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Nonlinear Superposition*

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1. INTRODUCTION

The purpose of this paper is to introduce a concept of nonlinear superposition. We define the term "connecting function" and give several examples. Many significant successes in constructing effective theories for physical phenomena can be traced to the linear principle of superposition. The ideas contained herein induce a similar tool for some nonlinear equations.

2. CONNECTING FUNCTIONS

For simplicity we restrict our attention to cases for two independent variables. However, our remarks will apply to cases with more independent variables or to ordinary differential equations.

DEFINITION. Let $u_1 = u_1(x, y)$, $u_2 = u_2(x, y), \ldots, u_k = u_k(x, y)$ be solutions of some partial differential equation $L(u) = 0$. Then a function $F = F(u_1, u_2, \ldots, u_k, x, y)$ is called a connecting function for $L(u) = 0$ if F is also a solution. This constitutes a nonlinear superposition principle.

It is clear that a connecting function for a given equation is not necessarily unique. Consider the equation

$$u_x + u_y = u. \tag{2.1}$$

This equation has the obvious connecting function

$$F = u_1 + u_2 \tag{2.2}$$

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since it is linear; however,

$$F = [u_1^n + u_2^n]^{1/n} \tag{2.3}$$

is also a connecting function for any real number $n \neq 0$.

3. Basic Results

Connecting functions for a class of quasilinear equations may be developed from a linear equation by dependent variable transformation. Consider the equation

$$a(x, y)\, u_x + b(x, y)\, u_y = c(x, y)\, u. \tag{3.1}$$

If we set $u = F(v(x, y))$, then (3.1) becomes

$$a(x, y)\, v_x + b(x, y)\, v_y = c(x, y)\, \frac{F(v)}{F'(v)}. \tag{3.2}$$

Since (3.1) is linear, if u_1, u_2,..., u_k are solutions, then

$$U = u_1 + u_2 + \cdots + u_k$$

is also a solution. However, if $V = F^{-1}(U)$ and $u_1 = F(v_1)$, $u_2 = F(v_2)$,..., $u_k = F(v_k)$, we have

$$V = F^{-1}[F(v_1) + F(v_2) + \cdots + F(v_k)] \tag{3.3}$$

is a solution to (3.2). We naturally assume that F^{-1} exists. The basic equation (3.1) need not be first order or even linear. We could reason from a nonlinear equation for which a connecting function has been found.

Some very interesting combinations can be taken from (3.2) and (3.3). For example, if $a \equiv 1$, $b \equiv 1$, $c \equiv n \neq 0$, and $F = v^n$, then (3.2) becomes (2.1) and (3.3) gives us the connecting function (2.3). Another combination is $c \equiv 1$ and $F = e^v$, which gives

$$a(x, y)\, v_x + b(x, y)\, v_y = 1 \tag{3.4}$$

and the connecting function obtained from (3.3) is

$$F = \log\,[\exp\,(v_1) + \exp\,(v_2)]. \tag{3.5}$$

A more complicated equation is

$$a(x, y)\, v_x + b(x, y)\, v_y = c(x, y)\, v^n \tag{3.6}$$

where $n \neq 1$. For this equation

$$F = \left\{ \log \left[\exp \left(\frac{1}{1-n} v_1^{1-n} \right) + \exp \left(\frac{1}{1-n} v_2^{1-n} \right) \right]^{1-n} \right\}^{1/(1-n)} \tag{3.7}$$

is a connecting function. Some of the results of this paragraph apply in more generality.

4. Burgers' Equation

The equation

$$u_t + uu_x = \lambda u_{xx} \tag{4.1}$$

where λ is a parameter, has been utilized by Burgers [2] as a mathematical model of turbulence. We give in this section connecting functions for this equation and an associated equation.

By setting $u = v_x$, integrating once with respect to x, and discarding an arbitrary function of t, (4.1) becomes

$$v_t + \tfrac{1}{2} v_x{}^2 = \lambda v_{xx}, \tag{4.2}$$

an equation which has some significance in the burning of a gas in a rocket, see Forsythe and Wasow [3; p. 141]. If we now set $v = -2\lambda \log(w)$, (4.2) transforms to

$$w_t = \lambda w_{xx} \tag{4.3}$$

the one-dimensional diffusion equation. These transformations are already well-known. A complete discussion of them with proper priorities may be found in Ames [1; p. 24].

Reasoning with (4.3) as we did in Section 3, a connecting function for (4.2) is found to be

$$F = -2\lambda \log \left[\exp \left(-\frac{v_1}{2\lambda} \right) + \exp \left(-\frac{v_2}{2\lambda} \right) \right]. \tag{4.4}$$

Now, (4.4) can be used to find a connecting function for (4.1). This is seen to be

$$F = -2\lambda \frac{\partial}{\partial x} \left\{ \log \left[\exp \left(-\frac{1}{2\lambda} \int_{x_0}^{x} u_1(\xi, t)\, d\xi \right) + \exp \left(-\frac{1}{2\lambda} \int_{x_0}^{x} u_2(\xi, t)\, d\xi \right) \right] \right\} \tag{4.5}$$

where $x_0 < x$ is some lower limit.

5. Remarks

The method of Section 3 has been applied to equations other than (3.1) with success. In a private discussion, Professor J. R. Ferron has noted that as $n \to 0$ in (2.3) the connecting function $F = (u_1 u_2)^{1/2}$ is obtained.

References

1. W. F. Ames. "Nonlinear Partial Differential Equations in Engineering." Academic Press, New York, 1965.
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3. G. E. Forsythe and W. R. Wasow. "Finite Difference Methods for Partial Differential Equations." Wiley, New York, 1960.