Combinatorial proofs of identities in basic hypergeometric series

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Abstract

In this paper, the \(q\)-Pfaff-Saalschütz formula and the \(q\)-Sheppard \(\phi_2\) transformation formula are established combinatorially.

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1. Introduction

In 1987, Joichi and Stanton [10] established bijective proofs of basic hypergeometric series identities by systematic methods. Their strategy was to set up either a bijection or an involution for each step and to form a bijection for an identity by combining the bijections and involutions. Their method works very nicely for identities whose analytic proofs are simple, while many bijections and involutions are required to find bijections for identities whose analytic proofs are rather complicated. For instance, the \(\phi_1\) transformation

\[
\sum_{n=0}^{\infty} \prod_{j=0}^{n-1} \frac{(1 - aq^j)(1 - bq^j)}{(1 - q^{j+1})(1 - cq^j)} z^j = \prod_{n=0}^{\infty} \frac{(1 - abzq^n/c)}{(1 - zq^n)}
\]

\[
\times \sum_{n=0}^{\infty} \prod_{j=0}^{n-1} \frac{(1 - cq^j/a)(1 - cq^j/b)}{(1 - q^{j+1})(1 - cq^j)} (abz/c)^j
\]

has a bijective proof, which requires 6 applications of their bijection for the \(q\)-binomial theorem and 23 cancelling involutions. By multiplying the \(\phi_1\) transformation above by itself after making
some changes of variables, and equating coefficients of $z^n$ on both sides, we obtain a terminating Saalschützian (1-balanced) $\phi_3$ transformation, which is in the middle of their hierarchy of higher transformations, where Watson’s transformation of a $\phi_7$ to a $\phi_3$ is followed by the Rogers–Ramanujan identities. They remarked that it would be interesting if their bijective proofs of higher transformations could be explicitly identified.

Recently, Corteel and Lovejoy [5,6] have worked on various identities in basic hypergeometric series by combinatorial methods. In particular, in [4], Corteel employed particle seas to obtain explicit bijections for the higher transformations in the hierarchy of Joichi and Stanton.

In the sequel, we assume that $|q| < 1$ and use the customary notation for $q$-series

$$(a)_0 := (a; q)_0 := 1,$$

$$(a)_{\infty} := (a; q)_{\infty} := \prod_{k=0}^{\infty} (1 - aq^k),$$

$$(a)_n := (a; q)_n := \frac{(a; q)_{\infty}}{(aq^n; q)_{\infty}} \text{ for any } n.$$

A terminating Saalschützian (1-balanced) $\phi_3$ transformation [10] is given by

$$\sum_{n=0}^{\infty} \frac{(a)_{n} (b)_{j} (d)_{n-j} (e)_{n-j}}{(q)_{j}(c)_{j}(q)_{n-j}(f)_{n-j}} \left( \frac{ab}{c} \right)^{n-j} = \sum_{j=0}^{\infty} \frac{(c/a)_{j} (c/b)_{j} (f/d)_{n-j} (f/e)_{n-j}}{(q)_{j}(c)_{j}(q)_{n-j}(f)_{n-j}} \left( \frac{ab}{c} \right)^{j},$$

(1.1)

where $c f = abde$. By letting $f$ be equal to $e$ in (1.1), we obtain the $q$-Pfaff-Saalschütz formula [7] given by

$$\sum_{j=0}^{\infty} \frac{(a)_{j} (b)_{j} (q^{n-j+1})_{j}}{(q)_{j}(c)_{j}((c/ab)q^{n-j})_{j}} (c/ab)^{j} = \frac{(c/a)_{n} (c/b)_{n}}{(c)_{n}(c/ab)_{n}}. \tag{1.2}$$

There are several combinatorial proofs of (1.2) [2,8,12], but they are all for special cases when $a = q^{\alpha}$, $b = q^{\beta}$, $c = q^{\gamma}$ for appropriate $\alpha$, $\beta$, and $\gamma$. In Section 3, we present a combinatorial proof of (1.2) for any $a$, $b$, and $c$.

In [2], Andrews and Bressoud extended the method for the $q$-Pfaff-Saalschütz formula to the special case of the following $q$-analogue of Sheppard’s $\phi_2$ transformation formula

$$\sum_{j=0}^{n} \frac{(a)_{j} (b)_{j} (q^{-n})_{j}}{(q)_{j}(c)_{j}(d)_{j}} \left( \frac{cdq^{n}}{ab} \right)^{j} = \frac{(d/a)_{n} (d/b)_{n}}{(d)_{n}} \sum_{j=0}^{n} \frac{(a)_{j} (c/b)_{j} (q^{-n})_{j} q^{j}}{(q)_{j}(c)_{j}(aq^{1-n}/d)_{j}},$$

which is proved combinatorially in Section 4.

In the following section, we present lemmas to be used in the other sections. These lemmas are all combinatorial versions of the $q$-binomial theorem and can be found in [3]. In particular, the concept of Lemma 2.1 was first introduced in [11] to prove Ramanujan’s $\psi_1$ summation formula.

2. Preliminaries

We combinatorially state and prove the $q$-binomial theorem in Lemmas 2.1–2.3, namely,
For any nonnegative integers \( l \)

In this lemma, we put the parts of a partition in nondecreasing order. Let

\[ \pi \]

where

\[ \phi \]

and define

\[ \rho \]

Let \( D_{m, M} \) be the set of partitions into distinct parts \( s, m \leq s < M \), and \( D_{m, M}^0(k) \) be the subset of \( D_{m, M} \) such that partitions have exactly \( k \) parts. Let \( P_{m, M} \) be the set of partitions into parts \( s, m \leq s < M \), and \( P_{m, M}^0(k) \) be the subset of \( P_{m, M} \) such that partitions have exactly \( k \) parts.

**Lemma 2.1.** For any nonnegative integers \( l, m, \) and \( n \), there is a one to one correspondence between

\[ D_{m, m+n} \times P_{l, \infty}(n) \hspace{1em} \text{and} \hspace{1em} \bigcup_{k=0}^{n} D_{m+l, \infty}(k) \times P_{l, \infty}(n-k). \]

**Proof.** In this lemma, we put the parts of a partition in nondecreasing order. Let \( (\pi^1, \pi^2) \in D_{m, m+n} \times P_{l, \infty}(n) \) be given. Let \( k \) be the number of parts of \( \pi^1 \). Define \( \pi^3 \) by

\[ \pi^3_i = \pi^2_{i-m+1} + \pi^1_i \quad \text{for} \quad 1 \leq i \leq k, \]

and define \( \pi^4 \) as the partition of the remaining \( n-k \) parts of \( \pi^2 \). Since \( \pi^1 \) is a partition into \( k \) distinct parts \( \geq m \), \( \pi^3 \) is a partition in \( D_{m+l, \infty}(k) \). Meanwhile, \( \pi^4 \) is a partition in \( P_{l, \infty}(n-k) \).

The process is invertible. Let \( (\pi^3, \pi^4) \in D_{m+l, \infty}(k) \times P_{l, \infty}(n-k) \) be given. We begin with \( \pi^4 \), and progressively build partitions using the parts of \( \pi^3 \) beginning with the largest one until all parts of \( \pi^3 \) have been used. We define a map \( \phi \) that will be employed at each step. Let \( \rho \) be a partition. Then for any integer \( I \geq m \), we can find the unique \( j \) such that

\[ \rho_j \leq I - m - j \leq \rho_{j+1}, \]

where \( \rho_0 = -\infty \) and \( \rho_{\ell(\rho)+1} = \infty \). Define \( \phi \) as

\[ \phi(\rho, m) = (\rho^*, m + j), \]

where

\[ \rho^*_i = \begin{cases} \rho_i, & \text{if} \ 1 \leq i \leq j, \\ I - m - j, & \text{if} \ i = j + 1, \\ \rho_{i-1}, & \text{if} \ i > j + 1. \end{cases} \]

We successively apply \( \phi \) a total of \( k \) times as follows:

\[ \phi(\pi^4(i), \pi^3_i) = (\pi^4(i+1), \pi^1_i), \quad 1 \leq i \leq k, \]

where \( \pi^4(1) = \pi^4 \). Let \( \pi^2 = \pi^4(k+1) \) and \( \pi^1 \) be the partition consisting of integers \( \pi^1_i \) obtained at each step. □
Example. We describe the idea of Lemma 2.1 with an example. Let \( m = l = 0 \) and \( n = 5 \). Take \( \pi^1 = 4 2 1 \in D_{0,5} \) and \( \pi^2 = 4 4 3 1 1 \in \mathcal{P}_{0,\infty}(5) \). The proof of the lemma has a very simple graphical representation as in Fig. 1, where the dots in columns above the dashed line and below \( \ast \) indicate the partition \( \pi^1 \), and the dots in rows below the dashed line indicate the partition \( \pi^2 \). By reading the dots in columns, we obtain \( \pi^3 = 8 5 2 \) and \( \pi^4 = 4 1 \), where \( \pi^3 \) consists of the dots below \( \ast \).

**Lemma 2.2.** For any nonnegative integers \( l, m, \) and \( n \), there is a one to one correspondence between

\[
D_{m,m+n} \times D_{l,\infty}(n) \quad \text{and} \quad \bigcup_{k=0}^{n} D_{l,\infty}(n-k) \times \mathcal{P}_{l+m+n-1,\infty}(k).
\]

**Proof.** Let \( (\mu^1, \mu^2) \in D_{m,m+n} \times D_{l,\infty}(n) \) be given. Let \( k \) be the number of parts of \( \mu^1 \). Define \( \mu^4 \) as

\[
\mu_i^4 = \mu_i^2 + \mu_i^1 \quad \text{for} \quad 1 \leq i \leq k
\]

and define \( \mu^3 \) as the partition of the remaining \( n-k \) parts of \( \mu^2 \). Note that \( \mu^4 \) is a partition in \( \mathcal{P}_{l+m+n-1,\infty}(k) \), since \( \mu^1 \) is a partition into \( k \) distinct parts \( \geq m \), \( \mu^2 \) is a partition into distinct parts \( \geq l \), and in the process, \( \mu_i^1 \) is added to the \((\mu_i^1 - m + 1)\)st smallest part of \( \mu^2 \). Meanwhile, \( \mu^3 \) is a partition in \( D_{l,\infty}(n-k) \).

Since the inverse process can be defined similarly, as we showed in the proof of Lemma 2.1, we omit the proof of the inverse process. \( \square \)

Throughout this paper, we say that two sets are equivalent if there exists a one to one correspondence between the sets.

**Lemma 2.3.** For any nonnegative integers \( l, m, \) and \( n \), \( D_{l,l+n} \times D_{m,m+n} \times \mathcal{P}_{1,n+1} \) is equivalent to

\[
\bigcup_{0 \leq k_1,k_2,k_3,k_4 \leq n, k_1 + \cdots + k_4 = n} D_{m,\infty}(k_1) \times \mathcal{P}_{l+m+n-1,\infty}(k_2) \times D_{l,\infty}(k_3) \times \mathcal{P}_{0,\infty}(k_4).
\]

**Proof.** Note that taking the conjugate of a partition in \( \mathcal{P}_{1,n+1} \) gives a partition in \( \mathcal{P}_{0,\infty}(n) \). Thus \( D_{l,l+n} \times D_{m,m+n} \times \mathcal{P}_{1,n+1} \) is equivalent to \( D_{l,l+n} \times D_{m,m+n} \times \mathcal{P}_{0,\infty}(n) \). By applying the map defined in Lemma 2.1 to \( D_{m,m+n} \times \mathcal{P}_{0,\infty}(n) \), we see that there is a one to one correspondence
between

$$D_{l,l+n} \times D_{m,m+n} \times \mathcal{P}_0(n)$$

and

$$D_{l,l+n} \times \bigcup_{k=0}^{n} D_{m,\infty}(n-k) \times \mathcal{P}_0(k).$$

By dividing the parts of partitions in $D_{l,l+n}$ into parts less than $l+k$ and the others, we can say that $D_{l,l+n}$ is equivalent to $D_{l,l+k} \times D_{l+k,l+n}$. Thus, $D_{l,l+n} \times D_{m,m+n} \times \mathcal{P}_0(n)$ is equivalent to

$$\bigcup_{k=0}^{n} D_{l+k,l+n} \times D_{m,\infty}(n-k) \times D_{l,l+k} \times \mathcal{P}_0(k),$$

which is equivalent to

$$\bigcup_{0 \leq k_1, k_2, k_3, k_4 \leq n, k_1 + k_2 + k_3 + k_4 = n} D_{m,\infty}(k_1) \times \mathcal{P}_{l+m+n-1,\infty}(k_2) \times D_{l,\infty}(k_3) \times \mathcal{P}_{0,\infty}(k_4)$$

by Lemmas 2.1 and 2.2.

Note that

$$\ell(\rho^1) = k_2 + k_3, \quad \ell(\rho^2) = k_1 + k_2$$

for $(\rho^1, \rho^2, \rho^3) \in D_{l,l+n} \times D_{m,m+n} \times \mathcal{P}_{1,n+1}$. □

3. The $q$-Pfaff-Saalschütz formula

A terminating balanced $3\phi_2$ series [7] is given by

$$\sum_{j=0}^{n} \frac{(a)j(b)j(q^{-n})_j}{(q)_j(c)j(ab^{-1}q^{1-n})_j} q^j = \frac{(c/a)_n(c/b)_n}{(c)_n(c/ab)_n},$$

which is equal to (1.2). In (1.2), we multiply both sides by $(c)_n(c/ab)_n/(q)_n$, and replace $a$ and $b$ by $-a$ and $-b$, respectively, to obtain

$$\sum_{j=0}^{n} \frac{(-a)j(-b)j(cq^{j})_{n-j}(c/ab)_{n-j}}{(q)_j(q)_{n-j}} q^j \left( \frac{c}{ab} \right)^j = \frac{(-c/a)_n(-c/b)_n}{(c)_n}. \quad (3.1)$$

Then the left side of (3.1) generates vector partitions $(\sigma^1, \sigma^2, \sigma^3, \sigma^4, \sigma^5, \sigma^6)$ in $D_{0,j} \times D_{0,j} \times \mathcal{P}_{1,j+1} \times D_{j,n} \times D_{0,n-j} \times \mathcal{P}_{1,n-j+1}$ for $0 \leq j \leq n$ with signed weight

$$(-1)^\ell(\sigma^1+\ell(\sigma^5)) a^\ell(\sigma^1)-\ell(\sigma^5)-j b^\ell(\sigma^2)-\ell(\sigma^5)-j c^\ell(\sigma^4)+\ell(\sigma^6)+j q^{|\sigma^1|+|\sigma^2|+|\sigma^3|+|\sigma^4|+|\sigma^5|+|\sigma^6|}.$$  

(3.2)

Meanwhile, the right side generates vector partitions $(v^1, v^2, v^3)$ in $D_{0,n} \times D_{0,n} \times \mathcal{P}_{1,n+1}$ with weight

$$a^{-\ell(v^1)} b^{-\ell(v^2)} c^{\ell(v^1)+\ell(v^2)} q^{v^1+v^2+v^3}.$$  

(3.3)

By Lemma 2.3, we see that $D_{0,j} \times D_{0,j} \times \mathcal{P}_{1,j+1} \times D_{j,n} \times D_{0,n-j} \times \mathcal{P}_{1,n-j+1}$ is equivalent to

$$\bigcup_{0 \leq k_1, k_2, k_3, k_4 \leq j, k_1+k_2+k_3+k_4 = j} D_{0,\infty}(k_1) \times \mathcal{P}_{j-1,\infty}(k_2) \times D_{0,\infty}(k_3) \times \mathcal{P}_{0,\infty}(k_4)$$
In this proof, we denote the smallest part of a partition $\pi$ by $s(\pi)$. Given $j$, let $(\mu_1, \mu_2, \mu_3, \mu_4)$ be in $\mathcal{P}_{j-1, \infty}(n_1) \times \mathcal{P}_{0, \infty}(n_2) \times \mathcal{D}_{0, \infty}(n_3) \times \mathcal{D}_{j, \infty}(n_4)$. Suppose that $\mu_1$ and $\mu_4$ are even, i.e., the partition with no parts. If $s(\mu_2) < s(\mu_3)$, then move $s(\mu_2)$ from $\mu_2$ to $\mu_3$; if $s(\mu_2) \geq s(\mu_3)$, then move $s(\mu_3)$ from $\mu_3$ to $\mu_2$. Suppose that either of $\mu_1$ or $\mu_4$ has a part. Compare $s(\mu_1)$ and $s(\mu_4)$. If $s(\mu_1) < s(\mu_4)$, then move $s(\mu_1)$ from $\mu_1$ to $\mu_4$; if $s(\mu_1) \geq s(\mu_4)$, then move $s(\mu_4)$ from $\mu_4$ to $\mu_1$. It is clear that the map is a sign reversing involution. Thus the lemma holds. \hfill $\Box$

\begin{equation}
\times \left( \bigcup_{0 \leq k_5, k_6, k_7, k_8 \leq n-j \atop k_5+k_6+k_7+k_8=n-j} \mathcal{D}_{0, \infty}(k_5) \times \mathcal{P}_{n-1, \infty}(k_6) \times \mathcal{D}_{j, \infty}(k_7) \times \mathcal{P}_{0, \infty}(k_8) \right) \tag{3.4}
\end{equation}

and $\mathcal{D}_{0,n} \times \mathcal{D}_{0,n} \times \mathcal{P}_{1,n+1}$ is equivalent to

\begin{equation}
\bigcup_{0 \leq l_1, l_2, l_3, l_4 \leq n} \mathcal{D}_{0, \infty}(l_1) \times \mathcal{P}_{n-1, \infty}(l_2) \times \mathcal{D}_{0, \infty}(l_3) \times \mathcal{P}_{0, \infty}(l_4). \tag{3.5}
\end{equation}

Note that for $(\sigma^1, \sigma^2, \sigma^3, \sigma^4, \sigma^5, \sigma^6) \in \mathcal{D}_{0,j} \times \mathcal{D}_{0,j} \times \mathcal{P}_{1,j+1} \times \mathcal{D}_{j,n} \times \mathcal{D}_{0,n-j} \times \mathcal{P}_{1,n-j+1}$,

$$
\ell(\sigma^1) = k_2 + k_3, \quad \ell(\sigma^2) = k_1 + k_2, \quad \ell(\sigma^4) = k_6 + k_7, \quad \ell(\sigma^5) = k_5 + k_6,
$$

and for $(v^1, v^2, v^3) \in \mathcal{D}_{0,n} \times \mathcal{D}_{0,n} \times \mathcal{P}_{1,n+1}$,

\begin{equation}
\ell(v^1) = l_2 + l_3, \quad \ell(v^2) = l_1 + l_2. \tag{3.6}
\end{equation}

Thus, (3.2) becomes

\begin{equation}
(-1)^{k_5+k_7}a^{k_2+k_3-k_5-k_6-j}b^{k_1+k_2-k_5-k_6-j}c^{k_5+2k_6+k_7+j}d^{\big|\sigma^1\big|+\big|\sigma^2\big|+\big|\sigma^3\big|+\big|\sigma^4\big|+\big|\sigma^5\big|+\big|\sigma^6\big|}. \tag{3.7}
\end{equation}

To prove (3.1), we take the union of the sets defined in (3.4) for $0 \leq j \leq n$ and define an involution on the union under which the invariant set is equivalent to the set in (3.5).

**Lemma 3.1.** Consider the set

\begin{equation}
\bigcup_{j=0}^{m} \bigcup_{0 \leq n_1, n_2 \leq j} \bigcup_{0 \leq n_3, n_4 \leq m-j} \mathcal{P}_{j-1, \infty}(n_1) \times \mathcal{P}_{0, \infty}(n_2) \times \mathcal{D}_{0, \infty}(n_3) \times \mathcal{D}_{j, \infty}(n_4), \tag{3.8}
\end{equation}

where $n_1 + n_2 \leq j$ and $n_3 + n_4 \leq m - j$, and define a weight $w$ by

$$
w(\mu_1, \mu_2, \mu_3, \mu_4) = (-1)^{\ell(\mu_3)+\ell(\mu_4)} q^{\big|\mu_1\big|+\big|\mu_2\big|+\big|\mu_3\big|+\big|\mu_4\big|}
$$

for $(\mu_1, \mu_2, \mu_3, \mu_4)$ in (3.8). Then

$$
\sum_{(\mu_1, \mu_2, \mu_3, \mu_4)} q^{w(\mu_1, \mu_2, \mu_3, \mu_4)} = 1,
$$

where $(\mu_1, \mu_2, \mu_3, \mu_4)$ is in (3.8).
We apply the involution defined in Lemma 3.1 to \((\pi^2, \pi^4, \pi^5, \pi^7)\), where \((\pi^1, \pi^2, \pi^3, \pi^4, \\
\pi^5, \pi^6, \pi^7, \pi^8)\) is a vector partition in

\[
\bigcup_{j=0}^{n} \left( \bigcup_{0 \leq k_1, k_2, k_3, k_4 \leq j} \mathcal{D}_{0,\infty}(k_1) \times \mathcal{P}_{j-1,\infty}(k_2) \times \mathcal{D}_{0,\infty}(k_3) \times \mathcal{P}_{0,\infty}(k_4) \right) \times \left( \bigcup_{0 \leq k_5, k_6, k_7, k_8 \leq n-j} \mathcal{D}_{0,\infty}(k_5) \times \mathcal{P}_{n-1,\infty}(k_6) \times \mathcal{D}_{j,\infty}(k_7) \times \mathcal{P}_{0,\infty}(k_8) \right).
\]

Thus, after cancellation, there remain the vector partitions in

\[
\bigcup_{j=0}^{n} \left( \bigcup_{0 \leq k_1, k_2, k_3, k_4 \leq j} \mathcal{D}_{0,\infty}(k_1) \times \emptyset \times \mathcal{D}_{0,\infty}(k_3) \times \emptyset \right) \times \left( \bigcup_{0 \leq k_5, k_6, k_7, k_8 \leq n-j} \emptyset \times \mathcal{P}_{n-1,\infty}(k_6) \times \emptyset \times \mathcal{P}_{0,\infty}(k_8) \right),
\]

which is equivalent to (3.5) by the correspondence from \(k_1, k_3, k_6, k_8\) to \(l_3, l_1, l_2, l_4\), respectively. By (3.7), the remaining vector partitions have the weight

\[
a^{k_1-k_6-j}b^{k_1-k_6-j}c^{2k_6+j}d^{j}q^{|\sigma^1|+|\sigma^2|+|\sigma^3|+|\sigma^4|+|\sigma^5|+|\sigma^6|},
\]

which is equal to

\[
a^{-k_1-k_6}b^{k_3-k_6}c^{k_1+k_3+2k_6}d^{j}q^{|\sigma^1|+|\sigma^2|+|\sigma^3|+|\sigma^4|+|\sigma^5|+|\sigma^6|},
\]

since \(k_1 + k_3 = j\). By (3.6) and the correspondence above, (3.9) is equal to (3.3). Therefore, the \(q\)-Pfaff-Saalschütz formula (3.1) has been proved.

4. The \(q\)-Sheppard \(3\phi_2\) transformation formula

A \(q\)-analogue of Sheppard’s transformations for a \(3\phi_2\) series [7] is given by

\[
\sum_{j=0}^{n} \frac{(a)_{j}(b)(q^{-n})_{j}}{(q)_{j}(c)(d)_{j}} \left( \frac{cdq^n}{ab} \right)^j = \frac{(d/a)_{n}}{(d)_{n}} \sum_{j=0}^{n} \frac{(a)_{j}(c/b)(q^{-n})_{j} q^j}{(q)_{j}(c)(aq^{1-n}/d)_{j}}.
\]

With substitutions of \(-a, -b\), and \(-c^{-1}q^{1-n}\) for \(a, b,\) and \(c\), respectively, and some manipulations, the formula above becomes

\[
\sum_{j=0}^{n} \frac{(-a)_{j}(-b)(dqj)(d)(q^{-n})_{j}(-c)(d)_{j}}{(q)_{j}(q^{-n})_{j}} \left( \frac{d}{ab} \right)^j = \sum_{j=0}^{n} \frac{(bcq^{n-j})_{j}(-a)(d/a)(d)_{j}}{(q)_{j}(q^{-n})_{j}} \left( \frac{d}{ab} \right)^j.
\]

(4.1)
Then the left side of (4.1) generates vector partitions \( (\sigma^1, \sigma^2, \sigma^3, \sigma^4, \sigma^5, \sigma^6) \) in \( \mathcal{D}_{0,j} \times \mathcal{D}_{j,n} \times \mathcal{P}_{1,j+1} \times \mathcal{D}_{j,n} \times \mathcal{P}_{1,n-j} \times \mathcal{P}_{1,n-j+1} \) for \( 0 \leq j \leq n \) with signed weight

\[
(-1)^{\ell(\sigma^4)} - \sum_{i=0}^{m} j \sum_{i=0}^{m-j} (-1)^i q^{\ell(\sigma^4) + i(i+1)/2}.
\] (4.2)

By Lemma 2.3, we see that \( \mathcal{D}_{0,j} \times \mathcal{D}_{0,j} \times \mathcal{P}_{1,j+1} \times \mathcal{D}_{j,n} \times \mathcal{D}_{0,n-j} \times \mathcal{P}_{1,n-j+1} \) is equivalent to

\[
\left( \bigcup_{0 \leq k_1, k_2, k_3, k_4 \leq j} \mathcal{D}_{0, \infty}(k_1) \times \mathcal{P}_{j-1, \infty}(k_2) \times \mathcal{D}_{0, \infty}(k_3) \times \mathcal{P}_{0, \infty}(k_4) \right) 
\times \left( \bigcup_{0 \leq k_5, k_6, k_7, k_8 \leq n-j} \mathcal{D}_{0, \infty}(k_5) \times \mathcal{P}_{n-1, \infty}(k_6) \times \mathcal{D}_{j, \infty}(k_7) \times \mathcal{P}_{0, \infty}(k_8) \right)
\]

and

\[
\ell(\sigma^1) = k_2 + k_3, \quad \ell(\sigma^2) = k_1 + k_2, \quad \ell(\sigma^4) = k_6 + k_7, \quad \ell(\sigma^5) = k_5 + k_6.
\] (4.3)

To prove (4.1), we take the union of the sets defined in (4.3) for \( 0 \leq j \leq n \) and define an involution on the union as in the following lemma, which is equivalent to

\[
\sum_{j=0}^{m} \frac{q^{j(j-1)}}{(q^j)} \sum_{i=0}^{m-j} (-1)^i q^{i(i+1)/2} = 1.
\]

**Lemma 4.1.** Consider the set

\[
\bigcup_{j=0}^{m} \bigcup_{0 \leq n_1 \leq j} \bigcup_{0 \leq n_2 \leq m-j} \mathcal{P}_{j-1, \infty}(n_1) \times \mathcal{D}_{j, \infty}(n_2),
\] (4.4)

and define a weight \( w \) by

\[
w(\mu_1, \mu_2) = (-1)^{\ell(\mu_2)} q^{\mu_1 + \mu_2}
\]

for \( (\mu_1, \mu_2) \) in (4.4). Then

\[
\sum_{(\mu_1, \mu_2)} q^{w(\mu_1, \mu_2)} = 1,
\]

where \( (\mu_1, \mu_2) \) is in (4.4).

**Proof.** In this proof, we denote the smallest part of a partition \( \pi \) by \( s(\pi) \). Given \( j \), let \( (\mu_1, \mu_2) \) be in \( \mathcal{P}_{j-1, \infty}(n_1) \times \mathcal{D}_{j, \infty}(n_2) \). If \( s(\mu_1) < s(\mu_2) \), then move \( s(\mu_1) \) from \( \mu_1 \) to \( \mu_2 \); if \( s(\mu_1) \geq s(\mu_2) \), then move \( s(\mu_2) \) from \( \mu_2 \) to \( \mu_1 \). It is clear that the map is a sign reversing involution. Thus, the lemma has been shown. □
We apply the involution defined in Lemma 4.1 to \((\pi^2, \pi^7)\), where \((\pi^1, \pi^2, \pi^3, \pi^4, \pi^5, \pi^6, \\pi^7, \pi^8)\) is a vector partition in

\[
\bigcup_{j=0}^{n} \left( \bigcup_{0 \leq k_1, k_2, k_3, k_4 \leq j \atop k_1 + k_2 + k_3 + k_4 = j} D_{0, \infty}(k_1) \times \mathcal{P}_{j, \infty}(k_2) \times D_{0, \infty}(k_3) \times \mathcal{P}_{0, \infty}(k_4) \right) \times \left( \bigcup_{0 \leq k_5, k_6, k_7, k_8 \leq n-j \atop k_5 + k_6 + k_7 + k_8 = n-j} D_{0, \infty}(k_5) \times \mathcal{P}_{n-1, \infty}(k_6) \times D_{j, \infty}(k_7) \times \mathcal{P}_{0, \infty}(k_8) \right).
\]

Thus, after cancellation, there remain the vector partitions in

\[
\bigcup_{j=0}^{n} \left( \bigcup_{0 \leq k_1, k_2, k_3, k_4 \leq j \atop k_1 + k_2 + k_3 + k_4 = j} D_{0, \infty}(k_1) \times \mathcal{P}_{j, \infty}(k_2) \times D_{0, \infty}(k_3) \times \mathcal{P}_{0, \infty}(k_4) \right) \times \left( \bigcup_{0 \leq k_5, k_6, k_7, k_8 \leq n-j \atop k_5 + k_6 + k_7 + k_8 = n-j} D_{0, \infty}(k_5) \times \mathcal{P}_{n-1, \infty}(k_6) \times \mathcal{P}_{0, \infty}(k_8) \right).
\]

By (4.2) and (4.3), the remaining vector partitions have the following weight:

\[
(-1)^{k_6} a - k_1 - k_4 b - k_3 - k_4 c - k_5 + k_6 d + k_1 + k_3 + k_4 q |\sigma^1 + |\sigma^2| + |\sigma^3| + |\sigma^4| + |\sigma^5| + |\sigma^6|,
\]

since \(k_1 + k_3 + k_4 = j\).

Meanwhile, the right side of (4.1) generates vector partitions \((\nu^1, \nu^2, \nu^3, \nu^4, \nu^5, \nu^6)\) in \(\mathcal{D}_{n-j,n} \times \mathcal{D}_{0,j} \times \mathcal{P}_{1,j+1} \times \mathcal{D}_{0,n-j} \times \mathcal{D}_{0,n-j} \times \mathcal{P}_{1,n-j+1}\) for \(0 \leq j \leq n\) with signed weight

\[
(-1)^\ell(\nu^1) a - \ell(\nu^2) - j b - \ell(\nu^3) - j c - \ell(\nu^4) + j d + \ell(\nu^5) + j q |\nu^1| + |\nu^2| + |\nu^3| + |\nu^4| + |\nu^5| + |\nu^6|.
\]

(4.5)

By Lemma 2.3, we see that \(\mathcal{D}_{n-j,n} \times \mathcal{D}_{0,j} \times \mathcal{P}_{1,j+1} \times \mathcal{D}_{0,n-j} \times \mathcal{D}_{0,n-j} \times \mathcal{P}_{1,n-j+1}\) is equivalent to

\[
\bigcup_{0 \leq l_1, l_2, l_3, l_4 \leq j \atop l_1 + l_2 + l_3 + l_4 = j} \mathcal{D}_{0, \infty}(l_1) \times \mathcal{P}_{n-1, \infty}(l_2) \times \mathcal{D}_{n-j, \infty}(l_3) \times \mathcal{P}_{0, \infty}(l_4) \times \bigcup_{0 \leq l_5, l_6, l_7, l_8 \leq n-j \atop l_5 + l_6 + l_7 + l_8 = n-j} \mathcal{D}_{0, \infty}(l_5) \times \mathcal{P}_{n-j-1, \infty}(l_6) \times \mathcal{D}_{j, \infty}(l_7) \times \mathcal{P}_{0, \infty}(l_8)
\]

and

\[
\ell(\nu^1) = l_2 + l_3, \quad \ell(\nu^2) = l_1 + l_2, \quad \ell(\nu^4) = l_6 + l_7, \quad \ell(\nu^5) = l_5 + l_6.
\]
We apply the involution defined in Lemma 4.1 to \((\pi^3, \pi^6)\), where \((\pi^1, \pi^2, \pi^3, \pi^4, \pi^5, \pi^6, \pi^7, \pi^8)\) is a vector partition in

\[
\bigcup_{j=0}^{n} \left( \bigcup_{\substack{0 \leq l_1, l_2, l_3, l_4 \leq j \\ l_1 + l_2 + l_3 + l_4 = j}} \mathcal{D}_{0, \infty}(l_1) \times \mathcal{P}_{n-1, \infty}(l_2) \times \mathcal{D}_{n-j, \infty}(l_3) \times \mathcal{P}_{0, \infty}(l_4) \right) \\
\times \left( \bigcup_{\substack{0 \leq l_5, l_7, l_8 \leq n-j \\ l_5 + l_7 + l_8 = n-j}} \mathcal{D}_{0, \infty}(l_5) \times \mathcal{P}_{n-j-1, \infty}(l_6) \times \mathcal{D}_{0, \infty}(l_7) \times \mathcal{P}_{0, \infty}(l_8) \right).
\]

Thus, after cancellation, left are the vector partitions in

\[
\bigcup_{j=0}^{n} \left( \bigcup_{\substack{0 \leq l_1, l_2, l_4 \leq j \\ l_1 + l_2 + l_4 = j}} \mathcal{D}_{0, \infty}(l_1) \times \mathcal{P}_{n-1, \infty}(l_2) \times \emptyset \times \mathcal{P}_{0, \infty}(l_4) \right) \\
\times \left( \bigcup_{\substack{0 \leq l_5, l_7, l_8 \leq n-j \\ l_5 + l_7 + l_8 = n-j}} \mathcal{D}_{0, \infty}(l_5) \times \emptyset \times \mathcal{D}_{0, \infty}(l_7) \times \mathcal{P}_{0, \infty}(l_8) \right).
\]

By (4.5) and (4.6), the remaining vector partitions have the weight

\[(-1)^{l_1}b^{-l_4-l_7}c^{-l_1-l_4}d^{l_2+l_5}e^{l_1+l_2+l_4}q^{|v^1|+|v^2|+|v^3|+|v^4|+|v^5|+|v^6|},\]

since \(l_1 + l_2 + l_4 = j\). Identifying \(k_1, k_3, k_4, k_5, k_6, k_8\) with \(l_7, l_1, l_4, l_5, l_2, l_8\), respectively, shows that the sets of the remaining vector partitions from both sides of (4.1) are equal. Thus (4.1) has been proved.

5. Remarks

In this paper, two special transformations in basic hypergeometric series have been studied combinatorially. In the future research, however, we shall seek more general and common constructions to discover simple combinatorial proofs for further identities.

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References


