The first homotopy group of a finite $H$-space

James P. Lin

Department of Mathematics, University of California, San Diego, La Jolla, CA 92903, USA

Communicated by J.D. Stasheff
Received 30 November 1990
Revised 22 September 1992

Abstract

Let $X$ be a finite $H$-space with $H_*(X; \mathbb{Z}_2)$ associative. Then the first homotopy group of $X$ occurs in degrees 1, 3 or 7. This result generalizes earlier results of Adams, Clark and Thomas. The proof uses secondary operations which factor primary operations of length 2 as well as a $K$-theory result due to Jeanneret.

Introduction

A theorem of Adams [1] states that the only spheres that are $H$-spaces are $S^1$, $S^3$ and $S^7$. Let $X$ be a finite $H$-space. In this paper, we prove the following theorem:

Theorem A. If $H_*(X; \mathbb{Z}_2)$ is associative, then the first nonvanishing homotopy group of $X$ occurs in degrees 1, 3 or 7.

We note that for Lie groups, it is known that the first nonvanishing homotopy group lies in degree 1 or 3. For finite loop spaces, a theorem of Clark [4] shows that the homotopy groups begin in degree 1 or 3. We know, however, that the seven sphere is neither a Lie group or a loop space but is a 6-connected finite $H$-space. Hence Theorem A generalizes these results and is the best possible result. This improves on the paper [11] which states that the first nonvanishing homotopy group occurs in degrees 1, 3, 7 or 15. We prove here the following:

Correspondence to: J.P. Lin, Department of Mathematics, University of California, San Diego, La Jolla, CA 92903-0112, USA.

0022-4049/94/$06.00  © 1993 – Elsevier Science Publishers B.V. All rights reserved
Theorem B. A 7-connected finite $H$-space with mod 2 associative homology is acyclic.

The proof proceeds in several steps. From the beginning we assume we have a nonacyclic 7-connected finite $H$-space $X$ and work to arrive at a contradiction.

In Section 1, earlier results about the action of the Steenrod algebra are applied to $H^*(X; \mathbb{Z}_2)$. One notes that the action of $Sq^8$ plays a crucial role in the existence of other nontrivial Steenrod operations. In particular, over the Steenrod algebra, it is known that the module $QH^*(X; \mathbb{Z}_2)$ is generated by elements of degree $2^l - 1$ for $l > 0$ [11]. Further, since $X$ is 7-connected, $Sq^1$, $Sq^2$, and $Sq^4$ vanish on $QH^{2l-1}(X; \mathbb{Z}_2)$. The vanishing of $Sq^8$ on $QH^{2l-1}(X; \mathbb{Z}_2)$ implies certain factorizations of $Sq^{2k}$ through secondary operations which are shown to be incompatible.

In Section 2, we describe two factorizations of $Sq^{16}$ through secondary operations. Using various suspension theorems in $\Omega^* X$, we prove $QH^{43}(X; \mathbb{Z}_2) = 0$. In Section 3, some other factorizations of $Sq^{16}$ are used to prove $x^2 = 0$ if deg $x = 31$. This is needed to show $x$ lies in the domain of a tertiary operation.

In Section 4 we use a tertiary operation defined in [13] to prove that $Sq^8QH^{31}(X; \mathbb{Z}_2) = 0$. Roughly speaking, this tertiary operation has reduced co-product $Sq^8x_{31} \otimes Sq^8x_{31}$. It follows from the results of Section 1 that all Steenrod operations vanish on generators of degree 31 or greater. We conclude that $H^*(X; \mathbb{Z}_2)$ must have Borel decomposition

$$\bigotimes B_i \otimes \langle y_1, \ldots, y_m \rangle,$$

where

$$B_i = \mathbb{Z}_2 \frac{x_{15}}{x_{15}^4} \otimes \langle x_{23}, x_{27}, x_{29} \rangle$$

and degree $y_i = 2^i - 1$ for some $i \geq 5$. A recent $K$-theory result of Jeanneret [6] states that such a ring is not realizable as the mod 2 cohomology of a space. This completes the proof.

Throughout the paper we use $H^*(X)$ to denote $H^*(X; \mathbb{Z}_2)$. The symbol $X$ is reserved for a finite 7-connected $H$-space with associative mod 2 homology ring. By [11], we may assume $X$ is in fact 14-connected.

The author wishes to thank John Harper, Daciberg Goncalves, T.B. Ng and Michael Slack for various formulas which factor $Sq^{16}$ through secondary operations.

1. Steenrod actions on $H^*(X)$

In this section, we compile information about the action of $\mathcal{A}(2)$ on $H^*(X)$. This will be used later to define a tertiary operation. We review here some earlier results of [11].
The first homotopy group of a finite H-space

Consider the following $A(2)$ coalgebra:

$$R = \{ x \in H^*(X) \mid \Delta x \in \xi H^*(X) \otimes H^*(X) \}.$$  

There is an exact sequence

$$0 \to \xi H^*(X) \to R \to QH^*(X) \to 0. \quad (1.1)$$

It follows that every algebra generator has a representative in $R$, and that every odd decomposable in $R$ is trivial. Furthermore, we have the following theorem:

**Theorem 1.1** [11]. For $k > 0$, $r \geq 0$, we have:

1. $R^{2^r + 2^{r+1}k - 1} = Sq^{2^r} R^{2^r + 2^{r+1}k - 1}$.
2. $Sq^n R^{2^r + 2^{r+1}k - 1} = 0$.
3. $\sigma^* R^{2^r + 2^{r+1}k - 1} \subseteq Sq^n H^*(\Omega X)$.

In the following proposition we use the fact that $X$ is 14-connected.

**Corollary 1.2.** Let $n = 1 + 2^r_1 + \cdots + 2^r_s$, $1 \leq r_1 < r_2 < \cdots < r_s$. Then if $s \leq 2$, $R^n = 0$.

**Proof.** By repeated application of Theorem 1.1, $R^n$ is in the image of $R^3$ or $R^7$ via Steenrod operations. Therefore, since $X$ is 14-connected, $R^3 = R^7 = 0$. \qed

Henceforth we use the notation $Q^*$ for the module $QH^*(X; \mathbb{Z}_2)$.

**Theorem 1.3.** Let $x_{2i-1} \in R^{2^i-1}$. If $Sq^8 x_{2i-1} = 0$, then all Steenrod operations vanish on $x_{2i-1}$ except possibly $Sq^1$. Further, all Steenrod operations vanish on $Q^{2^i-1}$ for $r \geq 4$.

**Proof.** By Theorem 1.1, $Sq^1 x_{2i-1}$ is decomposable and by Corollary 1.2, $Sq^2 x_{2i-1} = 0 = Sq^4 x_{2i-1}$. Therefore, if $y = \sigma^*(x_{2i-1})$, then $y \in \ker Sq^i$ for $i \leq 3$. By induction assume $Sq^{2^s} y = 0$ for $s < k$. Then, for $k \geq 4$, by [1],

$$Sq^{2^k} y = \sum a_{ij} \phi_{ij}(y).$$

For $i > 0$, $\phi_{ij}(y) \in H^{2i} (\Omega X) = 0$ by [12], so

$$Sq^{2^k} y = \sum_{i < k} a_{0j} \phi_{0j}(y),$$

$$\deg \phi_{0j}(y) = 2^j + 2^j - 2, 2 \leq j.$$ If $\phi_{0j}(y)$ is indecomposable, it must have a primitive representative. This primitive cannot be a transpotence because if it was,
\( \phi_{0,j}(y) = \phi_{2,j}(x_{2^{j-1}+1}) \) modulo decomposables. But \( QH^{2^{2j-1}+1}(X) = 0 \) by Corollary 1.2. So \( \phi_{0,j}(y) = \sigma^2(x_{0,j}), x_{0,j} \in R^{2j+2^{j-1}}. \)

Now \( \deg a_{0,j} = 2^k - 2^j \), so \( a_{0,j} \) is a linear combination of terms of the form

1. \( b_1 \text{Sq}^{2m} b_2 \text{Sq}^{2m} b_3 \) where \( m < j \) and \( \deg b_i \equiv 0 \mod 2^{n+1} \),

2. \( b_1 \text{ Sq}_{-1}^d b_2 \) where \( \deg b_i \equiv 0 \mod 2^d+1. \)

By Theorem 1.1, \( a_{0,j} x_{0,j} = 0 \). Hence \( \text{Sq}^{2k} y = 0 \) and \( \text{Sq}^{2k} x = 0 \), since suspension is monic in odd degrees. This completes the induction.

We have shown that all Steenrod operations vanish on \( Q^{2^{j-1}}. \)

Now for \( r > l \), by Theorem 1.1,

\[ \text{Sq}^8 Q^{2^{r-1}} = \text{Sq}^{2^{r-1}} \cdots \text{Sq}^{2l} \text{Sq}^8 Q^{2^{l-1}} = 0. \]

Applying the same argument as above to \( R^{2^{r-1}}, \) we have that all Steenrod operations vanish on \( Q^{2^{r-1}}. \) \( \Box \)

The action of \( \text{Sq}^8 \) plays a central role in our study. Eventually, we will prove \( \text{Sq}^8 Q^{31} = 0 \). This will show that above degree 31, all generators will be concentrated in degrees \( 2^l - 1. \)

We conclude this section by proving the following:

**Theorem 1.4.** The squaring map \( \zeta : H^{15}(X) \rightarrow H^{30}(X) \) is monic.

**Proof.** Consider the projective plane \( P_2 X. \) There exists an exact triangle [3]

\[
\begin{array}{ccc}
H^*(P_2 X) & \xrightarrow{i} & IH^*(X) \\
\downarrow \iota & & \downarrow \delta \\
IH^*(X) \otimes IH^*(X) & & \\
\end{array}
\]  

(1.2)

\( i \) has degree \(-1, \lambda \) has degree \(+2. \)

Now given \( x \in H^{15}(X) \simeq PH^{15}(X) \) there exists a \( y \in H^{16}(P_2 X) \) with \( i(y) = x. \) By exactness,

\[ \text{Sq}^{15} y \neq 0 \text{ if and only if } x^2 = \text{Sq}^{15} x \neq 0. \]  

(1.3)

Further by (1.2) and Corollary 1.2, \( y \in \ker \text{Sq}^1, \text{Sq}^2, \text{Sq}^4. \) It follows that \( \text{Sq}^8 y \neq 0 \) and \( (\text{Sq}^8 y) y \neq 0. \) To see this, if \( \text{Sq}^8 y = 0 \) then by [1]

\[ y^2 = \text{Sq}^{16} y = \sum a_{ij} \phi_{ij}(y). \]

The same argument as in [16] applies to show that \( y^2 = 0. \) But \( y^2 = \lambda(x \otimes x) \) in (1.2). Hence there would be an element \( z \) with \( \Delta z = x \otimes x \) by (1.2). It is well known
The first homotopy group of a finite $H$-space

that for $X$ a finite $H$-space such an element $z$ does not exist [12]. Similarly if $(\text{Sq}^8 y)y = \lambda (\text{Sq}^8 x \otimes x) = 0$ there would be an element $w$ with $\lambda w = \text{Sq}^8 x \otimes x$. Such elements also do not exist for finite $H$-spaces $X$ [12].

Now suppose $x^2 = 0$. Then $y \in \ker \text{Sq}^{15}$. By [13, Theorem 1.2], there exists a factorization

$$(\text{Sq}^8 y)y = \text{Sq}^2 y_3 + \text{Sq}^{4,2} y_2 + \text{Sq}^8 y_1 + (\text{Sq}^9 + \text{Sq}^{4,5})y_0 + (y_{0,2})^2.$$ 

Checking degrees,

$\deg y_3 = 38, \, \deg y_2 = 34, \, \deg y_1 = 32, \, \deg y_0 = 31, \, \deg y_{0,2} = 20.$

By exactness of the triangle (1.2), for $j = 2, 3, i(y_j) = 0$ and $y_j \not\in \im \lambda$, so $y_j = 0$. Further, by Theorem 1.1, $y_0 = \text{Sq}^{15} z_0$ for some $z_0 \in H^{16}(P_2 X)$. Hence, since $z_0 \in \ker \text{Sq}^{2i}$ for $i \leq 2$, $(\text{Sq}^9 + \text{Sq}^{4,5})y_0 = 0$.

Similarly by Theorem 1.1, $y_{0,2} = 0$ since $i(y_{0,2}) = 0$ and $y_{0,2} \not\in \im \lambda$. We conclude

$$(\text{Sq}^8 y)y = \text{Sq}^8 y_1.$$ 

If $y_1 = \sum z_i z'_i$ where $\deg z_i = \deg z'_i = 16$, then

$$(\text{Sq}^8 y)y = \sum_{z_i \neq z'_i} (\text{Sq}^8 z_i) z'_i + z_i (\text{Sq}^8 z'_i).$$ 

This implies there exists an element $w \in H^*(X)$ with $\lambda w = \text{Sq}^8 x \otimes x + \sum \text{Sq}^8 x_i \otimes x'_i + \sum x_i \otimes \text{Sq}^8 x'_i$ where $i(z_i) = x_i$ and $i(z'_i) = x'_i$. This is a contradiction since it is known [2] that if $s, t \in PH_{\text{odd}}(X)$, then $[s, t] = 0$. Hence $y_1$ is indecomposable. Hence by (1.2), $0 \neq i(y_1) = x_1 \in PH^{31}(X)$.

By Corollary 1.2 and the exact triangle $y_1 \in \ker \text{Sq}^{2i}$ for $i \leq 2$. But then

$$\text{Sq}^8 [(\text{Sq}^8 y)y] = (\text{Sq}^8 y)^2 = \text{Sq}^8 \text{Sq}^8 y_1$$ 

$$= (\text{Sq}^{15,1} + \text{Sq}^{14,2} + \text{Sq}^{12,4})y_1 = 0.$$ 

By exactness, this implies there exists a $w \in H^*(X)$ with $\lambda w = \text{Sq}^8 \otimes \text{Sq}^8 x$. This is also a contradiction. We conclude $x^2 \neq 0$. \[\square\]

2. $Q^{43}$

In this section we introduce a stable factorization of $\text{Sq}^{16}$ that will be used to prove that $Q^{43} = 0$. Note that by Theorem 1.1 and Corollary 1.2, $Q^{43} = \text{Sq}^{20} \text{Sq}^8 Q^{15} = \text{Sq}^{16} \text{Sq}^{12} Q^{15}$. 
The following formulas are due to M. Slack (unpublished). Now consider the following matrices:

\[
B = \begin{bmatrix}
    \text{Sq}^1 & 0 & 0 & 0 & 0 & 0 \\
    0 & \text{Sq}^1 & 0 & 0 & 0 & 0 \\
    \text{Sq}^4 & \text{Sq}^2 & \text{Sq}^1 & 0 & 0 & 0 \\
    \text{Sq}^7 & \text{Sq}^5 & \text{Sq}^4 & \text{Sq}^2 & 0 & 0 \\
    \text{Sq}^8 & 0 & \text{Sq}^4\text{Sq}^1 & \text{Sq}^3 & \text{Sq}^4 & 0 \\
    0 & 0 & \text{Sq}^7 & 0 & 0 & 0 \\
    \text{Sq}^8\text{Sq}^3 & \text{Sq}^7\text{Sq}^2 & 0 & \text{Sq}^4\text{Sq}^2 & 0 & \text{Sq}^2 \\
    \text{Sq}^{12} & \text{Sq}^{10} & \text{Sq}^6\text{Sq}^3 & 0 & \text{Sq}^5 & \text{Sq}^3 \\
(\text{Sq}^{11}\text{Sq}^2 + \text{Sq}^{10} + \text{Sq}^3) & \text{Sq}^{11} & 0 & 0 & \text{Sq}^4\text{Sq}^2 & \text{Sq}^4 \\
\text{Sq}^{15} & \text{Sq}^{13} & \text{Sq}^{12} & 0 & \text{Sq}^8 & \text{Sq}^6 \\
\end{bmatrix}
\]

(2.1)

\[
C = \begin{bmatrix}
    \text{Sq}^1 \\
    \text{Sq}^4 \\
    \text{Sq}^8 \\
    \text{Sq}^{12} \\
\end{bmatrix}
\]

(2.2)

\[
A = (\text{Sq}^{13.2} + \text{Sq}^{12.3} + \text{Sq}^{11.4} + \text{Sq}^{10.5}, \text{Sq}^{11.2}, \text{Sq}^{11.1} + \text{Sq}^{9.3} + \text{Sq}^{4.8}, \\
\text{Sq}^{8.1} + \text{Sq}^{7.2} + \text{Sq}^{6.3}, \text{Sq}^{8}, \text{Sq}^{4.2}, \text{Sq}^4, \text{Sq}^4, \text{Sq}^2, \text{Sq}^1).
\]

(2.3)

Here \(\text{Sq}^{i,j} = \text{Sq}^i\text{Sq}^j\). Then \(AB = 0\). If

\[
\Omega_{K_1}
\]

\[
\downarrow
\]

\[
\begin{array}{c}
    j \\
    E_0 \\
    p \\
    K(\mathbb{Z}_2, n) \\
\end{array}
\]

\[
\xrightarrow{w} K_1
\]

is the induced bundle with \(k\)-invariants defined by the matrix \(C\), then \(BC = 0\) implies there exist stable cohomology classes \(v_i \in H^*(E_0), i = 1, \ldots, 10\).
The first homotopy group of a finite H-space

The relation $AB = 0$ implies

$$A = \begin{pmatrix} v_1 \\ \vdots \\ v_{10} \end{pmatrix}$$

lies in $PH^{16+n}(E_0) \cap \ker j^* = \operatorname{im} p^* PH^{16+n}(K(\mathbb{Z}_2, n))$. (2.4)

**Theorem 2.1.**

$$A = \begin{pmatrix} v_1 \\ \vdots \\ v_{10} \end{pmatrix} = (\text{Sq}^{3,2} + \text{Sq}^{12,3} + \text{Sq}^{11,4} + \text{Sq}^{10,5}) v_1 + \text{Sq}^{11,2} v_2 + \text{Sq}^{11,1} v_3 + \text{Sq}^{9,3} + \text{Sq}^{4,8}) v_4 + \text{Sq}^{8,1} + \text{Sq}^{7,2} + \text{Sq}^{6,3}) v_5 + \text{Sq}^{8} v_6 + \text{Sq}^{5} v_7 + \text{Sq}^{4} v_8 + \text{Sq}^{3} v_9 + \text{Sq}^{1} v_{10}$$

- $\text{Sq}^{16} p^*(i_n)$.

**Proof.** $p^* PH^{16+n}(K(\mathbb{Z}_2, n))$ is generated by $\text{Sq}^{16} p^*(i_n)$ since all other admissibles in $H^{16+n}(K(\mathbb{Z}_2, n))$ lie in $\ker p^*$.

Let $n = 16$. Then there is a commutative diagram

$$\begin{array}{c}
K(\mathbb{Z}_2, 2) \\ f \downarrow \\
K(\mathbb{Z}_2, 16) \xrightarrow{w} K_1
\end{array}$$

where $f^*(i_{16}) = i_2^3 = \text{Sq}^{8,4,2} i_2$. $f$ is a loop map. With the exception of $\text{Sq}^1$, all the other $k$-invariants vanish on $Bf: K(\mathbb{Z}, 3) \to K(\mathbb{Z}_2, 17)$.

We have $\text{Sq}^1 \text{Sq}^{6,4,2} i_3 = (\text{Sq}^{4} i_3)^2$. It follows by [18, Proposition 3.2.2] that $f_0$ may be chosen such that the $H$-deviation $D_{f_0}$ factors

$$[D_{f_0}]: K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 2) \xrightarrow{\hat{D}} K(\mathbb{Z}, 16) \xrightarrow{j} E_0$$

with $[\hat{D}] = i_2^2 \otimes i_2^2$.

Using the formula $\tilde{D} f_0^*(v_i) = D_{f_0}^*(v_i) = \hat{D} j^*(v_i)$ we get

$$\tilde{D} f_0^*(v_i) = \begin{cases} 0 & \text{for } i \neq 5, \\ \text{Sq}^8(i_2^4 \otimes i_2^4) & \text{for } i = 5. \end{cases}$$

Since $H^*(K(\mathbb{Z}, 2))$ has only one nontrivial element in degree 24, $f_0^*(v_5) = i_2^4$.
All other \( f^n_\delta(v_i) \) are primitive. But the only primitives of \( H^\ast(K(\mathbb{Z}, 2)) \) are \( i^2 \). Hence there are no primitives of \( H^\ast(K(\mathbb{Z}, 2)) \) in degrees between 16 and 32. It follows that \( f^n_\delta(v_i) = 0 \) for \( i \neq 5 \).

Now

\[
\text{Sq}^8 f^n_\delta(v_3) - \text{Sq}^8 (i_1^{12}) - i_1^{16} - \text{Sq}^{16} (i_3^8).
\]

Hence, this proves Theorem 2.1. \( \square \)

Let \( \tilde{A} \) be the matrix with the left entry of \( A \) excluded, \( \tilde{B} \) the matrix with the left column of \( B \) excluded and \( \tilde{C} \) the matrix with the top entry of \( C \) excluded.

Then if \( \tilde{E} \) is the bundle

\[
\begin{array}{ccc}
\Omega \tilde{K}_1 & \xrightarrow{\Omega} & \Omega K_1 \\
\downarrow & & \downarrow \\
\tilde{E} & \xrightarrow{h} & E_0 \\
\tilde{\rho} & \downarrow & \rho \\
K(\mathbb{Z}, n) & \xrightarrow{\tilde{\nu}} & \tilde{K}_1 \\
\nu & \downarrow & \nu \\
\tilde{K}_1 & \xrightarrow{\tilde{w}} & K_1
\end{array}
\]

where \( \tilde{\nu} \) is defined by the matrix \( \tilde{C} \), there is a commutative diagram of infinite loop spaces and maps

If \( \tilde{v}_i = h^\ast(v_i) \), we obtain the "integral" formula by applying \( h^\ast \) to the relation in Theorem 2.1.

Corollary 2.2.

\[
\begin{align*}
\text{Sq}^{11,2} \tilde{v}_2 + (\text{Sq}^{11,1} + \text{Sq}^{9,3} + \text{Sq}^{4,8}) \tilde{v}_3 + (\text{Sq}^{8,1} + \text{Sq}^{7,2} + \text{Sq}^{6,3}) \tilde{v}_4 \\
+ \text{Sq}^{8} \tilde{v}_5 + \text{Sq}^{4,2} \tilde{v}_6 + \text{Sq}^{3} \tilde{v}_7 + \text{Sq}^{4} \tilde{v}_8 + \text{Sq}^{3} \tilde{v}_9 + \text{Sq}^{1} \tilde{v}_{10}
= \text{Sq}^{16} \tilde{p}^\ast(i_n). \quad \square
\end{align*}
\]
Now, let us analyze the cohomology of the loop spaces of $X$ in low degrees. The following diagram (2.6) is a schematic diagram of generators and nontrivial Steenrod connections. The symbol $x_j$ represents a possible nonzero generator in degree $j$. The arrows represent Steenrod connections or secondary operations connecting generators. If there is no $x_j$ this implies there cannot be any generators in degree $j$.

For example, the diagram for $\Omega X$ uses Corollary 2.2. If $x_{42}$ is nonzero, we must have $x_{42} = \text{Sq}^{16}x_{26}$. By Corollary 2.2, $\text{Sq}^{16}$ factors. Now $H^*(\Omega X)$ is even-dimensional, so there are only a few possible nonzero operations. If $\tilde{\phi}_3$ and $\tilde{\phi}_8$ represent $\tilde{v}_3$ and $\tilde{v}_8$, one of them must be nontrivial if $x_{42} \neq 0$. 

\begin{align*}
\text{(2.6)}
\end{align*}
We make use of several known facts to produce these diagrams:

1. The Eilenberg-Moore spectral sequence with \( E_2 = \text{Tor}_{H^*(\Omega^1 X)}(\mathbb{Z}_2, \mathbb{Z}_2) \) and \( E_{\infty} = \text{gr} H^*(\Omega^{j+1} X) \) has \( E_\infty \cong H^*(\Omega^{j+1} X) \) as coalgebras [8] and is a spectral sequence of differential Hopf algebras. Further, \( \text{Tor}_{H^*(\Omega^1 X)}(\mathbb{Z}_2, \mathbb{Z}_2) \cong H^*(\Omega X) \) as coalgebras [12].

2. By Theorem 1.4, the squaring map \( \xi : H^15(X) \to H^30(X) \) is monic.

3. The kernel \( \sigma^*: QH(\Omega^j X) \to PH(\Omega^{j+1} X) \) lies in degrees \( \equiv 2 \text{ mod } 4 \) and also lies in the image of some Bockstein by [9].

We now verify the details of the above diagrams. The diagram for \( H^*(X) \) follows from Theorem 1.1 and Proposition 1.2.

Applying the Eilenberg-Moore spectral sequence and using the fact that \( x_{15}^2 \neq 0 \), we note that the first possible nonprimitive generator of \( H^*(\Omega X) \) would be \( y^2(x_2) \), but this lies in degree greater than 42. Now in \( H^*(\Omega X) \), by Theorem 1.1 and Corollary 1.2,

\[
\text{Sq}^{12} x_{14} = x_{26} \in \ker \text{Sq}^1, \text{Sq}^3, \text{Sq}^4, \text{Sq}^{4.2}, \text{Sq}^8, \text{Sq}^{8.2}.
\]

By Theorem 2.1, \( \text{Sq}^{16} x_{26} \) factors through secondary operations. If \( \tilde{\phi}_i \) is the secondary operation defined by the \( \tilde{v}_i \) of Corollary 2.2, then since \( H^*(\Omega X) \) is even-dimensional [12], we must have

\[
\text{Sq}^{16} x_{26} = x_{42} = \text{Sq}^8 \tilde{\phi}_5(x_{26}) + \text{Sq}^4 \tilde{\phi}_8(x_{26}) + \text{Sq}^{4.2} \tilde{\phi}_6(x_{26}) + \text{Sq}^{4.8} \tilde{\phi}_3(x_{26}).
\]

Now \( x_{42} \) is indecomposable primitive. Further, \( H^j(\Omega X) = 0 \) for \( 14 < j < 22, 22 < j < 26, \)

\[
\tilde{\phi}_5(x_{26}) \in H^{24}(\Omega X) \cong PH^{24}(\Omega X) = \sigma^* QH^{25}(X) = 0 \quad \text{by (1).}
\]

\[
\tilde{\phi}_6(x_{26}) \in H^{36}(\Omega X) \cong IH^{14}(\Omega X)I H^{22}(\Omega X),
\]

since \( QH^{36}(\Omega X) \cong PH^{36}(\Omega X) \cong \sigma^* QH^{37}(X) = 0 \). Now \( \text{Sq}^2 H^{22}(\Omega X) = 0 \). So \( \text{Sq}^2 \tilde{\phi}_6(x_{26}) = 0 \). We conclude

\[
x_{42} = \text{Sq}^4 \tilde{\phi}_8(x_{26}) + \text{Sq}^{4.8} \tilde{\phi}_3(x_{26})
\]

and

\[
\tilde{\phi}_8(x_{26}) \in H^{38}(\Omega X) = PH^{38}(\Omega X).
\]

So if \( QH^{43}(X) \neq 0 \), then either

\[
0 \neq x_{38} = \tilde{\phi}_8(\text{Sq}^{12} x_{14}) \text{ is indecomposable in } H^*(\Omega X) \quad \text{or}
\]

\[
\tilde{\phi}_3(x_{26}) = x_{30} \text{ is indecomposable in } H^*(\Omega X).
\]

(2.7)
The first homotopy group of a finite $H$-space

Now by statement (3),

$$(\sigma^*)^3(x_{38}) = x_{35} \neq 0 \quad \text{and} \quad (\sigma^*)^3(x_{30}) = x_{27} \neq 0.$$  

Hence by (2.7),

$$x_{35} = \tilde{\phi}_8(Sq^{12}x_{11}) = \tilde{\phi}_8(0) \quad \text{and} \quad x_{27} = \tilde{\phi}_3(0).$$  

Therefore, $x_{35}$ lies in the indeterminacy of $\tilde{\phi}_8$ and $x_{27}$ lies in the indeterminacy of $\tilde{\phi}_3$.

By Corollary 2.2, $x_{35} \in \text{im}(Sq^1 + Sq^6, + Sq^6 + Sq^3)$. Using $PH^{odd}(\Omega^4X) \cong \text{QH}^{odd}(\Omega^4X)$, $\sigma^*: \text{QH}^{l+1}(\Omega^lX) \to PH^{l}(\Omega^{l+1}X)$ is onto if $l \neq 2 \mod 4$ and the exactness of the following sequence [15]:

$$0 \to P\xi H^*(\Omega^lX) \to PH^*(\Omega^lX) \to \text{QH}^*(\Omega^lX) \to Q(\lambda H^*(\Omega^lX)) \to 0.$$  

We have

$$\text{QH}^{34}(\Omega^4X) \cong PH^{34}(\Omega^4X). \quad (2.8)$$

Since $PH^{34}(\Omega^4X)$ cannot be transpotence elements, they must be suspension elements. Therefore,

$$PH^{34}(\Omega^4X) = \sigma^* \text{QH}^{35}(\Omega^3X) = \sigma^* \text{PH}^{33}(\Omega^3X)$$

$$= (\sigma^*)^2 \text{QH}^{36}(\Omega^2X) = (\sigma^*)^2 \text{PH}^{36}(\Omega^2X)$$

$$= (\sigma^*)^3 \text{QH}^{37}(\Omega X) = 0,$$

Since $H^*(\Omega X)$ is even-dimensional. Therefore, $x_{35} \notin \text{im} Sq^4 + \text{im} Sq^6 \subseteq \text{im} Sq^1 \text{PH}^{34}(\Omega^4X)$. Similarly,

$$\text{QH}^{25}(\Omega^4X) = PH^{25}(\Omega^4X) = \sigma^* \text{QH}^{26}(\Omega^3X)$$

$$= \sigma^* \text{PH}^{26}(\Omega^3X) = (\sigma^*)^2 \text{QH}^{27}(\Omega^2X)$$

$$= (\sigma^*)^3 \text{QH}^{28}(\Omega X) - (\sigma^*)^3 \text{PH}^{28}(\Omega X) = 0.$$

So $x_{35} \notin \text{im} \text{Sq}^{10} \text{QH}^{25}(\Omega^4X)$,

$$\text{QH}^{26}(\Omega^4X) = PH^{26}(\Omega^4X) = \sigma^* \text{QH}^{27}(\Omega^3X)$$

$$= \sigma^* \text{PH}^{27}(\Omega^3X) = (\sigma^*)^2 \text{PH}^{28}(\Omega^2X)$$

$$= (\sigma^*)^3 \text{QH}^{29}(\Omega X) = 0.$$

So, $x_{35} \notin \text{im} \text{Sq}^{6,3} \text{QH}^{26}(\Omega^4X)$. It follows that $x_{35} = 0$. 
Now suppose $x_{27} \in H^*(\Omega^4 X)$ lies in the indeterminacy of $\tilde{\phi}_3$ which is $\text{im } \text{Sq}^2 + \text{im } \text{Sq}^1$.

We have

\[ QH^{26}(\Omega^4 X) \cong PH^{26}(\Omega^4 X) = \sigma^* QH^{27}(\Omega^3 X) = 0, \]
\[ QH^{25}(\Omega^4 X) \cong PH^{25}(\Omega^4 X) = \sigma^* QH^{26}(\Omega^3 X) = 0. \]

Therefore, $x_{27} \notin \text{im } \text{Sq}^2 + \text{im } \text{Sq}^1$ so $x_{27} = 0$. We have shown the following theorem:

**Theorem 2.3.** $QH^{43}(X) = 0$. \(\square\)

### 3. $R^{31}$

In this section, we prove the following theorem:

**Theorem 3.1.** If $x \in R^{31}$ then $x^2 = 0$. In fact, $\text{Sq}^{24} x = 0$.

The proof uses several factorizations of $\text{Sq}^{16}$ which we will describe using matrices.

**Proposition 3.2.** (Goncalves [5]). There exists a stable factorization

\[ \text{Sq}^{16} = \phi_{0,3} \text{Sq}^8 + \text{Sq}^{15} \phi_{0,0} + \text{Sq}^{12} \phi_{0,2} + \text{Sq}^{6,3} \phi_{2,2} \]

defined on classes in the kernel of $\text{Sq}^1$, $\text{Sq}^2$, and $\text{Sq}^4$.

**Proof.** Consider the following commutative diagram

\[
\begin{array}{ccc}
K(\mathbb{Z}_2, n) & \overset{\text{Sq}^8}{\longrightarrow} & K(\mathbb{Z}_2, n + 8) \\
\downarrow \text{Sq}^1 & & \downarrow \text{Sq}^1 \\
\downarrow \text{Sq}^2 & & \downarrow \text{Sq}^2 \\
\downarrow \text{Sq}^4 & & \downarrow \text{Sq}^8 \\
K(\mathbb{Z}_2, n + 1, n + 2, n + 4) & \overset{A}{\longrightarrow} & K(\mathbb{Z}_2, n + 9, n + 10, n + 16) \\
\downarrow B & & \downarrow (\text{Sq}^8 + \text{Sq}^{6,2}, \text{Sq}^7 + \text{Sq}^{4,2,1}, \text{Sq}^4) \\
K(\mathbb{Z}_2, n + 2, n + 5, n + 8) & \overset{(\text{Sq}^{13}, \text{Sq}^{12}, \text{Sq}^{6,3})}{\longrightarrow} & K(\mathbb{Z}_2, n + 17) \\
\end{array}
\]

(3.1)
where $A$ and $B$ are the following two matrices:

$$A = \begin{pmatrix}
    Sq^8 & 0 & Sq^5 + Sq^{4,1} \\
    Sq^{7,2} & Sq^8 & Sq^{4,2} \\
    Sq^{15} & Sq^{14} & Sq^{12}
\end{pmatrix}, \quad B = \begin{pmatrix}
    Sq^1 & 0 & 0 \\
    Sq^4 & Sq^{2,1} & Sq^4 \\
    Sq^7 & Sq^6 & Sq^4
\end{pmatrix}.$$

The composite column maps of (3.1) are null homotopic. The right column composition defines the secondary operation $\phi_{0,0}$. The left column composition defines three secondary operations, $\phi_{0,0}$, $\phi_{0,2}$ and $\phi_{2,2}$. By [5, Proposition 1.2] the proposition follows. □

Applying Proposition 3.2 to $R^3$ we have if $x_{31} \in R$, $x_{31} \in \ker Sq^1 \cap \ker Sq^2 \cap \ker Sq^4$ by Corollary 1.2 and the fact that $x_{31}$ must be primitive. Therefore, by Proposition 3.2,

$$Sq^{16}x_{31} = \phi_{0,3} Sq^8 x_{31} + Sq^{15} \phi_{0,0}(x_{31})$$

$$+ Sq^{12} \phi_{0,2}(x_{31}) + Sq^{6,3} \phi_{2,2}(x_{31}).$$

By Theorem 1.1, and Corollary 1.2, $\phi_{0,0}(x_{31})$, $\phi_{0,2}(x_{31})$ and $\phi_{2,2}(x_{31})$ must be decomposable. Since

$$H^l(X) = 0 \quad \text{for } l < 15 \text{ or } 15 < l < 23,$$

we get $\phi_{0,0}(x_{31}) = 0 = \phi_{0,2}(x_{31})$ and $\phi_{2,2}(x_{31}) \in R^{15} \cdot R^{23}$. But then $Sq^{6,3} \phi_{2,2}(x_{31}) = 0$. Hence Proposition 3.2 implies

$$Sq^{16}x_{31} = \phi_{0,3} Sq^8 x_{31}. \quad (3.3)$$

By Corollary 1.2 and the Adem relations

$$Sq^{24} x_{31} = Sq^8 Sq^{16} x_{31} = Sq^8 \phi_{0,3} Sq^8 x_{31}.$$

We now describe a factorization of $Sq^{16}$ on elements in the kernel of $Sq^1$, $Sq^2$, $Sq^4$, $Sq^8$. This factorization is due to T.B. Ng (unpublished):
Consider the following matrix relations:

\[
\begin{pmatrix}
\phi_{0,0} & \text{Sq}^1 & 0 & 0 & 0 \\
\phi_{1,1} & \text{Sq}^3 & \text{Sq}^2 & 0 & 0 \\
\phi_{0,2} & \text{Sq}^4 & \text{Sq}^2\text{Sq}^1 & \text{Sq}^4 & 0 \\
\phi_{2,2} & \text{Sq}^7 & \text{Sq}^6 & \text{Sq}^4 & 0 \\
\phi_{3,3} & \text{Sq}^{15} & \text{Sq}^{14} & \text{Sq}^{12} & \text{Sq}^8 \\
\phi_{1,3} & (\text{Sq}^9 + \text{Sq}^4\text{Sq}^8) & \text{Sq}^8 & \text{Sq}^4\text{Sq}^2 & \text{Sq}^2 \\
\tilde{\phi}_{0,3} & (\text{Sq}^8 + \text{Sq}^4\text{Sq}^4) & (\text{Sq}^4\text{Sq}^2\text{Sq}^1 + \text{Sq}^7) & 0 & \text{Sq}^1
\end{pmatrix} = \begin{pmatrix} \text{Sq}^1 \\ \text{Sq}^2 \\ \text{Sq}^4 \\ \text{Sq}^8 \end{pmatrix} = 0.
\]

**Proposition 3.3.**

\[
\text{Sq}^1\phi_{3,3} + (\text{Sq}^7 + \text{Sq}^4\text{Sq}^2\text{Sq}^1)\phi_{1,3} + (\text{Sq}^8 + \text{Sq}^4\text{Sq}^4)\tilde{\phi}_{0,3} \\
+ (\text{Sq}^6\text{Sq}^3 + \text{Sq}^9)\phi_{2,2} + (\text{Sq}^8\text{Sq}^4 + \text{Sq}^{12})\phi_{0,2} \\
+ (\text{Sq}^{15} + \text{Sq}^{13}\text{Sq}^2 + \text{Sq}^{12}\text{Sq}^3 + \text{Sq}^{10}\text{Sq}^4)\phi_{0,0} = \text{Sq}^{16}.
\]

**Proof.** This result is easily verified by evaluating the operation on \(i^8 \in H^*(K(\mathbb{Z}, 2); \mathbb{Z}_2)\). □

Note that the operations \(\phi_{0,3}\) and \(\tilde{\phi}_{0,3}\) differ slightly. \(\phi_{0,3}\) has indeterminacy \(\text{Sq}^8 + \text{Sq}^{6.2}\) on the \(\text{Sq}^1\) factor and \(\phi_{0,3}\) has indeterminacy \(\text{Sq}^8 + \text{Sq}^{4.4} = \text{Sq}^8 + \text{Sq}^{6.2} + \text{Sq}^{7.1}\) on the \(\text{Sq}^1\) factor. An easy calculation shows

\[
\phi_{0,3}(z) = \tilde{\phi}_{0,3}(z) + \text{Sq}^7\beta_2(z),
\]

where \(\beta_2\) is the second Bockstein. By (3.2) and the fact \(H^*(X; \mathbb{Z})\) has no 4 torsion [12], we have

\[
\beta_2\text{Sq}^8x_{31} = 0 \quad \text{(with zero indeterminacy)}.
\]

By (3.4) and Proposition 3.3,

\[
\text{Sq}^{24}x_{31} = (\text{Sq}^8\tilde{\phi}_{0,3})\text{Sq}^8x_{31} \\
= [\text{Sq}^1\phi_{3,3} + (\text{Sq}^7 + \text{Sq}^{4.2.1})\phi_{1,3} + \text{Sq}^{4.4}\tilde{\phi}_{0,3} \\
+ (\text{Sq}^{6.3} + \text{Sq}^9)\phi_{2,2} + (\text{Sq}^{8.4} + \text{Sq}^{12})\phi_{0,2} \\
+ (\text{Sq}^{15} + \text{Sq}^{13.2} + \text{Sq}^{12.3} + \text{Sq}^{10.5})\phi_{0,0}](\text{Sq}^8x_{31}),
\]

since \(\text{Sq}^{16}\text{Sq}^8x_{31} = 0\) by Theorem 1.1.
By Theorem 1.1,

\[ Q^{2i} \cap \text{im} \ Sq^i = 0 \quad \text{for} \ i < 3, \]
\[ Q^{4i} \cap \text{im} \ Sq^i = 0 \quad \text{for} \ i < 4. \]

Hence \( Sq^{24} x_{31} \) is decomposable in \( R^{odd} \). But by the exact sequence (1.1), it follows that \( Sq^{24} x_{31} = 0 \). Finally,

\[ x_{31}^2 = Sq^7 Sq^{24} x_{31} = 0. \]

This proves Theorem 3.1. \( \square \)

Our goal now is to apply a tertiary operation defined in [13] to \( x_{31} \). We proceed here to build the universal example.

Let

\[ K_0 = \prod_{j=0}^{2} K(\mathbb{Z}_2, 32 + 2^j) \times K(\mathbb{Z}_2, 63), \quad K = K(\mathbb{Z}_2, 32), \]

\( w: K \to K_0 \) be defined by \( w^*(i_{32+2^j}) = Sq^{2^j}i_{32}, w^*(i_{63}) = Sq^{31}i_{32} \). Let \( E_0 \) be the fibre of \( w \)

\[
\begin{array}{ccc}
\Omega K_0 & \to & \Omega K_0 \\
\updownarrow & & \updownarrow \\
E_0 & \to & \Omega K_0 \\
\updownarrow p_0 & & \updownarrow \\
K & \xrightarrow{w} & K_0
\end{array}
\]

As in [13], there exist elements \( v_i \subset H^*(E_0) \), \( v_{0,2} \subset H^*(E_0) \) with the following properties:

If \( u_0 = p^*(i_{32}) \) then

\[ (\text{Sq}^4 v_{0,2} + \text{Sq}^8 u_0)^2 \]
\[ = \text{Sq}^8 v_3 + \text{Sq}^{8,4,2} v_2 \]
\[ + (\text{Sq}^{15,1} + \text{Sq}^{14,2} + \text{Sq}^{12,4}) v_1 + \text{Sq}^8 (\text{Sq}^{8,1} + \text{Sq}^{7,2}) v_0. \]

\[ \Omega^2 E_0 \simeq \Omega^2 E'_0 \times K(\mathbb{Z}_2, 60), \] where \( E_0 \) has the same \( k \)-invariants as \( E_0 \) with the exception of \( \text{Sq}^{31} \) and

\[ \sigma^* (v_i) = 1 \otimes \text{Sq}^{2i} \ i_{60} \quad \text{for} \ i < 3, \]
\[ \sigma^* (v_3) = 1 \otimes \text{Sq}^8 \ i_{60} + i_{30} (\text{Sq}^8 i_{30}) \otimes 1. \]
Define

\[ K_1 = \prod_{i=2}^{3} K(\mathbb{Z}_2, 62 + 2^i) \times \prod_{j=0}^{2} K(\mathbb{Z}_2, 64 + 2^j) \times \prod_{k=0}^{1} K(\mathbb{Z}_2, 63 + 2^k), \]

and \( w_1 : E_0 \to K_1 \) by

- \( w_1^*(i_{62 + 2^i}) = v_i \) for \( i = 2, 3 \),
- \( w_1^*(i_{64 + 2^j}) = \text{Sq}^{2^j} v_1 \) for \( j = 0, 1, 2 \),
- \( w_1^*(i_{63 + 2^k}) = \text{Sq}^{2^k} v_0 \) for \( k = 0, 1 \).

Let \( E_1 \) be the fibre of \( E_0 \),

\[
\begin{array}{c}
\Omega K_1 \\
\downarrow j_1 \\
E_1 \\
\downarrow p_1 \\
E_0 \longrightarrow w_1 K_1
\end{array}
\]

Define \( u_{30} = (\sigma^*)^2 p_1^* p_0^* (i_{32}) \). By abuse of notation let \( v_{0,2} = k^* (v_{0,2}) \). It clearly represents the stable secondary operation \( \phi_{0,2} \).

**Theorem 3.4.** There exists an element \( v \in PH^{77}(\Omega^2 E_1) \) with the following properties:

(a) \( c(v) = (\text{Sq}^8 u_{30} + \text{Sq}^4 v_{0,2}) \otimes (\text{Sq}^8 u_{30} + \text{Sq}^4 v_{0,2}) \), where \( c \) denotes the obstruction to homotopy commutativity defined in [17].

(b) \( (\Omega^2 j_1)^* (v) = \text{Sq}^{8,2} i_{67} + \text{Sq}^{8,4,2} i_{63} + \text{Sq}^{15} i_{62} + \text{Sq}^{14} i_{63} + \text{Sq}^{12} i_{65} + \text{Sq}^{8,8} i_{61} + \text{Sq}^{8,7} i_{62} \).

**Proof.** By construction and (3.5), \( [p_1^*(\text{Sq}^8 v_{0,2} + \text{Sq}^8 u_0)]^2 = 0 \) in \( H^*(E_1) \). Hence by the exact triangle

\[
\begin{array}{c}
H^*(E_1) \\
\downarrow i \quad \downarrow \lambda \\
H^*(P_2 \Omega E_1) \\
\downarrow \delta \quad \downarrow \Lambda \\
IH^*(\Omega E_1) \otimes IH^*(\Omega E_1)
\end{array}
\]

It follows there is an element \( Bv \in H^*(\Omega E_1) \) with \( \Lambda Bv = \sigma^*(\text{Sq}^4 v_{0,2} + \text{Sq}^8 u_0) \otimes \sigma^*(\text{Sq}^4 v_{0,2} + \text{Sq}^8 u_0) \). It follows that \( v = \sigma^*(Bv) \) has the desired properties. For details, see [17]. \( \square \)
Our strategy now is to produce a commutative diagram of $H$-maps of the form

\[
\begin{array}{ccc}
\Omega^2 E_1 & \xrightarrow{f_1} & \Omega^2 E_0 \\
\downarrow & & \downarrow \\
\Omega X & \xrightarrow{\Omega f_0} & \Omega^2 K
\end{array}
\]

We then analyze $c(f^+(v))$.

4. Computing a tertiary operation

In this section, we apply the tertiary operation defined in Section 3 to an element in $R^{31}$. We show that $\text{Sq}^8 R^{31} = 0$. By the results of Section 1, this will prove there are very few cohomology generators in $H^*(X)$ (see Corollary 4.4).

Given $x \in R^{31}$, note that $x \in \ker \text{Sq}^{2i}$ for $i \leq 2$ and $x^2 = \text{Sq}^{31} x = 0$ by Theorem 3.1. The main theorem of this section is the following:

**Theorem 4.1.** $\text{Sq}^8 R^{31} = 0$. All Steenrod operations vanish on $Q^{2l-1}$ for $l \geq 5$.

The proof will proceed in several steps. We first show there is a commutative diagram (3.8) of $H$ maps.

Note by Theorem 3.1 and Corollary 1.2, that there is a commutative diagram

\[
\begin{array}{ccc}
\Omega^2 K_0 & \xrightarrow{f_0} & \Omega E_0 \\
\downarrow & & \downarrow \\
X & \xrightarrow{f} & K(\mathbb{Z}_2, 31) \\
\end{array}
\xrightarrow{\omega} \Omega K_0
\]

(4.1)

where $f^*(i_{31}) = x$. Since $f$ is an $H$-map by (1.1), $D_{f_0}$ factors through the fibre

\[
\begin{array}{ccc}
\Omega^2 K_0 & \xrightarrow{j} & \Omega^2 E_0 \\
\downarrow & & \downarrow \\
X \wedge X & \xrightarrow{D_{f_0}} & \Omega E_0
\end{array}
\]

(4.2)

\[\Omega^2 K_0 \simeq \prod_{i=0}^{2} K(\mathbb{Z}_2, 30 + 2^i) \times K(\mathbb{Z}_2, 61).\]
Since $QH^{even}(X) = 0$ by Theorem 1.1, we have that $[\hat{D}_g(i_{61})]$ and $[\hat{D}_g(i_{31})]$ lie in $D \otimes H^* + H^* \otimes D$ where $D$ is the module of decomposables.

Since $c(\Omega f_0)$ is adjoint to $(\sigma^* \otimes \sigma^*)\tilde{D}_0$ by [17], we have that $c(\Omega f_0)$ actually factors through $K(\mathbb{Z}_2, 30) \times K(\mathbb{Z}_2, 32)$:

\[
\begin{array}{ccc}
K(\mathbb{Z}_2, 30, 32) & \longrightarrow & \Omega^4 K_0 \\
\epsilon \downarrow & & \downarrow \\
\Omega X \wedge \Omega X & \longrightarrow & \Omega^3 E_0 \\
c(\Omega f_0) & & \\
\end{array}
\]

We have the following proposition:

**Proposition 4.2.** $0 = [\epsilon]$ and $\Omega f_0$ is a $c$-map.

**Proof.** $\epsilon = (\sigma^* \otimes \sigma^*)\tilde{D}_0$. Since $\sigma^*(D) = 0$, $\epsilon$ lies in degrees 30 and 32. But $H^*(\Omega X)$ is 13 connected and $(\text{im } \sigma^*) = 0$ for $14 < l < 22, l < 14$. □

So we have a commutative diagram of loop maps and $c$-maps,

\[
\begin{array}{ccc}
\Omega^3 K_0 & \longrightarrow & \Omega^2 E_0 \\
\downarrow & & \downarrow \\
\Omega f_0 & \longrightarrow & K(\mathbb{Z}_2, 30) \\
f \downarrow & & \Omega^2 \longrightarrow \Omega^2 K_0 \\
\Omega X & \longrightarrow & \\
\end{array}
\]

To lift $\Omega f_0$ to $\Omega^2 E_1$, we need to show $(\Omega^2 w_1)(\Omega f_0)$ is null homotopic. $(\Omega^2 w_1)(\Omega f_0)$ consists of the following primitive suspension classes in $H^*(\Omega X)$:

\begin{enumerate}
  \item[(a)] $(\Omega f_0)^*(\sigma^* \sigma^*(v_i))$ for $i = 2, 3$, degrees $60 + 2^i$.
  \item[(b)] $(\Omega f_0)^*(\text{Sq}^{2j} \sigma^* \sigma^*(v_i))$ for $j = 0, 1, 2$, degrees $62 + 2^j$.
  \item[(c)] $(\Omega f_0)^*(\text{Sq}^{2k} \sigma^* \sigma^*(v_0))$ for $k = 0, 1$, degrees $61 + 2^k$.
\end{enumerate}

By Corollary 1.2, elements (a) and (b) are trivial. $f_0^*(\sigma^*(v_0))$ is the second Bockstein on $\text{Sq}^{30}X$. Hence $H^*(X; \mathbb{Z})$ has no 4-torsion [8, 12], it lies in im $\text{Sq}^1 \subset D$. Hence $\sigma^*[f_0^* \sigma^*(v_0)] - \Omega f_0^* \sigma^* \sigma^*(v_0) - 0$. We conclude

$(\Omega^2 w_1)(\Omega f_0)$ is null homotopic and $f_1$ exists. □
We have a commutative diagram

\[
\begin{array}{ccc}
\Omega^3 K_1 \\
\downarrow \\
\Omega^2 E_1 \\
\downarrow f_1 \\
\Omega X \\
\downarrow \Omega f_0 \\
\Omega^2 E_0.
\end{array}
\]

(4.6)

Since \( \Omega f_0 \) is an \( H \)-map, \( Df_1 \) factors through the fibre

\[
\begin{array}{ccc}
\hat{D}_1 \\
\downarrow \\
\Omega X \wedge \Omega X \\
\downarrow D_{f_1} \\
\Omega^2 E_1
\end{array}
\]

Since \( H^*(\Omega X) \) is even-dimensional, \( [\hat{D}_1] \) actually lies in degree 62 since \( \Omega^3 K_1 \) has even-dimensional \( K(\mathbb{Z}_2, n) \) only in degree 62.

Since \( \Omega f_0 \) is a \( c \) and \( a_3 \)-map, we have by [17] that \( [\hat{D}_1] \in P \text{Ext}^{2,62}_{H^*(\Omega X)}(\mathbb{Z}_2, \mathbb{Z}_2) \). Now elements of \( H_*(\Omega X) \) of finite height have height 2 by [10]. Hence there are no “transpotence” elements in this degree. Since \( H_*(\Omega X) \) is even-dimensional, \( [\hat{D}_1] = 0 \), and \( f_1 \) can be chosen to be an \( H \)-map.

We have proved the following:

**Proposition 4.3.** There is a commutative diagram of \( H \)-maps

\[
\begin{array}{ccc}
\Omega^2 K_1 \\
\downarrow \\
\Omega^2 E_1 \\
\downarrow e \\
K(\mathbb{Z}_2, 77)
\end{array}
\]

Further, \( \Omega f_0 \) is a \( c \)-map.

It follows that \( c(f_1) \) factors through the fibre

\[
\begin{array}{ccc}
\Omega^4 K_1 \\
\downarrow c(f_1) \\
\Omega^3 E_1 \\
\downarrow \Omega v
\end{array}
\]

\( K(\mathbb{Z}_2, 76) \)
Proof of Theorem 4.1. We have
\[ c(vf_1) = c(v)(f_1 \wedge f_1) + (\Omega v)c(f_1) \]
\[ \approx c(v)(f_1 \wedge f_1) + (\Omega v)(\Omega^3 j_1) e_1 \quad \text{by (4.7).} \]

On cohomology,
\[ c(f_1^*(v)) = f_1^*(\text{Sq}^8 u_{30} + \text{Sq}^4 v_{0,2}) \otimes f_1^*(\text{Sq}^8 u_{30} + \text{Sq}^4 v_{0,2}) \]
\[ + c_1(\Omega^3 j_1)^*(\sigma^* v) \quad \text{by Theorem 3.4.} \quad (4.8) \]

Now \( f_1^*(v_{0,2}) = (\Omega^2 p_1)^*(f_1)^*(\sigma^*(v_{0,2})) = (\Omega f_0)^*(\sigma^*(v_{0,2})) \in H^{35}(X) \) which is decomposable by Corollary 1.2. So
\[ f_1^*(v_{0,2}) = 0 \quad \text{and} \quad f_1^*(u_{30}) = \sigma^*(x). \]

By Theorem 3.4(b), (4.8) now reads
\[ c(f_1^*(v)) \in \text{Sq}^8 \sigma^*(x) \otimes \text{Sq}^8 \sigma^*(x) + \text{im} \text{Sq}^{8,2} + \text{im} \text{Sq}^{8,4,2} \]
\[ + \text{im} \text{Sq}^{15} + \text{im} \text{Sq}^{14} + \text{im} \text{Sq}^{12} + \text{im} \text{Sq}^{8,8} + \text{im} \text{Sq}^{8,7}. \quad (4.9) \]

Now \( f_1^*(v) \) is odd-dimensional, so it must be trivial. Hence \( c(f_1^*(v)) \) lies in image \((1 + T^*)\) which is the indeterminacy of the \( c \) operation. But \( \sigma^*: R^{\text{odd}} \to PH^{\text{even}}(\Omega X) \) is monic so \( \text{Sq}^8 \sigma^*(x) \neq 0 \) if \( \text{Sq}^8 x \neq 0 \). It follows that \( z = \text{Sq}^8 \sigma^*(x) \otimes \text{Sq}^8 \sigma^*(x) \) must be "cancelled" by the primary operations in (4.9).

Now there are no transpotence elements in degrees less than 38. We claim
\[ \text{Sq}^8 \sigma^*(x) \notin \text{im} \text{Sq}^{2i} \quad \text{for} \quad i \leq 2. \quad (4.10) \]

To see this, \( \text{Sq}^8 \sigma^*(x) \notin \text{im} \text{Sq}^1 \) since \( H^*(\Omega X) \) is even-dimensional. If \( \text{Sq}^8 \sigma^*(x) = \text{Sq}^y y \) then \( y \) is either a divided power or suspension. If \( y \) is a suspension, Theorem 1.1 implies \( y = y_1^1 \) where degree \( y_1 = 18 \). But Theorem 1.1 implies \( y_1 = 0 \). If \( y \) is a divided power, again there must be a \( y_1 \) in degree 18. Hence \( \text{Sq}^8 \sigma^*(x) \) cannot lie in \( \text{im} \text{Sq}^7 \).

If \( \text{Sq}^8 \sigma^*(x) = \text{Sq}^4 y \), then \( y = \sigma^*(x_{35}) \) but \( x_{35} = 0 \) by Corollary 1.2. Now by the Adem relations all the Steenrod operations in (4.9) except \( \text{Sq}^{8,4,2} \) lie in the left ideal generated by \( \text{Sq}^{2i} \) for \( i \leq 2 \). Hence we conclude
\[ c(f_1^*(v)) \in \text{Sq}^8 \sigma^*(x) \otimes \text{Sq}^8 \sigma^*(x) + \text{im} \text{Sq}^{8,4,2}. \quad (4.11) \]

This implies \( \text{Sq}^8 \sigma^*(x) \in \text{Sq}^{8,4,2} \) by the Cartan formulae and the fact that \( H^*(\Omega X) \) is even-dimensional. Hence \( \text{PH}^{30}(\Omega X) \cap \text{im} \text{Sq}^4 \neq 0 \). But this implies \( R^{31} \cap \text{Sq}^8 R^{27} \neq 0 \) which contradicts Theorem 1.1.
We conclude \( \text{Sq}^8 x = 0 \). By Theorem 1.3, this completes the proof of Theorem 4.1.

**Corollary 4.4.** In degrees greater than 30, \( R \) is concentrated in degrees \( 2^l - 1 \) for \( l \geq 5 \).

**Proof.** By Theorem 1.1, all \( Q^* \) is generated as a module over \( \mathcal{A}(2) \) by elements in degree \( 2^l - 1 \). By Theorem 4.1 and Theorem 1.3, the only nontrivial action of \( \mathcal{A}(2) \) occurs when one applies Steenrod operations to \( Q^{15} \). We have \( \text{Sq}^i Q^{15} = 0 \) for \( i \leq 2 \). Theorem 1.1 implies the only nontrivial operations are Steenrod operations applied to

\[
\text{Sq}^8 Q^{15}, \quad \text{Sq}^{12} Q^{15}, \quad \text{Sq}^{14} Q^{15}.
\]

Using Theorem 1.1(3), \( \sigma^*[\text{Sq}^{16,8} Q^{15}] = \text{Sq}^8 PH^{30}(\Omega X) \). But \( \text{Sq}^8 PH^{30}(\Omega X) = \text{Sq}^8 \sigma^* R^{31} = 0 \). Using the fact that there are no transposition elements up to degree 58 in \( H^*(\Omega X) \) one checks the corollary by using the Adem relations.

**Corollary 4.5.** \( H^*(X) \) is primitively generated, and has a Borel decomposition of the form

\[
\bigotimes B_i \otimes \bigwedge (y_1, \ldots, y_i)
\]

where

\[
B_i = \mathbb{Z}_2 \frac{[x_{15}]}{x_{15}} \otimes \bigwedge (x_{23}, x_{27}, x_{29})
\]

and \( \deg y_i = 2^l - 1 \) for some \( l_i \geq 5 \).

**Proof.** Suppose it is not. Let \( x \in R \) be a nonprimitive generator of lowest degree. Then \( \deg x = 2^l - 1 \) for some \( l \geq 5 \) and since \( \Delta x \in \xi H^*(X) \otimes R^{\text{odd}} \), we may assume \( \Delta x \in P(\xi H^*(X)) \otimes PH^{\text{odd}}(X) \), by co-associativity. But \( P\xi H^*(X) = \xi H^{15}(X) \) is concentrated in degree 30, so

\[
\Delta x \in P(\xi H^*(X)) \otimes PH^{2^l - 30 - 1}(X).
\]

But \( PH^{2^l - 31}(X) = 0 \) for \( l \geq 5 \) by Corollary 4.4. Therefore, \( x \) must have been primitive. By Theorem 1.4, each 15-dimensional generator produces a subHopf algebra of the form \( B_i \) and by Corollary 4.4 the \( y_i \)'s are exterior with trivial Steenrod action.

**Corollary 4.6 (Theorem B).** A seven connected \( H \)-space \( X \) with associative mod 2 homology is acyclic.
Proof. By a $K$-theory result of Jeanneret [6], there is no space $X$ with the mod 2 cohomology described in Corollary 4.5. □

References