A Lower Bound for the Smallest Singular Value of a Matrix

J. M. Varah
Computer Science Department,
University of British Columbia,
Vancouver, B. C., Canada

Recommended by H. Schneider

ABSTRACT

For a matrix A which is diagonally dominant both by rows and by columns, we give bounds for $\|A^{-1}\|_1$ and $\|A^{-1}\|_\infty$, which then can be used to give a lower bound for the smallest singular value. We also show that these bounds can be attained, and show how the result can be extended to block matrices.

Very often in numerical analysis, one needs a bound for the condition number of a square $n \times n$ matrix A, $\kappa(A) = \|A\| \cdot \|A^{-1}\|$, for some norm. Bounding $\|A\|$ is not usually difficult, but a bound for $\|A^{-1}\|$ is not usually available in any norm unless A^{-1} is known explicitly. For the l_2 norm, this is equivalent to finding a lower bound for $\sigma_n(A)$; here we denote the singular values of A by $\sigma_1(A) \geqslant \sigma_2(A) \geqslant \cdots \geqslant \sigma_n(A) \geqslant 0$.

However, if A is diagonally dominant by rows (i.e., $|a_{kk}| > \Sigma_{j \neq k} |a_{kj}|$, $1 \leqslant k \leqslant n$), we can bound the l_∞ norm of A^{-1} quite easily by the following theorem, the proof of which is a generalization of a proof of Keller [2, p. 77].

THEOREM 1. *Assume A is diagonally dominant by rows and set $\alpha = \min_k (|a_{kk}| - \Sigma_{j \neq k} |a_{kj}|)$. Then $\|A^{-1}\|_\infty < 1/\alpha$.*

Proof. Since

$$\|A^{-1}\|_\infty^{-1} = \inf_x \frac{\|Ax\|_\infty}{\|x\|_\infty},$$

we need only show that $\alpha \|x\|_\infty \leqslant \|Ax\|_\infty$ for all x. Take some vector x and let

$|x_k| = \|x\|_\infty$. Then

$$0 < \alpha \leqslant |a_{kk}| - \sum_{j \neq k} |a_{kj}|$$

$$0 < \alpha |x_k| \leqslant |a_{kk} x_k| - \sum_{j \neq k} |a_{kj}|\,|x_j|$$

$$\leqslant |a_{kk} x_k| - |\sum_{j \neq k} a_{kj} x_j|$$

$$\leqslant |\sum_j a_{kj} x_j| \leqslant \max_k |\sum_j a_{kj} x_j| = \|Ax\|_\infty.$$

Corollary 1. *If A is diagonally dominant by columns, and*

$$\beta = \min_k \left(|a_{kk}| - \sum_{j \neq k} |a_{jk}| \right),$$

then $\|A^{-1}\|_1 \leqslant 1/\beta$.

Corollary 2. *If A is diagonally dominant both by rows and by columns,*

$$\|A^{-1}\|_2^{-1} = \sigma_n(A) \geqslant \sqrt{\alpha\beta} \, .$$

Proof. Immediate, using the inequality $\|B\|_2^2 \leqslant \|B\|_1 \|B\|_\infty$.

Notice that for A a real symmetric matrix, this bound on σ_n is the same as that obtained by the Gerschgorin disks for the smallest eigenvalue of A. Also notice that the bound is independent of n, and is attained (asymptotically) for the matrix

$$\begin{pmatrix} 1 & x & & & \\ & 1 & x & & \\ & & 1 & x & \\ & & & \ddots & \ddots \end{pmatrix}_{n \times n}$$

for $|x| < 1$, as $n \to \infty$.

Now assume A is partitioned into blocks A_{ij} with the diagonal blocks square. Then A is *block* diagonally dominant by rows if for all k, $\|A_{kk}^{-1}\|_\infty^{-1} > \sum_{j \neq k} \|A_{kj}\|_\infty$. See, for example, Feingold and Varga [1].

Theorem 2. *Assume A is block diagonally dominant by rows, and set $\alpha = \min_k (\|A_{kk}^{-1}\|_\infty^{-1} - \sum_{j \neq k} \|A_{kj}\|_\infty)$. Then $\|A^{-1}\|_\infty \leqslant 1/\alpha$.*

The proof follows that of Theorem 1, replacing $|a_{kj}|$ by $\|A_{kj}\|_\infty$ and $|a_{kk}|$ by $\|A_{kk}^{-1}\|_\infty^{-1}$ and using the fact that

$$\|A_{kk}^{-1}\|_\infty^{-1} = \inf_{y_k} \frac{\|A_{kk}y_k\|_\infty}{\|y_k\|_\infty}.$$

COROLLARY 3. *Again, if* $\beta = \min(\|A_{kk}^{-1}\|_\infty^{-1} - \Sigma_{j\neq k}\|A_{jk}\|_\infty) > 0$, $\sigma_n(A) \geqslant \sqrt{\alpha\beta}$.

To show how we can again get equality, consider the block Toeplitz matrix

$$A = \begin{pmatrix} B & C & & \\ C^T & B & C & \\ & \ddots & \ddots & \ddots \end{pmatrix}$$

with $B = B^T$. For n large, the eigenvalues (and singular values) approach those of the infinite matrix, namely $\lambda(B + Ce^{i\theta} + C^T e^{-i\theta})$, $0 \leqslant \theta \leqslant 2\pi$. Thus

$$\sigma_n(A) \to \min_\theta \lambda[B + (C + C^T)\cos\theta + i(C - C^T)\sin\theta],$$

which for $C = cI$ gives

$$\sigma_n(A) \to \min\lambda(B) - 2c = \|B^{-1}\|_2^{-1} - 2c.$$

Since $\alpha = \|B^{-1}\|_\infty^{-1} - 2\|C\|_\infty$, we have $\sigma_n(A) \to \alpha$ as $n \to \infty$ if $\|B^{-1}\|_2 = \|B^{-1}\|_\infty$ and $C = cI$.

REFERENCES

1 D. Feingold and R. S. Varga, Block diagonally dominant matrices and generalizations of the Gerschgorin circle theorem, *Pac. J. Math.* **12** (1962), 1241–1250.

2 H. B. Keller, *Numerical Methods for Two-Point Boundary Value Problems*, Blaisdell, Waltham, Mass., 1968.

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