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Note on a Model of a Biochemical Reaction*

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1. INTRODUCTION

The autonomous system

$$\frac{dX}{dt} = A - (B + 1) X + X^2 Y,$$
$$\frac{dY}{dt} = BX - X^2 Y \tag{1}$$

proposed as the chemical kinetic description of a model biochemical reaction, has been the subject of several recent studies [1, 2, 3, 4; see 3 and 5 for a derivation and discussion of these equations]; X, Y represent the concentrations of two intermediate product reactants, A, B constant (initial) concentrations, and t the time.

The only singular point of (1) is at $X_0 = A$, $Y_0 = B/A$. One finds, after a standard calculation, that the singular point is unstable, but not a saddle point when $B > A^2 + 1$. We thus consider only the first quadrant $Q \overset{\text{def}}{=} \{X \geqslant 0, Y \geqslant 0\}$ in the X–Y plane, and take $A > 0$, $B > A^2 + 1$ throughout.

We transform (1) to a system that has been investigated in considerable detail [6, 7]. We show, for the first time to our knowledge, and as an immediate consequence of this earlier work, that (1) has a unique stable limit cycle, when $B > A^2 + 1$. Furthermore, the amplitude, period, and "wave shape" can also be given readily when $B \gg 1$ [7], thereby circumventing the elaborate asymptotic calculations of previous studies [2, 4].

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2. The Transformed System

Let

$$X = \frac{A}{1 + (B - 1)\,x}$$

and

$$t = \frac{(B - 1)^{1/2}}{A}\,\tau,$$

then (1) becomes, on eliminating Y,

$$\ddot{x} + \mu\left[2x - 1 + \frac{1}{\mu^2(x + \lambda)^2}\right]\dot{x} + \frac{x}{x + \lambda} = 0, \tag{2}$$

where $\mu = (B - 1)^{3/2}/A$, $\lambda = 1/(B - 1)$ and the dot indicates differentiation with respect to τ.

Equation (2) is a special case of

$$\ddot{x} + \mu f(x)\dot{x} + g(x) = 0 \tag{3}$$

which, in turn, is equivalent to the first order Lienard system

$$\begin{aligned} \dot{x} &= \mu[y - F(x)], \\ \dot{y} &= -g(x)/\mu \end{aligned} \tag{4}$$

with

$$F(x) = x^2 - x + \frac{1}{\mu^2}\left[\frac{1}{\lambda} - \frac{1}{x + \lambda}\right] = \int_0^x f(s)\, ds$$

and

$$g(x) = \frac{x}{x + \lambda}$$

for (2).

The above functions satisfy the conditions listed in [6]; thus the proof given there of uniqueness and stability of the priodic orbit applies. One concludes that (1) has a unique stable periodic solution when $B > A^2 + 1$.

The values $A = 8.2$, $B = 77$ have been used in previous numerical calculations [2, 4] and we adopt them. One gets that $\mu = 80.8$ and $\lambda = 0.01316$. A phase portrait of (4) using these values, is sketched in Fig. 1, which also portrays the limit cycle.

Let the zeros of $F'(x) = f(x)$ be denoted by a and b, as shown in Fig. 1, and let α, β be defined by $F(\alpha) = F(b)$, $F(\beta) = F(a)$, respectively.

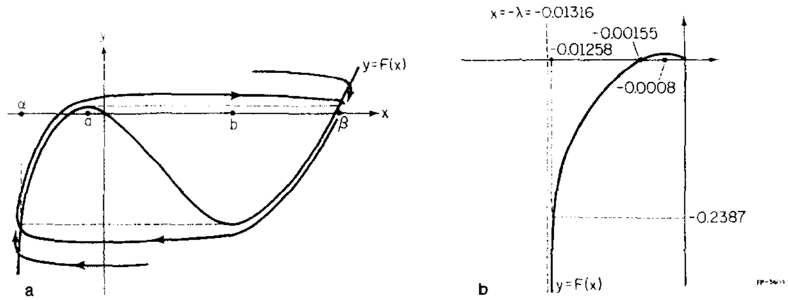

FIG. 1. (a) A phase portrait of the solutions of Eq. (4) is given in the x–y plane. The unique stable limit cycle is shown. The quantities a, α, b, β are defined in the text. The drawing is not to scale. (b) A plot of $y = F(x)$ in the second and third quadrants of the x–y plane is shown. The drawing is not to scale.

Expressions for the amplitude and period are, when $\mu \gg 1$, [7]:

$$\text{Negative amplitude} = \alpha + \frac{2.33811}{f(\alpha)}\left[-\frac{2g^2(b)}{f'(b)}\right]^{1/3}\mu^{-4/3} + \cdots,$$

$$\begin{aligned}\text{Period} = {} & -\mu\int_\alpha^a \frac{f(x)}{g(x)}\,dx - \mu\int_\beta^b \frac{f(x)}{g(x)}\,dx \\ & + 2.33811\left\{\left[\frac{2}{g(a)f'(a)}\right]^{1/3} + \left[\frac{2}{g(b)f'(b)}\right]^{1/3}\right. \\ & \left. + \frac{1}{g(\beta)}\left[-\frac{2g^2(a)}{f'(a)}\right]^{1/3} + \frac{1}{g(\alpha)}\left[-\frac{2g^2(b)}{f'(b)}\right]^{1/3}\right\}\mu^{-1/3} + \cdots.\end{aligned}$$

Using the above functions and numerical values one finds that $a = -0.0008$, $\alpha = -0.01258$, $b = 0.4997$, and $\beta = 0.9884$.

Thus

$$x_{\min} = \text{Negative amplitude} \doteq -0.01258$$
$$\text{Period} \doteq 24.00.$$

Converting to the original variables, one has

$$X_{\max} = \frac{A}{1 + (B-1)\,x_{\min}} \qquad \text{and} \qquad \Delta T = \frac{(B-1)^{1/2}}{A}\,\Delta\tau$$

or

$$X_{\max} \doteq 186.7, \qquad \Delta T = 25.5,$$

in excellent agreement with the estimates given in [2, 4].

We observe, finally, that other biochemical models, or any autonomous system transformable to (4) and satisfying the conditions on F and g given in [6], may be treated similarly. A sketch of one such possibility is depicted in Fig. 2.

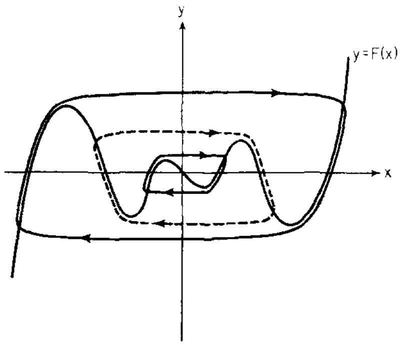

FIG. 2. A possible set of nested limit cycles, the solid curves stable, and the dotted curve unstable, is shown in the x–y plane.

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