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On the notion of canonical dimension for algebraic groups

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Abstract

We define and study a numerical invariant of an algebraic group action which we call the canonical dimension. We then apply the resulting theory to the problem of computing the minimal number of parameters required to define a generic hypersurface of degree d in $\mathbb{P}^{n-1}$.

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Keywords: Algebraic group; G-variety; Generic splitting field; Essential dimension; Canonical dimension; Homogeneous forms

Contents

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1. Introduction

Many important objects in algebra can be parametrized by a non-abelian cohomology set of the form $H^1(K, G)$, where K is a field and G is a linear algebraic group defined over K. For example, elements of $H^1(K, \mathrm{O}_n)$ can be identified with isomorphism classes of n-dimensional quadratic forms over K, elements of $H^1(K, \mathrm{PGL}_n)$ with isomorphism classes of central simple algebras of degree n, elements of $H^1(K, G_2)$ with isomorphism classes of octonion algebras, etc.; cf. [Se$_3$] or [KMRT]. Recall that $H^1(K, G)$ has a marked (split) element but usually no group structure. Thus, a priori there are only two types of elements in $H^1(K, G)$, split and non-split. However, it is often intuitively clear that some non-split elements are closer to being split than others. This intuitive notion can be quantified by considering degrees or Galois groups of splitting field extensions L/K for α; see, e.g., [T,RY$_2$]. Another "measure" of how far α is from being split is its essential dimension (here and in the sequel we assume that K is a finitely generated extension of an algebraically closed base field of characteristic zero, and G is defined over k); for details and further references, see Section 2.1 and the first paragraph of Section 14.

In this paper, we introduce and study yet another numerical invariant that "measures" how far α is from being split. We call this new invariant the *canonical dimension* and denote it by $\mathrm{cd}(\alpha)$. We give several equivalent descriptions of $\mathrm{cd}(\alpha)$; one of them is that $\mathrm{cd}(\alpha) = \min \operatorname{tr}\deg_K(L)$, where the minimum is taken over all generic splitting fields L/K for α (see Section 9). Generic splitting fields have been the object of much research in the context of central simple algebras (i.e., for $G = \mathrm{PGL}_n$; see, e.g., [Am,Ar,Roq$_1$,Roq$_2$]) and quadratic forms (i.e., for $G = \mathrm{O}_n$ or SO_n; see, e.g., [Kn$_1$,Kn$_2$,KS]); related results for Jordan pairs can be found in [Pe]. Kersten and Rehmann [KR], who, following on the work of Knebusch, studied generic splitting fields in a setting rather similar to ours (cf. Remark 9.5), remarked, on p. 61, that the question of determining the minimal possible transcendence degree of a generic splitting field (or $\mathrm{cd}(\alpha)$, in our language) appears to be difficult in general. Much of this paper may be viewed as an attempt to address this question from a geometric point of view.

Recall that we are assuming k to be an algebraically closed base field of characteristic zero, and K/k to be a finitely generated field extension. In this context every $\alpha \in H^1(K, G)$ is represented by a (unique, up to birational isomorphism) generically free G-variety X, with $k(X)^G = K$; see e.g., [Po, (1.3.3)]. We will often work with X, rather than α, writing $\mathrm{cd}(X, G)$ instead of $\mathrm{cd}(\alpha)$ and using the language of invariant theory, rather than Galois cohomology. An advantage of this approach is that $\mathrm{cd}(X, G)$ is well defined for G-varieties X that are not necessarily generically free (see Definition 3.5), and the interplay between generically free and non-generically free varieties can sometimes be used to gain insight into their canonical dimensions; cf., e.g., Lemma 6.1. If S is the stabilizer in general position for a G-variety X, then $\mathrm{cd}(X, G)$ can be related to the essential dimension of S. This connection is explored in Sections 5 and 6.

In Sections 7–13 we study canonical dimensions of generically free G-varieties or, equivalently, of classes $\alpha \in H^1(K, G)$. We will be particularly interested in the maximal possible value of $\mathrm{cd}(\alpha)$ for a given group G; we call this number *the canonical dimension of* G and denote it by $\mathrm{cd}(G)$. The canonical dimension $\mathrm{cd}(G)$, like the essential dimension $\mathrm{ed}(G)$, is a numerical invariant of G; if G is connected, both measure, in different ways, how far G is from being "special" (for the definition and a brief discussion of special groups, see Section 2.6 below). While $\mathrm{cd}(G)$ and $\mathrm{ed}(G)$ share some common properties (note, in particular, the similarity between the results of Section 7 in this paper and those of [R, Sections 3.1, 3.2]), their numerical values do not appear to be related to each other. For example, since $\mathrm{cd}(G) = 0$ for every finite group G (see Lemma 7.5(b)), the rich theory of essential dimension for finite groups (see [BR,BR$_2$,JLY, Section 8]) has no counterpart in the setting of canonical dimension. On the other hand, our classification of simple groups of canonical dimension 1 in Section 13 has no counterpart in the context of essential dimension, because connected groups of essential dimension 1 do not exist; see [R, Corollary 5.7].

In Section 8 we prove a strong necessary condition for $\alpha \in H^1(K, G)$ to be of canonical dimension $\leqslant 2$. A key ingredient in our proof is the Enriques–Manin–Iskovskih classification of minimal models for rational surfaces; see the proof of Proposition 8.2. In Sections 11 and 12 we study canonical dimensions of the groups GL_n/μ_d, SL_n/μ_e, SO_n and Spin_n. Our arguments there heavily rely on the recent results of Karpenko and Merkurjev [K$_1$,KM,M$_2$].

Our definition of canonical dimension naturally extends to the setting of functors $\mathcal{F}$ from the category of field extensions of k to the category of pointed sets; $\mathrm{cd}(G)$ is then a special case of $\mathrm{cd}(\mathcal{F})$, with $\mathcal{F} = H^1(\_, G)$ (see Section 10). A similar notion in the context of essential dimension is due to Merkurjev [M1]; see also [BF2] and the beginning of Section 14.

In Sections 14–16 we apply our results on canonical dimension to the problem of computing the minimal number $\mathrm{ed}\,[H_{n,d}]$ of independent parameters, required to define the general degree d hypersurface in $\mathbb{P}^{n-1}$. (For a precise statement of the problem, see Section 14.) We show that if $d \geqslant 3$ and $(n, d) \neq (2, 3), (2, 4)$ or $(3, 3)$, our problem reduces to that of computing the canonical dimension of the group $\mathrm{SL}_n/\mu_{\gcd(n,d)}$. In particular, combining Theorem 15.1 with Corollary 11.5, we obtain following theorem:

1.1. Theorem. *Let n and d be positive integers such that $d \geqslant 3$ and $(n, d) \neq (2, 3)$, $(2, 4)$ or $(3, 3)$. Suppose $\gcd(n, d)$ is a prime power p^j for some $j \geqslant 0$. Then*

$$\mathrm{ed}(H_{n,d}) = \binom{n+d-1}{d} - n^2 + \begin{cases} 0 & \text{if } j = 0, \\ p^i - 1 & \text{if } j \geqslant 1 \end{cases}$$

where p^i is the highest power of p dividing n.

If $d \leqslant 2$ or $(n, d) = (2, 3), (2, 4), (3, 3)$, then our problem reduces to computing canonical dimensions for certain group actions that are not generically free; this is done in Section 16. Related results for $(n, d) = (2, 3)$ and $(3, 3)$ can be found in [BF1]; cf. Remark 14.8.

2. Notation and preliminaries

Throughout this paper we will work over an algebraically closed base field k of characteristic zero. Unless otherwise specified, all algebraic varieties, algebraic groups, group actions, fields and all maps between them are assumed to be defined over k, all algebraic groups are assumed to be linear (but not necessarily connected), and all fields are assumed to be finitely generated over k.

By a G-variety we shall mean an algebraic variety X with a (regular) action of an algebraic group G. We will usually assume that X is irreducible and focus on properties of X that are preserved by (G-equivariant) birational isomorphisms. In particular, we will call a subgroup $S \subset G$ *a stabilizer in general position* for X if $\mathrm{Stab}(x)$ is conjugate to S for $x \in X$ in general position; cf. [PV, Section 7]. As usual, if $S = \{1\}$, i.e., G acts freely on a dense open subset of X, then we will say that the G-variety X (or equivalently, the G-action on X) is *generically free*.

2.1. Essential dimension

Let X be a generically free G-variety. The essential dimension $\mathrm{ed}(X, G)$ of X is the minimal value of $\dim(Y) - \dim(G)$, where the minimum is taken over all dominant rational maps $X \dashrightarrow Y$ of G-varieties with Y generically free. For a given algebraic group G, $\mathrm{ed}(X, G)$ attains its maximal value in the case where $X = V$ is a (generically free) linear representation of G. This value is called the essential dimension of G and is denoted by $\mathrm{ed}(G)$ (it is independent of the choice of V). For details, see [R, Section 3].

2.2. Rational quotients

A rational quotient for a G-variety X is an algebraic variety Y such that $k(Y) = k(X)^G$. The inclusion $k(Y) \hookrightarrow k(X)$ then induces a rational quotient map $\pi\colon X \dashrightarrow Y$. Note that Y and π are only defined up to birational isomorphism; one usually writes X/G in place of Y. We shall say that *G-orbits in X are separated by regular invariants* if π is a regular map and $\pi^{-1}(y)$ is a single G-orbit for every k-point $y \in Y$. By a theorem of Rosenlicht, X has a G-invariant dense open subset U, where G-orbits are separated by regular invariants. For a detailed discussion of the rational quotient and Rosenlicht's theorem, see [PV, Section 2.4].

2.3. Generically free actions and Galois cohomology

Let X be a generically free variety. Then X may be viewed as a torsor over the generic point of X/G via the rational quotient map $X \dashrightarrow X/G$. Let α be the class of this torsor in $H^1(K, G)$, where $K = k(X)^G$. This class is explicitly constructed in [Po, (1.3.1)]. Moreover, every $\alpha \in H^1(K, G)$ can be obtained in this way, and the G-variety X can be uniquely reconstructed from α, up to a (G-equivariant) birational isomorphism; see [Po, (1.3.2) and (1.3.3)]. In the sequel we shall say that $\alpha \in H^1(K, G)$ *represents* the generically free G-variety X.

2.4. Split generically free varieties

Let X be a generically free G-variety, where the G-orbits are separated by regular invariants. We will call a rational map $s\colon X/G \dashrightarrow X$ *a rational section* for π if $s \circ \pi = \mathrm{id}$ on X/G. (Note that since the fibers of π are precisely the G-orbits in X, $G \cdot s(X/G)$ is dense in X. Consequently, some translate of s will "survive'' if X is replaced by a birationally equivalent G-variety.) We shall say that X is *split* if one of the following equivalent conditions holds:

(i) X is birationally isomorphic to $G \times X/G$,
(ii) π has a rational section,
(iii) X represents the trivial class in $H^1(K, G)$,
(iv) $\mathrm{ed}(X, G) = 0$.

For a proof of equivalence of these four conditions, see [Po, (1.4.1), R, Lemma 5.2].

2.5. The groups GL_n/μ_d *and* SL_n/μ_e

In this section we will review known results about the Galois cohomology sets $H^1(K,G)$, where $G=\mathrm{GL}_n/\mu_d$ or SL_n/μ_e, μ_d is the unique central cyclic subgroup of GL_n of order d, and e divides n.

2.6. Lemma. *Let* $G=\mathrm{GL}_n/\mu_d$ (*respectively*, $G=\mathrm{SL}_n/\mu_e$), $f\colon G\longrightarrow \mathrm{PGL}_n$ *be the canonical projection*, *and* K/k *be a field extension. Then*

(a) *The map* $f_*\colon H^1(K,G)\to H^1(K,\mathrm{PGL}_n)$ *has trivial kernel.*
(b) *The image of* f_* *consists of those classes which represent central simple algebras of degree* n *and exponent dividing* d (*respectively*, *dividing* e).

Lemma 2.6 can be deduced from [Sal$_1$, Theorem 3.2]; for the sake of completeness, we supply a direct proof below.

Proof. (a) The exact sequence $1\longrightarrow \mathrm{Ker}(f)\overset{i}{\hookrightarrow} G\overset{f}{\longrightarrow}\mathrm{PGL}_n\longrightarrow 1$ of algebraic groups gives rise to an exact sequence

$$H^1(K,\mathrm{Ker}(f))\overset{i_*}{\longrightarrow}H^1(K,G)\overset{f_*}{\longrightarrow}H^1(K,\mathrm{PGL}_n)$$

of pointed sets; cf. [Se$_2$, pp. 123–126]. It is thus enough to show that i_* is the trivial map (i.e., its image is $\{1\}$). If $G=\mathrm{GL}_n/\mu_d$ this is an immediate consequence of the fact that $\mathrm{Ker}(f)=\mathbb{G}_m/\mu_d$ is isomorphic to $\mathbb{G}_m$ and thus $H^1(K,\mathrm{Ker}(f))=\{1\}$. If $G=\mathrm{SL}_n/\mu_e$ then $\mathrm{Ker}(f)=\mu_{\frac{n}{e}}$, and the commutative diagram

$$\begin{tikzcd} 1 \arrow[r] & \mu_n \arrow[r] \arrow[d, "\times e"] & \mathrm{SL}_n \arrow[r] \arrow[d] & \mathrm{PGL}_n \arrow[r] \arrow[d, equal] & 1 \\ 1 \arrow[r] & \mu_{\frac{n}{e}} \arrow[r, "i"] & \mathrm{SL}_n/\mu_e \arrow[r] & \mathrm{PGL}_n \arrow[r] & 1, \end{tikzcd} \tag{2.7}$$

of group homomorphisms induces the commutative diagram

$$\begin{tikzcd} H^1(K,\mu_n) \arrow[r] \arrow[d] & H^1(K,\mathrm{SL}_n) \arrow[d] \\ H^1(K,\mu_{\frac{n}{e}}) \arrow[r, "i_*"] & H^1(K,\mathrm{SL}_n/\mu_e) \end{tikzcd}$$

of maps of pointed sets. Since the left vertical map is surjective (it is the natural projection $K^\times/(K^\times)^n \longrightarrow K^\times/(K^\times)^{\frac{n}{e}}$), and $H^1(K, \mathrm{SL}_n) = \{1\}$ (see [Se$_2$, p. 151]), we see that the image of i_* is trivial, as claimed.

(b) We will assume $G = \mathrm{SL}_n/\mu_e$; the case $G = \mathrm{GL}_n/\mu_d$ is similar and will be left to the reader. We now focus on the connecting maps

$$\begin{tikzcd} & H^1(K, \mathrm{PGL}_n) \arrow[r, "\delta"] \arrow[d, equal] & H^2(K, \mu_n) \arrow[d, "\times e"] \\ H^1(K, \mathrm{SL}_n/\mu_e) \arrow[r, "f_*"] & H^1(K, \mathrm{PGL}_n) \arrow[r, "\delta'"] & H^2(K, \mu_{\frac{n}{e}}) \end{tikzcd}$$

induced by diagram (2.7). It is well known that $H^2(K, \mu_n)$ is the n-torsion part of the Brauer group of K and δ sends a central simple algebra A to its Brauer class $[A]$; see [Se$_2$, Section X.5]. Hence, $\delta'(A) = e \cdot [A]$, and $\mathrm{Im}(f_*) = \mathrm{Ker}(\delta')$ consists of algebras A of degree n and exponent dividing e, as claimed. □

2.6. Special groups

An algebraic group G is called *special* if $H^1(E, G) = \{1\}$ for every field extension E/k. Equivalently, G is special if every generically free G-variety is split. Special groups were introduced by Serre [Se$_1$] and classified by Grothendieck [Gro, Theorem 3] as follows: G is special if and only if its maximal semisimple subgroup is a direct product of simply connected groups of type SL or Sp; cf. also [PV, Theorem 2.8]. The following lemma can be easily deduced from Grothendieck's classification; we will instead give a proof based on Lemma 2.6.

2.9. Lemma. GL_n/μ_d *is special if and only if* $\gcd(n, d) = 1$.

Proof. If n and d are relatively prime then every central simple algebra of degree d and exponent dividing n is split. By Lemma 2.6, f_* has trivial image and trivial kernel, showing that $H^1(E, \mathrm{GL}_n/\mu_d) = \{1\}$ for every E, i.e., GL_n/μ_d is special. Conversely, suppose $e = \gcd(n, d) > 1$. Let $E = k(a, b)$, where a and b are algebraically independent variables over k, and $D = (a, b)_e$ = generic symbol algebra of degree e. Then $A = M_{\frac{n}{e}}(D)$ is a central simple algebra of degree n and exponent e, with center E. This algebra defines a class in $H^1(E, \mathrm{PGL}_n)$; since e divides d, Lemma 2.6 tells us that this class is the image of some $\alpha \in H^1(E, \mathrm{GL}_n/\mu_d)$. Since A is not split, $\alpha \neq 1$, and hence GL_n/μ_d is not special, as claimed. □

3. The canonical dimension of a G-variety

3.1. Definition. Let X be an irreducible G-variety (not necessarily generically free). We shall say that a rational map $F\colon X \dashrightarrow X$ is a *canonical form* map if $F(x) = f(x) \cdot x$

for some rational map $f: X \dashrightarrow G$. Here we think of $F(x)$ as a "canonical form'' of x. Note that F and f will usually not be G-equivariant.

3.2. Remark. If the G-action on X is generically free and $F: X \dashrightarrow X$ is a rational map then the following conditions are equivalent:

(a) F is a canonical form map,
(b) $F(x) \in G \cdot x$ for $x \in X$ in general position,
(c) $\pi \circ F = \pi$, where $\pi: X \dashrightarrow X/G$ is the rational quotient map.

The equivalence of (a) and (b) follows from the fact that the rational quotient map $\pi: X \dashrightarrow X/G$ is a G-torsor over a dense open subset of X/G; cf. Section 2.3. The equivalence of (b) and (c) is a consequence of the theorem of Rosenlicht mentioned in Section 2.2.

3.3. Remark. If the G-action on X is generically free then the argument we used to prove that (a) $\Leftrightarrow$ (b) in Remark 3.2 also shows that the rational map $f: X \dashrightarrow G$ in Definition 3.1 is uniquely determined by F. On the other hand, if the G-action on X is not generically free then this may no longer be the case. For example, if the G-action on X is trivial then every $f: X \dashrightarrow G$ gives rise to the trivial canonical form map $F = \mathrm{id}_X: X \dashrightarrow X$. This means that in working with canonical form maps, we will want to keep track of both F and f. In the sequel we will often say that $f: X \dashrightarrow G$ *induces* $F: X \dashrightarrow X$ if $F(x) = f(x) \cdot x$ for $x \in X$ in general position.

3.4. Example. Let $X = \mathrm{M}_n$, with the conjugation action of $G = \mathrm{GL}_n$. We claim that the rational map $F: \mathrm{M}_n \dashrightarrow \mathrm{M}_n$, taking A to its companion matrix

$$F(A) = \begin{pmatrix} 0 & 0 & \dots & 0 & -c_n \\ 1 & 0 & \dots & 0 & -c_{n-1} \\ 0 & 1 & \dots & 0 & -c_{n-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & -c_1 \end{pmatrix},$$

is a canonical form map. Here $t^n + c_1 t^{n-1} + \cdots + c_n = \det(tI - A)$ is the characteristic polynomial of A.

To prove the claim, fix a non-zero column vector $v \in k^n$ and define $f(A)$ as the matrix whose columns are $v, Av, \dots, A^{n-1}v$. It is now easy to see that $f: A \mapsto f(A)$ is a rational map $\mathrm{M}_n \dashrightarrow \mathrm{GL}_n$, and $f(A) \cdot A = f(A)^{-1} A f(A)$ is the companion matrix $F(A)$.

Our definition of a canonical form map is quite general; for example it includes the trivial case, where $f(x) = 1_G$ and thus $F(x) = x$ for every $x \in X$. Usually we would like to choose f so that the canonical form of every element lies in some subvariety of X of small dimension. With this in mind, we give the following:

3.5. Definition. The canonical dimension $\mathrm{cd}(X, G)$ of a G-variety X is defined as

$$\mathrm{cd}(X, G) = \min \{\dim F(X) - \dim(X/G)\},$$

where the minimum is taken over all canonical form maps $F\colon X \dashrightarrow X$. If the G-action on X is generically free, and X represents $\alpha \in H^1(K, G)$ (see Section 2.3), we will also write $\mathrm{cd}(\alpha)$ in place of $\mathrm{cd}(X, G)$.

Note that the symbol cd does not stand for and should not be confused with cohomological dimension.

3.6. Lemma. *The integer* $\mathrm{cd}(X, G)$ *is the minimal value of* $\dim F(G \cdot x)$ *for* $x \in X$ *in general position. Here the minimum is taken over all canonical form maps* $F\colon X \dashrightarrow X$.

Proof. Let $\pi\colon X \dashrightarrow X/G$ be the rational quotient map for the G-action on X. Then for any canonical form map $F\colon X \dashrightarrow X$, we have $\pi = \pi \circ F$. In particular, $\pi F(X)$ is dense in X/G. Applying the fiber dimension theorem to

$$\pi_{|F(X)}\colon F(X) \dashrightarrow X/G,$$

we see that

$$\dim F(X) - \dim X/G = \dim \overline{F(X)} \cap G \cdot x = \dim F(G \cdot x)$$

for $x \in X$ in general position. By Definition 3.5, $\mathrm{cd}(X, G)$ is the minimal value of this quantity, as F ranges over all canonical form maps $F\colon X \dashrightarrow X$. $\square$

3.7. Example. Let $X = \mathrm{M}_n$, with the conjugation action of $G = \mathrm{GL}_n$. Then the canonical form map F constructed in Example 3.4 takes every orbit in X to a single point. This shows that $\mathrm{cd}(\mathrm{M}_n, \mathrm{GL}_n) = 0$. The same argument shows that $\mathrm{cd}(\mathrm{M}_n, \mathrm{PGL}_n) = 0$; cf. also Lemma 4.10 below.

4. First properties

4.1. Subgroups

4.2. Lemma. *If X is a G-variety and H is a closed subgroup of G then*

$$\mathrm{cd}(X, G) + \dim X/G \leqslant \mathrm{cd}(X, H) + \dim X/H.$$

Proof. The left-hand side is the minimal value of dim $F(X)$, as F ranges over canonical form maps $F\colon X\dashrightarrow X$ induced by $f\colon X\dashrightarrow G$. The right-hand side is the same, except that f is only allowed to range over rational maps $X\dashrightarrow H$. Since there are more rational maps from X to G than from X to H, the inequality follows. $\square$

4.2. Connected components

4.4. Lemma. *Let X be a G-variety and let G^0 be the connected component of G. Then* $\mathrm{cd}(X, G^0) = \mathrm{cd}(X, G)$.

Proof. The inequality $\mathrm{cd}(X, G) \leqslant \mathrm{cd}(X, G^0)$ follows from Lemma 4.2, with $H = G^0$. To prove the opposite inequality, let $F\colon X\dashrightarrow X$ be a canonical form map such that $\dim F(G \cdot x) = \mathrm{cd}(X, G)$ for $x \in X$ in general position. Suppose F is induced by a rational map $f\colon X\dashrightarrow G$, as in Definition 3.1. Since X is irreducible, the image of f lies in some irreducible component of G. Let g be an element of this component. Then we can replace f by $f'\colon X\dashrightarrow G^0$, where $f'(x) = g^{-1}f(x)$, and F by $F'\colon X\dashrightarrow X$ given by $F'(x) = f'(x)\cdot x = g^{-1}\cdot F(x)$. (Note that here g is independent of x.) Since $F'(G\cdot x)$ is a translate of $F(G\cdot x)$, we conclude that $\mathrm{cd}(X, G^0) \leqslant \dim F'(G^0\cdot x) \leqslant \dim F'(G\cdot x) = \dim F(G \cdot x) = \mathrm{cd}(X, G)$. $\square$

4.3. Direct products

4.6. Lemma. *Let X_i be a G_i-variety for $i = 1, 2$, $G = G_1 \times G_2$ and $X = X_1 \times X_2$. Then* $\mathrm{cd}(X, G) \leqslant \mathrm{cd}(X_1, G_1) + \mathrm{cd}(X_2, G_2)$.

Proof. If $F_i\colon X_i\dashrightarrow X_i$ are canonical form maps induced by $f_i\colon X_i\dashrightarrow G_i$ (for $i = 1, 2$) then $F = (F_1, F_2)\colon X\dashrightarrow X$ is a canonical form map induced by $f = (f_1, f_2)\colon X = X_1 \times X_2\dashrightarrow G_1 \times G_2$. Clearly,

$$F(G \cdot x) = F_1(G_1 \cdot x_1) \times F_2(G_2 \cdot x_2)$$

for any $x = (x_1, x_2)$ and thus $\dim(F \cdot x) = \dim(F_1 \cdot x_1) + \dim(F_2 \cdot x_2)$. The desired inequality now follows from Lemma 3.6. $\square$

4.4. Split varieties

4.8. Lemma. *Let X be a generically free G-variety and let $\pi\colon X\dashrightarrow X/G$ be the rational quotient map.*

(a) *If X is split* (*cf.* Section 2.4) *then* $\mathrm{cd}(X, G) = 0$.
(b) *Suppose G is connected. Then the converse to part* (*a*) *holds as well.*

Proof. (a) Since X is split, we may assume $X = G \times X_0$, where $X_0 = X/G$; see Section 2.4(i). The map $F\colon X \longrightarrow X$, given by $F\colon (g, x_0) \mapsto (1_G, x_0)$ is clearly a canonical form map (see Remark 3.2), with $\dim F(X) = \dim(X/G)$, and the desired equality follows.

(b) After replacing X be a G-invariant dense open subset, we may assume that the G-orbits in X are separated by regular invariants. Suppose $\mathrm{cd}(X, G) = 0$, i.e., $\dim F(X) = \dim(X/G)$ for some canonical form map $F: X \dashrightarrow X$. It is enough to show that $\pi_{|F(X)}: F(X) \dashrightarrow X/G$ is a birational isomorphism. Indeed, if we can prove this then

$$\pi^{-1}_{|F(X)}: X/G \overset{\simeq}{\dashrightarrow} F(X) \hookrightarrow X$$

will be a rational section (as defined in Section 2.4).

To prove that $\pi_{|F(X)}$ is a birational isomorphism, consider the fibers of this map. If $x \in X$ is a point in general position and $y = \pi(x) \in X/G$ then $\pi^{-1}_{|F(X)}(y) = F(G \cdot x)$. Since G is connected, $G \cdot x$ is irreducible, and so is $F(G \cdot x)$. On the other hand, since $\mathrm{cd}(X, G) = 0$, $\pi^{-1}_{|F(X)}(y)$ is zero-dimensional. We thus conclude that $\pi^{-1}_{|F(X)}(y)$ is a single k-point for $y \in X/G$ in general position. Hence, $\pi_{|F(X)}$ is a birational isomorphism (cf., e.g., [Hu, Section I.4.6]), and the proof is complete. □

4.5. Normal subgroups

4.10. Lemma. *Let* $\alpha: G \longrightarrow \overline{G}$ *be a surjective map of algebraic groups and* $H = \mathrm{Ker}(\alpha)$. *Suppose* X *is a* $\overline{G}$*-variety or, equivalently, a* G*-variety with* H *acting trivially. Then*

(a) $\mathrm{cd}(X, G) \geqslant \mathrm{cd}(X, \overline{G})$.
(b) *If* H *is special then* $\mathrm{cd}(X, G) = \mathrm{cd}(X, \overline{G})$.

Proof. Part (a) follows from Definition 3.5, because $f: X \dashrightarrow G$ and

$$\overline{f} = f \pmod{H} : X \dashrightarrow \overline{G}$$

give rise to the same canonical form map $F: X \dashrightarrow X$.

(b) Reversing the argument of part (a), it suffices to show that every rational map $\overline{f}: X \dashrightarrow \overline{G}$ can be lifted to $f: X \dashrightarrow G$.

Since $\alpha: G \longrightarrow \overline{G}$ separates the orbits for the right H-action on G, it is a rational quotient map for this action: cf. [PV, Lemma 2.1]. If H is special then α has a rational section $\beta: \overline{G} \dashrightarrow G$. (Note that β is a rational map of varieties but not necessarily a group homomorphism.) Moreover for any $g_0 \in G$, the map $\beta_{g_0}: \overline{G} \dashrightarrow G$, given by $\beta_{g_0}: \mapsto g_0^{-1}\beta(\alpha g_0 \overline{g})$ is also a rational section of α. After replacing β by β_{g_0}, for a suitable $g_0 \in G$, we may assume that $\overline{f}(X)$ does not lie entirely in the indeterminacy locus of β. Now $f = \beta \circ \overline{f}: X \dashrightarrow G$ is the desired lifting of $\overline{f}: X \dashrightarrow \overline{G}$. □

4.11. Proposition. *Let* X *be a* G*-variety and* H *be a closed normal subgroup of* G. *If* H *is special and the* (*restricted*) H*-action on* X *is generically free then* $\mathrm{cd}(X, G) = \mathrm{cd}(X/H, G) = \mathrm{cd}(X/H, G/H)$.

Strictly speaking, the rational quotient variety X/H is only defined up to birational isomorphism. However, there exists a birational model Y of X/H, such that the G-action on X descends to a (regular) G/H-action (or equivalently, a regular G-action) on Y; see [PV, Proposition 2.6 and Corollary to Theorem 1.1]. The symbol X/H in the statement of Proposition 4.11 denotes Y as above; any two such models will are birationally isomorphic as G-varieties.

Proof of Proposition 4.11. The equality $\mathrm{cd}(X/H, G) = \mathrm{cd}(X/H, G/H)$ follows from Lemma 4.6(b); we shall thus focus on proving that $\mathrm{cd}(X, G) = \mathrm{cd}(X/H, G)$.

After replacing X by an G-invariant open subset, we may assume that the quotient map $\pi: X \longrightarrow X/H$ is regular. If $\pi': X/H \dashrightarrow Z$ is a rational quotient map for the G-action on X/H then $\pi' \circ \pi: X \dashrightarrow Z$ is a rational quotient map for the G-action on X. In particular, X/G and $Z = (X/H)/G$ have the same dimension. By Definition 3.5 we only need to show that:

(i) given a canonical form map $F: X \dashrightarrow X$ there exists a canonical form map $F': X/H \dashrightarrow X/H$ such that $\dim F(X) \geqslant \dim F'(X)$ and conversely, that
(ii) given a canonical form map $F': X/H \dashrightarrow X/H$, there exists a canonical form map $F: X \dashrightarrow X$ such that $\dim F(X) = \dim F'(X)$.

Since H is special and its action on X is generically free, we can choose a rational section $\alpha: X/H \dashrightarrow X$ for the quotient map $\pi: X \longrightarrow X/H$.

We now proceed to prove (i). Suppose F is induced by a rational map $f: X \dashrightarrow G$. After translating α by an element of H, we may assume that the image of α meets the domain of f. Now let $F': X/H \dashrightarrow X/H$ be the canonical form map induced by $f' = f \circ \alpha: X/H \dashrightarrow G$. Then the diagram

$$\begin{array}{ccc} X & \overset{F}{\dashrightarrow} & X \\ \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} \\ X/H & \overset{F'}{\dashrightarrow} & X/H \end{array} \tag{4.12}$$

commutes, and (i) follows.

To prove (ii), choose a Zariski open subset $U \subset X/H$, such that α is regular in U and $\pi: \pi^{-1}(U) \longrightarrow U$ is an H-torsor. In particular, there is a morphism $s: \pi^{-1}(U) \longrightarrow H$ such that $s(x) \cdot x = \alpha(\pi(x))$. Suppose the canonical form map $F': X/H \dashrightarrow X/H$ is induced by $f': X/H \dashrightarrow G$. After replacing f' by gf' for a suitable $g \in G$, we may assume without loss of generality (and without changing $\dim F'(X/H)$) that $F'(\overline{x}) \in U$ for $\overline{x} \in X/H$ in general position.

We will now construct the canonical form map $F: X \dashrightarrow X$, whose existence is asserted by (ii). To motivate this construction, we remark that it is easy to define a canonical form map F so that diagram (4.12) commutes; such a map is induced by $f = f' \circ \pi: X \dashrightarrow G$. However, this diagram only shows that $\dim F(X) \geqslant \dim F'(X/H)$;

equality will not hold in general. On the other hand, we are free to modify $f(x)$ by multiplying $f'(\pi(x))$ by any element of H on the left (this element of H may even depend on x); the resulting canonical form map F will still give rise to a commutative diagram (4.12). With this in mind, we define $F: X \dashrightarrow X$ as the canonical form map induced by $f: X \dashrightarrow G$, where

$$f(x) = s(f'(\pi(x)) \cdot x) f'(\pi(x))$$

for $x \in X$ in general position. As we mentioned above, diagram (4.12) commutes; moreover, by our choice of s, $F(X) \subset \alpha(X/H)$. Hence, π restricts to an isomorphism between $F(X)$ and $\pi(F(X)) = F'(X/H)$. This proves (ii). $\square$

4.13. Remark. Lemma 4.10 and Proposition 4.11 may fail if H is not special; see Example 5.9. If H is special, Proposition 5.9 may still fail if the H-action on X is not generically free; see Remark 15.5.

5. A lower bound

5.1. Definition. Let S be an algebraic group and Y be a generically free S-variety. We define $e(Y, S)$ as the smallest integer e with the following property: given a point $y \in Y$ in general position, there is an S-equivariant rational map $f: Y \dashrightarrow Y$ such that $f(Y)$ contains y and $\dim f(Y) \leqslant e + \dim(S)$.

5.2. Remark. Note that this definition is similar to the definition of the essential dimension $\mathrm{ed}(Y, S)$ of Y; cf. Section 2.1. The difference is that $\mathrm{ed}(Y, S)$ is the minimal value of $\dim f(Y) - \dim(S)$, where f is allowed to range over a wider class of rational S-equivariant maps. In particular, $e(Y, S) \geqslant \mathrm{ed}(Y, S)$. Note also that $e(Y, S)$ depends only on the birational class of Y, as an S-variety.

5.3. Remark. In the sequel we will be particularly interested in the case where Y is itself an algebraic group, S is a closed subgroup of Y, and the S-action on Y is given by translations (say, by right translations, to be precise). In this situation, $e(Y, S)$ is simply the minimal possible value of $\dim f(Y) - \dim(S)$, where f ranges over all S-equivariant rational maps $Y \dashrightarrow Y$. Indeed, after composing f with a suitable left translation $g: Y \longrightarrow Y$, we may assume that $f(Y)$ contains any given $y \in Y$.

5.4. Lemma. *Let Y be a generically free S-variety.*

(a) *If Y is split (cf. Section 2.4) then $e(Y, S) = 0$.*
(b) *Suppose there exists a dominant rational S-equivariant map $\alpha: V \dashrightarrow Y$, where V is a vector space with a linear S-action. Then $e(Y, S) = \mathrm{ed}(S)$.*
(c) *If $Y = G$ is a special algebraic group, S is a subgroup of G and the S-action on Y is given by translations then $e(Y, S) = \mathrm{ed}(S)$.*

Note that the condition of part (a) is always satisfied if S is a special group.

Proof of Lemma 5.4. (a) If Y is split, it is birationally isomorphic to $S \times Z$, where S acts by translations on the first factor and trivially on the second. In fact, we may assume without loss of generality that $Y = S \times Z$. Now for any $z_0 \in Z$ consider $f_{z_0}: S \times Z \dashrightarrow S \times Z$, given by $(s, z) \mapsto (s, z_0)$. As z_0 ranges over Z, the images of f_{z_0} cover Y. Each of these images has the same dimension as S; this yields $e(Y, S) = 0$.

(b) Let $\beta: Y \dashrightarrow Y_0$ be the dominant S-equivariant rational map from Y to a generically free S-variety Y_0 of minimal possible dimension, $\mathrm{ed}(S) + \dim(S)$; cf. Remark 5.3. Then for any $v \in V$, there is a rational G-equivariant map $\gamma: Y_0 \dashrightarrow V$ such that v lies in the image of γ; see [R, Proposition 7.1]. Taking $f = \alpha \circ \gamma \circ \beta: Y \dashrightarrow Y$ in Definition 5.1 and varying v over V, we see that $e(Y, S) \leqslant \dim(Y_0) - \dim(S) = \mathrm{ed}(S)$. The opposite inequality was noted in Remark 5.2.

(c) Let V be a generically free linear representation of G (and thus of S). Since G is special, V is split; cf. Section 2.4. Consequently, there is a dominant rational map $V \dashrightarrow G$ of G-varieties (and hence, of S-varieties). The desired conclusion now follows from part (b). $\square$

5.5. Proposition. *Let G be a connected group and X be an irreducible G-variety with a stabilizer S in general position. Then*

(a) $\mathrm{cd}(X, G) \geqslant e(G, S)$, *where S acts on G by translations.*

In particular,

(b) $\mathrm{cd}(X, G) \geqslant \mathrm{ed}(G, S)$, *and*
(c) *if G is special then* $\mathrm{cd}(X, G) \geqslant \mathrm{ed}(S)$.

Proof. (b) and (c) follow from (a) by Remark 5.2 and Lemma 5.4(c), respectively.

To prove part (a), choose a canonical form map $F: X \dashrightarrow X$ such that $\dim F(G \cdot x) = \mathrm{cd}(X, G)$ for x in general position; cf. Lemma 3.6. Suppose that F is induced by a rational map $f: X \dashrightarrow G$, as in Definition 3.1, and consider the commutative diagram

$$\begin{tikzcd} G \arrow[r, dashed, "F'"] \arrow[d, "\phi"] & G \arrow[d, "\phi"] \\ G \cdot x \arrow[r, dashed, "F"] & G \cdot x \end{tikzcd}$$

of rational maps, where $\phi: G \longrightarrow G \cdot x$ is the orbit map, $\phi(g) = g \cdot x$, and $F'(g) = f(g \cdot x) \cdot x$.

Now set $S = \mathrm{Stab}_G(x)$ and observe that $F'(gs^{-1}) = F'(g)s^{-1}$ for every $s \in S$. In view of Remark 5.3, this implies

$$\dim\, F'(G) - \dim\, S \geqslant e(G, S).$$

On the other hand, $F'(G)$ is an S-invariant subvariety of G and (because the above diagram in commutative) $F(G\cdot x) = \phi(F'(G))$. Finally, since for any $g \in G$, $\phi(gs^{-1}) = \phi(g)$ if and only if $s \in S$, we see that the fibers of the map $\phi_{|F'(X)}: F'(X) \longrightarrow F(G\cdot x)$ are precisely the S-orbits in $F'(X)$. Consequently,

$$\mathrm{cd}(X, G) = \dim\ F(G \cdot x) = \dim\ F'(G) - \dim\ S \geqslant e(G, S),$$

as claimed. □

5.6. Remark. Proposition 5.5 assumes that the G-action on X has a stabilizer in general position, i.e., there exists a subgroup $S \subset G$ such that $\mathrm{Stab}(x)$ is conjugate to S for $x \in X$ in general position. This condition is satisfied by many but not all group actions; see [PV, Section 7]. For an arbitrary G-action on X, our proof of part (a) shows that if $e(G, \mathrm{Stab}(x)) \geqslant d$ for x in a Zariski dense open subset of X then $\mathrm{cd}(X, G) \geqslant d$.

Note also that if L_x is a Levi subgroup of $\mathrm{Stab}(x)$ then by a theorem of Richardson (see [Ri, Theorem 9.3.1] or [PV, Theorem 7.1]), there exists a non-empty Zariski open subset $U \subset X$ such that L_x and L_y are conjugate in G. Since $\mathrm{ed}(L_x) = \mathrm{ed}(\mathrm{Stab}(x))$ (this is an immediate consequence of [San, Lemma 1.13]; for a direct geometric proof, see [Ko]), $\mathrm{ed}(\mathrm{Stab}(x))$ assumes the same value for every $x \in U$. In particular, Proposition 5.5(c) remains valid for an arbitrary G-action, provided that we replace the inequality $\mathrm{cd}(X, G) \geqslant \mathrm{ed}(S)$ by $\mathrm{cd}(X, G) \geqslant \mathrm{ed}(\mathrm{Stab}(x))$ for $x \in U$.

5.7. Corollary. *Let G be a connected group, S be a closed subgroup, and $X = G/S$ be a homogeneous space. Then*

(a) $\mathrm{cd}(X, G) = e(G, S)$, *where S acts on G by translations.*
(b) *If G is special then* $\mathrm{cd}(X, G) = \mathrm{ed}(S)$.

Proof. Part (b) follows from part (a) and Lemma 5.4(c).

To prove (a), note that by Proposition 5.5, we only need to show that $\mathrm{cd}(X, G) \leqslant e(G, S)$, i.e., to construct a canonical form map $F: X \dashrightarrow X$ such that

$$\dim\ F(G \cdot x) = e(G, S) \tag{5.8}$$

for x in general position. We will define F by reversing the construction in Proposition 5.5. Let $F': G \dashrightarrow G$ be an S-equivariant rational map (with respect to the right translation action of S on G), such that $\dim\ F'(G)$ assumes its minimal possible value, $e(G, S) + \dim(S)$; cf. Remark 5.3. Then $f': G \dashrightarrow G$ given by $f'(g) = F'(g)g^{-1}$ is S-invariant (with respect to the right translation action of S on G). Hence, f' descends to $f: G/S \dashrightarrow G$. Thus we have a commutative diagram

$$\begin{array}{ccc} G & \overset{F'}{\dashrightarrow} & G \\ \downarrow & & \downarrow \\ G/S & \overset{F}{\dashrightarrow} & G/S \end{array}$$

where $F(x) = f(x) \cdot x$. Here F is, by construction, a canonical form map, and

$$\dim\ F(G/S) = \dim\ F'(G) - \dim\ S = e(G, S),$$

as desired. $\square$

5.9. Example. Let G be a special group and H be a non-special closed normal subgroup of G. (For example, $G = \mathrm{SL}_n$ and $H = \mu_n$ is the center of G.)

(i) Let $X = G/H$. Then by Corollary 5.7(b), $\mathrm{cd}(X, G) = \mathrm{ed}(H)$, which is $\geqslant 1$; cf. [R, Proposition 5.3]. On the other hand, $\mathrm{cd}(X, G/H) = 0$; (cf. Lemma 4.8(a)). This shows that the equality $\mathrm{cd}(X, G) = \mathrm{cd}(X, G/H)$ in Lemma 4.10(b) may fail if H is not special.

(ii) Now let $X = G$ (viewed as a G-variety with the translation action). Then $\mathrm{cd}(X, G) = 0$ (cf. Lemma 4.8(a)) but, $\mathrm{cd}(X/H, G) = \mathrm{ed}(H)$. This shows that the equality $\mathrm{cd}(X, G) = \mathrm{cd}(X/H, G)$ in Proposition 4.11 may also fail if H is not special.

6. A comparison lemma

6.1. Lemma. *Let $\alpha: X \dashrightarrow Y$ be a dominant rational map of irreducible G-varieties. Suppose $\dim(G \cdot x) = d$ and $\dim(G \cdot y) = e$ for $x \in X$ and $y \in Y$ in general position. Then $\mathrm{cd}(X, G) \leqslant \mathrm{cd}(Y, G) + d - e$.*

Proof. Let $F: Y \dashrightarrow Y$ be a canonical form map such that $\dim\ F(G \cdot y) = \mathrm{cd}(Y, G)$ for Y in general position. Suppose F is induced by $f: Y \dashrightarrow G$, i.e., $F(y) = f(y) \cdot y$, as in Definition 3.1.

Now consider $f' = f \circ \alpha: X \dashrightarrow G$ and the induced canonical form map $F': X \dashrightarrow X$ given by $F'(x) = f'(x) \cdot x$. The relationship between F and F' is illustrated by the following commutative diagram, where x is a point in general position in X and $y = \alpha(x) \in Y$.

$$\begin{array}{ccc} G \cdot x & \overset{F'}{\dashrightarrow} & F'(G \cdot x) \\ \alpha \downarrow & & \downarrow \alpha \\ G \cdot y & \overset{F}{\dashrightarrow} & F(G \cdot y). \end{array}$$

Each fiber of $\alpha: G \cdot x \longrightarrow G \cdot y$ has dimension $d - e$. Hence, each fiber of the right vertical map $\alpha_{|F'(G \cdot x)}$ has dimension $\leqslant d - e$. Applying the fiber dimension theorem to this map, we obtain

$$\dim\ F'(G \cdot x) \leqslant \dim\ F(G \cdot y) + d - e = \mathrm{cd}(Y, G) + d - e,$$

and the proposition follows; cf. Lemma 3.6. $\square$

Let X be a G-variety and H be a closed subgroup of G. Recall that an H-invariant (not necessarily irreducible) subvariety $Y \subset X$ is called a (G, H)-section if (i) $G \cdot Y$ is dense in X and (ii) for $y \in Y$ in general position, $g \cdot y \in Y \Leftrightarrow g \in H$. Note that in some papers a (G, H)-section is called a *relative section* (cf. [PV, Section 2.8]) or a *standard relative section with normalizer H* (cf. [Po, (1.7.6)]).

6.2. Corollary. *Let X be an irreducible G-variety.*

(a) *If X has a* (G, H)*-section then* $\mathrm{cd}(X, G) \leqslant e(G, H) + d - \dim(G) + \dim(H)$, *where* $d = \dim(G \cdot x)$ *for* $x \in X$ *in general position.*
(b) *If X has a stabilizer S in general position then*

$$e(G, S) \leqslant \mathrm{cd}(X, G) \leqslant e(G, N) - \dim(S) + \dim(N),$$

where N is the normalizer of S in G.

Proof. (a) The existence of a (G, H)-section is equivalent to the existence of a G-equivariant rational map $X \dashrightarrow G/H$; see [Po, Theorem 1.7.5]. Thus by Lemma 6.1, $\mathrm{cd}(X, G) \leqslant \mathrm{cd}(G/H, G) - d + \dim(G/H)$. By Corollary 5.7(a) $\mathrm{cd}(G/H, G) = e(G, H)$, and part (a) follows.

(b) The inequality $e(G, S) \leqslant \mathrm{cd}(X, G)$ follows from Proposition 5.5(a). To prove the inequality

$$\mathrm{cd}(X, G) \leqslant e(G, N) - \dim(S) + \dim(N), \tag{6.3}$$

note that by [Po, (1.7.8)], X has a (G, N)-section. Substituting $H = N$ and $d = \dim(G) - \dim(S)$ into the inequality of part (a), we obtain (6.3). $\square$

7. The canonical dimension of a group

In this section we will define the canonical dimension of an algebraic group G. We begin with a simple lemma.

7.1. Lemma. *Let X be an irreducible G-variety, and let Z be an irreducible variety with trivial action of G. Then* $\mathrm{cd}(X \times Z, G) = \mathrm{cd}(X, G)$.

Proof. The inequality $\mathrm{cd}(X \times Z, G) \leqslant \mathrm{cd}(X, G)$ follows from Lemma 6.1, applied to the projection map $\alpha: X \times Z \longrightarrow X$. To prove the opposite inequality, let $c = \mathrm{cd}(X \times Z, G)$ and choose a canonical form map $F: X \times Z \dashrightarrow X \times Z$ such that $\dim F(G \cdot (x, z)) = c$. Suppose F is induced by $f: X \times Z \dashrightarrow G$, i.e., $F(x, z) = f(x, z) \cdot (x, z)$, as in Definition 3.1. It is now easy to see that for $z_0 \in Z$ in general position, the map $f_{z_0}: X \dashrightarrow G$ given by $f_{z_0}(x) = f(x, z_0)$ gives rise to a canonical form map $F_{z_0}: X \dashrightarrow X$ such that $\dim F_0(G \cdot x) = c$. In other words, $\mathrm{cd}(X, G) \leqslant c$, as claimed. $\square$

7.2. Proposition. *Let V be a generically free linear representation of G.*

(a) *If X is an irreducible generically free G-variety then* $\mathrm{cd}(X, G) \leqslant \mathrm{cd}(V, G)$.
(b) *If W is another generically free G-representation, then* $\mathrm{cd}(V, G) = \mathrm{cd}(W, G)$.

Proof. (a) By [R, Corollary 2.20], there is a dominant rational map $\alpha: X \times \mathbb{A}^d \dashrightarrow V$ of G-varieties, where $d = \dim(V)$, and G acts trivially on $\mathbb{A}^d$. Now

$$\mathrm{cd}(X, G) \overset{\text{by Lemma 7.1}}{=} \mathrm{cd}(X \times \mathbb{A}^d, G) \overset{\text{by Lemma 6.1}}{\leqslant} \mathrm{cd}(V, G),$$

as claimed.

(b) $\mathrm{cd}(W, G) \leqslant \mathrm{cd}(V, G)$ by part (a). To prove the opposite inequality, interchange the roles of V and W. $\square$

7.3. Definition. We define the canonical dimension $\mathrm{cd}(G)$ of an algebraic group G to be $\mathrm{cd}(V, G)$, where V is a generically free linear representation of G. By Proposition 7.2(b) this number is independent of the choice of V. Moreover, by Proposition 7.2(a), $\mathrm{cd}(G) = \max\{\mathrm{cd}(X, G)\}$, as X ranges over all irreducible generically free G-varieties.

7.4. Corollary. *Suppose W is a linear representation of G such that* $\mathrm{Stab}_G(w)$ *is finite for $w \in W$ in general position. Then* $\mathrm{cd}(G) \leqslant \mathrm{cd}(W, G)$.

Proof. Let V be a generically free linear representation of G. Then so is $X = V \times W$. The desired inequality is now a consequence of Lemma 6.1, applied to the projection map $\alpha: V \times W \longrightarrow W$. $\square$

7.5. Lemma. (*a*) $\mathrm{cd}(G) \leqslant \mathrm{cd}(H) + \dim(G) - \dim(H)$, *for any closed subgroup $H \subset G$.*
(*b*) $\mathrm{cd}(G) = \mathrm{cd}(G^0)$.
(*c*) $\mathrm{cd}(G) = 0$ *if and only if G^0 is special.*
(*d*) $\mathrm{cd}(G_1 \times G_2) \leqslant \mathrm{cd}(G_1) + \mathrm{cd}(G_2)$.

Proof. (a) Follows from Lemma 4.2, with $X = V$ = generically free linear representation of G.

(b) Immediate from Lemma 4.4.

(c) By part (b), we may assume $G = G^0$ is connected. The desired conclusion now follows from Lemma 4.8.

(d) Follows from Lemma 4.6, by taking $X_i = V_i$ to be a generically free representation of G_i for $i = 1, 2$. $\square$

7.6. Example. Consider the subgroup

$$H = \left\{ \left(\begin{array}{cc} & b_1 \\ A & \vdots \\ & b_{n-1} \\ 0 \ldots 0 & 1 \end{array} \right) \middle| \, A \in \mathrm{GL}_{n-1},\ b_1, \ldots, b_{n-1} \in k \right\},$$

of $G = \mathrm{PGL}_n$. The Levi subgroup of H is special (it isomorphic to GL_{n-1}); hence, H itself is special. (This follows from the theorem of Grothendieck stated in Section 2.6 or alternatively, from [San, Lemma 1.13].) Since $\dim(H) = n^2 - n$, Lemma 7.5(a) yields $\mathrm{cd}(\mathrm{PGL}_n) \leqslant n - 1$. In particular, $\mathrm{cd}(\mathrm{PGL}_2) = 1$. (Note that $\mathrm{cd}(\mathrm{PGL}_2) \geqslant 1$ by Lemma 7.5(c)).

Alternatively, we can deduce the inequality $\mathrm{cd}(\mathrm{PGL}_n) \leqslant n - 1$ by applying Lemma 6.1 to the projection map $\mathrm{M}_n \times \mathrm{M}_n \longrightarrow \mathrm{M}_n$ to the first factor, where PGL_n acts on M_n by conjugation. The PGL_n-action on $\mathrm{M}_n \times \mathrm{M}_n$ is generically free; hence, $\mathrm{cd}(\mathrm{PGL}_n) = \mathrm{cd}(\mathrm{M}_n \times \mathrm{M}_n, \mathrm{PGL}_n)$. On the other hand, $\mathrm{cd}(\mathrm{M}_n, \mathrm{PGL}_n) = 0$; see Example 3.7. Now Lemma 6.1 tells us that

$$\mathrm{cd}(\mathrm{PGL}_n) = \mathrm{cd}(\mathrm{M}_n \times \mathrm{M}_n, \mathrm{PGL}_n) \leqslant \mathrm{cd}(\mathrm{M}_n, \mathrm{PGL}_n) + n - 1 - 0 = n - 1.$$

For a third proof of this inequality, see Example 9.9.

8. Splitting fields

Throughout this section we will assume that G/k is a connected linear algebraic group. Unless otherwise specified, the fields E, K, L, etc., are assumed to be finitely generated extensions of the base field k.

Let X be a generically free irreducible G-variety, $E = k(X)^G = k(X/G)$, $\pi: X \dashrightarrow X/G$ be the rational quotient map and $F: X \dashrightarrow X$ be a canonical form map. Recall that F commutes with π, so that $F(X)$ may be viewed as an algebraic variety over E.

8.1. Lemma. *Let X be a generically free G-variety such that G-orbits in X are separated by regular invariants and let $F: X \dashrightarrow X$ be a canonical form map. Suppose $\alpha \in H^1(E, G)$ is the class represented by X. Then for any field extension K/E the following conditions are equivalent*:

(a) $\alpha_K = 1$,
(b) *X is rational over K*,
(c) *$F(X)$ is unirational over K*,

(d) *K-points are dense in* $F(X)$,
(e) $F(X)$ *has a K-point*.

Proof. We begin by proving the lemma in the case where $K = E$.

(a) $\Rightarrow$ (b): If $\alpha = 1$ then X is birationally isomorphic to $X/G \times G$ (over X/G). Now recall that the underlying variety of a connected algebraic group G is rational over k. Hence, $X/G \times G$ is rational over X/G, i.e., X is rational over E.
(b) $\Rightarrow$ (c): The rational map $F\colon X \dashrightarrow F(X)$ is, by definition, dominant. If X is rational, this makes $F(X)$ unirational.
(c) $\Rightarrow$ (d) and (d) $\Rightarrow$ (e) are obvious.
(e) $\Rightarrow$ (a): An E-point in $F(X)$ is a rational section $s\colon X/G \dashrightarrow F(X) \subset X$ for π. The existence of such a section implies that X is split, and hence, so is α; see Section 2.4.

To prove the general case, note that since the G-orbits in X are separated by regular invariants, we can choose a regular model of the rational quotient variety X/G, so that the rational quotient map $\pi\colon X \longrightarrow X/G$ is regular and its fibers are exactly the G-orbits in X. After making X/G smaller if necessary, we may also assume that our field extension K/E is represented by a surjective morphism $Y \longrightarrow X/G$ of algebraic varieties. Then α_K is represented by the G-variety $X_K = X \times_{X/G} Y$.

We claim that the morphism $\pi_K\colon X_K \longrightarrow Y$ (projection to the second component) separates the G-orbits in X_K. Indeed, if for some $x_1, x_2 \in X$, $z_1 = (x_1, y)$ and $z_2 = (x_2, y) \in X_K$ have the same second component then $\pi(x_1) = \pi(x_2)$. We conclude that x_1 and x_2 are in the same G-orbit in X and consequently, z_1 and z_2 are in the same G-orbit in X_K, as claimed. This shows that π_K is a rational quotient map for the G-action on X_K; cf. [PV, Lemma 2.1].

We now define a rational map $F_K\colon X_K \dashrightarrow X_K$ by $F_K(x, y) = (F(x), y)$. Since $\pi_K \circ F_K = \pi_K$, F_K is a canonical form map; see Remark 3.2. Moreover, $F_K(X_K) = F(X) \times_{X/G} Y$. Replacing X by X_K and F by F_K, we reduce the lemma to the case we settled at the beginning of the proof (where $K = E$). $\square$

Let $\alpha \in H^1(E, G)$. As usual, we will call a field extension K/E a *splitting field* for α if the image α_K of α under the natural map $H^1(E, G) \longrightarrow H^1(K, G)$ is split. If α is represented by a generically free G-variety X, with $k(X)^G = E$ then we will also sometimes say that K is a splitting field for X.

8.2. Proposition. *Suppose* $\alpha \in H^1(E, G)$.

(a) *If* $\mathrm{cd}(\alpha) = 1$ *then there exist* $0 \neq a, b \in E$ *such that a field extension* K/E *splits* α *if and only if the quadratic form* $q(x, y, z) = x^2 + ay^2 + bz^2$ *is isotropic over* K. *In particular*, α *has a splitting field* K/E *of degree* 2.
(b) *If* $\mathrm{cd}(\alpha) = 2$ *then* α *has a splitting field* K/E *of degree* 2, 3, 4, *or* 6.

Note that if K/E is a splitting field for α then $[K : E] = 1$ is impossible in either part. Indeed, otherwise α itself is split, and $\mathrm{cd}(\alpha) = 0$ by see Lemma 4.8(a).

Proof of Proposition 8.2. Choose a canonical form map $F\colon X \dashrightarrow X$, such that $\dim F(X) - \dim(X/G) = \mathrm{cd}(\alpha)$. By Lemma 8.1, $F(X)$ is unirational over every splitting field K of α; in particular, it is unirational over the algebraic closure $\overline{E}$ of E.

(a) Here $F(X)$ is a curve over E, and Lüroth's theorem tells us that $F(X)$ is rational over $\overline{E}$. It is well known that any such curve is birationally isomorphic to a conic Z in $\mathbb{P}^2_E$ (see, e.g., [MT, Proposition 1.1.1]) and that K-points are dense in Z if and only if $Z(K) \neq \emptyset$ (see, e.g., [MT, Theorem 1.2.1]). Writing the equation of $Z \subset \mathbb{P}^2_E$ in the form $x^2 + ay^2 + bz^2 = 0$, we deduce the first assertion of part (a). The second assertion is an immediate consequence of the first; for example, $K = E(\sqrt{-a})$ is a splitting field for α.

(b) Here $F(X)$ is a surface over E, which becomes unirational over the algebraic closure $\overline{E}$. By a theorem of Castelnuovo, $F(X)$ is, in fact, rational over $\overline{E}$. Let Z be a complete smooth minimal surface, defined over E, which is birationally isomorphic to $F(X)$ via $\phi\colon F(X) \overset{\sim}{\dashrightarrow} Z$ and let $U \subset Z$ be an open subset such that ϕ is an isomorphism over U. Part (b) now follows from Lemma 8.1 and Lemma 8.3 below. $\square$

8.3. Lemma. *Let E be a field of characteristic zero, Z be a complete minimal surface defined over E and rational over $\overline{E}$, and let U be a dense open subset of Z (defined over E). Then U contains a K-point for some field extension K/E of degree* 1, 2, 3, 4 *or* 6.

Proof. By the Enriques–Manin–Iskovskih classification Z is a conic bundle or a del Pezzo surface; see [I, Theorem 1] or [MT, Theorem 3.1.1]. Note that $Z = \mathbb{P}^2$, listed as a separate case in [I, Theorem 1], is, in fact, a del Pezzo surface. (We remark however, that the lemma is obvious in this case, since E-points are dense in $\mathbb{P}^2_E$.)

If $f\colon Z \longrightarrow C$ is a conic bundle over a rational curve C, then after replacing E by a quadratic extension E', we may assume that $C_{E'} \simeq \mathbb{P}^1_{E'}$. For every E'-point $z \in C$, $f^{-1}(z)$ is a rational curve over E'. Taking $z \in C_{E'}$ so that $f^{-1}(z) \cap U \neq \emptyset$, we can choose an extension K/E' of degree 1 or 2 so that $f^{-1}(z)_K \simeq \mathbb{P}^1_K$. Now $[K:E] = 1$, 2 or 4, and K-points are dense in $f^{-1}(z)$, so that one of them will lie in U.

From now on we may assume that Z is a del Pezzo surface. Recall that the anti-canonical divisor $-\Omega_Z$ on a del Pezzo surface is ample, and the degree $d = \Omega_Z \cdot \Omega_Z$ can range from 1 to 9.

If $d = 1$ the linear system $|-2\Omega_Z|$ defines a (ramified) double cover $f\colon Z \longrightarrow Q$, where Q is a quadric cone in $\mathbb{P}^3_E$; see [I, p. 30]. Then $Q_{E'} \simeq \mathbb{P}^2_{E'}$ for some extension E'/E of degree 1 or 2. Now choose an E'-point $x \in f(U) \subset Q$ and split $f^{-1}(x)$ over a field extension K/E' of degree 1 or 2. Then $[K:E] = 1$, 2 or 4 and U contains a K-point.

If $d = 2$ then the linear system $|-\Omega_Z|$ defines a (ramified) double cover $Z \longrightarrow \mathbb{P}^2_E$ (see [I, p. 30]), and points of degree 2 are dense in Z.

If $3 \leqslant d \leqslant 9$ then it is enough to show that $Z(K) \neq \emptyset$ for some field extension K/E of degree 1, 2, 3, 4 or 6. Indeed, if $Z(K) \neq \emptyset$ then Z_K is unirational over K (see [MT, Theorem 3.5.1]) and thus K-points are dense in Z. Note also that for $3 \leqslant d \leqslant 9$, Z is isomorphic to a surface in $\mathbb{P}^d$ of degree d. Intersecting this surface by two hyperplanes in general position, we see that Z has a point of degree dividing d. This proves the lemma for $d = 3$, 4, and 6.

For $d = 5$ and 7, Z always has an E-point (see [MT, Theorem 7.1.1]), so the lemma holds trivially in these cases. For $d = 8$, Z has a point of degree dividing 4 and for $d = 9$, Z has a point of degree dividing 3 (see [MT, p. 80]). The proof of the lemma is now complete. □

8.4. Example. Suppose $\alpha \in H^1(K, \mathrm{PGL}_n)$ is represented by a central simple algebra of index d. Then the degree of every splitting field for α is divisible by d (cf. e.g, [Row, Theorem 7.2.3]); hence,

$$\mathrm{cd}(\alpha) \geqslant \begin{cases} 2, & \text{if } d \geqslant 3, \\ 3, & \text{if } d \neq 1, 2, 3, 4 \text{ or } 6. \end{cases}$$

In particular,

$$\mathrm{cd}(\mathrm{PGL}_n) \geqslant \begin{cases} 2, & \text{if } n \geqslant 3, \\ 3, & \text{if } n \neq 1, 2, 3, 4 \text{ or } 6. \end{cases}$$

For sharper results on $\mathrm{cd}(\mathrm{PGL}_n)$, see Section 11.

8.5. Example. Let V be a generically free linear representation of $G = F_4$, E_6 or E_7 (adjoint or simply connected), $K = k(V)^G$ and $\alpha \in H^1(K, G)$ be the class represented by the G-variety V. Then the degree of any splitting field L/K for $\alpha \in H^1(k(V)^G, G)$ is divisible by 6; [RY_2, p. 223]. We conclude that $\mathrm{cd}(G) \geqslant 2$ for these groups.

8.6. Example. $\mathrm{cd}(G) \geqslant 3$, if $G = E_8$ or adjoint E_7; see [RY_2, Corollaries 5.5 and 5.8].

8.7. Remark. Let G be a connected linear algebraic group defined over k and let H be a finite abelian p-subgroup of G, where p is a prime integer. Recall that the *depth* of H is the smallest value of i such that $[H : H \cap T] = p^i$, as T ranges over all maximal tori of G; see [RY_2, Definition 4.5]. Note that H has depth 0 if and only if it lies in a torus of G. A prime p is called a *torsion prime* for G if and only if G has a finite abelian p-subgroup of depth $\geqslant 1$. (This is one of many equivalent definitions of torsion primes; see [St, Theorem2.28].) The inequalities of Examples 8.4–8.6 may be viewed as special cases of the following assertion:

Suppose a connected linear algebraic group G has a p-subgroup H of depth d.

(a) *If* $\mathrm{cd}(G) \leqslant 1$ *then* $p^d = 1$ *or* 2.
(b) *If* $\mathrm{cd}(G) \leqslant 2$ *then* $p^d = 1$, 2, 3 *or* 4.

The proof is immediate from Proposition 8.2 and [RY_2, Theorem 4.7] (where we take X to be a generically free linear representation of G).

9. Generic splitting fields

9.1. Definition. Let K/E be a (finitely generated) field extension. A (finitely generated) field extension L/E is said to be

(a) a *specialization* of K/E if there is a place $\phi: K \longrightarrow L \cup \{\infty\}$, defined over E,
(b) a *rational specialization* of K/E if there is an embedding $K \hookrightarrow L(t_1, \ldots, t_r)$, over E, for some $r \geqslant 0$.

In geometric language, (a) and (b) can be restated as follows. Suppose the field extensions K/E and L/E are induced by dominant rational maps $V \dashrightarrow Z$ and $W \dashrightarrow Z$ of irreducible algebraic k-varieties, respectively. Then

(a′) W is a specialization of V if there is a rational map $W \dashrightarrow V$, such that the diagram

$$\begin{tikzcd} W \arrow[r, dashed] \arrow[dr, dashed] & V \arrow[d, dashed] \\ & Z \end{tikzcd}$$

commutes.
(b′) W is a rational specialization of V if for some $r \geqslant 0$ there is a dominant map $W \times \mathbb{A}^r \dashrightarrow V$ such that the diagram

$$\begin{tikzcd} W \times \mathbb{A}^r \arrow[r, dashed, two heads] \arrow[dr, dashed] & V \arrow[d, dashed] \\ & Z \end{tikzcd}$$

commutes.

9.2. Remark. In the definition of rational specialization we may assume without loss of generality that $r = \max\{0, \operatorname{tr}\deg_E(L) - \operatorname{tr}\deg_E(K)\}$; see [Roq$_2$, Lemma 1].

9.3. Definition. Let $\alpha \in H^1(E, G)$. A splitting field K/E for α is called *generic* (respectively, *very generic*) if every splitting field L/E for α is a specialization (respectively, a rational specialization) of K/E.

9.4. Remark. It is easy to see that a rational specialization is a specialization (cf. [Sal$_2$, Lemma 11.1]). Consequently, a very generic splitting field is generic.

9.5. Remark. The generic splitting field of α in Definition 9.3 is the same as the generic splitting field for the twisted group ${}_\alpha G$ defined by Kersten and Rehmann [KR].

9.6. Lemma. *Let G be a connected algebraic group, X be an irreducible generically free G-variety, $E = k(X)^G = k(X/G)$ and $F\colon X \dashrightarrow X$ be a canonical form map. Then $k(F(X))/E$ is a very generic splitting field for the class $\alpha \in H^1(E, G)$ represented by X.*

Proof. After replacing X by a G-invariant open subset, we may assume that G-orbits in X are separated by regular invariants. The generic point of $F(X)$ is a $k(F(X))$-point; hence, by Lemma 8.1, $F(X)/E$ is a splitting field for α.

It remains to show that every splitting field L/E for α is a rational specialization of $k(F(X))/E$. After replacing X by a smaller G-invariant dense open subset, we may assume that L/E is induced by a surjective morphism $Y \longrightarrow X/G$ of algebraic varieties. Then $\alpha_L = 1_L$ is represented by the generically free G-variety $X_L = X \times_{X/G} Y$. Since X_L is split, it is rational over L; see Lemma 8.1. The morphisms

$$X_L \xrightarrow{\mathrm{pr}_1} X \xrightarrow{F} F(X) \xrightarrow{\pi} X/G$$

now tell us that $k(F(X)) \hookrightarrow k(X_L) = L(t_1, \ldots, t_r)$, over E, where $t_1, \ldots, t_r$ are independent variables and $r = \dim(G)$. (Here pr_1 is the projection $X_L = X \times_{X/G} Y \longrightarrow X$ to the first factor.) This shows that L/E is a rational specialization of $k(F(X))/E$, as claimed. $\square$

9.7. Proposition. *Let E/k be a finitely generated field extension. Then for every $\alpha \in H^1(E, G)$,*

$$\begin{aligned}\mathrm{cd}(\alpha) &= \min\{\mathrm{tr\,deg}_E(K) \mid K/E \textit{ is a generic splitting field for } \alpha\}\\ &= \min\{\mathrm{tr\,deg}_E(L) \mid L/E \textit{ is a very generic splitting field for } \alpha\}.\end{aligned}$$

Proof. Let X be a generically free G-variety representing α; in particular, $E = k(X)^G = k(X/G)$. Since $\mathrm{cd}(X, G)$ is, by definition, the minimal value of $\dim F(X) - \dim(X/G) = \mathrm{tr\,deg}_E\, k(F(X))$, as F ranges over all canonical form maps $X \dashrightarrow X$, Lemma 9.6 tells us that

$$\mathrm{cd}(\alpha) = \mathrm{cd}(X, G) \geqslant \min\{\mathrm{tr\,deg}_E(L) \mid L/E \text{ is a very generic splitting field for } \alpha\}.$$

Now let K/E be a generic splitting field for α. It remains to show that

$$\mathrm{cd}(X, G) \leqslant \mathrm{tr\,deg}_E(K). \tag{9.8}$$

Choose a variety Y whose function field $k(Y)$ is K; the inclusion $E \subset K$ then gives rise to a rational map $Y \dashrightarrow X/G$. By Lemma 9.6 (with $F = id$), $k(X)/E$ is a very generic (and hence, a generic; cf. Remark 9.4) splitting field of α. Since $k(X)/E$ and K/E are both generic splitting fields, each is a specialization of the other. Geometrically, this means that there exist rational maps $f_1: X \dashrightarrow Y$ and $f_2: Y \dashrightarrow X$ such that the diagram

$$\begin{tikzcd} X \arrow[r, dashed, "f_1"] \arrow[dr, dashed] & Y \arrow[r, dashed, "f_2"] \arrow[d, dashed] & X \arrow[dl, dashed] \\ & X/G & \end{tikzcd}$$

commutes. After replacing Y by the closure of the graph of f_2 in $Y \times X$, and f_2 by the projection from this graph to X, we may assume that f_2 is a morphism. Now $F = f_2 \circ f_1$ is a well defined rational map $X \dashrightarrow X$ which commutes with the rational quotient map $\pi: X \dashrightarrow X/G$. By Remark 3.2, F is a canonical form map. Thus

$$\begin{aligned} \mathrm{cd}(X, G) &\leqslant \dim F(X) - \dim X/G \leqslant \dim f_2(Y) - \dim X/G \\ &\leqslant \dim(Y) - \dim(X/G) = \mathrm{tr\,deg}_k(K) - \mathrm{tr\,deg}_k(E) = \mathrm{tr\,deg}_E(K). \end{aligned}$$

Thus completes the proof of (9.8) and thus of Proposition 9.7. $\square$

9.9. Example. Let $\alpha \in H^1(E, \mathrm{PGL}_n)$ be represented by a central simple E-algebra A and let K be the function field of the Brauer–Severi variety of A. Then K/E is a very generic splitting field for α (see, e.g., [Sal$_2$, Corollary 13.9]) and $\mathrm{tr\,deg}_E(K) = n - 1$. By Proposition 9.7, $\mathrm{cd}(\alpha) \leqslant n - 1$. This gives yet another proof of the inequality $\mathrm{cd}(\mathrm{PGL}_n) \leqslant n - 1$ of Example 7.6.

10. The canonical dimension of a functor

The results of the previous section naturally lead to the following definitions. Let $\mathcal{F}$ be a functor from the category $\mathbf{Fields}_k$ of finitely generated extensions of the base field k to the category $\mathbf{Sets}^*$ of pointed sets. We will denote the marked element in $\mathcal{F}(E)$ by 1_E (and sometimes simply by 1, if the reference to the field E is clear from the context). Given a field extension L/E, we will denote the image of $\alpha \in \mathcal{F}(E)$ in $\mathcal{F}(L)$ by α_L.

The notions of splitting field and generic splitting field naturally extend to this setting. That is, given $\alpha \in \mathcal{F}(E)$, we will say that L/E is a *splitting field* for α if $\alpha_L = 1_L$. We will call a splitting field K/E for α *generic* (respectively, *very generic*) if every splitting field L/E for α is a specialization (respectively, a rational specialization) of K/E. Moreover, we can now define $\mathrm{cd}(\alpha)$ by

$$\mathrm{cd}(\alpha) = \min\{\mathrm{tr\,deg}_E(K) \mid K/E \text{ is a generic splitting field for } \alpha\}$$

and $\mathrm{cd}(\mathcal{F})$ by

$$\mathrm{cd}(\mathcal{F}) = \max\{\mathrm{cd}(\alpha) \mid E/k \text{ is a finitely generated extension, } \alpha \in \mathcal{F}(E)\}.$$

Proposition 9.7 says that if G is a connected linear algebraic group and $\mathcal{F} = H^1(\_, G)$ then the above definition of $\mathrm{cd}(\alpha)$ agrees with Definition 3.5. Moreover, Definition 7.3 tells us that for this $\mathcal{F}$, $\mathrm{cd}(\mathcal{F}) = \mathrm{cd}(G)$.

Note that none of the above definitions require k to be algebraically closed. In particular, it now makes sense to talk about the canonical dimension of an algebraic group defined over a non-algebraically closed field. This opens up interesting directions for future research, but we shall not pursue them in this paper. Instead we will continue to assume that k is algebraically closed, and our main focus will remain on the functors $H^1(\_, G)$. However, even in this (more limited but already very rich) context, we will take advantage of the notion of canonical dimension for a functor by considering certain subfunctors of $H^1(\_, G)$.

We also remark that it is a priori possible that for some functors $\mathcal{F}$, some fields E/k and some $\alpha \in \mathcal{F}(E)$ there will not exist a generic splitting field; if this happens, then, according to our definition, $\mathrm{cd}(\alpha) = \mathrm{cd}(\mathcal{F}) = \infty$. However, Proposition 9.7 tells us that this does not occur for any functor of the form $H^1(\_, G)$, where G is a linear algebraic group, and consequently, for any of its subfunctors.

10.1. Example. Isomorphic functors clearly have the same canonical dimension. In particular, suppose G is a linear algebraic group and U is a normal unipotent subgroup of G. Then the natural map $H^1(\_, G) \longrightarrow H^1(\_, G/U)$ is an isomorphism (see, e.g., [San, Lemma 1.13]) and hence, $\mathrm{cd}(G) = \mathrm{cd}(G/U)$. Taking U to be the unipotent radical of G, we see that $\mathrm{cd}(G) = \mathrm{cd}(G_{\mathrm{red}})$, where G_{red} is the Levi subgroup of G.

The following simple lemma slightly extends the observation that isomorphic functors have the same canonical dimension. This lemma will turn out to be surprisingly useful in the sequel.

10.2. Lemma. *Suppose $\tau : \mathcal{F}_1 \longrightarrow \mathcal{F}_2$ is a morphism of functors with trivial kernel. Then for every finitely generated field extension E/k,*

(a) $\mathrm{cd}(\alpha) = \mathrm{cd}(\tau(\alpha))$ *for any* $\alpha \in \mathcal{F}_1(E)$.
(b) $\mathrm{cd}(\mathcal{F}_1) \leq \mathrm{cd}(\mathcal{F}_2)$.
(c) *Moreover, if τ is surjective then* $\mathrm{cd}(\mathcal{F}_1) = \mathrm{cd}(\mathcal{F}_2)$.

Proof. Since τ has trivial kernel, α and $\tau(\alpha)$ have the same splitting fields and hence, the same generic splitting fields. This proves part (a). Parts (b) and (c) follow from part (a) and the definition of $\mathrm{cd}(\mathcal{F})$. $\square$

10.3. Example. Recall that the cohomology set $H^1(\_, \mathrm{PSO}_{2n})$ classifies pairs (A, σ), where A is a central simple algebras of degree $2n$ with an orthogonal involution σ of determinant 1; see [KMRT, p. 405]. (Note that [KMRT] uses the symbol PGO^+

instead of PSO.) Consider the morphism of functors $f\colon H^1(\_, \mathrm{SO}_{2n}) \longrightarrow H^1(\_, \mathrm{PSO}_{2n})$ sending a quadratic form q of dimension $2n$ to the pair $(\mathrm{M}_{2n}(K), \sigma_q)$, where σ_q is the involution of $\mathrm{M}_{2n}(K)$ associated to q. We claim that f has trivial kernel. Indeed, $q \in \mathrm{Ker}(K) \Longleftrightarrow q$ gives rise to the standard (transposition) involution on $\mathrm{M}_{2n}(K)$ $\Longleftrightarrow q$ is the $2n$-dimensional form $\langle a, a, \ldots, a, a\rangle$ for some $a \in K^*$; cf. e.g., [KMRT, p.14]. On the other hand, since we are assuming that K contains an algebraically closed base field k of characteristic zero (and in particular, K contains a primitive 4th root of unity), the form $\langle a, a\rangle$ is hyperbolic (cf., e.g., [Lam, Theorem I.3.2]), and hence, so is q. This shows that f has trivial kernel, as claimed. Lemma 10.2(b) now tells us that $\mathrm{cd}(\mathrm{PSO}_{2n}) \geqslant \mathrm{cd}(\mathrm{SO}_{2n})$.

10.4. Example. The exact sequence

$$1 \longrightarrow \mu_2 \longrightarrow \mathrm{Spin}_n \xrightarrow{\pi} \mathrm{SO}_n \longrightarrow 1$$

of algebraic groups gives rise to the exact sequence

$$\mathrm{SO}_n(\_) \xrightarrow{\delta} H^1(\_, \mu_2) \longrightarrow H^1(\_, \mathrm{Spin}_n) \xrightarrow{\pi_*} H^1(\_, \mathrm{SO}_n)$$

of cohomology sets, where δ is the spinor norm; see, e.g., [Gar, p. 688]. Since -1 is a square in k, the unit form is hyperbolic, hence δ is surjective and thus π_* has trivial kernel. On the other hand, the image of π_* consists of quadratic forms q of discriminant 1 such that

$$q' = \begin{cases} q & \text{if } n \text{ is even}, \\ q \oplus \langle 1\rangle & \text{if } n \text{ is odd} \end{cases}$$

has trivial Hasse–Witt invariant. Thus $\mathrm{cd}(\mathrm{Spin}_n) = \mathrm{cd}(\mathrm{HW}_n)$, where HW_n is the set of n-dimensional quadratic forms q such that q' has trivial discriminant and trivial Hasse–Witt invariant.

10.5. Example. Define the functors Pf_r and GPf_r by $\mathrm{Pf}_r(E) = r$-fold Pfister forms defined over E and $\mathrm{GPf}_r(E) =$ scaled r-fold Pfister forms defined over E. In other words,

$$\mathrm{GPf}_r(E) = \{\langle c\rangle \otimes q \mid c \in E^*,\ q \in \mathrm{Pf}_r(E)\}.$$

Taking $c = 1$ above, we see that Pf_r is a subfunctor of GPf_r; hence, $\mathrm{cd}(\mathrm{Pf}_r) \leqslant \mathrm{cd}(\mathrm{GPf}_r)$. On the other hand, since q and $\langle c\rangle \otimes q$ have the same splitting fields for every $q \in \mathrm{Pf}_r(E)$ and every $c \in E^*$, we actually have equality $\mathrm{cd}(\mathrm{Pf}_r) = \mathrm{cd}(\mathrm{GPf}_r)$.

Now suppose $q \in \mathrm{Pf}_r(E)$. Let q' be a subform of q of dimension $2^{r-1} + 1$. The argument in [KS, p. 29] shows that $K = E(q')$ is a generic splitting field for α. Recall that $E(q')$ is defined as the function field of the quadric hypersurface $q' = 0$ in $\mathbb{P}_E^{2^{r-1}}$; in particular, $\operatorname{tr deg}_E E(q) = 2^{r-1} - 1$. Proposition 9.7 now tells us that $\operatorname{cd}(q) \leqslant 2^{r-1} - 1$. On the other hand, if q is anisotropic, a theorem of Karpenko and Merkurjev [KM, Theorem 4.3] tells us that, in fact $\operatorname{cd}(q) = 2^{r-1} - 1$. We conclude that

$$\operatorname{cd}(\mathrm{GPf}_r) = \operatorname{cd}(\mathrm{Pf}_r) = 2^{r-1} - 1. \tag{10.6}$$

We remark that the setting considered by Karpenko and Merkurjev [KM] is a bit different from ours in that they call a field K/E splitting for a quadratic form q/E if q_K is isotropic, where as we use this term to indicate that q_K is hyperbolic. However, if q is a Pfister form then the two definitions coincide; cf., e.g., [Lam, Corollary 10.1.6].

10.7. Example. Consider the exceptional group G_2. The functors $H^1(\_, G_2)$ and Pf_3 are isomorphic; see, e.g., [KMRT, Corollary 33.20]. Hence, $\operatorname{cd}(G_2) = \operatorname{cd}(\mathrm{Pf}_3) = 3$; see 10.6.

11. Groups of type A

In this section we will study canonical dimensions of the groups GL_n/μ_d and SL_n/μ_e, where e divides n. We define the functor

$$C_{n,e} \colon \mathbf{Fields}_k \longrightarrow \mathbf{Sets}^* \tag{11.1}$$

by $C_{n,e}(E/k) = \{$isomorphism classes of central simple E-algebras of degree n and exponent dividing $e\}$. The marked element in $C_{n,e}(E/k)$ is the split algebra $\mathrm{M}_n(E)$. Clearly, $C_{n,e}$ is a subfunctor of $H^1(\_, \mathrm{PGL}_n)$.

11.2. Lemma. *Let n and d be positive integers and e be their greatest common divisor. Then* $\operatorname{cd}(\mathrm{GL}_n/\mu_d) = \operatorname{cd}(\mathrm{GL}_n/\mu_e) = \operatorname{cd}(\mathrm{SL}_n/\mu_e) = \operatorname{cd}(C_{n,d}) = \operatorname{cd}(C_{n,e})$.

Proof. By Lemma 2.6, there are surjective morphisms of functors

$$\begin{aligned} &H^1(\_, \mathrm{GL}_n/\mu_d) \longrightarrow C_{n,d}, \\ &H^1(\_, \mathrm{GL}_n/\mu_e) \longrightarrow C_{n,e} \quad \text{and} \\ &H^1(\_, \mathrm{SL}_n/\mu_e) \longrightarrow C_{n,e} \end{aligned}$$

with trivial kernels. Basic properties of the index and the exponent of a central simple algebra tell us that $C_{n,d} = C_{n,e}$; the rest follows from Lemma 10.2(c). $\square$

11.3. Lemma. *Let n and e be positive integers such that e divides n,*

(a) *If $e' \mid e$ and $n' \mid n$ then* $\operatorname{cd}(\mathrm{SL}_n/\mu_e) \geqslant \operatorname{cd}(\mathrm{SL}_{n'}/\mu_{e'})$.

(b) *Suppose* $n = n_1 n_2$ *and* $e = e_1 e_2$, *where* $e_i \mid n_i$ *and* n_1, n_2 *are relatively prime. Then*

$$\mathrm{cd}(\mathrm{SL}_n/\mu_e) = \mathrm{cd}(\mathrm{SL}_{n_1}/\mu_{e_1} \times \mathrm{SL}_{n_2}/\mu_{e_2}) \leqslant \mathrm{cd}(\mathrm{SL}_{n_1}/\mu_{e_1}) + \mathrm{cd}(\mathrm{SL}_{n_2}/\mu_{e_2})$$

(c) *Let* $n = \prod p^{a_p}$ *be the prime factorization of* n (*here the product is taken over all primes* p *and* $a_p = 0$ *for all but finitely many primes*) *and* $m = \prod_{p|e} p^{a_p}$. *Then* $\mathrm{cd}(\mathrm{SL}_n/\mu_e) = \mathrm{cd}(\mathrm{SL}_m/\mu_e)$.

Proof. (a) The morphism of functors $C_{n',e'} \longrightarrow C_{n,e}$ given by $A \mapsto \mathrm{M}_{\frac{n}{n'}}(A)$ has trivial kernel. By Lemma 10.2(b), $\mathrm{cd}(C_{n,e}) \geqslant \mathrm{cd}(C_{n',e'})$. The desired inequality now follows from Lemma 11.2.

(b) First note that the functors

$$H^1(\_, \mathrm{SL}_{n_1}/\mu_{e_1} \times \mathrm{SL}_{n_2}/\mu_{e_2}) \text{ and } H^1(\_, \mathrm{SL}_{n_1}/\mu_{e_1}) \times H^1(\_, \mathrm{SL}_{n_2}/\mu_{e_2})$$

are isomorphic. Thus by Lemma 2.6, there is a surjective morphism of functors

$$H^1(\_, \mathrm{SL}_{n_1}/\mu_{e_1} \times \mathrm{SL}_{n_2}/\mu_{e_2}) \longrightarrow C_{n_1,e_1} \times C_{n_2,e_2}$$

with trivial kernel. By Lemma 10.2(c),

$$\mathrm{cd}(\mathrm{SL}_{n_1}/\mu_{e_1} \times \mathrm{SL}_{n_2}/\mu_{e_2}) = \mathrm{cd}(C_{n_1,e_1} \times C_{n_2,e_2})$$

and by Lemma 11.2, $\mathrm{cd}(\mathrm{SL}_n/\mu_e) = \mathrm{cd}(C_{n,e})$. The equality in part (b) now follows from the fact that for relatively prime n_1 and n_2 the functors $C_{n_1,e_1} \times C_{n_2,e_2}$ and $C_{n,e}$ are isomorphic via $(A_1, A_2) \mapsto A_1 \otimes A_2$. The inequality in part (b) is a special case of Lemma 7.5(d).

(c) By Lemma 11.2, $\mathrm{cd}(\mathrm{SL}_n/\mu_e) = \mathrm{cd}(C_{n,e})$ and $\mathrm{cd}(\mathrm{SL}_m/\mu_e) = \mathrm{cd}(C_{m,e})$. On the other hand, basic properties of the index and the exponent of a central simple algebra tell us that the functors $C_{m,e}$ and $C_{n,e}$ are isomorphic via $A \mapsto \mathrm{M}_{n/m}(A)$. □

11.4 Theorem. *Let* $\alpha \in H^1(E, \mathrm{PGL}_n)$ *be the class of a division algebra* A *of degree* $n = p^i$, *where* p *is a prime. Then* $\mathrm{cd}(\alpha) = n - 1$.

Let X be the Brauer–Severi variety of A. By a theorem of Karpenko [K_1, Theorem 2.1] every rational map $X \dashrightarrow X$ defined over E is necessarily dominant. (For a related stronger result due to Merkurjev, see [M_2, Section 7.2].) Theorem 11.4 is an easy consequence of this fact; we outline the argument below.

Proof of Theorem 11.4. The function field $K = E(X)$ is a generic splitting field for A; in particular, as we pointed out in Example 9.9, $\mathrm{cd}(\alpha) \leqslant n-1$. To prove the opposite inequality, assume the contrary: A has a generic splitting field L/E of transcendence degree $< n - 1$. Let Y be a variety (defined over E) with function field L. Since

$k(X)/E$ and K/E are both generic splitting fields for α, each is a specialization of the other. Arguing as in the proof of Proposition 9.7, we see that there exist rational maps $f_1: X \dashrightarrow Y$ and $f_2: Y \dashrightarrow X$ such that the diagram

$$\begin{tikzcd} X \arrow[r, dashed, "f_1"] \arrow[dr, dashed] & Y \arrow[r, dashed, "f_2"] \arrow[d, dashed] & X \arrow[dl, dashed] \\ & X/G & \end{tikzcd}$$

commutes. Moreover, after replacing Y by the closure of the graph of f_2 in $Y \times X$, we may assume that f_2 is regular. Now $f_2 \circ f_1$ is a well-defined rational map $X \dashrightarrow X$, and

$$\dim\ f_2 \circ f_1(X) \leqslant \dim(Y) < n - 1 = \dim(X),$$

contradicting Karpenko's theorem. This concludes the proof of Theorem 11.4. □

11.5. Corollary. *Suppose* $n = p^i n_0$ and $e = p^j$, *where* $\gcd(p, n_0) = 1$ *and* $i \geqslant j$. *Then* $\mathrm{cd}(\mathrm{SL}_n/\mu_e) = \begin{cases} 0 & \text{if } j = 0, \\ p^i - 1 & \text{if } j \geqslant 1. \end{cases}$

Proof. If $j = 0$ then $\mathrm{SL}_n/\mu_e = \mathrm{SL}_n$ is special and hence, has canonical dimension 0. Thus we only need to consider the case where $j \geqslant 1$. By Lemma 11.3(c), $\mathrm{cd}(\mathrm{SL}_n/\mu_e) = \mathrm{cd}(\mathrm{SL}_{p^i}/\mu_{p^j})$. Thus we may also assume that $n_0 = 1$, i.e., $n = p^i$.

By Lemma 11.2, $\mathrm{cd}(\mathrm{SL}_n/\mu_e) = \mathrm{cd}(C_{n,e})$. Since $C_{n,e}$ is, by definition, a subfunctor of $H^1(\_, \mathrm{PGL}_n)$, Example 7.6 tells us that $\mathrm{cd}(\mathrm{SL}_n/\mu_e) \leqslant n - 1 = p^i - 1$. To prove the opposite inequality, let A be a division algebra of degree p^i and exponent p^j (such algebras are known to exist; see, e.g., [Row, Appendix 7C]) and let α be the class of A in $H^1(E, \mathrm{PGL}_n)$. Then by Theorem 11.4, $\mathrm{cd}(\mathrm{SL}_n/\mu_e) \geqslant \mathrm{cd}(\alpha) = n - 1$, as desired. □

11.6. Corollary. *Let* $n = 2^i n_0$, *where* $i \geqslant 1$ *and* n_0 *is odd. Then* $\mathrm{cd}(\mathrm{PSp}_n) = 2^i - 1$.

Here PSp stands for the projective symplectic group. Note that these groups are sometimes denoted by the symbol PGSp; see, e.g., [KMRT, p. 347].

Proof of Corollary 11.6. Recall that every $\alpha \in H^1(\_, \mathrm{PSp}_n)$ is represented by a pair (A, σ), where A is a central simple algebra of degree $2n$ and exponent $\leqslant 2$, and σ is a symplectic involution on A. A central simple algebra has a symplectic involution if and only if its exponent is 1 or 2; moreover, a symplectic involution of a split algebra is necessarily hyperbolic. In other words, the morphism of functors

$$H^1(\_, \mathrm{PSp}_n) \longrightarrow C_{n,2},$$

given by $\alpha \mapsto A$ is surjective and has trivial kernel. Here $C_{n,2}$ is the functor of central simple algebras of degree n and exponent dividing 2, as in (11.1). Thus

$$\mathrm{cd}(\mathrm{PSp}_n) = \mathrm{cd}(C_{n,2}) \overset{\text{by Lemma 11.2}}{=} \mathrm{cd}(\mathrm{SL}_n/\mu_2) \overset{\text{by Corollary 11.5}}{=} 2^i - 1,$$

as claimed. $\square$

12. Orthogonal and spin groups

12.1. Lemma. (a) $\mathrm{cd}(\mathrm{SO}_{n-1}) \leqslant \mathrm{cd}(\mathrm{SO}_n)$ *for every* $n \geqslant 2$. *Moreover, equality holds if* n *is even.*

(b) $\mathrm{cd}(\mathrm{Spin}_{n-1}) \leqslant \mathrm{cd}(\mathrm{Spin}_n)$ *for every* $n \geqslant 2$. *Moreover, equality holds if* n *is even.*

(c) $\mathrm{cd}(\mathrm{SO}_n) \geqslant \mathrm{cd}(\mathrm{Spin}_n)$ *for every* $n \geqslant 2$. *Moreover, if* $n \geqslant 2^r$, *where* $r \geqslant 3$ *is an integer, then* $\mathrm{cd}(\mathrm{Spin}_n) \geqslant 2^{r-1} - 1$.

Proof. (a) The morphism $\tau\colon H^1(\_, \mathrm{SO}_{n-1}) \to H^1(\_, \mathrm{SO}_n)$, sending a quadratic form q to $\langle 1 \rangle \oplus q$ has trivial kernel. Lemma 10.2(b) now tells us that $\mathrm{cd}(\mathrm{SO}_{n-1}) \leqslant \mathrm{cd}(\mathrm{SO}_n)$.

To prove the opposite inequality for n is even, let $q = \langle a_1, \ldots, a_n \rangle \in \mathrm{SO}_n$. Then $q = \langle a_1 \rangle \otimes \tilde{q}$, where $\tilde{q} = \langle 1, a_1 a_2, \ldots, a_1 a_n \rangle$ lies in the image of τ. (Note that here we use the assumption that n is even to conclude that $\tilde{q}$ has discriminant 1.) Since q and $\tilde{q}$ have the same splitting fields, $\mathrm{cd}(q) = \mathrm{cd}(\tilde{q})$. On the other hand, since $\tilde{q}$ lies in the image of τ, Lemma 10.2(a) tells us that $\mathrm{cd}(\tilde{q}) \leqslant \mathrm{cd}(\mathrm{SO}_{n-1})$. Thus $\mathrm{cd}(q) \leqslant \mathrm{cd}(\mathrm{SO}_{n-1})$ and consequently, $\mathrm{cd}(\mathrm{SO}_n) \leqslant \mathrm{cd}(\mathrm{SO}_{n-1})$, as desired.

(b) is proved by the same argument as (a), using the identity $\mathrm{cd}(\mathrm{Spin}_n) = \mathrm{cd}(\mathrm{HW}_n)$ of Example 10.4. The first assertion follows from the fact that τ restricts to a morphism $\mathrm{HW}_{n-1} \longrightarrow \mathrm{HW}_n$. In the proof of the second assertion, the key point is that if a quadratic form $q = \langle a_1 \rangle \otimes \tilde{q}$, of even dimension n, has trivial discriminant and trivial Hasse–Witt invariant then so does $\tilde{q}$; the rest of the argument goes through unchanged.

(c) The first inequality follows from the fact that HW_n is a subfunctor of $H^1(\_, \mathrm{SO}_n)$. To prove the second inequality, note that by part (b) we may assume $n = 2^r$. Since the discriminant and the Hasse–Witt invariant of an r-fold Pfister form are both trivial for any $r \geqslant 3$, we see that Pf_r is a subfunctor of HW_n and thus

$$\mathrm{cd}(\mathrm{Spin}_n) = \mathrm{cd}(\mathrm{HW}_n) \geqslant \mathrm{cd}(\mathrm{Pf}_r) = 2^{r-1} - 1;$$

see Example 10.5. $\square$

12.2. Example. (a) $\mathrm{cd}(\mathrm{SO}_n) = \begin{cases} 0 & \text{if } n = 1 \text{ or } 2, \\ 1 & \text{if } n = 3 \text{ or } 4, \\ 3 & \text{if } n = 5 \text{ or } 6. \end{cases}$

(b) $\mathrm{cd}(\mathrm{Spin}_n) = 0$ for $n = 3$, 4, 5 or 6.

(c) $\mathrm{cd}(\mathrm{Spin}_n) = 3$ for $n = 7$, 8, 9 or 10.

Proof. (a) Note that $\mathrm{SO}_1 = \{1\}$, $\mathrm{SO}_3 \simeq \mathrm{PGL}_2$, and $\mathrm{SO}_6 \simeq \mathrm{SL}_4/\mu_2$. Hence, $\mathrm{cd}(\mathrm{SO}_1) = 0$, and (by Corollary 11.5) $\mathrm{cd}(\mathrm{SO}_3) = 1$ and $\mathrm{cd}(\mathrm{SO}_6) = 3$. The remaining cases follow from Lemma 12.1(a).

(b) In view of Lemma 12.1(b), it is enough to show that $\mathrm{cd}(\mathrm{Spin}_6) = 0$. By the Arason–Pfister theorem [Lam, Theorem 10.3.1], the only six-dimensional form with trivial discriminant and trivial Hasse–Witt invariant is the split form. In other words, HW_6 is the trivial functor and thus

$$\mathrm{cd}(\mathrm{Spin}_6) = \mathrm{cd}(\mathrm{HW}_6) = 0.$$

Alternative proof of (b): Exceptional isomorphisms of simply connected simple groups tell us that $\mathrm{Spin}_3 \simeq \mathrm{SL}_2$, $\mathrm{Spin}_5 \simeq \mathrm{Sp}_4$ and $\mathrm{Spin}_6 \simeq \mathrm{SL}_4$ are all special and hence, have canonical dimension 0; cf. Lemma 7.5(c).

(c) Using the Arason–Pfister theorem once again, we see that every eight-dimensional quadratic form with trivial discriminant and trivial Hasse–Witt invariant, is a scaled Pfister form; see [Lam, Corollary 10.3.3]. Thus

$$\mathrm{cd}(\mathrm{Spin}_7) = \mathrm{cd}(\mathrm{Spin}_8) = \mathrm{cd}(\mathrm{GPf}_3) = 3;$$

see Example 10.5. On the other hand, by a theorem of Pfister every $q \in \mathrm{HW}_{10}$ is isotropic, i.e., has the form $\langle 1, -1 \rangle \oplus q'$, where $q' \in \mathrm{HW}_8$; see [Pf, Proof of Satz 14] (cf. also [KM, Theorem 4.4]).

Applying Lemma 10.2(a) to the morphism $\mathrm{HW}_8 \longrightarrow \mathrm{HW}_{10}$ given by $q' \longrightarrow \langle 1, -1 \rangle \oplus q'$, we see that

$$\mathrm{cd}(q) = \mathrm{cd}(q') \leqslant \mathrm{cd}(\mathrm{HW}_8) = \mathrm{cd}(\mathrm{Spin}_8).$$

This shows that $\mathrm{cd}(\mathrm{Spin}_9) = \mathrm{cd}(\mathrm{Spin}_{10}) \leqslant \mathrm{cd}(\mathrm{Spin}_8) = 3$. The opposite inequality is given by Lemma 12.1(b). □

12.3. Proposition. $\mathrm{cd}(\mathrm{SO}_{2m}) \leqslant \frac{m(m-1)}{2}$ *for every* $m \geqslant 1$.

Proof. Write $\mathrm{SO}_{2m} = \mathrm{SO}(q)$, where q is a non-degenerate quadratic form on k^{2m}. Let $X = \mathrm{Gr}_{iso}(m, 2m)$ be the Grassmannian of maximal (i.e., m-dimensional) q-isotropic subspaces of k^{2m}, i.e., of m-dimensional subspaces contained in the quadric $Q \subset k^{2m}$ given by $q = 0$. It is well known that X is a projective variety with two irreducible components X_1 and X_2, each of dimension $\frac{m(m-1)}{2}$; see e.g., [GH, Section 6.1]. Using the Witt Extension Theorem (see, e.g., [Lam, p. 26]), it is easy to see that the full orthogonal group $O(q)$ acts transitively on X and $SO(q)$ acts transitively on each component X_i $(i = 1, 2)$. Fix an isotropic subspace $L \in X_1$ and let $P = \mathrm{Stab}_{SO(q)}(L)$. By Lemma 7.5(a), with $G = SO(q)$ and $H = P$, we have

$$\begin{aligned}\mathrm{cd}(\mathrm{SO}_{2m}) &\leqslant \mathrm{cd}(P) + \dim(\mathrm{SO}_{2m}) - \dim(P)\\ &= \mathrm{cd}(P) + \dim(X_1) = \mathrm{cd}(P) + \frac{m(m-1)}{2}.\end{aligned}$$

It remains to show that $\mathrm{cd}(P) = 0$. We claim that the Levi subgroup of P is naturally isomorphic to $\mathrm{GL}(L) \simeq \mathrm{GL}_m$ via $f : P \longrightarrow \mathrm{GL}(L)$, where $f(g) = g_{|L}$. Once this claim is established, Example 10.1 tells us that $\mathrm{cd}(P) = \mathrm{cd}(\mathrm{GL}_m) = 0$. (The last equality follows from the fact that GL_m is special.)

To prove the claim, note that by the Witt Extension Theorem, f is a surjective homomorphism. It remains to show that $\mathrm{Ker}(f)$ is unipotent. Indeed, choose a basis $e_1, \ldots, e_{2m}$ of k^{2m} so that

$$q(x_1e_1 + \cdots + x_{2m}e_{2m}) = x_1x_{m+1} + \cdots + x_mx_{2m}$$

and L is the span of $e_1, \ldots, e_m$. Then every $g \in \mathrm{Ker}(f)$ has the form

$$g = \begin{pmatrix} I_m & A \\ O_m & B \end{pmatrix}, \tag{12.4}$$

for some $m \times m$-matrices A and B. (Here O_m and I_m are, respectively, the zero and the identity $m \times m$-matrices.) The condition that $g \in O(q)$ translates into

$$g \begin{pmatrix} O_m & I_m \\ I_m & O_m \end{pmatrix} g^{\mathrm{Transpose}} = \begin{pmatrix} O_m & I_m \\ I_m & O_m \end{pmatrix}. \tag{12.5}$$

Substituting (12.4) into (12.5), we see that $B = I_m$. Formula (12.4) now shows that g is unipotent; consequently, $\mathrm{Ker}(f)$ is a unipotent group, as claimed. $\square$

12.6. Conjecture. $\mathrm{cd}(\mathrm{SO}_{2m-1}) = \mathrm{cd}(\mathrm{SO}_{2m}) = \frac{m(m-1)}{2}$ *for every* $m \geqslant 1$.

12.7. Remark. Conjecture 12.6 was recently proved by Karpenkoe [K_2]; for an alternative proof due to Vishik, see [V].

13. Groups of low canonical dimension

13.1. Theorem. *Assume that G is simple. Then* $\mathrm{cd}(G) = 1$ *if and only if* $G \simeq \mathrm{SL}_{2m}/\mu_2$ *or* PSp_{2m}, *where m is an odd integer.*

Proof. First of all, observe that if $G \simeq \mathrm{SL}_{2m}/\mu_2$ or PSp_{2m}, with m odd, then indeed, $\mathrm{cd}(G) = 1$; see Corollary 11.5 and Corollary 11.6. Thus we only need to show that no other simple group has this property. Our proof relies on the classification of simple algebraic groups; cf., e.g., [KMRT, §24 and 25]. (Note that [KMRT] uses the symbols O^+ and PGSp instead of SO and PSp. Recall also that we are working over an algebraically closed base field k of characteristic zero.) We begin by observing that $\mathrm{cd}(G) \geqslant 2$ for every simple group of exceptional type; see Examples 8.5, 8.6 and 10.7.

Now suppose $\mathrm{cd}(G) = 1$ and G is of type A. Then $G \simeq \mathrm{SL}_n/\mu_e$, where e divides n. Let p be a prime dividing e. Then we can write $e = p^j e_0$ and $n = p^i n_0$, where $i \geqslant j \geqslant 1$, and $\gcd(p, e_0) = \gcd(p, n_0) = 1$. Then

$$1 = \mathrm{cd}(G) \overset{\text{by Lemma 11.3(a)}}{\geqslant} \mathrm{SL}_{p^i}/\mu_{p^j} \overset{\text{by Corollary 11.5}}{\geqslant} p^i - 1,$$

which is only possible if $p = 2$ and $i = 1$. This implies that e cannot be divisible by 4 or by any prime $p \geqslant 3$; in other words, $e = 1$ or 2. If $e = 1$ then $G = \mathrm{SL}_n$ is special and thus $\mathrm{cd}(G) = 0$. If $e = 2$ then $i = 1$ implies that $n \equiv 2 \pmod 4$, as claimed.

Next suppose G is of type C. Then G is isomorphic to Sp_{2m} or PSp_{2m}. The groups Sp_{2m} are special and thus have canonical dimension 0. By Corollary 11.6, $\mathrm{cd}(\mathrm{PSp}_{2m}) = 1$ if and only if m is odd. This completes the proof of Theorem 13.1 for groups of type A or C.

Now suppose G is of type B or D. We have already considered some of these groups. In particular,

- $\mathrm{cd}(\mathrm{SO}_n) \geqslant \mathrm{cd}(\mathrm{SO}_5) = 3$ for any $n \geqslant 5$ (see Lemma 12.1(a) and Example 12.2(a)),
- $\mathrm{cd}(\mathrm{PSO}_{2n}) \geqslant \mathrm{cd}(\mathrm{SO}_{2n}) \geqslant 3$ for any $n \geqslant 3$ (see Example 10.3),
- $\mathrm{cd}(\mathrm{Spin}_n) = 0$ for $n = 3, 4, 5, 6$ (see Example 12.2(b)), and
- $\mathrm{cd}(\mathrm{Spin}_n) \geqslant \mathrm{cd}(\mathrm{Spin}_7) = 3$ for any $n \geqslant 7$ (see Lemma 12.1 and Example 12.2(c)).

We also remark that $\mathrm{SO}_2 \simeq \mathbb{G}_m$ and $\mathrm{SO}_4 \simeq (\mathrm{SL}_2 \times \mathrm{SL}_2)/\mu_2$ are not simple, and $\mathrm{SO}_3 \simeq \mathrm{PGL}_2 = \mathrm{SL}_2/\mu_2$ was considered above.

This covers every simple group of type B; the only simple groups of type D we have not yet considered are $G = \mathrm{Spin}_{4n}^{\pm}$ $(n \geqslant 2)$; cf., e.g., [KMRT, Theorems 25.10 and 25.12]. The natural projection $\pi\colon \mathrm{Spin}_{4n} \longrightarrow \mathrm{SO}_{4n}$ factors through $\mathrm{Spin}_{4n}^{\pm}$:

$$\pi\colon \mathrm{Spin}_{4n} \xrightarrow{f} \mathrm{Spin}_{4n}^{\pm} \longrightarrow \mathrm{SO}_{4n}.$$

Since $\pi_*\colon H^1(\_, \mathrm{Spin}_{4n}) \longrightarrow H^1(\_, \mathrm{SO}_{4n})$ has trivial kernel (see Example 10.4), so does $f_*\colon H^1(\_, \mathrm{Spin}_{4n}) \longrightarrow H^1(\_, \mathrm{Spin}_{4n}^{\pm})$. Now for any $n \geqslant 2$,

$$\mathrm{cd}(\mathrm{Spin}_{4n}^{\pm}) \overset{\text{by Lemma 10.2(b)}}{\geqslant} \mathrm{cd}(\mathrm{Spin}_{4n}) \overset{\text{by Lemma 12.1(b)}}{\geqslant} \mathrm{cd}(\mathrm{Spin}_8) \overset{\text{by Example 12.2(c)}}{=} 3.$$

This completes the proof of Theorem 13.1. $\square$

13.2. Remark. The above argument also shows that if a simple classical group G has canonical dimension 2 then either (i) $G \simeq \mathrm{SL}_{3m}/\mu_3$, where m is prime to 3 or possibly (ii) $G \simeq \mathrm{SL}_{6m}/\mu_6$, where m is prime to 6. In case (i), we know that $\mathrm{cd}(G) = \mathrm{cd}(\mathrm{PGL}_3) = 2$; see Corollary 11.5. In case (ii), $\mathrm{cd}(G) = \mathrm{cd}(\mathrm{PGL}_6)$ (see Corollary 11.3(c)); we do not know whether this number is 2 or 3.

14. The functor of orbits and homogeneous forms

We now briefly recall the definition of essential dimension of a functor, due to Merkurjev [M1].

Let $\mathcal{F}$ be a functor from the category of all field extensions K of k to the category of sets. (For our purposes, it is sufficient to consider only finitely generated extensions K/k.) Given $\alpha \in \mathcal{F}(K)$, we define $\mathrm{ed}(\alpha)$ as the minimal value of $\mathrm{tr\,deg}_k(K_0)$, where $k \subset K_0 \subset K$ and α lies in the image of the natural map $\mathcal{F}(K_0) \longrightarrow \mathcal{F}(K)$. The essential dimension $\mathrm{ed}(\mathcal{F})$ of the functor $\mathcal{F}$ is then defined as the maximal value of $\mathrm{ed}(\alpha)$, as α ranges over $\mathcal{F}(K)$ and K ranges over all field extensions of k. In the special case, where G is an algebraic group and $\mathcal{F} = H^1(\_, G)$, we recover the numbers defined in Section 2.1: $\mathrm{ed}(\alpha) = \mathrm{ed}(X, G)$, where $\alpha \in H^1(K, G)$ and X is a generically free G-variety representing α. Moreover, $\mathrm{ed}(H^1(\_, G)) = \mathrm{ed}(G)$. For details, see [BF2].

Now to each G-variety X we will associate the functor $\mathbf{Orb}_{X,G}$ given by $\mathbf{Orb}_{X,G}(L) = X(L)/\sim$, where $a \sim b$ for $a, b \in X(L)$, if $a = g \cdot b$ for some $g \in G(L)$. Given an L-point $a \in X(L)$, we shall denote $a \pmod{\sim}$ by $[a] \in \mathbf{Orb}_{X,G}(L)$. Using this terminology, Definition 3.5 can be rewritten as follows.

14.1. Proposition. $\mathrm{ed}\,[\eta] = \mathrm{cd}(X, G) + \dim X/G$, *where* $\eta \in X(k(X))$ *is the generic point of* X.

Proof. Let Y be a variety with function field $k(Y) = L$. Then $z \in X(L)$ may be viewed as a rational map $\phi_z \colon Y \dashrightarrow X$ and $g \in G(L)$ as a rational map $f_g \colon Y \dashrightarrow G$. The point $g \cdot z$ of $X(L)$ corresponds to the map $F_{z,g} \colon Y \dashrightarrow X$ given by $F_{z,g}(y) = f_g(y) \cdot \phi_z(y)$. Consequently, the definition of $\mathrm{ed}\,[z]$ can be rewritten as

$$\mathrm{ed}\,[z] = \min_{g \in G(L)} \{\, \mathrm{tr\,deg}_k\, k(F_{z,g}(Y)) \,\}. \tag{14.2}$$

Now set $z = \eta$, $L = k(X)$, $Y = X$, and $\phi = id_X$. The element $g \in G(L)$ is then a rational map $f = f_g \colon X \dashrightarrow G$, $F = F_{z,g} \colon X \dashrightarrow X$ is, by definition, a canonical form map (see Definition 3.1) and the proposition follows from (14.2) and Definition 3.5. $\square$

14.3. Remark. By the definition of $\mathrm{ed}(\mathbf{Orb}_{X,G})$, we have

$$\mathrm{ed}(\mathbf{Orb}_{X,G}) \geqslant \mathrm{ed}\,[\eta].$$

We do not know whether or not equality holds in general.

For the rest of this paper we will focus on the following example. Let $N = \binom{n+d-1}{d}$ and let $X = \mathbb{A}^N$ be the space of degree d forms in n variables $x = (x_1, \ldots, x_n)$. That is, elements of $\mathbb{A}^N$ are forms $p(x_1, \ldots, x_n)$ of degree d and elements of $\mathbb{P}^{N-1}$ are hypersurfaces $p(x_1, \ldots, x_n) = 0$. The generic point of $\mathbb{A}^N$ is the "general'' degree d form in n variables as

$$\phi_{n,d}(x) = \sum_{i_1+\cdots+i_n=d} a_{i_1,\ldots,i_n} x_1^{i_1} \cdots x_n^{i_n} \in K[x_1, \ldots, x_n],$$

where $a_{i_1,\ldots,i_n}$ are independent variables, $K = k(\mathbb{A}^N)$ is the field these variables generate over k. Then

$$\mathrm{ed}\,[\phi_{n,d}] = \min_{g \in \mathrm{GL}_n(K)} \operatorname{tr\,deg}_k k(b_{i_1,\ldots,i_d} \,|\, i_1 + \cdots + i_n = d), \tag{14.4}$$

where

$$\phi_{n,d}(g \cdot x) = \sum_{i_1+\cdots+i_n=d} b_{i_1,\ldots,i_n} x_1^{i_1} \ldots x_n^{i_n}.$$

The generic point of $\mathbb{P}^{N-1}$ is the "general'' degree d hypersurface $\phi_{n,d}(x) = 0$ in $\mathbb{P}^{N-1}(K)$, which we denote by $H_{n,d}$.

14.5. Lemma. Let $N = \binom{n+d-1}{d}$ *and*

$$D = \dim(\mathbb{A}^N/\mathrm{GL}_n) = \dim(\mathbb{P}^{N-1}/\mathrm{PGL}_n) = \dim(\mathbb{A}^N/(\mathbb{G}_m \times \mathrm{GL}_n)).$$

(*Here* $\mathbb{G}_m$ *acts on* $\mathbb{A}^N$ *by scalar multiplication.*) *Then*

(a) $\mathrm{ed}\,[\phi_{n,d}] = D + \mathrm{cd}(\mathbb{A}^N, \mathrm{GL}_n)$.

(b) $\mathrm{ed}\,[H_{n,d}] = D + \mathrm{cd}(\mathbb{P}^{N-1}, \mathrm{GL}_n) = D + \mathrm{cd}(\mathbb{P}^{N-1}, \mathrm{PGL}_n) = D + \mathrm{cd}(\mathbb{A}^N, \mathbb{G}_m \times \mathrm{GL}_n)$.

Proof. Part (a) and the first equality in part (b) are immediate consequences of Proposition 14.1. To complete the proof of (b), note that

$$\mathrm{cd}(\mathbb{P}^{N-1}, \mathrm{GL}_n) = \mathrm{cd}(\mathbb{P}^{N-1}, \mathrm{PGL}_n)$$

by Lemma 4.10(b), and

$$\mathrm{cd}(\mathbb{P}^{N-1}, \mathrm{GL}_n) = \mathrm{cd}(\mathbb{A}^N, \mathbb{G}_m \times \mathrm{GL}_n)$$

by Proposition 4.11 (with $H = \mathbb{G}_m \times \{1\}$). $\square$

14.6. Corollary. *Let $K = k(\mathbb{A}^N)$ (as above) and $K_0 = k(\mathbb{P}^N)$. Given $g \in \mathrm{GL}(K)$, let F_g be the field extension of k generated by elements of the form*

$$\frac{b_{i_1,\ldots,i_n}}{b_{j_1,\ldots,j_n}}, \quad \textit{where } \phi_{n,d}(g \cdot x) = \sum_{i_1+\cdots+i_n=d} = b_{i_1,\ldots,i_n} x_1^{i_1} \ldots x_n^{i_n}$$

and $i_1, \ldots, i_n, j_1, \ldots, j_n \geqslant 0$ satisfy *$i_1 + \cdots + i_n = j_1 + \cdots + j_n = d$ and $b_{j_1,\ldots,j_n} \neq 0$. Then*

$$\mathrm{ed}\,[H_{n,d}] = \min_{g \in \mathrm{GL}_n(K_0)} \mathrm{tr\,deg}_k(F_g) = \min_{g \in \mathrm{GL}_n(K)} \mathrm{tr\,deg}_k(F_g).$$

Proof. The first equality is an immediate consequence of the definition of $\mathrm{ed}\,[H_{n,d}]$. To prove the second equality, we use the identity $\mathrm{ed}\,[H_{n,d}] = D + \mathrm{cd}(\mathbb{A}^N, \mathbb{G}_m \times \mathrm{GL}_n)$ of Lemma 14.5. Proposition 14.1 now tells us that

$$\mathrm{ed}\,[H_{n,d}] = \min_{g \in \mathrm{GL}_n(K),\, c \in K^*} \mathrm{tr\,deg}_k(cb_{i_1,\ldots,i_n} \mid i_1 + \cdots + i_n = d).$$

The minimum is clearly attained if $c = b_{j_1,\ldots,j_n}^{-1}$, for some (and thus any) $j_1, \ldots, j_n$ such that $b_{j_1,\ldots,j_n} \neq 0$, and the corollary follows. $\square$

In view of (14.4), it is natural to think of $\mathrm{ed}\,[\phi_{n,d}]$ as the minimal number of independent parameters required to define the general form of degree d in n variables. Corollary 14.6 says that $\mathrm{ed}\,[H_{n,d}]$ can be similarly interpreted as the minimal number of independent parameters required to define the general degree d hypersurface in $\mathbb{P}^{n-1}$. Comparing the expressions for these numbers given by (14.4) and Corollary 14.6, we see that they are closely related.

14.7. Corollary. $\mathrm{ed}\,[H_{n,d}] \leqslant \mathrm{ed}\,[\phi_{n,d}] \leqslant \mathrm{ed}\,[H_{n,d}] + 1$.

14.8. Remark. Lemma 14.5(b) shows that the number $\mathrm{ed}\,[H_{n,d}]$ is the essential dimension of the generic form $\phi_{n,d}$ in the sense of [BF$_1$], i.e., the essential dimension of the $\mathbb{G}_m \times \mathrm{GL}_n$-orbit of the generic point of $\mathbb{A}^N$. Note that the emphasis in [BF$_1$] is on the essential dimension of the functor $\mathbf{Hypersurfaces}_{n,d} = \mathbf{Orb}_{\mathbb{A}^N, \mathbb{G}_m \times \mathrm{GL}_n}$ (which is denoted there by $\mathcal{F}_{d,n}$), and, more specifically, on the functor $\mathbf{Hypersurfaces}_{3,3}$ (which is denoted there by $\mathbf{Cub}_3$). As we pointed out in Remark 14.3, $\mathrm{ed}(\mathbf{Hypersurfaces}_{n,d}) \geqslant \mathrm{ed}\,[H_{n,d}]$ but we do not know whether or not equality holds.

15. Essential dimensions of homogeneous forms I

15.1. Theorem. *Let n and d be positive integers such that $d \geqslant 3$ and $(n, d) \neq (2, 3)$, $(2, 4)$ or $(3, 3)$. Then*

$$\mathrm{ed}\,[H_{n,d}] = N - n^2 + \mathrm{cd}(\mathrm{GL}_n/\mu_d) = N - n^2 + \mathrm{cd}(\mathrm{SL}_n/\mu_{\gcd(n,d)}),$$

where $N = \begin{pmatrix} n+d-1 \\ d \end{pmatrix}$.

Proof. First observe that by Lemma 11.2, $\mathrm{cd}(\mathrm{GL}_n/\mu_d) = \mathrm{cd}(\mathrm{SL}_n/\mu_{\gcd(n,d)})$, so only the first equality needs to be proved.

Secondly, under our assumption on n and d, the PGL_n-action on $\mathbb{P}^{N-1}$ is generically free. For $n = 2$ this is classically known (cf., e.g., [PV, p. 231]), for $n = 3$, this is proved in [B] and for $n \geqslant 4$ in [MM]. Substituting

$$D = \dim(\mathbb{P}^{N-1}/\mathrm{PGL}_n) = \dim(\mathbb{P}^{N-1}) - \dim(\mathrm{PGL}_n) = N - n^2$$

into Lemma 14.5(b), we reduce the theorem to the identity

$$\mathrm{cd}(\mathbb{A}^N, \mathbb{G}_m \times \mathrm{GL}_n) = \mathrm{cd}(\mathrm{GL}_n/\mu_d). \tag{15.2}$$

To prove 15.2, observe that the normal subgroup

$$S = \{(t^{-d}, t) \mid t \in \mathbb{G}_m\} \subset \mathbb{G}_m \times \mathbb{G}_m \subset \mathbb{G}_m \times \mathrm{GL}_n$$

acts trivially on $\mathbb{A}^N$. Since S is special, we have

$$\begin{aligned} &\mathrm{cd}(\mathbb{A}^N, \mathbb{G}_m \times \mathrm{GL}_n) \overset{\text{by Lemma 4.10(b)}}{=} \\ &\mathrm{cd}(\mathbb{A}^N, (\mathbb{G}_m \times \mathrm{GL}_n)/S) \overset{\text{by Definition 7.3}}{=} \mathrm{cd}((\mathbb{G}_m \times \mathrm{GL}_n)/S), \end{aligned} \tag{15.3}$$

where the last equality is a consequence of the fact that the PGL_n-action on $\mathbb{P}^{N-1}$ (and hence, the $(\mathbb{G}_m \times \mathrm{GL}_n)/S$-action on $\mathbb{A}^N$) is generically free.

Finally, consider the homomorphism $\mathrm{GL}_n \longrightarrow (\mathbb{G}_m \times \mathrm{GL}_n)/S$ given by $g \mapsto (1, g)$, modulo S. Since we are working over an algebraically closed field k of characteristic zero, this homomorphism is surjective, and its kernel is exactly μ_d. Thus $(\mathbb{G}_m \times \mathrm{GL}_n)/S \simeq \mathrm{GL}_n/\mu_d$. Combining this with (15.3), we obtain (15.2). $\square$

The results of Section 11 can now be used to determine $\mathrm{ed}\,[H_{n,d}]$ for many values of n and d (and produce estimates for others). In particular, combining Theorem 15.1 with Corollary 11.5, we deduce Theorem 1.1 stated in the Introduction.

The number $\mathrm{ed}\,[\phi_{n,d}]$ appears to be harder to compute than $\mathrm{ed}\,[H_{n,d}]$. By Corollary 14.7, $\mathrm{ed}\,[\phi_{n,d}] = \mathrm{ed}\,[H_{n,d}]$ or $\mathrm{ed}\,[\phi_{n,d}] = \mathrm{ed}\,[H_{n,d}] + 1$, but for general n and d, we do not know which of these cases occurs. One notable exception is the case where n and d are relatively prime.

15.4. Corollary. *Suppose* $d \geqslant 3$, $\gcd(n, d) = 1$ *and* $(n, d) \neq (2, 3)$. *Then*

(a) $\mathrm{ed}\,[H_{n,d}] = \binom{n+d-1}{d} - n^2$ *and*

(b) $\mathrm{ed}\,[\phi_{n,d}] = \binom{n+d-1}{d} - n^2 + 1.$

Proof. Part (a) is a special case of Theorem 1.1 (with $j = 0$). We can also deduce it directly from Theorem 15.1 by noting that SL_n is a special group and thus $\mathrm{cd}(\mathrm{SL}_n) = 0$.

(b) In view of Corollary 14.7, we only need to prove that $\mathrm{ed}\,[\phi_{n,d}] \geqslant \mathrm{ed}\,[H_{n,d}] + 1$ or equivalently, $\mathrm{cd}(\mathbb{A}^N, \mathrm{GL}_n) \geqslant 1$; see Lemma 14.5(a). Recall that the central subgroup μ_d of GL_n acts trivially on $\mathbb{A}^N$, and (under our assumptions on n and d) the induced GL_n/μ_d-action is generically free. Thus the stabilizer in general position for the GL_n-action on $\mathbb{A}^N$ is μ_d, and

$$\mathrm{cd}(\mathbb{A}^N, \mathrm{GL}_n) \overset{\text{by Proposition 5.5(c)}}{\geqslant} \mathrm{ed}(\mu_d) \overset{\text{by [BR, Theorem 6.2]}}{=} 1,$$

as claimed. $\square$

15.5. Remark. We have proved that if n and d are relatively prime then $\mathrm{cd}(\mathbb{P}^{N-1}, \mathrm{GL}_n) = 0$ but $\mathrm{cd}(\mathbb{A}^N, \mathrm{GL}_n) = 1$. In particular, this shows that the equality $\mathrm{cd}(X, G) = \mathrm{cd}(X/H, G)$ of Proposition 4.11 fails for $X = \mathbb{A}^N$, $G = \mathrm{GL}_n$ and $H = \mathbb{G}_m$. Note that Proposition 4.11 does not apply in this situation because the H-action on X is not generically free (the subgroup μ_d acts trivially).

16. Essential dimensions of homogeneous forms II

In this section we will study $\mathrm{ed}\,[\phi_{n,d}]$ and $\mathrm{ed}\,[H_{n,d}]$ for the pairs (n, d) not covered by Theorem 15.1. We begin with a simple lemma.

16.1. Lemma. $\mathrm{ed}\,[H_{2,d}] \leqslant d - 2$ *for any* $d \geqslant 3$.

In the sequel we will only need this lemma for $d = 3$ and 4. For $n \geqslant 5$, Theorem 1.1 (with $n = 2$) gives a stronger result, namely,

$$\mathrm{ed}\,[H_{2,d}] = \begin{cases} d-2 & \text{if } d \text{ is even,} \\ d-3 & \text{if } d \text{ is odd.} \end{cases}$$

However, the proof of the lemma below is valid for any $d \geqslant 3$.

Proof of Lemma 16.1. The linear transformation $g \in \mathrm{GL}_2(K)$ given by

$$x_1 \mapsto x_1 - \frac{a_{d-1,1}}{n\, a_{d,0}}, \quad x_2 \mapsto x_2$$

reduces the generic binary form

$$\phi_{2,d}(x_1, x_2) = a_{d,0}x_1^d + a_{d-1,1}x_1^{d-1}x_2 + \cdots + a_{0,d}x_2^d$$

to

$$\phi_{2,d}(g \cdot (x_1, x_2)) = b_{d,0}x_1^d + b_{d-2,2}x_1^{d-2}x_2^2 + \cdots + b_{1,d-1}x_1x_2^{d-1} + b_{0,d}x_2^d$$

for some $b_{i,d-i} \in K = k(a_{0,d}, \dots, a_{d,0})$. Composing this linear transformation with

$$x_1 \mapsto \frac{b_{0,d}}{b_{1,d-1}}x_1, \quad x_2 \mapsto x_2$$

(and, by abuse of notation, denoting the composition by g once again), we may further assume $b_{1,d-1} = b_{0,d}$. The field $F_g = k(b_{i,d-i}/b_{0,d} \,|\, i = 1, \dots, d)$, defined in the statement of Corollary 14.6, now has transcendence degree $\leqslant d-2$. By Corollary 14.6 we conclude that $\mathrm{ed}\,[H_{2,d}] \leqslant n-2$. $\square$

We are now ready to proceed with the main result of this section.

16.2. Proposition. (a) $\mathrm{ed}\,[\phi_{n,1}] = \mathrm{ed}\,[H_{n,1}] = 0$.
(b) $\mathrm{ed}\,[\phi_{n,2}] = n$ *and* $\mathrm{ed}\,[H_{n,2}] = n-1$.
(c) $\mathrm{ed}\,[\phi_{2,3}] = 2$ *and* $\mathrm{ed}\,[H_{2,3}] = 1$.
(d) $\mathrm{ed}\,[\phi_{2,4}] = 3$ *and* $\mathrm{ed}\,[H_{2,4}] = 2$.
(e) $\mathrm{ed}\,[H_{3,3}] = 3$.

We do not know whether $\mathrm{ed}\,[\phi_{3,3}]$ is 3 or 4.

Proof of Proposition 16.2. (a) A linear form $l(x_1, \dots, x_n)$ over K can be reduced to just x_1 by applying a linear transformation $g \in \mathrm{GL}_n(K)$. Thus $\mathrm{ed}\,[\phi_{n,1}] = \mathrm{ed}\,[H_{n,1}] = 0$.

(b) Here $d = 2$, $N = n(n+1)/2$, and elements of $\mathbb{A}^N$ are quadratic forms in n variables. Diagonalizing the generic quadratic form $\phi_{n,2}$ over K, we see that $\mathrm{ed}\,[\phi_{n,2}] \leqslant n$ and $\mathrm{ed}\,[H_{n,2}] \leqslant n-1$. In view of Corollary (14.7) it suffices to show that $\mathrm{ed}\,[\phi_{n,2}] = n$.

The GL_n-action on $\mathbb{A}^N$ has a dense orbit, consisting of non-singular forms. In particular, $D = \dim(\mathbb{A}^N/\mathrm{GL}_n) = 0$, so that by Lemma 14.5(a)

$$\mathrm{ed}\,[\phi_{n,2}] = \mathrm{cd}(\mathbb{A}^N, \mathrm{GL}_n).$$

Since the stabilizer of a non-singular form is the orthogonal group O_n, $\mathbb{A}^N$ is birationally G-equivariantly isomorphic to $\mathrm{GL}_n/\mathrm{O}_n$. Thus

$$\mathrm{ed}\,[\phi_{n,2}] = \mathrm{cd}(\mathbb{A}^N, \mathrm{GL}_n) = \mathrm{cd}(\mathrm{GL}_n/\mathrm{O}_n, \mathrm{GL}_n) \overset{\text{by Corollary 5.7(b)}}{=} \mathrm{ed}(\mathrm{O}_n) \overset{\text{by [R, Theorem 10.3]}}{=} n.$$

This completes the proof of part (b).

(c) By Lemma 16.1, $\mathrm{ed}\,[H_{2,3}] \leqslant 1$. Thus in view of Corollary 14.7, we only need to show that $\mathrm{ed}\,[\phi_{2,3}] \geqslant 2$.

Here $N = 4$, and the GL_2-action on $\mathbb{A}^4$ has a dense orbit consisting of binary cubic forms with three distinct roots. Applying Lemma 14.5(a), with $D = \dim(\mathbb{A}^4/\mathrm{GL}_2) = 0$, as in part (b), we obtain

$$\mathrm{ed}\,[\phi_{2,3}] = \mathrm{cd}(\mathbb{A}^4, \mathrm{GL}_2) = \mathrm{cd}(\mathrm{GL}_2/S, \mathrm{GL}_2) \overset{\text{by Corollary 5.7(b)}}{=} \mathrm{ed}(S),$$

where $S \subset \mathrm{GL}_2$ is the stabilizer of a binary cubic form with three roots, say of $x^3 + y^3$. Note that S is a finite group and that matrices that multiply x and y by third roots of unity form a subgroup of S isomorphic to $(\mathbb{Z}/3\mathbb{Z})^2$. Thus $\mathrm{ed}(S) \geqslant \mathrm{ed}\,(\mathbb{Z}/3\mathbb{Z})^2 = 2$ (cf. [BR, Lemma 4.1(a) and Theorem 6.1]), as desired.

(d) By Lemma 16.1, $\mathrm{ed}\,[H_{2,4}] \leqslant 2$. In view of Corollary 14.7, it remains to prove the inequality $\mathrm{ed}\,[\phi_{2,4}] \geqslant 3$. Note that since the invariant field $k(\mathbb{A}^5)^{\mathrm{GL}_2}$ is generated by one element (namely, the cross-ratio of the four roots of the quartic binary form), we have $D = \dim(\mathbb{A}^5/\mathrm{GL}_2) = 1$. Thus we only need to show that

$$\mathrm{cd}(\mathbb{A}^5, \mathrm{GL}_2) \geqslant 2. \tag{16.3}$$

Let S be the stabilizer of $f \in \mathbb{A}^5$ (i.e., of a degree 4 binary form) in general position. By Proposition 5.5(c),

$$\mathrm{cd}(\mathbb{A}^5, \mathrm{GL}_2) \geqslant \mathrm{ed}(S).$$

To compute $\mathrm{ed}(S)$, recall that the stabilizer of f in PGL_2 is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$; cf. e.g., [PV, p. 231]. It is now easy to see that S fits into the sequence

$$\{1\} \longrightarrow \mathbb{Z}/4\mathbb{Z} \longrightarrow S \longrightarrow (\mathbb{Z}/2\mathbb{Z})^2 \longrightarrow \{1\}.$$

In particular, S is a finite group which admits a surjective homomorphism onto $(\mathbb{Z}/2\mathbb{Z})^2$. Thus S is neither cyclic nor odd dihedral and consequently, $\mathrm{ed}(S) \geqslant 2$; see [BR, Theorem 6.2(a)]. This concludes the proof of (16.3) and thus of part (d). For the sake of completeness, we remark that since S is a finite subgroup of GL_2, we also have $\mathrm{ed}(S) \leqslant 2$ and thus $\mathrm{ed}(S) = 2$.

(e) Here $N = 10$, and the rational quotient $\mathbb{P}^9/\mathrm{GL}_3$ is the j-line, so that $D = \dim \mathbb{P}^9/\mathrm{GL}_3 = 1$. Thus we only need to show

$$\mathrm{cd}(\mathbb{P}^9, \mathrm{GL}_3) = 2. \tag{16.4}$$

An element of $\mathbb{P}^9$ (i.e., a plane cubic curve) in general position can be written as $F_\lambda = x^3 + y^3 + z^3 + 3\lambda xyz$. Denote the stabilizer of F_λ by $S \subset \mathrm{GL}_3$. We will deduce 16.4 from Corollary 6.2(b). Indeed, let N be the normalizer of S in GL_3. Since GL_3 is a special group, $e(\mathrm{GL}_3, S) = \mathrm{ed}(S)$, $e(\mathrm{GL}_3, N) = \mathrm{ed}(N)$ (see Lemma 5.4(c)), and Corollary 6.2(b) assumes the following form:

$$\mathrm{ed}(S) \leqslant \mathrm{cd}(\mathbb{P}^9, \mathrm{GL}_3) \leqslant \mathrm{ed}(N) - \dim(S) + \dim(N).$$

Let $\overline{S}$ and $\overline{N}$ be the images of S and N in PGL_3, under the natural projection $\mathrm{GL}_3 \longrightarrow \mathrm{PGL}_3$. Note that $\overline{S}$ is a finite group (this follows from the fact that $D = \dim \mathbb{P}^9/\mathrm{PGL}_3 = 1$). In particular, $\dim(S) = 1$. It thus suffices to show:

(e_1) $\mathrm{ed}(S) \geqslant 2$,
(e_2) $\overline{N}$ is a finite subgroup of PGL_3 (and consequently, $\dim(N) = 1$).
(e_3) $\mathrm{ed}(N) \leqslant 2$.

The inequality (e_1) is a consequence of [RY_1, Corollary 7.3], with $G = S$ and

$$H = <\mathrm{diag}(1, \zeta, \zeta^2), \sigma> \simeq (\mathbb{Z}/3\mathbb{Z})^2, \tag{16.5}$$

where σ is a cyclic permutation of the variables x, y, z, and ζ is a primitive third root of unity. (Note that [RY_1, Corollary 7.3] applies because S has no non-trivial unipotent elements, and the centralizer of H in S is finite.)

To prove (e_2), note that $\overline{N}$ is the normalizer of $\overline{S}$ in PGL_n. The natural three-dimensional representation of $S \subset \mathrm{GL}_3$ is irreducible (to see this, restrict to the subgroup H of S defined in 16.5). Hence, by Schur's lemma, the centralizer $C_{\mathrm{PGL}_n}(\overline{S}) = \{1\}$, so that $\overline{N} = N_{\mathrm{PGL}_n}(\overline{S})/C_{\mathrm{PGL}_n}(\overline{S})$. The last group is naturally isomorphic to a subgroup of $\mathrm{Aut}(\overline{S})$, which is a finite group. This proves (e_2).

To prove (e_3), consider the natural representation of $N \subset \mathrm{GL}_3$ on $\mathbb{A}^3$. (e_2) implies that this representation is generically free; cf. [BF_1, Section 1]. Consequently, $\mathrm{ed}(N) \leqslant 3 - \dim(N) = 2$. $\square$

This completes the proof of part (e). $\square$

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