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A Note on Groups with "Large Extraspecial" Subgroups of Width 4

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Recently, F. Timmesfeld investigated finite simple groups G in which the generalized Fitting group of the centralizer M of some involution is extraspecial [3]. The purpose of this paper is to eliminate a special configuration of M which occurs in the case of width 4. More precisely, we prove the following

PROPOSITION. *There exists no nonabelian simple group G which has an involution z such that the centralizer M of z in G satisfies the following conditions:*

(i) $Q = F^*(M)$ *is extraspecial of width* 4.

(ii) $\bar{M} = M/Q = (\bar{F}_0 \times \bar{F}_1) \cdot \bar{I}$ *with* $\bar{F}_0 \simeq Z_3$, $\bar{F}_1 \simeq A_6$, $\bar{F}_0 \cdot \bar{I} \simeq \Sigma_3$ *and* $\bar{F}_1 \cdot \bar{I} \simeq \Sigma_6$.

According to [3] we set $\tilde{M} = M/\langle z \rangle$ and $\bar{M} = M/Q$ and we use the bar conventions. In addition we write $C(X) = C_G(X)$ and $N(X) = N_G(X)$ for any subset X of G.

From now on we assume G is simple. Let $\bar{F}_0 = \langle \bar{\rho}_0 \rangle$ and let $X = N_M(\rho_0)$. Then X covers $\bar{M}$. As $\bar{M}$ acts irreducibly on $\tilde{Q}$ by [2], so $X \cap Q = \langle z \rangle$. We have $\tilde{X} \simeq \bar{M}$. Set $\tilde{X} = (\tilde{F}_0 \times \tilde{F}_1) \cdot \tilde{I}$. By [1] there is an involution $a \in Q - \langle z \rangle$ with $a \sim_G z$. Let $L = Q(Q_a \cap M)$ where $Q_a = F^*(C(a))$. Then by [3, (3.11)] we have $\bar{L} \simeq E_{2^3}$ and $C_{\tilde{Q}}(\bar{L}) = \langle \tilde{a} \rangle$. This implies that we can assume $[\tilde{\rho}_1, \tilde{a}] = 1$ for an element $\tilde{\rho}_1$ of order 3 of $C_{\tilde{F}_0 \times \tilde{F}_1}(\tilde{\tau}) = C_{\tilde{F}_1}(\tilde{\tau})$ where $\tilde{\tau}$ is some involution in $(\tilde{F}_0 \times \tilde{F}_1) \cdot \tilde{I} - (\tilde{F}_0 \times \tilde{F}_1)$. In other words $C_Q(\rho_1) \simeq Q_8 \times Q_8$ or $Q_8 \times Q_8 \times Q_8$. Suppose $C_Q(\rho_1) \simeq Q_8 \times Q_8 \times Q_8$, then $[Q, \rho_1] \simeq Q_8$. As $\tilde{\tau}$ inverts $\tilde{F}_0 = \langle \tilde{\rho}_0 \rangle$ and $C_{\tilde{Q}}(\tilde{\rho}_0) = 1$, so $|C_{\tilde{Q}}(\tilde{\tau})| = 2^4$. On the other hand $\tilde{\tau}$ centralizes $\widetilde{[Q, \langle \rho_1 \rangle]}$ and $|C_{\widetilde{C_Q(\rho_1)}}(\tilde{\tau})| \geqslant 2^3$, so $|C_{\tilde{Q}}(\tilde{\tau})| \geqslant 2^5$, a contradiction. Thus $C_Q(\rho_1) \simeq Q_8 * Q_8$. Let $\tilde{P}_1 = \langle \tilde{\rho}_1, \tilde{\rho}_2 \rangle$ be an S_3-subgroup of $\tilde{F}_1$ with $\tilde{\rho}_1 \sim_{\tilde{F}_1} \tilde{\rho}_2$. Then $C_Q(\rho_2) \simeq Q_8 \times Q_8$. There is an involution $\tilde{\tau}_0 \in \tilde{F}_1$ such that $\tilde{\tau}_0$ acts invertingly on $\tilde{P}_1$. Suppose $C_Q(\rho_1) \cap C_Q(\rho_2) \neq \langle z \rangle$. As $|C_M(a)|_3 = 3$, so $C_Q(\rho_1) \cap C_Q(\rho_2) \simeq Q_8$.

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Then $\tilde{\tau}_0$ induces an outer automorphism on $C_{[Q,\langle\delta_1\rangle]}(\rho_2) \simeq Q_8$. Since $\tilde{\rho}_1$ acts fixed-point-free on $[\widetilde{Q, \langle\rho_1\rangle}]$, thus $|C_{[\widetilde{Q,\langle\rho_1\rangle}]}(\tilde{\tau}_0)| = 2^2$ and $\tilde{\tau}_0$ induces an outer automorphism on Q_1 and on Q_2 where $[Q, \langle\rho_1\rangle] = Q_1 * Q_2$ and $Q_1 \simeq Q_2 \simeq Q_8$. Hence $C_{[Q,\rho_1\rangle]}(\tilde{\tau}_0) \simeq E_{2^2}$, contradicting the fact that $C_{[Q,\langle\rho_1\rangle]}(\tilde{\tau}_0)$ is $\tilde{\rho}_0$-invariant and $C_{\tilde{Q}}(\tilde{\rho}_0) = 1$. Thus we have $C_Q(\rho_1) \cap C_Q(\rho_2) = \langle z\rangle$. Therefore $\widetilde{\rho_1\rho_2}$ and $\widetilde{\rho_1{}^2\rho_2}$ act fixed-point-free on $\tilde{Q}$. Let $\tilde{\tau}_1$ be an involution in $\tilde{F}_1 \cdot \tilde{I} \simeq \Sigma_6$.

Case (1). $\tilde{\tau}_1 \in \tilde{F}_1$. Then we may assume $\tilde{\tau}_1$ inverts $\tilde{P}_1 = \langle\tilde{\rho}_1, \tilde{\rho}_2\rangle$. In particular $\tilde{\tau}_1$ inverts $\tilde{\rho}_1\tilde{\rho}_2$. As $C_{\tilde{Q}}(\widetilde{\rho_1\rho_2}) = 1$, it follows that $|C_{\tilde{Q}}(\tilde{\tau}_1)| = 2^4$. Acting on Q with $\langle\tilde{\rho}_0, \tilde{\rho}_1\tilde{\rho}_2\rangle$, we see that $Q = C_Q(\rho_0\rho_1\rho_2) \cdot C_Q(\rho_0\rho_1{}^2\rho_2{}^2)$ and so $C_Q(\rho_0\rho_1\rho_2) \simeq^2 C_Q(\rho_0\rho_1{}^2\rho_2{}^2) \simeq Q_8 * Q_8$ and thus $C_Q(\tau_1) \simeq E_{2^5}$.

Case (2). $\tau_1 \in \tilde{F}_1 \cdot \tilde{I} - \tilde{F}_1$. Then $\tilde{\rho}_0^{\tilde{\tau}_1} = \tilde{\rho}_0{}^2$. Let $\tilde{\rho} \in C_{\tilde{F}_1}(\tilde{\tau}_1)$ be an element of order 3. We have either $C_{\tilde{Q}}(\tilde{\rho}) = 1$ or $C_Q(\tilde{\rho}) \simeq Q_8 * Q_8$.

(a) $C_{\tilde{Q}}(\tilde{\rho}) = 1$. Then $C_Q(\tau) \simeq E_{2^5}$ by arguing as above.

(b) $C_Q(\tilde{\rho}) \simeq Q_8 * Q_8$. Acting with $\langle\tilde{\rho}_0, \tilde{\rho}\rangle$ on $[Q, \tilde{\rho}] \simeq Q_8 * Q_8$, we get $[Q, \tilde{\rho}] = C_{[Q,\langle\tilde{\rho}\rangle]}(\tilde{\rho}_0\tilde{\rho})\cdot C_{[Q,\langle\tilde{\rho}\rangle]}(\tilde{\rho}_0{}^2\tilde{\rho})$. As $(\tilde{\rho}_0\tilde{\rho})^{\tilde{\tau}_1} = \tilde{\rho}_0{}^2\tilde{\rho}$, so $C_{[Q,\langle\tilde{\rho}\rangle]}(\tilde{\rho}_0\tilde{\rho}) \simeq C_{[Q,\langle\tilde{\rho}\rangle]}(\tilde{\rho}_0{}^2\tilde{\rho}) \simeq Q_8$. This implies that $C_{[Q,\langle\tilde{\rho}\rangle]}(\tau_1) \simeq E_{2^3}$. From the structure of Σ_6 $\tilde{\tau}_1$ inverts an element $\tilde{\rho}^* \sim_{\tilde{F}_1\cdot\tilde{I}} \tilde{\rho}$ and $\langle\tilde{\rho}, \tilde{\rho}^*\rangle \in \mathrm{Syl}_3(\tilde{F}_1 \cdot \tilde{I})$. So $C_Q(\tilde{\rho}) = [Q, \tilde{\rho}^*]$. Acting with $\langle\tilde{\rho}_0, \tilde{\rho}^*\rangle$ on $C_Q(\tilde{\rho})$ we have $C_Q(\tilde{\rho}) = C_{C_Q(\tilde{\rho})}(\tilde{\rho}_0\tilde{\rho}^*)\, C_{C_Q(\tilde{\rho})}(\tilde{\rho}_0{}^2\tilde{\rho}^*)$ and $C_{C_Q(\tilde{\rho})}(\tilde{\rho}_0\tilde{\rho}^*) \simeq C_{C_Q(\tilde{\rho})}(\tilde{\rho}_0{}^2\tilde{\rho}^*) \simeq Q_8$. Thus $\tilde{\tau}_1$ induces an outer automorphism on $C_{C_Q(\tilde{\rho})}(\tilde{\rho}_0\tilde{\rho}^*)$ and on $C_{C_Q(\tilde{\rho})}(\tilde{\rho}_0{}^2\tilde{\rho}^*)$ and so $C_{C_Q(\tilde{\rho})}(\tau_1) \simeq E_{2^2}$. Hence $C_Q(\tau_1) \simeq E_{2^4}$.

Now we remark that there are involutions in the coset $Q\tau_1$. Suppose $o(\tau_1) = 4$. Then there exists an element $u \in Q$ with $o(\tau_1 u) = 2$, so $\tau_1 u \tau_1 u = \tau_1\tau_2\tau_1^{-1} u \tau_1 u = z u^{\tau_1} \cdot u = 1$ and then $u^{\tau_1} = z u^{-1}$, contradicting the fact that the inverse image of $C_{\tilde{Q}}(\tilde{\tau}_1)$ in Q is isomorphic either to E_{2^5} or $(2, 2, 2, 4)$. Thus we have $o(\tau_1) = 2$. Hence X splits over Q and M splits also over Q. We have proved the following result

LEMMA 1. *The group M is a splitting extension of Q by $(F_0 \times F_1) \cdot I$ with $F_0 \simeq Z_3$, $F_1 \simeq A_6$, $I \simeq Z_2$, $F_0 \cdot I \simeq \Sigma_3$ and $F_1 \cdot I \simeq \Sigma_6$. Let $F_0 = \langle\rho_0\rangle$. Then $C_{\tilde{Q}}(\tilde{\rho}_0) = 1$. Let P_1 be an S_3-subgroup of F_1. Then $P_1 = \langle\rho_1, \rho_2\rangle$ such that $C_Q(\rho_1) \simeq C_Q(\rho_2) \simeq Q_8 * Q_8$ and $C_{\tilde{Q}}(\widetilde{\rho_1\rho_2}) = C_{\tilde{Q}}(\widetilde{\rho_1{}^2\rho_2}) = 1$. Furthermore $C_Q(\rho_0\rho_1) \simeq C_Q(\rho_0\rho_2) \simeq Q_8$ and $C_Q(\rho_0\rho_1\rho_2) \simeq C_Q({}_0{}^2\rho_1\rho_2) \simeq Q_8 * Q_8$. Let τ be an involution in $F_0 \times F_1$. Then $C_Q(\tau) \simeq E_{2^5}$. Thus $Q\tau$ contains exactly 32 involutions with $|\tau^Q| = |(\tau z)^Q| = 16$ and $\tau \not\sim_M \tau z$. Let τ_1 be an involution in $(F_0 \times F_1) \cdot I - F_0 \times F_1$ such that $[\tau_1, \rho_1\rho_2] = 1$. Then $C_Q(\tau_1) \simeq E_{2^5}$. Thus $Q\tau_1$ contains 32 involutions with $|\tau_1{}^Q| = |(\tau_1 z)^Q| = 16$ and $\tau_1 \not\sim_M \tau_1 z$. Let τ_2 be an involution in $(F_0 \times F_1) \cdot I - F_0 \times F_1$ such that $[\tau_2, \rho_1] = 1$. Then $C_Q(\tau_2) \simeq E_{2^4}$. Thus $Q\tau_2$ contains 32 involutions which are all conjugate under Q.*

Let $P = F_0 \times P_1$ be an S_3-subgroup of M. We have $C_Q(\rho_1) \simeq Q_8 * Q_8$ and $P/\langle\rho_1\rangle$ acts faithfully on $C_Q(\rho_1)$ hence QP acts transitively on 18 noncentral involutions of $C_Q(\rho_1)$. Similarly $C_Q(\rho_0\rho_1\rho_2) \simeq C_Q(\rho_0\rho_1^2\rho_2^2) \simeq Q_8 * Q_8$ and $P/\langle\rho_0\rho_1\rho_2\rangle$ acts faithfully on $C_Q(\rho_0\rho_1\rho_2)$, so QP acts transitively on 18 noncentral involutions of $C_Q(\rho_0\rho_1\rho_2)$. On the other hand $\rho_0\rho_1 \sim_M \rho_0\rho_1^2$ and $C_Q(\rho_0\rho_1) \simeq C_Q(\rho_0\rho_1^2) \simeq Q_8$. Thus $9 \nmid |C_M(u)|$ for every involution $u \in Q - \langle z\rangle$. Further as Q has 270 noncentral involutions and as $C_{\bar{Q}}(\bar{\lambda}) = 1$ for an element λ of order 5 of M, it follows that $3 \mid |C_M(u)|$. Since M has exactly 2 conjugacy subgroups of order 3 with the representatives ρ_1 and $\rho_0\rho_1\rho_2$ which centralize some noncentral involutions of Q, thus M has precisely 2 conjugacy classes of involutions with the representatives a and b where $[a, \rho_1] = [b, \rho_0\rho_1\rho_2] = 1$. By [3, (3.11)], we have $|M : C_M(a)| = 2 \cdot 3^2 \cdot 5 = 90$ and $|M : C_M(b)| = 2^2 \cdot 3^2 \cdot 5 = 180$ and $a \sim_G z$. By [3, (3.13)] $a \not\sim_G b$. We have proved

LEMMA 2. *M has precisely 2 conjugacy classes of involutions contained in $Q - \langle z\rangle$ with the representatives a and b. We have $|a^M| = 2 \cdot 3^2 \cdot 5 = 90$ and $|b^M| = 2^2 \cdot 3^2 \cdot 5 = 180$. Furthermore $a \sim_G z$ and $b \not\sim_G z$.*

Let $L = Q(Q_a \cap M)$ with $a \in Q - \langle z\rangle$ and $a \sim_G z$. Let $T = Q \cdot S \supseteq L$ be an S_2-subgroup of M where $S \in \mathrm{Syl}_2((F_0 \times F_1) \cdot I)$. Set $V = S \cap L$. Then $V \simeq E_{2^3}$ and $Z(S) \subseteq V$. Let $\tau \in Z(S)^\# \cap F_1$ and put $A = C_Q(\tau) \times \langle\tau\rangle$. Then $A \simeq E_{2^6}$ and $A \trianglelefteq T$. Further $C(A) = A$. Let $x \in Q_{a z} - M$, then $z^x = a$. Hence $A^x \subseteq L^x = (Q \cdot (Q_a \cap M)^x = Q_a \cdot (Q \cap M_a)$ where $M_a = C(a)$. On the other hand $A^x \subseteq M$, thus $A^x \subseteq L$. It follows that $A^x \cap Q \simeq E_{2^5}$ and $A^x \cap V = \langle v\rangle$. Now there is an element $y \in N_{(F_0 \times F_1)I}(V)$ such that $v^y \in Z(S)$. Thus $A^{xy} \leq T$. Since T is self-normalizing, so $A = A^{xy}$. As $xy \notin M$, so we have proved $N(A) \not\subseteq M$. It is easy to see that A contains 6 classes of involutions under the action of $N_M(A)$ with the representatives z (1 conjugate), a (6 conj.), a_1 (12 conj.) ($a_1 \sim_M a$), b (12 conj.), (16 conj.) and z (16 conj.). Let $d = |N(A) : N_M(A)|$, the length of the orbit of z in $N(A)$. Since $N(A)/A$ is isomorphic to a subgroup of $GL(6, 2)$, so we have $d = 7$ or $5 \cdot 7$. Assume $d = 7$. Then $z \sim_{N(A)} a$ and $|a_1^{N(A)}| = 28$. Furthermore $|b^{N(A)}| = 28$, since otherwise $|b^{N(A)}| = 12$ and $|\tau z^{N(A)}| = 16$ (say); let R be a subgroup of order 7 of $N(A)$ then $C_A(R) \simeq E_{2^3}$ contains 5 conjugates of b and 2 conjugates of τz; on the other hand $3 \mid |N_{N(A)}(R)|$; let P be a subgroup of order 3 of $N_{N(A)}(R)$, then P centralizes $C_A(R)$, contradicting the fact that $C_A(P) \simeq E_{2^2}$, as $P \sim_{N(A)} F_0 = \langle\rho_0\rangle$. It follows that an S_7-subgroup of $N(A)$ acts fixed-point-free on A. As $C_A(F_0) = \langle z, \tau\rangle$ so $C_{N(A)}(F_0) = C_{N_M(A)}(F_0)$ thus $C_{N(A)}(F_0) \simeq Z_6 \times D_8$. Now let $\bar{H}$ be a minimal normal subgroup of $\bar{N} = N(A)/A$. Suppose $\bar{H}$ is solvable, then $\bar{H}$ is either 2-group or 7-group. If $\bar{H}$ is a 2-group, then $C_A(\bar{H}) = A_0 \simeq E_{2^3}$. Let $\bar{W} = C_{\bar{N}}(A_0)$, then $\bar{W} \trianglelefteq \bar{N}$. As $Z(T) = \langle z\rangle$, so $|\bar{W}| \leqslant 2^5$. Now $\bar{N}/\bar{W}$ is isomorphic to a subgroup of $GL(3, 2)$. Since $2, 3, 7 \mid |\bar{N}/\bar{W}|$, so $\bar{N}/\bar{W} \simeq GL(3, 2)$. Thus $|\bar{W}| = 2^4$. Let $R_0 \cdot F_0$ be a subgroup of order 21 of $\bar{N}$, where

$|R_0| = 7$, then $C_{\bar{W}}(R_0 \cdot F_0) = 1$. Let $\bar{w} \in C_{\bar{W}}(R_0 \cdot F_0)$ be an involution, then $C_A(\bar{w}) \simeq E_{2^4}$, contradicting $C_A(R_0) = 1$. If $|\bar{H}| = 7$, then $|C_{\bar{N}}(\bar{H}) \cap C_{\bar{N}}(F_0)|_2 \neq 1$ and we get a contradiction as above. Thus $\bar{H}$ is not solvable. Hence $\bar{H} \simeq L_3(2)$, but then $1 \neq C_{\bar{N}}(\bar{H}) \trianglelefteq \bar{N}$, a contradiction. Thus we have $d = 5 \cdot 7$, so $|z_{\bar{N}}| = 35$ and $\bar{N}$ acts irreducibly on A. Ley $\bar{H}$ be a minimal normal subgroup of $\bar{N}$. Then $\bar{H}$ is either 7-group of $\bar{H}$ is simple. If $|\bar{H}| = 7$ then $C_A(\bar{H}) = 1$ but $\bar{H}$ centralizes a subgroup K of order 5 of $\bar{N}$ and $C_A(K) \simeq E_{2^2}$, a contradiction. Hence $\bar{H}$ is simple and $\bar{H} \simeq A_5$ or $L_3(2)$. In any case $C_{\bar{N}}(\bar{H}) \neq 1$, a contradiction.

The proof of the proposition is complete.

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