Thermal analysis of straight rectangular fin using homotopy perturbation method

Pranab Kanti Roy*, Ashis Mallick

Department of Mechanical Engineering, Indian School of Mines, Dhanbad 826004, Jharkand, India

Received 11 November 2015; revised 12 May 2016; accepted 18 May 2016
Available online 10 June 2016

Abstract In this study a simple and accurate semi-analytical method called Homotopy Perturbation Method (HPM) is used for solving nonlinear energy equation in a straight rectangular fin. The thermal conductivity and surface emissivity are considered as temperature dependent with some constant source of internal heat generation. The problem is solved for two main cases of thermogeometric fin parameter $N_c = 1$ and for $N_c = 0.25$. The results are presented for the temperature distribution, efficiency and optimum dimensionless parameters are effective and convenient for practical fins. Also it is found that this method can achieve more suitable results as compared to the other methods available in the literature.

1. Introduction

The fins are extended surface that are used to dissipate heat from the primary surface to the surrounding environment [1]. Heat transfer through the straight is common because of their low manufacturing cost and simplicity. The fins are sometimes subjected to both convective and radiative environments [2]. Under the circumstances of both convective and radiative modes of heat transfer the fin may generate heat internally due to passage of electric current as in electric filament or due to an atomic or chemical reaction as in an atomic reactor [3–6]. The internal heat generation may be assumed to be at constant rate with respect to the volume of fin. But to utilize the fin materials effectively the model includes the thermal conductivity varies linearly with temperature and in most cases the real surface of the fin material varies linearly with temperature. The energy equation for a convective–radiative fin along with heat generation with two variable thermal properties results in highly nonlinear terms. The resulting equation does not admit the exact solution. Consequently the energy equation with nonlinear terms is solved either numerically or using a variety of approximate analytical methods. Many different methods have been introduced to solve nonlinear problems, the Homotopy analysis method (HAM), Galerkin method (GM), Spectral collocation method (SCM), Adomian decomposition method (ADM). Homotopy perturbation method (HPM) is one of the semi analytical methods for solving nonlinear boundary value problem introduced by He [7]. An HPM possesses all the advantages of perturbation method and also it is independent of assumption of small parameter. As compared to the Adomian decomposition method, Homotopy perturbation method does not require the calculation of Adomian polynomials but it requires only the initial approximation [8–10]. Also the method does not require determination of ‘h-curves’ which

* Corresponding author. Tel.: +91 9609551679.
E-mail addresses: pranab38_skd@yahoo.com (P.K. Roy), mall123_us@yahoo.com (A. Mallick).
Peer review under responsibility of Faculty of Engineering, Alexandria University.

http://dx.doi.org/10.1016/j.aej.2016.05.020
1110-0168 © 2016 Faculty of Engineering, Alexandria University. Production and hosting by Elsevier B.V.
This is an open access article under the CC BY-NC-ND license (http://creativecommons.org/licenses/by-nc-nd/4.0/).
Nomenclature

\( N_r \)  radiation–conduction parameter
\( C \)  constant which represents the temperature
\( k \)  temperature dependent thermal conductivity (W/(mK))
\( k_a \)  thermal conductivity corresponding to ambient condition (W/(mK))
\( \varepsilon_s \)  the surface emissivity corresponding to radiation sinks temperature, \( T_s \)
\( T \)  temperature (K)
\( P \)  fin perimeter (m)
\( T_b \)  fin's base temperature (K)
\( T_a \)  convection sink temperature (K)
\( r \)  sink temperature for radiation (K)
\( b \)  length of the fin (m)
\( x \)  axial coordinate of the entire fin (m)
\( A_r \)  cross-sectional area of the entire fin (m\(^2\))
\( X \)  dimensionless axial coordinate
\( A \)  thermal conductivity parameters
\( B \)  the surface emissivity parameters

Greek symbols
\( \alpha \)  slope of the thermal conductivity-temperature curve (K\(^{-1}\))
\( \beta \)  slope of the surface emissivity-temperature curve (K\(^{-1}\))
\( \theta \)  dimensionless temperature of the fin
\( \theta_a \)  dimensionless convection sinks temperature
\( \theta_r \)  dimensionless radiation sinks temperature
\( \sigma \)  Stefan–Boltzmann constant (W/(m\(^2\)K\(^4\)))
\( \varepsilon \)  emissivity

makes it different from the Homotopy Analysis method. Bhowmik et al. [11] predicted the dimensions of rectangular and hyperbolic fins with variable thermal properties. Hazarika et al. [12] established an analytical method to predict the performance and design parameters of T-shape fin with simultaneous heat and mass transfer. Several researchers applied Collocation method (CM) to obtain the analytical solution of unsteady motion of fluid particles in conjunction with other numerical technique [13,14]. Rahimi-Gorji et al. [15] used Galerkin Method to obtain the heat transfer characteristics of micro channel heat sink cooled by different nanofluids in porous media. Pourmehran et al. [16] studied the thermal analysis of a fin shaped micro channel heat sink cooled by different nanofluids using least square method. Spectral collocation method has been used to study the performance parameter of simple and complex cross section of moving rod in thermal processing of continuous casting and rolling [17,18]. Sun et al. [19] used the Lagrange interpolation polynomials to approximate the temperature distribution at the spectral collocation points of nonlinear heat transfer problems. Ma et al. [20] presented the spectral collocation method to predict the thermal performance of porous fin with temperature dependent heat transfer coefficient, surface emissivity and internal heat generation.

From the above discussion it is clear that in fin design application the selection of choice of conductive–convective parameter is very important for practicing engineer. The practical fins work at low values of conductive convective fin parameter. HPM is used to study the variation of surface emissivity, thermal conductivity parameter, heat generation number on the temperature distribution and efficiency of fin working under practical range of operating condition.

2. Mathematical formulations

Let us consider rectangular fin geometry of various details as shown in Fig. 1. The thermal conductivity \( k \) and surface emissivity \( \varepsilon \) of the fin materials are temperature dependent. The heat generated inside the fin material is assumed to be at the constant rate \( q \). The fin is assumed to be insulated at the free end side and the effect of heat transfer in vertical direction is neglected. The steady state one dimensional energy equation is given by

\[
\frac{d}{dx} \left[ k(T) \frac{dT}{dx} \right] - \frac{hP(T - T_a)}{A_c} - \varepsilon(T)\sigma P(T^4 - T_s^4) = 0
\]

(1)

With the insulated boundary condition

\[
\frac{dT}{dx} = 0 \quad \text{at} \quad x = 0 \quad \text{(2a)}
\]

\[
T = T_b \quad \text{at} \quad x = b \quad \text{(2b)}
\]

For the better utilization of the fin materials the thermal conductivity of fin material is assumed to be linear function of temperature. Also the emissivity of all real surfaces is not constant, and it sometimes varies linearly with temperature. Therefore both of the parameters can written as below

\[
k(T) = k_a[1 + \alpha(T - T_s)]
\]

(3)

\[
\varepsilon(T) = \varepsilon_s[1 + \beta(T - T_s)]
\]

(4)

![Figure 1](image-url) The geometry of straight rectangular fin.
In order to express Eq. (1) in non-dimensional forms, the following dimensionless parameters are defined as

\[
\begin{align*}
\theta &= \frac{T}{T_b}, & \theta_a &= \frac{T_a}{T_b}, & \theta_b &= \frac{T_b}{T_b}, & X &= \frac{x}{b} \\
A &= \pi T_b, & B &= \beta T_b, & N_r^2 &= \frac{hPb^2}{kaAc}, & N_i &= \frac{\alpha T_i^2 P h^2}{kaAc}, & Q &= \frac{b^2q}{kaT_b} 
\end{align*}
\]

The formulation of the fin problem reduces to the following equations:

\[
\begin{align*}
\frac{d^2\theta}{dx^2} + A\frac{d\theta}{dx} + A \left( \frac{d\theta}{dx} \right)^2 - \frac{N_i^2}{N_i} (\theta - \theta_a) - \frac{N_i}{1 + B(\theta - \theta_a)}(\theta^0 - \theta^1) + Q &= 0 \\
\text{With the following boundary conditions:} \\
\frac{d\theta}{dx} &= 0 \text{ at } X = 0 \\
\theta &= 1 \text{ at } X = 1
\end{align*}
\]

3. Homotopy Perturbation Method (HPM)

Homotopy perturbation method (HPM) is a semi-numerical method for solving linear or nonlinear, homogeneous or inhomogeneous boundary value problem first introduced by He [7]. As compared to the Adomian decomposition method this method does not require the calculation of Adomian polynomial and leads to convergent solution rapidly. Moreover this method requires only initial condition as input for its solution.

To illustrate the basic idea of HPM according to He [7], consider the following nonlinear differential equation,

\[
A(\theta) - f(r) = 0, \quad r \in \Omega
\]

With the boundary conditions

\[
B \left( \theta, \frac{\partial \theta}{\partial x} \right) = 0 \quad r \in \Gamma
\]

where \( A \) is a general differential operator, \( B \) is a boundary operator, \( f(r) \) is a known analytic function, and \( \Gamma \) is the boundary of the domain \( \Omega \).

The operator \( A \) can be generally divided into linear and nonlinear parts say \( L(\theta) \) and \( N(\theta) \). Therefore, Eq. (8) can be written as

\[
L(\theta) + N(\theta) - f(r) = 0
\]

Define the Homotopy \( v(r,p) : \Omega \times [0,1] \rightarrow \mathbb{R} \) that satisfies

\[
H(\theta,p) = (1-p)L(\theta - \theta_0) + p[L(\theta) + N(\theta) - f(r)] = 0
\]

This can be rearranged as

\[
H(\theta,p) = L(\theta) - L(\theta_0) + p[L(\theta) + N(\theta) - f(r)] = 0
\]

where \( L = \frac{\partial}{\partial \theta} \) and \( \theta_0 \) is the initial approximation. Here \( \theta = f(X) \) and \( p \) is the embedding parameter such that \( p \in [0,1] \). It is clear that

\[
H(\theta,0) = [L(\theta) - L(\theta_0)]
\]

For \( p = 1 \) we obtain

\[
H(\theta,1) = [L(\theta) + N(\theta) - f(r)]
\]

Using the perturbation technique by taking into account of small values of \( p \), then the solution of Eq. (10) can be written as

\[
\theta = \theta_0 + p\theta + p^2\theta_2 + p^3\theta_3 + p^4\theta_4 + \cdots
\]

The series converges for \( p = 1 \) the solution for \( \theta \) can be given by

\[
\theta = \theta_0 + \theta_1 + \theta_2 + \theta_3 + \theta_4 + \cdots
\]

4. HPM formulation

Using Eq. (12), Eq. (6), can be written as in HPM form

\[
H(\theta,p) = (1-p)L(\theta - \theta_0) + pL(\theta) + N(\theta) - f(r)
\]

This can be arranged as

\[
H(\theta,p) = L(\theta) - L(\theta_0) + p[L(\theta) + N(\theta) - f(r)] = 0
\]

Substituting \( \theta \), from Eqs. (15) to (18), and separating the variables of identical power of \( p \).

\[
p^0 : \theta = \theta_0
\]

From the boundary condition (7) it is clear that the solution is becoming meaningless. Therefore in order to predict the solution physically meaningful the \( \theta(0) \) must be an arbitrary constant. This \( \theta_0 = C \) is taken as initial input for HPM.

And

\[
p^1 : \frac{d^2 \theta_1}{dx^2} + \left[ A \theta_0 \frac{d\theta_0}{dx} + A \left( \frac{d\theta_0}{dx} \right)^2 - A \theta_0 \frac{d\theta_0}{dx} - N_i^2 (\theta - \theta_a) \\
- N_i (\theta^0 - \theta^1) + B^{\theta_0,0} - B^{\theta_0,0} (\theta_1 - \theta^0) + B^{\theta_0,0} \right] = 0
\]

\[
\frac{d\theta_1}{dx} = 0 \text{ at } X = 0, \quad \theta_1 = 0 \text{ at } X = 0
\]

And

\[
p^2 : \frac{d^2 \theta_2}{dx^2} + \left[ A \theta_0 \frac{d\theta_0}{dx} + A \left( \frac{d\theta_0}{dx} \right)^2 - A \theta_0 \frac{d\theta_0}{dx} - N_i^2 (\theta - \theta_a) \\
- N_i (4\theta_0^0,0 + B^{\theta_0,0} \theta_1 - B^{\theta_0,0} \theta_1 - B^{\theta_0,0} \theta_1) \right] = 0
\]

\[
\frac{d\theta_2}{dx} = 0 \text{ at } X = 0, \quad \theta_2 = 0 \text{ at } X = 0
\]
Thermal analysis of straight rectangular fin

\[ \theta_1 = + \frac{1944\gamma_c c_c c_c c_c c_c}{40320} + \frac{28720\gamma_b c_b c_b c_b c_b c_b}{40320} + \frac{3880\gamma_0 c_0 c_0 c_0 c_0 c_0}{40320} - \frac{11868\gamma_0 c_0 c_0 c_0 c_0 c_0}{40320} \]

Substituting the values of \( \theta_0, \theta_1, \theta_2, \theta_3 \) and \( \theta_4, \ldots \) in Eq. (13), the formulation of non-dimensional temperature can be obtained as

\[ \theta = \sum \frac{\theta_m}{\theta_m} = 0 + \theta_1 + \theta_2 + \theta_3 + \theta_4 + \ldots \]  

(28)

Now the temperature field, \( \theta \), can be evaluated if the fin tip temperature \( C \) is known whose value lies in the interval (0, 1). Using an arbitrary initial guess value for \( C \), for the temperature field \( \theta \) computed from the above Eq. (26) and applying Newton–Raphson method satisfying the boundary conditions (6) the actual temperature field can be obtained.

4.1. Fin efficiency (\( \eta \))

The fin efficiency is defined as the ratio of actual heat transfer rate to the ideal heat transfer rate. The ideal heat transfer rate takes place in the fin if the whole fin is subjected to base temperature.

The actual rate of heat transfer can be obtained by computing the heat losses from the base of the fin

\[ Q_{\text{actual}} = K(T)A_1 \left[ \frac{dT}{dx} \right]_{x=b} \]  

(29)

The ideal heat transfer rate \( Q_{\text{ideal}} \) obtained if the entire fin is kept at the base temperature and can be expressed as

\[ Q_{\text{ideal}} = [hPb(T - T_a) + \nu_0 c_0 c_0 c_0 c_0 c_0] \frac{d\theta}{dX} \]  

(30)

The fin efficiency can be expressed using non-dimensional terms by Eq. (5)

\[ \eta = \frac{Q_{\text{actual}}}{Q_{\text{ideal}}} \]

\[ \eta = \frac{hPb(T - T_a) + \nu_0 c_0 c_0 c_0 c_0 c_0}{[1 + A(\theta - \theta_b)]\frac{d\theta}{dX} X_{x=1}} \]

\[ = \frac{1 + A(\theta - \theta_b)}{1 + A(\theta - \theta_b)\frac{d\theta}{dX} X_{x=1}} \]  

(31)

5. Optimization

The optimization of the fin can be made by either minimizing the volume for any required heat dissipation or maximizing the heat dissipation for any given fin volume [22,23].

Assuming that the volume of the fin is constant the fin volume is defined as

\[ V = A_1 b \]

The heat dissipated per unit volume is

\[ q = \frac{K(T)}{b} \left[ 1 + A(\theta - \theta_b) \right] \frac{d\theta}{dX} X_{x=1} \]  

(32)

The dimensionless form of the above equation [21] is

\[ N = \frac{q}{k_0 T_a V} = \frac{A_1}{b} \left[ 1 + A(\theta - \theta_b) \right] \frac{d\theta}{dX} A_1 \]  

(33)
The maximum heat dissipation value expressed the optimum fin characteristics and the dimensions are the optimum fin configuration. The optimization procedure is also done to fix the profile area $A_c$ to express $N$ as a function of $b$ and then search the optimum value of $b$.

6. Results and discussion

The validation of results is done by considering the both sink temperature, $\theta_s$, $\theta_a$ and surface emissivity parameter, $B$ and heat generation number, equating to zero Eq. (6) is converted into the previous work found in the literature [21]. The present works consider five terms and the accuracy results are compared with the previous work as available in the literature as shown in Tables 1 and 2. In fin design application the selection of the values of $N_c$ is very important. The fins are categorized as practical fins depending on the value of conductive–convective fin parameter $N_c$ whose value lies below 0.5. Depending on the surface condition of the fin materials the surface emissivity can increase with temperature ($B > 0$) or decrease with temperature ($B < 0$). In practice values of $B$ range from $-0.6$ to $+0.6$ [8]. In order to study the effect of surface emissivity on the temperature distribution and efficiency of the fin, highest, lowest and constant values of surface emissivity are taken i.e. $B = -0.6$, $0$, $+0.6$ respectively in the present analysis. The idea of maintaining nonzero environmental temperature ($\theta_s$ and $\theta_a$) is that in actual practice it is not possible for a fin to maintain one end at high temperature and dissipating heat to the absolute zero. The study of effect of the environmental temperature is also found effective on the thermal performance. The range values of environmental temperature and heat generation number are found in the literature [6].

![Figure 2](image-url) The tip temperatures function of heat generation number.

**Table 1** Comparison of HPM solutions with Adomian Decomposition Method (ADM), Galerkin Method (GM) and Boundary Value Problem (NM).

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.666858541</td>
<td>0.666858541</td>
<td>0.667013018</td>
<td>0.667013597</td>
</tr>
<tr>
<td>0.1</td>
<td>0.669984372</td>
<td>0.670004372</td>
<td>0.670132740</td>
<td>0.670132745</td>
</tr>
<tr>
<td>0.2</td>
<td>0.679386188</td>
<td>0.679376188</td>
<td>0.679514048</td>
<td>0.679514005</td>
</tr>
<tr>
<td>0.3</td>
<td>0.695136900</td>
<td>0.695136900</td>
<td>0.695228804</td>
<td>0.695229254</td>
</tr>
<tr>
<td>0.4</td>
<td>0.717357999</td>
<td>0.717357999</td>
<td>0.717399183</td>
<td>0.717399776</td>
</tr>
<tr>
<td>0.5</td>
<td>0.746220026</td>
<td>0.746220026</td>
<td>0.746198567</td>
<td>0.746198734</td>
</tr>
<tr>
<td>0.6</td>
<td>0.781944267</td>
<td>0.781944267</td>
<td>0.781855307</td>
<td>0.781855061</td>
</tr>
<tr>
<td>0.7</td>
<td>0.824806704</td>
<td>0.824806704</td>
<td>0.82469376</td>
<td>0.82465933</td>
</tr>
<tr>
<td>0.8</td>
<td>0.875145547</td>
<td>0.875145547</td>
<td>0.874971905</td>
<td>0.874972380</td>
</tr>
<tr>
<td>0.9</td>
<td>0.933237796</td>
<td>0.933237796</td>
<td>0.933237591</td>
<td>0.933237796</td>
</tr>
<tr>
<td>1.0</td>
<td>0.9999999999</td>
<td>0.9999999999</td>
<td>0.9999999999</td>
<td>0.9999999999</td>
</tr>
</tbody>
</table>

**Table 2** Effect of number of terms in HPM on the temperature distribution of rectangular fin.

<p>| $N_c = 1$, $N_r = 0.2$, $A = 0.2$, $Q = 0$, $\theta_s = 0$, $\theta_a = 0$, $B = 0$ |
|-----------------|-----------------|-----------------|-----------------|</p>
<table>
<thead>
<tr>
<th>Number of terms (m)</th>
<th>Distance, $X$</th>
<th>Distance, $X$</th>
<th>Distance, $X$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.0000</td>
<td>0.0100</td>
<td>0.2000</td>
</tr>
<tr>
<td>Absolute error difference With $m = 2$ and $m = 3$</td>
<td>0.0079</td>
<td>0.0081</td>
<td>0.0082</td>
</tr>
<tr>
<td>3</td>
<td>0.0687</td>
<td>0.0671</td>
<td>0.0612</td>
</tr>
<tr>
<td>Absolute error difference With $m = 3$ and $m = 4$</td>
<td>0.0019</td>
<td>0.0018</td>
<td>0.0018</td>
</tr>
</tbody>
</table>
of surface emissivity parameters, i.e. $B = -0.6, 0, +0.6$ while other parameter maintained at zero except $N_r = 1$. The surface emissivity lines increase linearly with increase in internal heat generation number $Q = 0$ to $Q = 0.4$. The internal heat source affects the surface of the fin materials. The relative spacing between the emissivity lines is higher when conduction–convection parameter, $N_c$, is maintained at low values. This is because at lower value conduction–convection parameter, $N_c$, the convective heat loss is higher and hence temperature gradient is higher.

Fig. 3 and 4 demonstrates the variation of temperature of the fin materials for different values of thermal emissivity parameters. The thermal conductivity of the fin materials is maintained at $A = -0.6$ and sinks temperature and heat generation maintained at some fixed value. It is evident from the both figures that the surface emissivity lines of fin materials are more affected when the values of $N_r$ are maintained below the range of $0.5$. The surface of the fin materials is more affected at higher values of heat generation number which is evident from Fig. 4.

Fig. 5 illustrates the variation of fin efficiency of the fin materials $\eta$ with respect to heat generation number $Q$. It is clear from the figure that the efficiency of the fin materials is very low without internal heat generation number $Q = 0$. Fin efficiency increases with the increase in internal heat generation number $Q$. The efficiency corresponding to lower values of conduction–convection parameter, $N_c$, is steeper as compared to other. That may be because of lower values of conduction–convection parameter, the simultaneous internal heat generation and surface heat loss enhances the fin efficiency.

Fig. 6 shows the variation of fin efficiency with respect to the radiation sink temperature $\theta_s$. The efficiency of surface emissivity curves corresponding to negative values of surface emissivity is higher. This observation has been found to be true in case of space radiative fin [9]. The efficiency is higher when fin maintained at zero sink temperature, and decreases beyond the radiation sink temperature 4.5.
Fig. 7 shows the variation of fin efficiency with respect to radiation conduction parameter $N_r$. As the radiation conduction parameter increases the efficiency corresponding to different surface emissivity lines decreases gradually. At higher values of radiation conduction parameter the radiation heat losses are higher but at negative values of surface emissivity parameter the efficiency is higher.

Fig. 8 shows the optimum dimensionless parameter $N$ as function surface emissivity parameter, $B$ for different values of conduction–convection parameter. Fig. 9 presents optimum dimensionless parameter $N$ is a function of fin length. The best fin length can be obtained under a given volume $V$, profile area $A_c$ and thermal parameter.

7. Conclusion

In this study Homotopy Perturbation Method (HPM) is used for solving nonlinear energy equation in a straight rectangular fin with variable thermal conductivity and surface emissivity with some constant source of heat generation. The problem is solved for two main real and industrial use of thermo geometric fin parameter $N_c = 0.25$ and $N_c = 1$. The results are presented for the temperature distribution, efficiency and optimum dimensionless parameters are effective and convenient. Also it is found that this method can achieve more suitable results as compared to the other methods available in the literature.

References

[10] Ashis Mallick, Sujeet Ghosal, Prabir Kr. Sarkar, Rajiv Ranjan, Homotopy perturbation method for thermal stresses in an
Thermal analysis of straight rectangular fin


