mathcal{F}$ and that $V_2 + V_3 \in \mathcal{J}_T$. Hence

$$\sigma_T(H) = \sigma_T(H_0 + V_1) = \sigma(H_0 + V_1).$$

This set relates to the states in which particle 2 moves away to infinity and also separates from particle 1, which may or may not stay close to the nucleus. If $A = -\overline{\Delta} + V_1$ acting in $L^2(\mathbf{R}^3)$ then by taking Fourier transforms with respect to x_2 it is seen that

$$\sigma(H_0 + V_1) = \sigma(A) + [0, \infty)$$

where $\sigma(A) = [0, \infty) \cup \{\lambda_n\}$, where λ_n are the possibly complex-valued discrete eigenvalues of the operator A.

8. Hyperbolic space

Let (X, d, μ) denote a complete non-compact Riemannian manifold X with bounded geometry, Riemannian metric d (in the sense of the triangle inequality) and Riemannian measure μ. The Laplace–Beltrami operator $H = -\Delta$ on $L^2(X, \mu)$ is essentially self-adjoint of $C_c^\infty(X)$ and the spectrum of its closure is contained in $[0, \infty)$. The one-parameter semigroup $\{e^{-Ht}\}_{t \geqslant 0}$ is associated with a positive C^∞ heat kernel K by

$$\bigl(e^{-Ht} f\bigr)(x) = \int\limits_X K(t, x, y) f(y)\, \mu(dy).$$

The kernel K satisfies

$$\int\limits_X K(t, x, y)\, \mu(dy) = 1$$

for all $x \in X$ and $t > 0$. We wish to show that e^{-Ht} and $(\lambda I + H)^{-1}$ lie in the C^*-algebra $\mathcal{A}$ for all $t, \lambda > 0$. Rather than proving this under the weakest possible conditions, we consider the hyperbolic space $\mathbf{H}^3$, in which all of the expressions involved may be written down explicitly. The proof that we given may be extended to $\mathbf{H}^d$ for arbitrary $d \geqslant 2$ with minimal effort.

The geometry of hyperbolic space is well-studied; see [18, Section 4.6] for the results listed below. In the upper half space model $\mathbf{H}^n$ is the set $\{x \in \mathbf{R}^n\colon\ x_n > 0\}$ with the local Riemannian metric

$$\mathrm{d}s^2 = \frac{\mathrm{d}^2 x_1 + \cdots + \mathrm{d}^2 x_n}{x_n^2}.$$

The global metric d is given by

$$\cosh\bigl(d(x,y)\bigr) = 1 + \frac{|x-y|^2}{2x_n y_n}$$

and the volume element is given by

$$\mu(dx) = \frac{\mathrm{d}x_1 \cdots \mathrm{d}x_n}{x_n^n}.$$

The area of the unit sphere $S(x,r)$ of radius $r > 0$ does not depend on $x \in X$ and is given by

$$\rho(r) = c_n \sinh^{n-1}(r)$$

where $c_3 = 4\pi$. If $f : (0,\infty) \to \mathbf{R}$ is any positive, measurable function then

$$\int_X f\bigl(d(x,y)\bigr)\,\mu(\mathrm{d}y) = \int_0^\infty f(r)\rho(r)\,\mathrm{d}r$$

for all $x \in X$.

If $X = \mathbf{H}^n$, the spectrum of $H = -\overline{\Delta}$ acting in $L^2(X,\mu)$ is equal to $[(n-1)^2/4,\infty)$, but the L^p spectrum depends on p; see [8]. The heat kernel may be written in the form $K(t,x,y) = k_t(d(x,y))$, where for $n = 3$ we have

$$k_t(r) = (4\pi t)^{-n/2} \frac{r}{\sinh(r)} \mathrm{e}^{-t-d(x,y)^2/4t}.$$

See [9]; see also [6] for relevant upper and lower bounds when $n \neq 3$. One verifies directly that

$$\begin{aligned}
\int_X K(t,x,y)\,\mu(\mathrm{d}y) &= \int_0^\infty k_t(r)\rho(r)\,\mathrm{d}r \\
&= \int_0^\infty (4\pi)^{-1/2} t^{-3/2} r \sinh(r) \mathrm{e}^{-t-r^2/4t}\,\mathrm{d}r \\
&= \int_{-\infty}^\infty (4\pi)^{-1/2} t^{-3/2} 2^{-1} r \mathrm{e}^{r-t-r^2/4t}\,\mathrm{d}r
\end{aligned}$$

$$= \int_{-\infty}^{\infty} (4\pi)^{-1/2} t^{-3/2} 2^{-1} r \mathrm{e}^{-(r-2t)^2/4t} \, \mathrm{d}r$$
$$= 1$$

for all $t > 0$.

If $\lambda > 0$ the operator $(H + \lambda I)^{-1}$ has a Green function G given explicitly by $G(\lambda, x, y) = g_\lambda(d(x, y))$, where

$$g_\lambda(r) = \int_0^\infty \mathrm{e}^{-\lambda t} k_t(r) \, \mathrm{d}t = \frac{\mathrm{e}^{-r\sqrt{\lambda+1}}}{4\pi \sinh(r)}.$$

A direct calculation establishes that

$$\int_X G(\lambda, x, y) \, \mu(\mathrm{d}y) = \int_0^\infty g_\lambda(r) \rho(r) \, \mathrm{d}r = 1/\lambda \tag{27}$$

for all $\lambda > 0$. We will need the following lemma.

Lemma 43. *If*

$$(Rf)(x) = \int_X r(x, y) f(y) \, \mu(\mathrm{d}y)$$

for all $x \in X$ and $f \in L^2(X, \mu)$, then

$$\|R\|^2_{L^2(X,\mu)} \leqslant \left\{ \sup_{x\in X} \int_X \left|r(x, y)\right| \mu(\mathrm{d}y) \right\} \left\{ \sup_{x\in X} \int_X \left|r(y, x)\right| \mu(\mathrm{d}y) \right\}.$$

See [5, Corollary 2.2.15] for the proof.

Theorem 44. *If $\lambda > 0$ and $t > 0$ then e^{-Ht} and $(H + \lambda I)^{-1}$ both lie in the standard C^*-algebra $\mathcal{A}$.*

Proof. The proof is almost the same in both cases so we only treat the resolvent operators. We have $(\lambda I + H)^{-1} = A_n + B_n$ where

$$(A_n f)(x) = \int_X a_n(x, y) f(y) \, \mu(\mathrm{d}y),$$
$$a_n(x, y) = \tilde{a}_n\big(d(x, y)\big),$$
$$\tilde{a}_n(r) = \begin{cases} g_\lambda(r) & \text{if } r \leqslant n, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$(B_n f)(x) = \int_X b_n(x,y) f(y)\, \mu(\mathrm{d}y),$$

$$b_n(x,y) = \tilde{b}_n\big(d(x,y)\big),$$

$$\tilde{b}_n(r) = \begin{cases} g_\lambda(r) & \text{if } r > n, \\ 0 & \text{otherwise.} \end{cases}$$

It follows from its definition that $A_n \in \mathcal{A}_n$ and from (27) and Lemma 43 that $\lim_{n\to\infty} \|B_n\| = 0$. □

Example 45. The ideas in the second part of Section 4 can be applied in the setting of hyperbolic space. In the upper half space model the natural compactification has $\partial \mathbf{H}^n \sim (\mathbf{R}^{n-1} \times \{0\}) \cup \{\infty\}$. If we put $S = \{x \in \mathbf{H}^n\colon\ 0 < x_n < 1\}$ then $S(m) = \{x \in \mathbf{H}^n\colon\ 0 < x_n < \mathrm{e}^m\}$. Moreover $\widehat{S(m)} = \widehat{S} = \mathbf{R} \times \{0\}$ for all $m \geqslant 1$. Therefore the quotient map $\pi : \mathcal{B} \to \mathcal{B}/(\mathcal{J}_S \cap \mathcal{B}) \simeq \mathbf{C}$ is given by $\pi(f) = f(\infty)$.

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