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On solutions of the matrix equations $X - AXB = C$ and $X - A\overline{X}B = C$ ☆

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Abstract

This paper studies the solutions of complex matrix equations $X - AXB = C$ and $X - A\overline{X}B = C$, and obtains explicit solutions of the equations by the method of characteristic polynomial and a method of real representation of a complex matrix respectively.

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Keywords: Matrix equation; Solution; Characteristic polynomial; Real representation

1. Introduction

The matrix equations $AX - XB = C$ and $X - AXB = C$ play important roles in the theories and applications of stability and control [1,2]. When $A = B^{\mathrm{H}}$ (the conjugate transpose of B), the equations are the well-known Lyapunov and Stein equations respectively.

In [3], Jameson studied the matrix equation $AX - XB = C$ by the method of characteristic polynomial, and derived explicit solution of the equation as follows.

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Lemma 1.1 (Jameson). *Let* $A \in \mathbf{C}^{m\times m}$, $B \in \mathbf{C}^{n\times n}$, $C \in \mathbf{C}^{m\times n}$. *Then*:

(1) *If* X *is a solution of* $AX - XB = C$, *then*

$$Xf_A(B) = -\sum_{k=1}^{m}\sum_{j=1}^{k-1} p_k A^j C B^{k-j-1},$$

where $f_A(\lambda) = \det(\lambda I - A) = p_m\lambda^m + p_{m-1}\lambda^{m-1} + \cdots + p_0$.

(2) *If* X *is the unique solution of* $AX - XB = C$, *then*

$$X = -\left(\sum_{k=1}^{m}\sum_{j=1}^{k-1} p_k A^j C B^{k-j-1}\right)(f_A(B))^{-1}.$$

In [4, Chapter 12.3] Lancaster and Tismenetsky studied the matrix equation $X - AXB = C$, and derived that the equation has a unique solution $X = \sum_{j=0}^{\infty} A^j C B^j$ when A and B have spectral radii μ_A and μ_B respectively with $\mu_A\mu_B < 1$.

In this paper, we extend the result of [4] concerning the matrix equation $X - AXB = C$, and obtain explicit solutions of matrix equation $X - AXB = C$ by the method of characteristic polynomial, and then we characterize the existence of solution to the equation $X - A\overline{X}B = C$, derive the solution of matrix equation $X - A\overline{X}B = C$ in explicit form by means of real representation, where $\overline{X}$ denotes the conjugate of the complex matrix X.

Let $\mathbf{R}$ denote the real number field, $\mathbf{C}$ the complex number field. $\mathbf{F}^{m\times n}$ denotes the set of $m \times n$ matrices on a field $\mathbf{F}$, and for any $A \in \mathbf{C}^{m\times n}$, $\mathrm{rank}(A)$, A^{T} and $A^{(1)}$ denote the rank, the transpose and $\{1\}$-inverse of matrix A respectively. Let $f_A(\lambda)$ denote the characteristic polynomial of matrix A.

2. Matrix equation $X - AXB = C$

In this section, we discuss the solution of the matrix equation

$$X - AXB = C \tag{2.1}$$

by the method of characteristic polynomial of matrix, where $A \in \mathbf{C}^{m\times m}$, $B \in \mathbf{C}^{n\times n}$ and $C \in \mathbf{C}^{m\times n}$.

By [4, Chapter 12], we have the following result.

Lemma 2.1. *Let* $A \in \mathbf{C}^{m\times m}$, $B \in \mathbf{C}^{n\times n}$ *and* $C \in \mathbf{C}^{m\times n}$. *Then*:

(1) *Eq.* (2.1) *has a solution if and only if*

$$\mathrm{rank}(I_{mn} - A \otimes B^{\mathrm{T}}) = \mathrm{rank}(I_{mn} - A \otimes B^{\mathrm{T}}, \mathrm{vec}(C)),$$

where $\otimes$ *denotes the Kronecker product and* vec *is the* “vec” *operation.*

(2) *Eq.* (2.1) *has a unique solution if and only if* $\lambda_i\mu_j \neq 1$, *where* $\lambda_1, \dots, \lambda_m$ *and* $\mu_1, \dots, \mu_n$ *are the characteristic values of A and B respectively.*
(3) *Eq.* (2.1) *has a unique solution if and only if* $f_A(B)$ *is nonsingular.*

For $A \in \mathbf{C}^{m\times m},\ B \in \mathbf{C}^{n\times n}$, define linear operators δ and $\rho : \mathbf{C}^{m\times n} \to \mathbf{C}^{m\times n}$ with $\delta(X) = AX$ and $\rho(X) = XB,\ X \in \mathbf{C}^{m\times n}$. Clearly $\delta\rho(X) = AXB = \rho\delta(X)$, and the Eq. (2.1) is equivalent to

$$(1 - \delta\rho)X = C. \tag{2.2}$$

It is easy to verify the following.

Lemma 2.2
(1) $\delta\rho = \rho\delta$.
(2) *If* $q(\lambda, \mu) = \sum_{i,j} a_{ij}\lambda^i\mu^j \in \mathbf{C}[\lambda, \mu]$, *then for any* $X \in \mathbf{C}^{m\times n}$,

$$q(\delta, \rho)X = \sum_{i,j} a_{ij}A^iXB^j.$$

(3) *If* $f(\lambda)$ *is a polynomial of* λ, *then* $f(\delta)X = f(A)X$ *and* $f(\rho)X = Xf(B)$.

For $A \in \mathbf{C}^{m\times m}$, let the characteristic polynomial of A be

$$f_A(\lambda) = \lambda^m + a_{m-1}\lambda^{m-1} + \cdots + a_0 \tag{2.3}$$

and define

$$h_A(\lambda) = \lambda^m f_A(\lambda^{-1}) = 1 + a_{m-1}\lambda + \cdots + a_0\lambda^m \tag{2.4}$$

then

$$q(\lambda, \mu) \equiv \mu^{m-1}\frac{f_A(\lambda) - f_A(\mu^{-1})}{\lambda - \mu^{-1}} = \sum_{k=1}^{m}\sum_{s=1}^{k} a_k\lambda^{k-s}\mu^{m-s} \tag{2.5}$$

in which $a_m = 1$ and

$$q(\lambda, \mu)(1 - \lambda\mu) = -\mu^m\big(f_A(\lambda) - f_A(\mu^{-1})\big) = -\mu^m f_A(\lambda) + h_A(\mu). \tag{2.6}$$

Lemma 2.3. $q(\delta, \rho)(1 - \delta\rho) = h_A(\rho)$.

Proof. By Lemma 2.2 and the Cayley–Hamilton theorem of matrices, for any $X \in \mathbf{C}^{m\times n},\ f_A(\delta)X = f_A(A)X = 0$, i.e. $f_A(\delta) = 0$. So the Lemma 2.3 follows from (2.6). □

Proposition 2.4. *Let* $f_A(\lambda)$ *and* $h_A(\lambda)$ *be given in Eqs.* (2.3) *and* (2.4). *Then*:

(1) $(h_A(\lambda), f_B(\lambda)) = 1$ *if and only if* $\lambda_i\mu_j \neq 1$, *where* $\lambda_1, \dots, \lambda_m$ *and* $\mu_1, \dots, \mu_n$ *are the characteristic values of A and B respectively.*

(2) *Eq.* (2.1) *has a unique solution if and only if* $(h_A(\lambda), f_B(\lambda)) = 1$.
(3) *Eq.* (2.1) *has a unique solution if and only if* $h_A(B)$ *is nonsingular.*

Proof. It is easy to see that (1) follows from the construction of $h_A(\lambda)$ in Eq. (2.4), and by Lemma 2.1, (2) and (3) come from (1). □

Theorem 2.5. *Let* $A \in \mathbf{C}^{m\times m}$, $B \in \mathbf{C}^{n\times n}$ *and* $C \in \mathbf{C}^{m\times n}$. *Then*:

(1) *If Eq.* (2.1) *has a solution* X, *then*

$$Xh_A(B) = \sum_{k=1}^{m}\sum_{s=1}^{k} a_k A^{k-s} C B^{m-s},$$

where $h_A(\lambda)$ *is given in Eq.* (2.4), *and*

$$X = Fh_A^{(1)}(B) + Y(I_n - h_A(B)h_A^{(1)}(B))$$

in which $h_A^{(1)}(B) = (h_A(B))^{(1)}$, *and* $F = \sum_{k=1}^{m}\sum_{s=1}^{k} a_k A^{k-s} C B^{m-s}$, *some* $Y \in \mathbf{C}^{m\times n}$.

(2) *If Eq.* (2.1) *has a unique solution* X, *then*

$$X = \left(\sum_{k=1}^{m}\sum_{s=1}^{k} a_k A^{k-s} C B^{m-s}\right)(h_A(B))^{-1}.$$

Proof. Since Eq. (2.1) has a solution X if and only if Eq. (2.2) has a solution X, and by Lemma 2.3 we have

$$\begin{aligned} Xh_A(B) &= h_A(\rho)X = q(\delta,\rho)(1-\delta\rho)X \\ &= q(\delta,\rho)C = \sum_{k=1}^{m}\sum_{s=1}^{k} a_k A^{k-s} C B^{m-s}. \end{aligned}$$

So by [5, Chapter 2.1], we easily know that (1) holds, and (2) follows clearly from (3) in Proposition 2.4. □

3. Real representation

Let $A \in \mathbf{C}^{m\times n}$, then A can be uniquely written as $A = A_1 + A_2\mathrm{i}$, $A_1, A_2 \in \mathbf{R}^{m\times n}$, $\mathrm{i}^2 = -1$. Define real representation σ

$$A_\sigma = \begin{pmatrix} A_1 & A_2 \\ A_2 & -A_1 \end{pmatrix} \in \mathbf{R}^{2m\times 2n},$$

A_σ is called the real representation matrix of A.

For a $m \times m$ complex matrix A, define $A_\sigma^i = (A_\sigma)^i$, and

$$P_j = \begin{pmatrix} I_j & 0 \\ 0 & -I_j \end{pmatrix}, \quad Q_j = \begin{pmatrix} 0 & I_j \\ -I_j & 0 \end{pmatrix},$$

where I_j is the $j \times j$ identity matrix.

Proposition 3.1

(1) *If* $A, B \in \mathbf{C}^{m\times n}$, $a \in \mathbf{R}$, *then*

$$(A+B)_\sigma = A_\sigma + B_\sigma,\ (aA)_\sigma = aA_\sigma.$$

(2) *If* $A \in \mathbf{C}^{m\times n}$, $B \in \mathbf{C}^{n\times r}$, *then*

$$(AB)_\sigma = A_\sigma P_n B_\sigma = A_\sigma (\overline{B})_\sigma P_r.$$

(3) *If* $A \in \mathbf{C}^{m\times m}$, *then A is nonsingular if and only if A_σ is nonsingular.*

(4) *If* $A \in \mathbf{C}^{m\times m}$, *then* $A_\sigma^{2k} = \left((A\overline{A})^k\right)_\sigma P_m$.

(5) *If* $A \in \mathbf{C}^{m\times m}$, $B \in \mathbf{C}^{n\times n}$, $C \in \mathbf{C}^{m\times n}$, *and $k+l$ is even, then*

$$A_\sigma^k C_\sigma B_\sigma^l = \begin{cases} \left((A\overline{A})^s (A\overline{C}B)(\overline{B}B)^t\right)_\sigma, & k = 2s+1,\ l = 2t+1, \\ \left((A\overline{A})^s C(\overline{B}B)^t\right)_\sigma, & k = 2s,\ l = 2t. \end{cases}$$

(6) *If* $A \in \mathbf{C}^{m\times n}$, *then* $Q_m A_\sigma Q_n = A_\sigma$.

Proof. By direct calculation, we easily know (1) and (2) hold, and (3) follows from (2). By (2), $A_\sigma^{2k} = (A_\sigma)^{2k} = (A_\sigma)^{2(k-1)}(A_\sigma)^2 = (A_\sigma)^{2(k-1)}(A\overline{A})_\sigma P_m$, so (4) is proved by induction. Finally (5) follows clearly from (2) and (4). □

For $A \in \mathbf{C}^{m\times m}$, if

$$A_\sigma \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \lambda \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix},$$

then by the definition of real representation A_σ we easily have

$$A_\sigma \left(\alpha_2\ \alpha_1\right) = -\lambda \left(\alpha_2\ \alpha_1\right), \quad A_\sigma \begin{pmatrix} \overline{\alpha}_1 \\ \overline{\alpha}_2 \end{pmatrix} = \overline{\lambda} \begin{pmatrix} \overline{\alpha}_1 \\ \overline{\alpha}_2 \end{pmatrix},$$

$$A_\sigma \left(\overline{\alpha}_2\ \overline{\alpha}_1\right) = -\overline{\lambda} \left(\overline{\alpha}_2\ \overline{\alpha}_1\right),$$

so we have following result.

Proposition 3.2. *If λ is a characteristic value of A_σ, then so are $\pm\lambda$, $\pm\overline{\lambda}$.*

For any $A \in \mathbf{C}^{m\times m}$, let $f_{A_\sigma}(\lambda) = \det(\lambda I_{2m} - A_\sigma) = \sum_{k=0}^{2m} a_k \lambda^k$ be the characteristic polynomial of the real matrix A_σ, we have

Proposition 3.3. *Let* $A \in \mathbf{C}^{m\times m}$, $B \in \mathbf{C}^{n\times n}$. *Then*

(1) $f_{A_\sigma}(\lambda)$ *is a real polynomial, and* $f_{A_\sigma}(\lambda) = \sum_{k=0}^{m} a_{2k}\lambda^{2k}$;

(2) $h_{A_\sigma}(\lambda)$ *is a real polynomial, and* $h_{A_\sigma}(\lambda) = \sum_{k=0}^{m} a_{2k}\lambda^{2(m-k)}$;

(3) $h_{A_\sigma}(B_\sigma) = (g_{A_\sigma}(B\overline{B}))_\sigma P_n,\ f_{A_\sigma}(B_\sigma) = (p_{A_\sigma}(B\overline{B}))_\sigma P_n$

in which $g_{A_\sigma}(\lambda) = \sum_{k=0}^{m} a_{2k}\lambda^{m-k},\ p_{A_\sigma}(\lambda) = \sum_{k=0}^{m} a_{2k}\lambda^k \in \mathbf{R}[\lambda]$.

Proof. By Proposition 3.2, we know a_k are real numbers, and $a_{2k+1} = 0$, so (1) and (2) follow from Eqs. (2.3) and (2.4). For any k, by Proposition 3.1, we know $B_\sigma^{2k} = ((B\overline{B})^k)_\sigma P_n$, so (3) is valid. □

4. The matrix equation $X - A\overline{X}B = C$

In this section, we study the solution of matrix equation

$$X - A\overline{X}B = C \tag{4.1}$$

by the method of real representation, where $A \in \mathbf{C}^{m\times m}$, $B \in \mathbf{C}^{n\times n}$ and $C \in \mathbf{C}^{m\times n}$.

We first define the real representation matrix equation of Eq. (4.1) by

$$Y - A_\sigma Y B_\sigma = C_\sigma. \tag{4.2}$$

By (2) in Proposition 3.1, Eq. (4.1) is equivalent to the equation

$$X_\sigma - A_\sigma X_\sigma B_\sigma = C_\sigma. \tag{4.3}$$

Proposition 4.1. *Eq.* (4.1) *has a solution* X *if and only if its real representation Eq.* (4.2) *has a real matrix solution* $Y = X_\sigma$.

Theorem 4.2. *Let* $A \in \mathbf{C}^{m\times m}$, $B \in \mathbf{C}^{n\times n}$ *and* $C \in \mathbf{C}^{m\times n}$. *Then Eq.* (4.1) *has a solution* $X \in \mathbf{C}^{m\times n}$ *if and only if Eq.* (4.2) *has a solution* $Y \in \mathbf{R}^{2m\times 2n}$, *in which case, if* Y *is a solution to Eq.* (4.2), *then the following matrix*

$$X = \frac{1}{4}(I_m, \mathrm{i}I_n)(Y + Q_m Y Q_n)\begin{pmatrix} I_m \\ \mathrm{i}I_n \end{pmatrix} \tag{4.4}$$

is a solution to Eq. (4.1).

Proof. We only show that if

$$Y = \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix}, \quad Y_{uv} \in \mathbf{R}^{m\times n},\ u, v = 1, 2 \tag{4.5}$$

is a solution to Eq. (4.2), then the matrix given in Eq. (4.4) is a solution to Eq. (4.1). In fact, according to Proposition 3.1, since $Q_m A_\sigma Q_n = A_\sigma$, so

$$Q_m Y Q_n - A_\sigma Q_m Y Q_n B_\sigma = C_\sigma. \tag{4.6}$$

This equation shows that if Y is a solution to Eq. (4.2), then $Q_m Y Q_n$ is also a solution to Eq. (4.2). Thus the following real matrix

$$\hat{Y} = \frac{1}{2}(Y + Q_m Y Q_n) \tag{4.7}$$

is also a solution to Eq. (4.2). Now substituting Eq. (4.5) in Eq. (4.8) and then simplifying the expression, we easily get

$$\hat{Y} = \begin{pmatrix} Z_0 & Z_1 \\ Z_1 & -Z_0 \end{pmatrix}, \tag{4.8}$$

where

$$Z_0 = \frac{1}{2}(Y_{11} - Y_{22}), \qquad Z_1 = \frac{1}{2}(Y_{12} + Y_{21}).$$

From Eq. (4.8) we construct a complex matrix as follows

$$X = Z_0 + Z_1\mathrm{i} = \frac{1}{2}(I_m, \mathrm{i}I_n)\hat{Y}\begin{pmatrix} I_m \\ \mathrm{i}I_n \end{pmatrix}. \tag{4.9}$$

clearly the real representation of complex matrix X is $\hat{Y}$, i.e. $X_\sigma = \hat{Y}$. By Proposition 4.1, X is a solution to Eq. (4.1). □

Remark. Theorem 4.2 gives a practical technique for us to find a solution to Eq. (4.1) by a solution to Eq. (4.2) by means of real representation matrices.

By Lemma 2.1 and Theorem 4.2 we have

Theorem 4.3. *Let $A \in \mathbf{C}^{m\times m}$, $B \in \mathbf{C}^{n\times n}$ and $C \in \mathbf{C}^{m\times n}$. Then Eq.* (4.1) *has a solution if and only if*

$$\operatorname{rank}\bigl(I_{mn} - A_\sigma \otimes B_\sigma^{\mathrm{T}}\bigr) = \operatorname{rank}\bigl(I_{mn} - A_\sigma \otimes B_\sigma^{\mathrm{T}}, \operatorname{vec}(C_\sigma)\bigr),$$

where $\otimes$ denotes the Kronecker product and vec *is the* "vec" *operation.*

Theorem 4.4. *Let $A \in \mathbf{C}^{m\times m}$, $B \in \mathbf{C}^{n\times n}$, $C \in \mathbf{C}^{m\times n}$. Then*:

(1) *If Eq.* (4.1) *has a solution X, then*

$$Xg_{A_\sigma}(\overline{B}B) = \sum_{k=1}^{m} a_{2k}\left[\sum_{j=0}^{k-1}(A\overline{A})^{k-j-1}(A\overline{C}B)(\overline{B}B)^{m-j-1} + \sum_{j=1}^{k}\Bigl[(A\overline{A})^{k-j}C(\overline{B}B)^{m-j}\Bigr]\right]$$

and

$$X = Fg_{A_\sigma}^{(1)}(\overline{B}B) + Y\bigl(I_n - g_{A_\sigma}(\overline{B}B)g_{A_\sigma}^{(1)}(\overline{B}B)\bigr)$$

in which

$$g_{A_\sigma}^{(1)}(\overline{B}B) = \left(g_{A_\sigma}(\overline{B}B)\right)^{(1)},$$

$$F = \sum_{k=1}^{m} a_{2k}\left[\sum_{j=1}^{k-1}(A\overline{A})^{k-j-1}(A\overline{C}B)(\overline{B}B)^{m-j-1} + \sum_{j=1}^{k}\left[(A\overline{A})^{k-j}C(\overline{B}B)^{m-j}\right]\right],$$

and some $Y \in \mathbf{C}^{m\times n}$.

(2) *If Eq.* (4.1) *has a solution* X, *and* $h_{A_\sigma}(\lambda)$ *and* $f_{B_\sigma}(\lambda)$ *are relatively prime*, *then Eq.* (4.1) *has a unique solution*

$$\begin{aligned} X &= F(g_{A_\sigma}(\overline{B}B))^{-1} \\ &= \sum_{k=1}^{m} a_{2k}\left[\sum_{j=1}^{k-1}(A\overline{A})^{k-j-1}(A\overline{C}B)(\overline{B}B)^{m-j-1} + \sum_{j=1}^{k}(A\overline{A})^{k-j}C(\overline{B}B)^{m-j}\right](g_{A_\sigma}(\overline{B}B))^{-1}. \end{aligned}$$

Proof. (1) If Eq. (4.1) has a solution X, then Eq. (4.2) has a solution $Y = X_\sigma$. By Lemma 1.1 and Proposition 3.3, we have

$$X_\sigma h_{A_\sigma}(B_\sigma) = \sum_{k=1}^{m}\sum_{s=1}^{2k} a_{2k} A_\sigma^{2k-s} C_\sigma B_\sigma^{2m-s}, \tag{4.10}$$

where $h_{A_\sigma}(\lambda)$ is given in Eq. (2.4).

By Proposition 3.3, $g_{A_\sigma}(\lambda)$ is a real polynomial and $h_{A_\sigma}(B_\sigma) = (g_{A_\sigma}(B\overline{B}))_\sigma P_n$. So from Proposition 3.1 and Eq. (4.10), we have

$$\begin{aligned} \left[Xg_{A_\sigma}(\overline{B}B)\right]_\sigma &= X_\sigma\overline{\left(g_{A_\sigma}(\overline{B}B)\right)}_\sigma P_n = X_\sigma\left(g_{A_\sigma}(B\overline{B})\right)_\sigma P_n = X_\sigma h_{A_\sigma}(B_\sigma) \\ &= \sum_{k=1}^{m}\sum_{s=1}^{2k} a_{2k} A_\sigma^{2k-s} C_\sigma B_\sigma^{2m-s} \\ &= \sum_{k=1}^{m} a_{2k}\left(\sum_{j=0}^{k-1} A_\sigma^{2(k-j-1)+1} C_\sigma B_\sigma^{2(m-j-1)+1} + \sum_{j=1}^{k} A_\sigma^{2(k-j)} C_\sigma B_\sigma^{2(m-j)}\right) \end{aligned}$$

$$= \sum_{k=1}^{m} a_{2k} \left\{ \sum_{j=0}^{k-1} \left[(A\overline{A})^{k-j-1} (A\overline{C}B)(\overline{B}B)^{m-j-1} \right]_\sigma + \sum_{j=1}^{k} \left[(A\overline{A})^{k-j} C(\overline{B}B)^{m-j} \right]_\sigma \right\}$$

$$= \left\{ \sum_{k=1}^{m} a_{2k} \left[\sum_{j=0}^{k-1} (A\overline{A})^{k-j-1} (A\overline{C}B)(\overline{B}B)^{m-j-1} + \sum_{j=1}^{k} \left[(A\overline{A})^{k-j} C(\overline{B}B)^{m-j} \right] \right] \right\}_\sigma$$

so we have

$$Xg_{A_\sigma}(\overline{B}B) = \sum_{k=1}^{m} a_{2k} \left[\sum_{j=0}^{k-1} (A\overline{A})^{k-j-1} (A\overline{C}B)(\overline{B}B)^{m-j-1} + \sum_{j=1}^{k} \left[(A\overline{A})^{k-j} C(\overline{B}B)^{m-j} \right] \right]$$

so by [5, Chapter 2.1], we know that (1) holds.

(2) Since $h_{A_\sigma}(\lambda)$ and $f_{B_\sigma}(\lambda)$ are relatively prime, so by Proposition 2.4, Eq. (4.2) has a unique solution, this implies that Eq. (4.1) has a unique solution, and $h_{A_\sigma}(B_\sigma)$ is a nonsingular matrix. Because $\left(g_{A_\sigma}(B\overline{B})\right)_\sigma P_n = h_{A_\sigma}(B_\sigma)$, so by (3) in Proposition 3.1, $g_{A_\sigma}(\overline{B}B)$ is a nonsingular matrix, and (2) follows directly from (1). □

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