Physics Letters B 705 (2011) 273–278

Contents lists available at SciVerse ScienceDirect

Physics Letters B

www.elsevier.com/locate/physletb

Effective potential in curved space and cut-off regularizations

Flávia Sobreira [a], Baltazar J. Ribeiro [b], Ilya L. Shapiro [b,*]

[a] IFT-UNESP, São Paulo, SP, Brazil
[b] Departamento de Fisica, UFJF, Juiz de Fora, MG, Brazil

ARTICLE INFO

Article history:
Received 16 July 2011
Received in revised form 7 October 2011
Accepted 10 October 2011
Available online 13 October 2011
Editor: M. Cvetič

Keywords:
Effective potential
Curved space
Normal coordinates
Renormalization group
Renormalization schemes

ABSTRACT

We consider derivation of the effective potential for a scalar field in curved space–time within the physical regularization scheme, using two sorts of covariant cut-off regularizations. The first one is based on the local momentum representation and Riemann normal coordinates and the second is operatorial regularization, based on the Fock–Schwinger–DeWitt proper-time representation. We show, on the example of a self-interacting scalar field, that these two methods produce equal results for divergences, but the first one gives more detailed information about the finite part. Furthermore, we calculate the contribution from a massive fermion loop and discuss renormalization group equations and their interpretation for the multi-mass theories.

1. Introduction

Recently there was a growing interest in the more physical regularization and renormalization schemes in curved space–time. In particular, one can mention the papers on deriving the energy–momentum tensor of vacuum in momentum cut-off regularization [1–7] from one side and intensive discussions of physical interpretation of renormalization group from another one [8–17].

One of the outputs of the works on the cut-off approach is that this regularization may produce an explicit breaking of the local Lorentz invariance [18] and also of general covariance. Therefore, it would be interesting to have an example of the cut-off-based calculations which preserve both symmetries explicitly.

The effective potential of a scalar field in curved space–time has been studied in a number of papers starting from [19–22] (see [23] for further references). In particular, a very general expression for such an effective potential has been obtained in [22] via the renormalization group method. Indeed, this means the Minimal Subtraction scheme of renormalization, when the effect of masses of the quantum fields is either ignored or taken into account through the heuristic method. An additional motivation for a more physical renormalization and regularization schemes comes from inflationary side. In the recent paper [24] it was shown (see also previous works [25] in this direction) that the Higgs-based inflation, originally invented by A. Guth [26], can be consistent with known observational tests if assuming that the Higgs field H couples non-minimally to scalar curvature. Let us remark that the corresponding term $\xi R H^* H$ is requested in order to make Standard Model of elementary particles multiplicatively renormalizable in curved space–time [23]. The value of ξ should be of the order of 10^4–10^5, but this does not pose a problem, because the dimensional quantity $|\xi R|$ does not exceed the square of the Higgs mass. The great difference between the Higgs-based and inflation-based inflationary models is that the Higgs field probably does exist. Therefore, the model of [26,24] should be considered as the first candidate to describe an inflationary paradigm [27]. According to the further works on Higgs inflation [28] and [29] (see also [30] and references therein), the renormalization group-based quantum correction to the Higgs potential plays an essential role in this inflationary model, such that taking them into account leads to important restrictions for the Higgs mass. This result was essentially based on the well-known renormalization group derivation of effective potential in curved space–time, completely equivalent to the one which was first performed in [22,23], and concerns both one- and two-loop contributions. However, as far as this derivation is based on the Minimal Subtraction scheme of renormalization, it would be interesting to verify what is the effect of the masses of quantum fields by direct calculation.

In order to address the issues of covariant cut-off and of the effect of masses on the Higgs potential in curved space, we perform direct calculation of the one-loop effective potential of a scalar field in curved space–time. We consider two sorts of covariant

* Corresponding author.
E-mail address: ilyashapiro2003@yahoo.com.br (I.L. Shapiro).

0370-2693 © 2011 Elsevier B.V. Open access under the Elsevier OA license.
doi:10.1016/j.physletb.2011.10.016

cut-off regularizations. The first one is based on the local momentum representation, which is due to the use of Riemann normal coordinates, and the second is the so-called operatorial regularization, based on the Fock–Schwinger–DeWitt proper-time representation. It was demonstrated recently in [31] that these two types of regularizations give equivalent results in flat space–time. In view of this, our calculations can be seen as an extension of the same statement to a curved space.

The Letter is organized as follows. In the next section we perform calculation for a self-interacting scalar field through the cut-off regularization in the local momentum representation. In Section 3 we consider a technically simpler scheme of operatorial regularization cut-off. In Section 4 we extend the previous results to the fermion contributions. In Section 5 the μ-dependence and renormalization group equations for the parameters of the theory are discussed. Finally, in Section 6 we draw our conclusions.

2. Covariant momentum cut-off calculation

The effective potential is defined as the zero-order term in the derivative expansion of the effective action of a mean scalar field,

$$\Gamma[\varphi, g_{\mu\nu}] = \int d^4x\sqrt{-g}\Big\{-V_{eff}(\varphi) + \frac{1}{2}Z(\varphi)g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi + \cdots\Big\}. \quad (1)$$

The calculation of $V_{eff}(\varphi)$ can be performed for constant φ, in different theories with different content of quantum fields. In this Letter we consider two examples, namely self-interacting scalar field and also fermion field with Yukawa coupling to the background scalar, both in curved space–time.

Our starting point will be the action of a real scalar field

$$S_0 = \int d^4x\sqrt{-g}\Big\{\frac{1}{2}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - \frac{1}{2}\big(m^2 - \xi R\big)\varphi^2 - V(\varphi)\Big\}, \quad (2)$$

where $V(\varphi)+m^2\varphi^2$ is the minimal potential term and $\xi R\varphi^2$ is the non-minimal addition, which is necessary for formulating renormalizable theory in curved space–time. In flat space $R = 0$ and hence the non-minimal term vanishes. Our purpose is to derive one-loop correction to Eq. (2) in the constant scalar case. We perform calculations in four space–time dimensions. Hence we are mainly interested in the renormalizable case $V = f\varphi^4/4$. However in this section we shall use general notation $V(\varphi)$, as being more compact and general. Let us emphasize that the theory of scalar field (2) is renormalizable in the framework of semiclassical gravity [23]. In this approach the metric is not quantized and represents a classical background for the quantum matter (in our case scalar) fields. The consistency and status of semiclassical approach have been recently discussed in [27].

In what follows we briefly consider the flat case first. One can see, e.g., [32] for a very pedagogical exposition with full details, despite there is some difference with our method. Recently, a similar calculation in flat space–time has been performed in [33], for a model of two scalar fields coupled to massless fermions. Since the main target of this work was an application to cosmology, it would be interesting to extend the result by taking curvature into account.

At the second stage of the work, we will take care about linear in curvature corrections. We stop at the first order because it is sufficient for our purposes and because calculations become too cumbersome in the next-order approximation. However, the normal coordinate method enables one, in principle, to perform calculations to any given order in curvature tensor and its derivatives and also can be helpful to evaluate higher loops contributions.

2.1. Flat space calculation

The result for the flat space is pretty well known [34]. The derivation for the massive case can be found, e.g., in the text-book [32], where it was obtained via Feynman diagrams. We can also arrive at the same result via the path integral functional methods. The starting point is the following expression:

$$V_{eff}(\varphi) = m^2\varphi^2 + V(\varphi) + \bar{V}_0(\varphi), \quad (3)$$

where

$$\bar{V}_0(\varphi) = \frac{1}{2}\,\mathrm{Tr}\ln S_2(\varphi) - \frac{1}{2}\,\mathrm{Tr}\ln S_2(\varphi = 0), \quad (4)$$

where $S_2(\varphi)$ is the bilinear form of the classical action in the background-field formalism [35]. The last term in (4) can be seen as normalization of a functional integral. This term arises naturally through the diagrammatic representation of effective potential (see, e.g., [32]). In curved space–time the second term gets dependent on the metric and hence becomes relevant. Here and below we omit an infinite volume factor. Let us note that the one-loop contribution (4) represents a quantum correction to the complete expression $V(\varphi) + m^2\varphi^2/2$ and not just for $V(\varphi)$. The same notations will be used in what follows.

By introducing four-dimensional momentum cut-off Ω, we arrive at the result[1]

$$\bar{V}_0(\varphi, \eta_{\mu\nu}) = \frac{1}{32\pi^2}\int_0^\Omega k^2\,dk^2 \ln\Big(\frac{k^2 + m^2 + V''}{k^2 + m^2}\Big). \quad (5)$$

After taking this integral we obtain

$$\bar{V}_0(\varphi, \eta_{\mu\nu}) = \bar{V}_0 = \bar{V}_0^{div} + \bar{V}_0^{fin}, \quad (6)$$

$$\bar{V}_0^{div} = \frac{1}{32\pi^2}\Big\{\Omega^2 V'' - \frac{1}{2}\big(m^2 + V''\big)^2 \ln\frac{\Omega^2}{m^2}\Big\}, \quad (7)$$

$$\bar{V}_0^{fin} = \frac{1}{32\pi^2}\Big\{\frac{1}{2}\big(m^2 + V''\big)^2 \ln\Big(1 + \frac{V''}{m^2}\Big) - \frac{1}{4}\big(m^2 + V''\big)^2\Big\}. \quad (8)$$

In the last expressions we have included the φ-independent m^4-type terms, which are indeed part of the second term in (4). The naive quantum contribution (6) must be supplemented by an appropriate local counterterm, which we choose in the form[2]

$$\Delta V_0 = \frac{1}{32\pi^2}\Big\{-\Omega^2 V'' + \frac{1}{2}\big(m^2 + V''\big)^2 \ln\frac{\Omega^2}{\mu^2} + \frac{1}{4}\big(m^2 + V''\big)^2\Big\}. \quad (9)$$

As a result we eliminate both quadratic and logarithmic divergences and arrive at the simple form of renormalized effective potential

$$\begin{aligned} V^{ren}_{eff,0}(\eta_{\mu\nu}, \varphi) &= m^2\varphi^2 + V + \bar{V}_0 + \Delta V_0 \\ &= m^2\varphi^2 + V + \frac{1}{64\pi^2}\big(m^2 + V''\big)^2 \ln\Big(\frac{m^2 + V''}{\mu^2}\Big). \end{aligned} \quad (10)$$

Looking at the counterterms (9) it is easy to see that the renormalizable theory is the one which has $V(\varphi) = \mathrm{const}\times\varphi^4$. The

[1] In all momentum integrals we assume that the Euclidean rotation is performed.

[2] For the sake of convenience we have included into ΔV the finite term, this can be easily compensated by changing μ.

reason is that for this potential the counterterms have the same form as the classical potential with an additional cosmological constant. At the next stage we will see that the same feature holds in curved space if the non-minimal term $\xi R\varphi^2$ is introduced.

2.2. Riemann normal coordinates

Riemann normal coordinates represent a useful tool for deriving local quantities, such as divergences or effective potential. These coordinates are based on the geodesic lines which link some fixed point $P'(x^{\mu\prime})$ with other points. We can always assume that $g_{\mu\nu}(P') = \eta_{\mu\nu}$. One can fix the initial conditions for the geodesic lines in such a way that the metric in the point $P(x^\mu)$ becomes a Taylor series in the deviation $y^\mu = x^{\mu\prime} - x^\mu$. The coefficients of such an expansion are curvature tensor, its contractions and covariant derivatives at the point P'. In the present work we will be interested only in the first order in curvature terms, and therefore all expansions will be taken in linear approximation.

For instance, for the metric tensor we meet [36]

$$g_{\alpha\beta}(x) = g_{\alpha\beta}(x') - \frac{1}{3}R_{\alpha\mu\beta\nu}(x')y^\mu y^\nu. \tag{11}$$

The bilinear operator of the action (2) is

$$\begin{aligned}-\hat{H} &= -\frac{1}{\sqrt{-g}}\frac{\delta^2 S_0}{\delta\varphi(x)\delta\varphi(x')} = \Box + m^2 - \xi R + V'' \\ &= \eta^{\mu\nu}\partial_\mu\partial_\nu + \frac{1}{3}R^\mu{}_\alpha{}^\nu{}_\beta y^\alpha y^\beta \partial_\mu\partial_\nu \\ &\quad - \frac{2}{3}R^\alpha_\beta y^\beta \partial_\alpha + m^2 - \xi R + V''.\end{aligned} \tag{12}$$

Of course, the term $-\xi R$ must be also expanded, but as far as we keep only first order in curvature, this is not relevant.

The main advantage of the local momentum representation is that all calculations can be performed in flat space–time (but with modified elements of Feynman technique) and the result for some local quantity can be always presented in a covariant way. For instance, the equation for the propagator of the scalar field has the form

$$\hat{H}G(x,x') = -g^{1/4}(x')\delta(x,x')g^{1/4}(x). \tag{13}$$

It proves better to work with the modified propagator [37] $\bar{G}(x,x')$, where

$$\hat{H}\bar{G}(x,x') = -\delta(x,x'). \tag{14}$$

It is important for us that the *r.h.s.* of the last relation does not depend on the metric, because we are going to use the relation $\mathrm{Tr}\ln\hat{H} = -\mathrm{Tr}\ln G(x,x')$ to obtain the dependence on curvature.

The explicit form of $\bar{G}(x,x')$ is known for a long time [37] for the free $V''=0$ case. As far as $V''=\text{const}$, we can simply replace m^2 by $\tilde{m}^2 = m^2 + V''$ and obtain, in the linear in curvature approximation,

$$\bar{G}(y) = \int \frac{d^4k}{(2\pi)^4}e^{iky}\left[\frac{1}{k^2+\tilde{m}^2} - \frac{(\xi-1/6)R}{(k^2+\tilde{m}^2)^2}\right]. \tag{15}$$

Now it is a simple exercise to expand $\mathrm{Tr}\ln\hat{H} = -\mathrm{Tr}\ln G(x,x')$ up to the first order in the scalar curvature. We define

$$\hat{H} = \hat{H}_0 + \hat{H}_1 R + \mathcal{O}(R^2), \qquad \bar{G} = \bar{G}_0 + \bar{G}_1 R + \mathcal{O}(R^2)$$

and consider

$$\begin{aligned}-\frac{1}{2}\mathrm{Tr}\ln\bar{G}(x,x') &= \frac{1}{2}\mathrm{Tr}\ln(\hat{H}_0 + \hat{H}_1 R) \\ &= \frac{1}{2}\mathrm{Tr}\ln\hat{H}_0 + \frac{1}{2}\mathrm{Tr}(\bar{G}_0\hat{H}_1 R).\end{aligned} \tag{16}$$

The first term in the last expression has been calculated in the previous subsection, and the second one can be transformed as follows:

$$\begin{aligned}\frac{1}{2}\mathrm{Tr}(\bar{G}_0\hat{H}_1 R) &= -\int d^4x\, V_1 R = \frac{1}{2}\mathrm{Tr}\big[\bar{G}_0^{-1}(x'',x')\bar{G}_1(x',x)\big]R \\ &= \frac{1}{2}\int d^4x \int d^4x' \big[\bar{G}_0^{-1}(x,x')\bar{G}_1(x',x)\big]R \\ &= \frac{1}{2}\int d^4x \int d^4x'\, R \int \frac{d^4k}{(2\pi)^4}e^{ik(x-x')} \\ &\quad \times \int \frac{d^4p}{(2\pi)^4}e^{ip(x'-x)}\bar{G}_0^{-1}(k)\bar{G}_1(p) \\ &= \frac{1}{2}\int d^4x\, R \int \frac{d^4k}{(2\pi)^4}\bar{G}_0^{-1}(k)\bar{G}_1(-k).\end{aligned} \tag{17}$$

The last integration is trivial due to a simple form of $\bar{G}_0(k)$ and $\bar{G}_1(k) = \bar{G}_1(-k)$ in (15), the final result reads

$$\bar{V}(\varphi, g_{\mu\nu}) = \bar{V}_0 + \bar{V}_1 R, \qquad \bar{V}_1 = \bar{V}_1^{div} + \bar{V}_1^{fin}, \tag{18}$$

$$\bar{V}_1^{div} = \frac{1}{2(4\pi)^2}\left(\xi - \frac{1}{6}\right)\left\{-\Omega^2 + (m^2 + V'')\ln\frac{\Omega^2}{m^2}\right\}, \tag{19}$$

$$\bar{V}_1^{fin} = -\frac{1}{2(4\pi)^2}\left(\xi - \frac{1}{6}\right)(m^2 + V'')\ln\left(\frac{m^2+V''}{m^2}\right). \tag{20}$$

Similar to the flat space case, the potential must be modified by adding a counterterm

$$\Delta V_1 = \frac{1}{2(4\pi)^2}\left(\xi - \frac{1}{6}\right)\left\{\Omega^2 - (m^2 + V'')\ln\frac{\Omega^2}{\mu^2}\right\}, \tag{21}$$

as a result one eliminates quadratic and logarithmic divergences and arrives at the renormalized expression

$$\begin{aligned}&V_{eff,1}^{ren}(g_{\mu\nu},\varphi) \\ &\quad = -\xi\varphi^2 - \frac{1}{2(4\pi)^2}\left(\xi - \frac{1}{6}\right)(m^2 + V'')\ln\left(\frac{m^2+V''}{\mu^2}\right).\end{aligned} \tag{22}$$

Obviously, the renormalizable theory is the one which has the non-minimal term in the classical expression (2), without this term we cannot deal with the corresponding counterterm (21).

Making covariant generalization of the flat-space result (10) and summing it up with (22), we arrive at the complete one-loop renormalized expression

$$\begin{aligned}V_{eff}^{ren}(g_{\mu\nu},\varphi) &= \rho_\Lambda + \frac{1}{2}(m^2 - \xi R)\varphi^2 + V \\ &\quad + \frac{\hbar}{2(4\pi)^2}\left[\frac{1}{2}(m^2 + V'')^2 \right. \\ &\quad \left. - \left(\xi - \frac{1}{6}\right)R(m^2 + V'')\right]\ln\left(\frac{m^2+V''}{\mu^2}\right),\end{aligned} \tag{23}$$

where we restored the loop expansion parameter $\hbar$ at its place and also included the classical density of the cosmological constant term, ρ_Λ, both for the sake of completeness.

Let us note that the ambiguity related to μ can be eliminated by imposing renormalization conditions. Furthermore, μ cancels automatically if we take into account the renormalization relations for the coupling f and mass m in the renormalizable case $V = f\varphi^4/4$. This follows from the overall μ-independence of the effective action. However, in curved space–time the dependence on μ may be a useful tool for exploring different limits of effective action, such as the limit of short distances, the limit of strong scalar field or their combination [22,23] (see further references therein). Furthermore, as it was recently discussed in [15] the μ-dependence can be an indication to the physical running of the cosmological constant.

The numerical evaluation of the relative importance of the gravitational term in (23) is strongly dependent on the mass m of the field under discussion, on the value of ξ and on the magnitude of curvature scalar in the given physical problem. It is easy to see that the relation between "flat" and "curved" terms in (23) is the same for classical and quantum one-loop terms. In the case when the scalar field is the Standard Model Higgs, we can assume the mass of the order of 100 GeV. The magnitude of ξ which is needed for the Higgs inflation model of [24] is about 4×10^4. Then it is easy to see that the value of curvature, when the gravitational term in (23) becomes dominating, is defined from the relation $\xi R = m^2$, hence the critical value is $R \propto 0.25\ \mathrm{GeV}^2$. In the cosmological setting the corresponding value of the Hubble parameter is, therefore, $H \propto$ GeV, which is much greater than the phenomenologically acceptable value. From one side, this shows that the requested value of ξ is not unnaturally large, because the dimensional product ξR remains small in at least most of the inflationary period. From another side, as it was discussed in [28,29] (see further references therein) the predictions of the theory are sufficiently sensible to the quantum corrections and this can lead to the constraints on the Higgs mass.

3. Operatorial cut-off regularization

Another possibility is to implement the cut-off regularization in a covariant manner via the Schwinger–DeWitt proper-time representation. Let us note that similar calculation for the massless case, using dimensional regularization, has been performed earlier in [38].

The effective action can be written in the form (in Euclidean case)

$$\bar{\Gamma}^{(1)} = \frac{1}{2}\operatorname{Tr}\ln\hat{H} = \frac{1}{2}\operatorname{Tr}\lim_{x'\to x}\int_{1/L^2}^{\infty}\frac{ds}{s}e^{-is\hat{H}}, \tag{24}$$

where L is the cut-off parameter. Let us remember that the heat-kernel can be presented as [39] (see also [40,41] and further references therein)

$$\hat{U}(x,x';s) = e^{-is\hat{H}}\delta(x,x') = \hat{U}_0(x,x';s)\sum_{k=0}^{\infty}(is)^k\hat{a}_k(x,x'), \tag{25}$$

where

$$\hat{U}_0(x,x';s) = \frac{1}{(4\pi i)^{n/2}}\frac{\mathcal{D}^{1/2}(x,x')}{s^{n/2}}e^{\frac{i\sigma(x,x')}{2s}-im^2s}. \tag{26}$$

Here $\sigma(x,x')$ is the geodesic distance between the two points x and x'. $\sigma(x,x')$ satisfies an identity $2\sigma = (\nabla\sigma)^2 = \sigma^\mu\sigma_\mu$ and vanishes in the coincidence limit $x' \to x$. $\mathcal{D}(x,x')$ is the Van Vleck–Morette determinant

$$\mathcal{D}(x,x') = \det\left[-\frac{\partial^2\sigma(x,x')}{\partial x^\mu\,\partial x'^\nu}\right], \tag{27}$$

which is a double tensor density, with respect to the both space–time arguments x and x'.

Taking into account the mentioned features of the geodesic distance $\sigma(x,x')$, it is easy to see that the divergences are concentrated in the coincidence limits of the first three Schwinger–DeWitt coefficients, namely

$$\bar{\Gamma}^{(1)}_{div} = -\frac{1}{2}\operatorname{Tr}\left[\frac{1}{2}a_0L^4 + a_1L^2 + a_2\ln\left(\frac{L^2}{\mu^2}\right)\right], \tag{28}$$

where

$$a_k = \lim_{x'\to x}\hat{a}_k(x,x').$$

The expressions for the a_0, a_1 and a_2 are well known [39] (see also Appendix B of [42] for the expressions with cut-off regularizations). In the scalar field case, for a constant background field we immediately obtain from (28) the expression

$$\begin{aligned}\bar{V}^{(1)}_{div}(\text{scalar}) = \frac{1}{2(4\pi)^2}\bigg\{&-\frac{L^4}{2} + \left[m^2 + V'' - \left(\xi - \frac{1}{6}\right)R\right]L^2 \\ &-\left[\frac{1}{2}(m^2+V'')^2 \\ &-(m^2+V'')\left(\xi-\frac{1}{6}\right)R\right]\ln\left(\frac{L^2}{\mu^2}\right)\bigg\},\end{aligned} \tag{29}$$

where we disregarded the higher-curvature terms. The comparison between the divergences calculated with the two types of cut-off shows that (29) is equivalent to the sum of (7) and (19) if we identify the two cut-off parameters Ω and L.

It is possible now to make a comparison between the two cut-off schemes. The both give equivalent φ-dependent divergent parts, however the local momentum cut-off method is capable to provide also complete expressions for the finite part of the one-loop effective potential (8) and (20). In the case of the proper-time cut-off scheme one can also arrive at the same renormalized expression (23) through the renormalization group approach [22]. However this requires an *ad hoc* identification of μ^2 with $m^2 + V''$. At the same time such an identification arises quite naturally within the local momentum cut-off method, because in this case we can work directly with the finite part of the renormalized effective potential (23).

Some additional remark would be in order. A natural tentation would be calculate the effective potential directly by using the method of summing up the Schwinger–DeWitt series [43]. However this idea meets an obstacle when it is used to calculate static quantities such as quantum corrections to the cosmological constant [44]. The reason is that the final output of this approach is a form factor which is given by an algebraic function of D'Alembert operator $\Box$ (covariant Laplacian in Euclidean case) acting on generalized curvature. In the static case, $\Box$ acting on a constant gives zero and hence this method in its original form is not efficient. The same applies also to the scalar field potential, because according to Eq. (1) the derivatives of a scalar go to the next term of the expansion of effective action. It would be an interesting exercise to modify the Schwinger–DeWitt series in such a way that the derivation of finite quantum corrections to the cosmological constant or to the potential of scalar field (these two are in fact closely related [15]) becomes possible, but at the moment the perspective of such calculation looks unclear. At the same time one can perfectly calculate the effective potential in the form of expansion in curvature tensor directly by using normal coordinates, as use did in the previous section.[3] We can conclude that the two methods perfectly complement each other, because they enable one to identify μ^2 as $m^2 + V''$ or other similar expression which shows up in other models. We will discuss an application of this idea in the next section.

[3] Unfortunately, this method is useless for the cosmological constant case.

4. Fermion contributions

As a practical application of the equivalence between the two cut-off schemes, let us consider the contribution of the fermion field with Yukawa interaction,

$$S_0 = \int d^4x \sqrt{-g} i\bar{\psi}\big(\gamma^\mu \nabla_\mu - im_f - ih\varphi\big)\psi. \tag{30}$$

As far as we are interested in the effective potential, the calculation can be done for a constant φ and hence we can denote

$$\tilde{m} = m_f + h\varphi. \tag{31}$$

Taking the Grassmann parity into account, the object of our interest is[4]

$$\bar{\Gamma}^{(1)}_f[\varphi, g_{\mu\nu}] = -\operatorname{Tr}\ln \hat{H}_f, \quad \text{where } \hat{H}_f = i\big(\gamma^\mu\nabla_\mu - i\tilde{m}\big). \tag{32}$$

As far as the result is expected to be even in $\tilde{m}$ (see, e.g., [45]), one can perform the transformation

$$\operatorname{Tr}\ln \hat{H}_f = \frac{1}{2}\operatorname{Tr}\ln\big(\hat{H}_f \hat{H}^*_f\big), \quad \text{where } \hat{H}^*_f = i\big(\gamma^\mu\nabla_\mu + i\tilde{m}\big). \tag{33}$$

The last product can be cast into the form

$$\hat{H}_f \hat{H}^*_f = -\left(\Box - \frac{1}{4}R + \tilde{m}^2\right). \tag{34}$$

Using the proper-time method we arrive at the expression for divergences

$$\bar{V}^{(1)}_{div}(\text{fer}) = -\frac{2}{(4\pi)^2}\left\{-\frac{L^4}{2} + \left(\tilde{m}^2 - \frac{1}{12}R\right)L^2 - \frac{1}{2}\left(\tilde{m}^4 - \frac{1}{6}R\tilde{m}^2\right)\ln\left(\frac{L^2}{\mu^2}\right)\right\}, \tag{35}$$

where we neglected the higher-curvature terms.

By using equivalence between the two cut-off schemes, we can easily write down the finite part of the renormalized one-loop contribution to the effective potential, namely

$$\bar{V}^{(1)}_{ren}(\text{fer}) = -\frac{1}{(4\pi)^2}\left(\tilde{m}^4 - \frac{1}{6}R\tilde{m}^2\right)\ln\left(\frac{\tilde{m}^2}{\mu^2}\right). \tag{36}$$

If we compare this result to the one of the Minimal Subtraction scheme of renormalization, it is clear that the correct identification of μ^2 is $\tilde{m}^2 = (m_f + h\varphi)^2$.

5. Renormalization group

Let us come back to the scalar result (23) and use it as an example of how the renormalization group equations for the parameters of the potential can be obtained. For this end we have to restrict our consideration by the renormalizable case $V = \lambda\varphi^4/4!$, such that $V'' = \lambda\varphi^2/2$ and the counterterms $\Delta V = \Delta V_0 + \Delta V_1 R$, with ΔV_0 from (9) and ΔV_1 from (21), have the same dependence on φ as the corresponding classical terms.

In order to obtain the renormalization group equations for the parameters one has to assume that the renormalized effective potential is equal to the bare effective potential. This statement is an intrinsic feature of the effective action which can be easily proved in a general form (see, e.g., [23]). For the finite part of effective potential this means that the apparent μ-dependence of the renormalized effective potential (23) must be compensated by the μ-dependence of the independent parameters of the theory, namely λ, m and ρ_Λ. Therefore one can easily find $\lambda(\mu)$, $m(\mu)$ and $\rho_\Lambda(\mu)$ directly from (23). We leave this calculation as an exercise for the interested reader and instead will obtain the corresponding β-functions from the infinite renormalization of the classical action, similar as it is done in the MS-scheme and dimensional regularization [23].

The classical (extended by mass and non-minimal terms) potential, with the added counterterm, form the renormalized classical potential, which should be equal to the bare one, hence[5]

$$\rho_\Lambda + \big(m^2 - \xi R\big)\varphi^2 + \frac{\lambda\varphi^4}{4!} + \Delta V_0 + R\Delta V_1 = \rho_{\Lambda(0)} + \big[m^2_{(0)} - \xi_{(0)}R\big]\varphi^2 + \frac{\lambda_{(0)}\varphi^4}{4!}. \tag{37}$$

The *l.h.s.* of this relation does depend on μ explicitly and the *r.h.s.* does not. This condition should be satisfied for all terms separately, because there are arbitrary quantities φ and R. Therefore, using (9) and (21), we arrive at the equations

$$\rho_{\Lambda(0)} = \rho_\Lambda + \frac{m^4}{2(4\pi)^2}\ln\frac{\Omega^2}{\mu^2},$$

$$m^2_{(0)} = m^2 + \frac{\lambda m^2}{2(4\pi)^2}\ln\frac{\Omega^2}{\mu^2},$$

$$\xi_{(0)} = \xi + \frac{\lambda}{2(4\pi)^2}\left(\xi - \frac{1}{6}\right)\ln\frac{\Omega^2}{\mu^2},$$

$$\lambda_{(0)} = \lambda + \frac{4!\lambda^2}{16(4\pi)^2}\ln\frac{\Omega^2}{\mu^2}.$$

At this stage we can apply the conventional wisdom to take derivatives $\mu\frac{d}{d\mu}$ of the bare quantities $\rho^{(0)}_\Lambda$, $m^2_{(0)}$, $\xi_{(0)}$ and $\lambda_{(0)}$ and set them to zero. As a result we arrive at the following β-functions:

$$\mu\frac{d\rho_\Lambda}{d\mu} = \frac{m^4}{2(4\pi)^2}, \qquad \rho_\Lambda(\mu_0) = \rho_{\Lambda,0};$$

$$\mu\frac{dm^2}{d\mu} = \frac{\lambda}{(4\pi)^2}m^2, \qquad m^2(\mu_0) = m^2_0;$$

$$\mu\frac{d\xi}{d\mu} = \frac{\lambda}{(4\pi)^2}\left(\xi - \frac{1}{6}\right), \qquad \xi(\mu_0) = \xi_0;$$

$$\mu\frac{d\lambda}{d\mu} = \frac{3\lambda^2}{(4\pi)^2}, \qquad \lambda(\mu_0) = \lambda_0, \tag{38}$$

where the initial points of the renormalization group trajectories are defined at some reference value (scale) μ_0. The solution of these equations is well known, e.g., in the leading-log approximation we have

$$\lambda(\mu) = \lambda_0 + \frac{3\hbar\lambda_0^2}{(4\pi)^2}\ln(\mu/\mu_0),$$

where we restored $\hbar$ for further convenience.[6]

It is easy to check that if we replace these solutions into the renormalized effective potential (23), the dependence on μ completely disappears in the $\mathcal{O}(\hbar)$-terms. Definitely, this does not mean that the effective potential becomes trivial, because the real

[4] The operation of Tr is defined without taking statistics into account.

[5] Here we mark bare parameters by the subscript (0). In the simple case of purely scalar theory one does not need to renormalize the field φ, but in general case it is not so, of course.

[6] We note that the solution in the momentum-subtraction scheme is much less simple, see [46].

content of quantum corrections is related to the dependence on φ, which did not change under the procedure described above.

We can conclude that the μ-dependence is nothing but a useful tool for obtaining the dependence on φ, or on the derivatives of φ (or other mean field). This tool becomes especially important in those cases when the derivation of explicit dependence on the fields and their derivatives is not possible, as it was discussed recently in [15] for the case of external gravitational field.

The last observation is that the relation between μ-dependence and real effective potential may be rather nontrivial in more complicated models. Consider, for example, a theory where the scalar field is coupled to different fermions with distinct masses. According to the result of the previous section, (36), there is no unique identification of μ in this case. Therefore, one need to be very careful when using the renormalization group results for the massive theories, especially when different masses are present.

6. Conclusions

We have performed an explicitly covariant calculation of effective potential in two types of cut-off regularizations. The divergences are identical within the two approaches, but the covariant local momentum cut-off has an advantage to provide also the finite part of effective potential for the massive case and, consequently, it indicates the physical interpretation for the renormalization group parameter μ. It would be interesting to apply the same method to the derivation of the "Energy–Momentum Tensor" of vacuum and, in this way, resolve the amazing puzzle with non-covariant power-like divergences which were described recently in [2,5,6]. The work in this direction is in progress and the results of the calculations presented here are going to be useful in this respect.

Another important conclusion of our work is the restricted sense of the renormalization group-based quantum corrections for the quantum theory of massive fields, especially if different masses are present.

Acknowledgements

Authors are very grateful to Neven Bilic, Silvije Domazet and Branco Guberina for carefully reading the paper and indicating some errors/misprints in signs. Authors are grateful to CAPES (B.R. and F.S.) for support, also to FAPEMIG (B.R. and I.Sh.), CNPq and ICTP (I.Sh.) for partial support of their work.

References

[1] B.S. DeWitt, Phys. Rep. 19 (1975) 295.
[2] E.K. Akhmedov, Vacuum energy and relativistic invariance, e-print: hep-th/0204048.
[3] P.J.E. Peebles, B. Ratra, Rev. Mod. Phys. 75 (2003) 559.
[4] S. Nobbenhuis, The cosmological constant problem, an inspiration for new physics, Ph.D. thesis, e-print: gr-qc/0609011.
[5] M. Maggiore, Phys. Rev. D 83 (2011) 063514.
[6] N. Bilic, Phys. Rev. D 83 (2011) 105003.
[7] M. Maggiore, L. Hollenstein, M. Jaccard, E. Mitsou, Early dark energy from zero-point quantum fluctuations, e-print: arXiv:1104.3797 [astro-ph.CO].
[8] C. Wetterich, Nucl. Phys. B 352 (1991) 529;
J. Berges, N. Tetradis, C. Wetterich, Phys. Rep. 363 (2002) 223 (see further references therein).
[9] A. Bonanno, M. Reuter, Phys. Rev. D 62 (2000) 043008;
A. Bonanno, M. Reuter, Phys. Rev. D 65 (2002) 043508;
E. Bentivegna, A. Bonanno, M. Reuter, JCAP 0401 (2004) 001;
M. Reuter, H. Weyer, JCAP 0412 (2004) 001.
[10] I.L. Shapiro, J. Solà, Phys. Lett. B 475 (2000) 236.
[11] I.L. Shapiro, J. Solà, JHEP 0202 (2002) 006.
[12] A. Babic, B. Guberina, R. Horvat, H. Štefančić, Phys. Rev. D 65 (2002) 085002;
B. Guberina, R. Horvat, H. Štefančić, Phys. Rev. D 67 (2003) 083001;
A. Babic, B. Guberina, R. Horvat, H. Štefančić, Phys. Rev. D 71 (2005) 124041.
[13] I.L. Shapiro, J. Solà, C. España-Bonet, P. Ruiz-Lapuente, Phys. Lett. B 574 (2003) 149;
I.L. Shapiro, J. Solà, C. España-Bonet, P. Ruiz-Lapuente, JCAP 0402 (2004) 006;
I.L. Shapiro, J. Solà, Nucl. Phys. Proc. Suppl. 127 (2004) 71.
[14] I.L. Shapiro, J. Solà, H. Štefančić, JCAP 0501 (2005) 012.
[15] I.L. Shapiro, J. Solà, Phys. Lett. B 682 (2009) 105, arXiv:0910.4925 [hep-th], see also the extended version arXiv:0808.0315 [hep-th].
[16] D.C. Rodrigues, P.S. Letelier, I.L. Shapiro, JCAP 1004 (2010) 020, e-print: arXiv:0911.4967.
[17] S. Domazet, H. Štefančić, Phys. Lett. B 703 (2011) 1, e-print: arXiv:1010.3585.
[18] B.S. DeWitt, Phys. Rep. C 19 (1975) 297.
[19] A.A. Grib, V.M. Mostepanenko, JETP Lett. 25 (1977) 302.
[20] V.G. Lapchinsky, V.A. Rubakov, Theor. Math. Phys. 42 (1980) 23;
V.G. Lapchinsky, V.A. Rubakov, Teor. Mat. Fiz. 42 (1980) 37.
[21] G.M. Shore, Ann. Phys. 128 (1980) 376.
[22] I.L. Buchbinder, S.D. Odintsov, Sov. Phys. J. 27 (1984) 554;
I.L. Buchbinder, S.D. Odintsov, Class. Quant. Grav. 2 (1985) 721;
See also further development of the renormalization group method, applied for other sectors of effective action in I.L. Buchbinder, J.J. Wolfengaut, Class. Quant. Grav. 5 (1988) 1127.
[23] I.L. Buchbinder, S.D. Odintsov, I.L. Shapiro, Effective Action in Quantum Gravity, IOP Publishing, 1992.
[24] F.L. Bezrukov, M. Shaposhnikov, Phys. Lett. B 659 (2008) 703.
[25] B.L. Spokoiny, Phys. Lett. B 147 (1984) 39;
T. Futamase, K. Maeda, Phys. Rev. D 39 (1989) 399;
D.S. Salopek, J.R. Bond, J.M. Bardeen, Phys. Rev. D 40 (1989) 1753;
R. Fakir, W.G. Unruh, Phys. Rev. D 41 (1990) 1783;
D.I. Kaiser, Phys. Rev. D 52 (1995) 4295;
E. Komatsu, T. Futamase, Phys. Rev. D 59 (1999) 064029;
S. Tsujikawa, B. Gumjudpai, Phys. Rev. D 69 (2004) 123523.
[26] A.H. Guth, Phys. Rev. D 23 (1981) 347.
[27] I.L. Shapiro, Class. Quant. Grav. 25 (2008) 103001.
[28] A.O. Barvinsky, A.Yu. Kamenshchik, A.A. Starobinsky, JCAP 0811 (2008) 021;
A.O. Barvinsky, A.Y. Kamenshchik, C. Kiefer, A.A. Starobinsky, C. Steinwachs, JCAP 0912 (2009) 003.
[29] F.L. Bezrukov, M. Shaposhnikov, JHEP 0907 (2009) 089;
F.L. Bezrukov, A. Magnin, M. Shaposhnikov, Phys. Lett. B 675 (2009) 88.
[30] A. De Simone, M.P. Hertzberg, F. Wilczek, Phys. Lett. B 678 (2009) 1, arXiv:0812.4946.
[31] S.-B. Liao, Phys. Rev. D 53 (1996) 2020.
[32] M.B. Voloshin, K.A. Ter-Martirosyan, Gauge Theory of Particle Interaction, Nauka, 1981 (in Russian).
[33] M. Bastero-Gil, A. Berera, B.M. Jackson, JCAP 1107 (2011) 010, arXiv:1003.5636.
[34] S.R. Coleman, E.J. Weinberg, Phys. Rev. D 7 (1973) 1888.
[35] B.S. DeWitt, Phys. Rev. 162 (1967) 1195, 1239;
G. 't Hooft, Nucl. Phys. B 62 (1973) 444;
For the introduction see, e.g., L.F. Abbott, Acta Phys. Pol. B 13 (1982) 33, and [23].
[36] A.Z. Petrov, Einstein Spaces, Oxford, Pergamon, 1969.
[37] T.S. Bunch, L. Parker, Phys. Rev. D 20 (1979) 2499.
[38] K. Ishikawa, Phys. Rev. D 28 (1983) 2445.
[39] B.S. DeWitt, Dynamical Theory of Groups and Fields, Gordon and Breach, 1965.
[40] A.O. Barvinsky, G.A. Vilkovisky, Phys. Rep. 119 (1985) 1.
[41] I.G. Avramidi, Covariant methods for the calculation of the effective action in quantum field theory and investigation of higher-derivative quantum gravity, Ph.D. thesis at Moscow State University, 1986, hep-th/9510140;
I.G. Avramidi, Heat Kernel and Quantum Gravity, Springer-Verlag, 2000.
[42] E.S. Fradkin, A.A. Tseytlin, Nucl. Phys. B 201 (1982) 469.
[43] A.O. Barvinsky, G.A. Vilkovisky, Nucl. Phys. B 333 (1990) 471;
I.G. Avramidi, Yad. Fiz. (Sov. J. Nucl. Phys.) 49 (1989) 1185.
[44] E.V. Gorbar, I.L. Shapiro, JHEP 0302 (2003) 021.
[45] G. de Berredo-Peixoto, Mod. Phys. Lett. A 16 (2001) 2463.
[46] G. de Berredo-Peixoto, E.V. Gorbar, I.L. Shapiro, Class. Quant. Grav. 21 (2004) 2281.