JOURNAL OF DIFFERENTIAL EQUATIONS **132**, 293–318 (1996)
ARTICLE NO. 0181

Linear Grading Function and Further Reduction of Normal Forms

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Received May 3, 1996

In this note an idea of quasi-homogeneous normal form theory using new grading functions is introduced, the definition of Nth order normal form is given and some sufficient conditions for the uniqueness of normal forms are derived. A special case of the unsolved problem in a paper of Baider and Sanders for the unique normal form of Bogdanov–Takens singularities is solved. © 1996 Academic Press, Inc.

1. INTRODUCTION

Normal forms are basic and powerful tools in bifurcation theory of vector fields. The classical normal form theory, known as Poincaré normal form (see, e.g., Arnold [Ar]), however, may not give the simplest form

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0022-0396/96 $18.00

since only linear parts are used for simplifying the nonlinear terms, and hence one can not apply Poincaré normal form theory to vector fields whose linear parts are identically zero. On the other hand, classical normal forms are not unique in general. In order to get unique normal forms so that formal classification could be made, further reduction of the classical normal forms is necessary and the concept of normal forms should be refined.

Many authors have discussed the further reduction of normal forms and some of them have discussed uniqueness of normal forms, see, e.g., [SM] and references therein. Ushiki [Us] introduced a systematic method by which nonlinear parts are also used to simplify higher order terms. In contrast to the classical method of normal form theory where only one Lie bracket is used to simplify the higher order terms, the Ushiki's method allows more Lie brackets for the simplification (see e.g. [CK] for examples of calculation). In particular, Ushiki obtained unique normal forms (simplest normal forms) up to some degree for given vector fields. Wang [Wa] gave a method to calculate coefficients of normal forms, which needs more parameters in the transformations due to the non-uniqueness of transformations and hence which may give simplest normal forms (up to some finite order) by suitable choice of parameters. In fact nonlinear terms play also a role in the reduction. Baider introduced the notion of "special form" [Ba], which is in fact a unique normal form in an abstract sense. Baider and Sanders [BS1] introduced new grading functions to get further reduction of normal forms. They introduced the concept of nth order normal forms related with the nth grading function and gave the definition of infinite order normal forms (which is unique). They gave unique normal forms for some nilpotent Hamiltonian vector field singularities. They also obtained unique normal forms for some cases of Bogdanov–Takens singularities ([BS2]), although some cases still remain unsolved. Results concerning uniqueness of normal forms for some other cases can be found in [BC2] and [SM].

In this paper we first introduce the concept of linear grading function in Section 2 and we give a method to define new grading functions. Then in Section 3 we develop a quasi-homogeneous normal form theory by using grading functions and define nth order normal forms, in which we combine methods of Ushiki and of Baider–Sanders. In fact we need only one grading function, but the nth order normal forms relate to n Lie brackets in the computation. In Section 4 we define infinite order normal forms and prove that the infinite order normal forms must be unique. In Section 5 we give a sufficient condition for a finite order normal form being unique. Finally in Section 6 we prove the uniqueness of a first order normal form of the special case $\mu = 2$, $\nu = 1$ of Bogdanov–Takens singularities, which solves a special case of the remaining problem in [BS2].

2. LINEAR GRADING FUNCTION

Let H be the linear space of all n dimensional real or complex formal vector fields. We define a bilinear operator $[\cdot, \cdot]: H \times H \to H$ by $[u, v] = Du \cdot v - Dv \cdot u$ for any $u, v \in H$. Then $\{H, [\ ,\ ]\}$ forms a Lie algebra. Now let us define a "grading function" such that $\{H, [\ ,\ ]\}$ becomes a graded Lie algebra.

For the purpose of computing normal forms of formal vector fields, the "grading function" should satisfy the following properties:

(i) The degree of any monomial is defined to be an integer. The dimension of the linear space H_k spanned by all monomials of degree k is finite for any integer k (in the case when there is no monomials of degree k for some integer k we define $H_k = \{0\}$);

(ii) $[H_m, H_n] \subset H_{m+n}$ for any integers m, n;

(iii) The grading function should be bounded below.

Let

$$D_n = \left\{ \prod_{i=1}^{n} x_i^{l_i} \mathbf{e}_j \;\middle|\; l_i \in \mathbb{Z}^+,\ x_i \in \mathbb{R}\ (\text{or } \mathbb{C}),\ i, j = 1, ..., n \right\},$$

where $\mathbf{e}_j$ is the jth standard unit vector in $\mathbb{R}^n$ (or $\mathbb{C}^n$). Consider the function δ: $D_n \to \mathbb{Z}$ defined by

$$\delta\left(\prod_{i=1}^{n} x_i^{l_i} \mathbf{e}_j \right) = \sum_{i=1}^{n} a_{ij} l_i + d_j, \tag{1}$$

where $a_{ij},\ d_j \in \mathbb{Z},\ i, j = 1, ..., n$. From the definition of δ, it is obvious that condition (i) for a grading function is satisfied. Now we look for conditions such that the function δ defined by (1) satisfies all other conditions of grading functions.

LEMMA 2.1. *The function δ defined by* (1) *is bounded below if and only if all $\{a_{ij}\}$ are non-negative integers.*

LEMMA 2.2. *Let the function δ be defined by* (1) *with non-negative coefficients $\{a_{ij}\}$ and let H_k be the linear space spanned by all monomials in $\delta^{-1}(k)$. Then* $\dim H_k$ *is finite* (*or zero*) *for any integer k if and only if all $\{a_{ij}\}$ are natural numbers.*

LEMMA 2.3. *Let the function δ be defined by* (1) *and H_k be defined as in Lemma* 2.2. *Then $[H_m, H_n] \subset H_{m+n}$ if and only if*

$$a_{i1} = \cdots = a_{in} = -d_i \qquad \textit{for any} \quad i = 1, ..., n.$$

Proof. Let $u=\prod_{i=1}^{n} x_i^{l_i}\mathbf{e}_j$, $v=\prod_{i=1}^{n} x_i^{l'_i}\mathbf{e}_k$ and $\delta(u)=m$, $\delta(v)=n$. Then

$$\begin{aligned}[u, v] &= Du\cdot v - Dv\cdot u\\ &= \frac{l_k}{x_k}\prod_{i=1}^{n} x_i^{l_i}\cdot\prod_{i=1}^{n} x_i^{l'_i}\mathbf{e}_j - \frac{l'_j}{x_j}\prod_{i=1}^{n} x_i^{l'_i}\cdot\prod_{i=1}^{n} x_i^{l_i}\mathbf{e}_k\\ &= \frac{l_k}{x_k}\prod_{i=1}^{n} x_i^{l_i+l'_i}\mathbf{e}_j - \frac{l'_j}{x_j}\prod_{i=1}^{n} x_i^{l_i+l'_i}\mathbf{e}_k. \end{aligned} \tag{2}$$

We first assume that $a_{i1}=a_{i2}=\cdots=a_{in}=-d_i$, $i, j=1, ..., n$. Then

$$\begin{aligned}\delta\left(\frac{1}{x_k}\prod_{i=1}^{n} x_i^{l_i+l'_i}\mathbf{e}_j\right) &= \sum_{i=1}^{n} a_{ij}(l_i+l'_i)-a_{kj}+d_j\\ &= \left(\sum_{i=1}^{n} a_{ij}l_i+d_j\right)+\left(\sum_{i=1}^{n} a_{ik}l'_i+d_k\right)=m+n,\end{aligned}$$

and

$$\begin{aligned}\delta\left(\frac{1}{x_j}\prod_{i=1}^{n} x_i^{l_i+l'_i}\mathbf{e}_k\right) &= \sum_{i=1}^{n} a_{ik}(l_i+l'_i)-a_{jk}+d_k\\ &= \left(\sum_{i=1}^{n} a_{ik}l_i+d_k\right)+\left(\sum_{i=1}^{n} a_{ij}l'_i+d_j\right)=n+m.\end{aligned}$$

Hence, from (2), we have $[u, v]\in H_{m+n}$. Note that the operator $[\cdot,\cdot]$ is bilinear. Therefore $[H_m, H_n]\subset H_{m+n}$.

Conversely, we suppose that $[H_m, H_n]\subset H_{m+n}$ holds for any integer m, n. For any $k\in\mathbb{N}$, we fix a $u\in H_m$ with $l_k>0$. Then from (2) we have

$$\sum_{i=1}^{n} a_{ij}(l_i+l'_i)-a_{kj}+d_j=\left(\sum_{i=1}^{n} a_{ij}l_i+d_j\right)+\left(\sum_{i=1}^{n} a_{ik}l'_i+d_k\right).$$

Hence

$$\sum_{i=1}^{n}(a_{ik}-a_{ij})\,l'_i+d_k+a_{kj}=0,$$

i.e.

$$d_k+a_{kj}=\sum_{i=1}^{n}(a_{ij}-a_{ik})\,l'_k.$$

If we take

$$l_i' = \begin{cases} l, & i = k, \\ 0, & i \neq k, \end{cases}$$

where $l \in \mathbb{N}$, then

$$a_{kj} - a_{kk} = \frac{d_k + a_{kj}}{l}. \tag{3}$$

Letting $l \to +\infty$, we have $a_{kj} = a_{kk}$. Note that j is arbitrary. Therefore $a_{k1} = \cdots = a_{kk} = \cdots = a_{kn}$, and hence from (3), $d_k = -a_{k1}$ follows.

DEFINITION 2.4. Let

$$D_n = \left\{ \prod_{i=1}^{n} x_i^{l_i} \mathbf{e}_j \,\middle|\, l_i \in \mathbb{Z}^+, x_i \in \mathbb{R} \text{ (or } \mathbb{C}), i, j = 1, ..., n \right\},$$

where $\mathbf{e}_j$ is the jth standard unit vector in $\mathbb{R}^n$ (or $\mathbb{C}^n$). Then the function $\delta: D_n \to \mathbb{Z}$ defined by

$$\delta\left(\prod_{i=1}^{n} x_i^{l_i} \mathbf{e}_j \right) = \sum_{i=1}^{n} a_i l_i - a_j, \tag{4}$$

where $a_i \in \mathbb{N}$, $i = 1, ..., n$, is called a linear grading function.

Remark 2.5. (1) Let δ be a linear grading function defined by a set of natural numbers $\{a_i\}$. If $\{a_i\}$ has a common factor c, then the function $(1/c)\,\delta$ is also a linear grading function. So we may always assume that every linear grading function is defined by a set of coprime natural numbers $\{a_i\}$.

(2) Any linear grading function δ satisfies

$$\delta(x_i \mathbf{e}_i) = 0 \qquad \text{for all} \quad i = 1, ..., n.$$

Hence for any grading function δ, $\min_p \delta(p) \leqslant 0$.

(3) If the linear grading function δ is defined by a set of successive natural numbers $\{a_1, ..., a_n\}$, then for $\forall k \geqslant 1 - n$, $\dim H_k \geqslant 1$.

EXAMPLE 2.6. If $\delta(\prod_{i=1}^{n} x_i^{l_i} \mathbf{e}_j) = \sum_{i=1}^{n} l_i - 1$, i.e. $a_1 = \cdots = a_n = 1$, then δ is a linear grading function. Note that the usual definition of the degree of $\prod_{i=1}^{n} x_i^{l_i} \mathbf{e}_j$ is $\sum_{i=1}^{n} l_i$ and hence the grading function defined above shifts by 1 with respect to the usual grading.

EXAMPLE 2.7. In $D_2 = \{\prod_{i=1}^{2} x_i^{l_i} \mathbf{e}_j, j = 1, 2\}$, define

$$\delta(x_1^{l_1} x_2^{l_2} \mathbf{e}_j) = \begin{cases} 2l_1 + 3l_2 - 2, & j = 1, \\ 2l_1 + 3l_2 - 3, & j = 2. \end{cases}$$

Then $\delta(x_2\,\mathbf{e}_1)=\delta(x_1^2\,\mathbf{e}_2)=1$. Note that $x_2\,\mathbf{e}_1$ is a linear term and $x_1^2\,\mathbf{e}_2$ is a nonlinear term in the sense of usual grading.

3. *N*th ORDER NORMAL FORMS

Let δ be a linear grading function and H_k be the linear space spanned by all monomials of degree k. Consider a formal vector field V defined by the following formal series

$$X=X_\mu+X_{\mu+1}+\cdots+X_{\mu+k}+\cdots, \tag{5}$$

where $X_k\in H_k$, $k\geqslant\mu$ and $X_\mu\neq 0$. We call (5) a zeroth order normal form and denote it as

$$V^{(0)}=V_\mu^{(0)}+X_{\mu+1}+\cdots+X_{\mu+k}+\cdots. \tag{6}$$

We may assume that X_μ is already in some simple or satisfactory form (e.g. X_μ may have been changed to a simpler form by classical normal form theory).

Let $Y_k\in H_k$ and let Φ_{Y_k} be its time one mapping given by the flow $\Phi^t_{Y_k}$ generated from the vector field corresponding to the equation $\dot{x}=Y_k(x)$, $x\in\mathbb{R}^n$. Then the transformation $y=\Phi_{Y_k}(x)$, which is a near identity change of variables, brings (5) to

$$\begin{aligned}(\Phi_{Y_k})_*\,X&=\exp(\text{ad }Y_k)\,X\\&=X+(\text{ad }Y_k)\,X+\cdots+\frac{1}{n!}(\text{ad }Y_k)^n\,X+\cdots,\end{aligned}$$

where $(\text{ad }Y_k)\,X=[Y_k,X]$ and $(\text{ad }Y_k)^n=(\text{ad }Y_k)^{n-1}\cdot(\text{ad }Y_k)$, $n=2,3,\ldots$.

For any $k\in\mathbb{N}$, define an operator

$$L_k^{(1)}\colon H_k\to H_{\mu+k};\ Y_k\mapsto[Y_k,V_\mu^{(0)}]. \tag{7}$$

It is obvious that $L_k^{(1)}$ is linear. Note that $L_k^{(1)}$ depends on $V_\mu^{(0)}$ and can be denoted by $L_k^{(1)}=L_k^{(1)}[V_\mu^{(0)}]$.

DEFINITION 3.1.

$$V=V_\mu+V_{\mu+1}+\cdots+V_{\mu+k}+\cdots$$

is called a *first order normal form*, if

$$V_{\mu+k}\in N_{\mu+k}^{(1)},\qquad k=1,2,\ldots,$$

where $N^{(1)}_{\mu+k}$ is a complement subspace to $\operatorname{Im} L^{(1)}_k$ in $H_{\mu+k}$ and $L^{(1)}_k = L^{(1)}_k[V_\mu]$.

It is easy to see that there is a sequence of near identity formal transformations such that (5) is transformed into a first order normal form which is called the first order normal form of (5) and can be denoted by

$$V^{(1)} = V^{(1)}_\mu + V^{(1)}_{\mu+1} + \cdots + V^{(1)}_{\mu+k} + \cdots. \tag{8}$$

Note that $V^{(1)}_\mu = V^{(0)}_\mu$.

In order to make further reduction of a first order normal form, we define a sequence of linear operators $L^{(m)}_k$, $m, k = 1, 2, 3, ...$ as follows. Let

$$V = V_\mu + V_{\mu+1} + V_{\mu+2} + \cdots + V_{\mu+k} + \cdots$$

be a formal series, where $V_m \in H_m$ for each $m \geqslant \mu$. Then we define $L^{(1)}_k = L^{(1)}_k[V_\mu]$ by (7) for any $k \in \mathbb{N}$; if $L^{(m)}_k = L^{(m)}_k[V_\mu, V_{\mu+1}, ..., V_{\mu+m-1}]$ is defined already for an $m \geqslant 1$ and any $k \in \mathbb{N}$, then we define $L^{(m+1)}_k = L^{(m+1)}_k[V_\mu, V_{\mu+1}, ..., V_{\mu+m}]$ by

$$\begin{aligned} L^{(m+1)}_k \colon \operatorname{Ker} L^{(m)}_k \times H_{m+k} &\to H_{\mu+m+k}: \\ ((Y_k, Y_{k+1}, ..., Y_{k+m-1}), Y_{k+m}) & \\ &\mapsto [Y_k, V_{\mu+m}] + \cdots + [Y_{k+m-1}, V_{\mu+1}] + [Y_{k+m}, V_\mu]. \end{aligned}$$

Remark 3.2. By definition, it is obvious that

$$\begin{aligned} \operatorname{Ker} L^{(m)}_k = \{ (Y_k, Y_{k+1}, ..., Y_{k+m-1}) \in H_k \times \cdots \times H_{k+m-1} \mid & \\ [Y_k, V_\mu] = 0, & \\ [Y_{k+1}, V_\mu] + [Y_k, V_{\mu+1}] = 0, & \\ \vdots & \\ [Y_{k+m-1}, V_\mu] + \cdots + [Y_k, V_{\mu+m-1}] = 0 \}. & \end{aligned}$$

DEFINITION 3.3. A formal vector field

$$V = V_\mu + V_{\mu+1} + V_{\mu+2} + \cdots + V_{\mu+N} + \cdots$$

where $V_m \in H_m$ for each $m \geqslant \mu$, is called an *Nth order normal form*, if

$$V_{\mu+i} \in N^{(i)}_{\mu+i} \quad (1 \leqslant i \leqslant N-1),$$

and

$$V_{\mu+j} \in N^{(N)}_{\mu+j} \quad (j \geqslant N),$$

where $N^{(m)}_{\mu+k}$ is a complement to the image of $L^{(m)}_{k-m+1}[V_\mu, V_{\mu+1}, ..., V_{\mu+m-1}]$ in $H_{\mu+k}$ for each $m \geqslant 1$ and $k \geqslant 1$.

THEOREM 3.4. *For any $N \in \mathbb{N}$, every formal vector field can be changed by a sequence of near identity formal transformations to an Nth order normal form.*

Proof. Consider a formal vector field (a zeroth order normal form)

$$V^{(0)} = V^{(0)}_\mu + X_{\mu+1} + \cdots + X_{\mu+k} + \cdots. \tag{9}$$

Define linear operator $L^{(1)}_1 = L^{(1)}_1[V^{(0)}_\mu]$ and let

$$H_{\mu+1} = \operatorname{Im} L^{(1)}_1 \oplus N^{(1)}_{\mu+1}.$$

Then there is a polynomial $Y^1 = Y^{(1)}_1 \in H_1$ such that (9) is converted to

$$V^{(1)} = \exp(\operatorname{ad} Y^1)\, V^{(0)} = V^{(0)}_\mu + V^{(1)}_{\mu+1} + X^{(1)}_{\mu+2} + \cdots, \tag{10}$$

where $V^{(1)}_{\mu+1} \in N^{(1)}_{\mu+1}$. We define linear operator $L^{(2)}_1 = L^{(2)}_1[V^{(0)}_\mu, V^{(1)}_{\mu+1}]$ and let

$$H_{\mu+2} = \operatorname{Im} L^{(2)}_1 \oplus N^{(2)}_{\mu+2}.$$

There is a polynomial $Y^2 = Y^{(2)}_1 + Y^{(2)}_2$, where $Y^{(2)}_1 \in \operatorname{Ker} L^{(1)}_1$ and $Y^{(2)}_2 \in H_2$ such that (10) is converted to

$$V^{(2)} = \exp(\operatorname{ad} Y^2)\, V^{(1)} = V^{(0)}_\mu + V^{(1)}_{\mu+1} + V^{(2)}_{\mu+2} + X^{(2)}_{\mu+3} + \cdots, \tag{11}$$

where $V^{(2)}_{\mu+2} \in N^{(2)}_{\mu+2}$. Step by step, for each $m = 2, 3, ..., N$, we define a linear operator $L^{(m)}_1 = L^{(m)}_1[V^{(0)}_\mu, ..., V^{(m-1)}_{\mu+m-1}]$ and then find a polynomial $Y^m = Y^{(m)}_1 + \cdots + Y^{(m)}_m$, where $(Y^{(m)}_1, ..., Y^{(m)}_{m-1}) \in \operatorname{Ker} L^{(m-1)}_1$ and $Y^{(m)}_m \in H_m$ such that

$$\begin{aligned} V^{(m)} &= \exp(\operatorname{ad} Y^m)\, V^{(m-1)} \\ &= V^{(0)}_\mu + \cdots + V^{(m)}_{\mu+m} + X^{(m)}_{\mu+m+1} + \cdots, \end{aligned}$$

where $V^{(k)}_{\mu+k} \in N^{(k)}_{\mu+k}$ for $k = 1, ..., m$, and where $N^{(k)}_{\mu+k}$ is a complement to $\operatorname{Im} L^{(k)}_1$ in $H_{\mu+k}$. Furthermore for $V^{(N)}$ (denoted also as $V^{(N,1)}$) and for each $k = 2, 3, ...,$ we consider linear operator $L^{(N)}_k = L^{(N)}_k[V^{(0)}_\mu, ..., V^{(N-1)}_{\mu+N-1}]$, and find $Y^{N,k} = Y^{(N,k)}_k + \cdots + Y^{(N,k)}_{k+N-1}$ where $(Y^{(N,k)}_k, ..., Y^{(N,k)}_{k+N-2}) \in \operatorname{Ker} L^{(N-1)}_k[V^{(0)}_\mu, ..., V^{(N-2)}_{\mu+N-2}]$ and $Y^{(N,k)}_{k+N-1} \in H_{k+N-1}$ such that

$$\begin{aligned} V^{(N,k)} &= \exp(\operatorname{ad} Y^{N,k})\, V^{(N,k-1)} \\ &= V^{(0)}_\mu + \cdots + V^{(N)}_{\mu+N} + V^{(N)}_{\mu+N+1} + \cdots + V^{(N)}_{\mu+k+N-1} + \text{h.o.t.} \end{aligned} \tag{12}$$

where $V^{(N)}_{\mu+N+j} \in N^{(N)}_{\mu+N+j}$ for each $j \geqslant 1$ and where $N^{(N)}_{\mu+N+j}$ is a complement to $\operatorname{Im} L^{(N)}_{j+1}$ in $H_{\mu+N+j}$. Now the sequence of time one mappings defined by the sequence of polynomial vector fields $Y^1, ..., Y^N (= Y^{N,1}), Y^{N,2}, ...$ change the given vector fields to an Nth order normal form. ∎

In what follows, we may always assume that all linear operators $L^{(m)}_k$ are defined by the same sequence of homogeneous polynomials $V_\mu, V_{\mu+1}, ...$.

LEMMA 3.5.

$$(0, Y_{k+1}, ..., Y_{k+m-1}) \in \operatorname{Ker} L^{(m)}_k \Leftrightarrow (Y_{k+1}, ..., Y_{k+m-1}) \in \operatorname{Ker} L^{(m-1)}_{k+1}$$

LEMMA 3.6.

$$\operatorname{Im} L^{(m)}_{k+1} \subset \operatorname{Im} L^{(m+1)}_k, \qquad \forall k, m \geqslant 1.$$

Proof. Note that

$$\begin{aligned} \operatorname{Im} L^{(m)}_{k+1} = \{ X_{\mu+m+k} \mid \exists (Y_{k+1}, ..., Y_{k+m-1}) \in \operatorname{Ker} L^{(m-1)}_{k+1} \\ \text{and } Y_{k+m} \in H_{k+m} \text{ such that } [Y_{k+1}, V_{\mu+m-1}] + \cdots \\ + [Y_{k+m}, V_\mu] = X_{\mu+m+k} \}. \end{aligned}$$

Take $Y_k = 0$. From Lemma 3.5, if

$$[Y_{k+1}, V_{\mu+m-1}] + \cdots + [Y_{k+m}, V_\mu] = X_{\mu+m+k},$$

then $(0, Y_{k+1}, ..., Y_{k+m-1}) \in \operatorname{Ker} L^{(m)}_k$ and

$$\begin{aligned} &[0, V_{\mu+m}] + [Y_{k+1}, V_{\mu+m-1}] + \cdots + [Y_{k+m}, V_\mu] \\ &\quad = [Y_{k+1}, V_{\mu+m-1}] + \cdots + [Y_{k+m}, V_\mu] \\ &\quad = X_{\mu+m+k}. \end{aligned}$$

Hence $X_{\mu+m+k} \in \operatorname{Im} L^{(m+1)}_k$. ∎

COROLLARY 3.7.

$$\dim N^{(m+1)}_{\mu+k+m} \leqslant \dim N^{(m)}_{\mu+k+m}, \qquad \forall k, m \in \mathbb{N}.$$

Remark 3.8. It is reasonable to set

$$N^{(m+1)}_{\mu+k+m} \subset N^{(m)}_{\mu+k+m}, \qquad \forall k, m \in \mathbb{N}.$$

Remark 3.9. It is obvious that for a given formal vector field its Nth order normal form is simpler than its mth order normal form if $m < N$.

4. UNIQUE NORMAL FORMS

DEFINITION 4.1.

$$V = V_\mu + V_{\mu+1} + \cdots + V_{\mu+m} + \cdots$$

is called an *infinite order normal form*, if $V_{\mu+m} \in N^{(m)}_{\mu+m}$ for $\forall m \in \mathbb{N}$, where $N^{(m)}_{\mu+m}$ is a complementary subspace to $\operatorname{Im} L^{(m)}_1$ in $H_{\mu+m}$ and where $L^{(m)}_1 = L^{(m)}_1[V_\mu, V_{\mu+1}, ..., V_{\mu+m-1}]$ for $\forall m \in \mathbb{N}$.

Though in general we have infinitely many choices for the complementary space to the image of $L^{(m)}_1$ in $H_{\mu+m}$, in what follows, we assume that the choice of the complementary space $N^{(m)}_{\mu+m}$ to $\operatorname{Im} L^{(m)}_1$ is fixed.

THEOREM 4.2. *Let*

$$V = V_\mu + V_{\mu+1} + \cdots + V_{\mu+m} + \cdots$$

and

$$W = V_\mu + W_{\mu+1} + \cdots + W_{\mu+m} + \cdots$$

be both infinite order normal forms. If there exists a formal series $Y = Y_1 + Y_2 + \cdots + Y_m + \cdots$ *with* $Y_m \in H_m$ $(\forall m \in \mathbb{N})$ *such that* $(\Phi_Y)_* V = W$, *then*

$$V_{\mu+m} = W_{\mu+m} \qquad \forall m \in \mathbb{N}.$$

Proof. Suppose it would not be the case. Then there exists an $m \in \mathbb{N}$ such that

$$V_{\mu+k} = W_{\mu+k}\ (1 \leqslant k \leqslant m-1), \qquad V_{\mu+m} \neq W_{\mu+m}.$$

Recalling

$$\begin{aligned} W &= \exp(\operatorname{ad} Y)\, V \\ &= V + [Y, V] + \frac{1}{2!}[Y, [Y, V]] + \cdots + \frac{1}{n!}[Y, ..., [Y, V]...] + \cdots, \end{aligned}$$

we have

$$W_k = V_k + [Y, V]_k + \frac{1}{2!}[Y, [Y, V]]_k + \cdots + \frac{1}{n!}[Y, ..., [Y, V]...]_k + \cdots,$$

where $[Y, V]_k = [Y, V] \cap H_k$. Similarly for $[Y, ..., [Y, V] ...]_k$. Notice that this infinite sum has in fact only finitely many nontrivial terms, and hence the summation is well-defined. Therefore we have

$$[Y, V]_k + \frac{1}{2!}[Y, [Y, V]]_k + \cdots + \frac{1}{n!}[Y, ..., [Y, V] ...]_k + \cdots = 0 \quad (13)$$

for $\mu + 1 \leqslant k \leqslant \mu + m - 1$. It is easy to see that if $[Y, V] \neq 0$ and if the lowest degree of terms in $[Y, V]$ is l, then the lowest degree of terms in $[Y, ..., [Y, V] ...]$ with n-fold bracket operations is $l + n - 1$. Hence from (13), we have

$$[Y, V]_{\mu+1} = 0,$$

$$[Y, V]_{\mu+2} + \frac{1}{2!}[Y, [Y, V]]_{\mu+2} = 0,$$

$$\vdots$$

$$[Y, V]_{\mu+m-1} + \frac{1}{2!}[Y, [Y, V]]_{\mu+m-1} + \cdots + \frac{1}{m!}[Y, ..., [Y, V] ...]_{\mu+m-1} = 0.$$

By induction, we have

$$[Y, V]_{\mu+1} = [Y, V]_{\mu+2} = \cdots = [Y, V]_{\mu+m-1} = 0,$$

and therefore

$$[Y_1, V_\mu] = 0,$$

$$[Y_2, V_\mu] + [Y_1, V_{\mu+1}] = 0,$$

$$\vdots$$

$$[Y_{m-1}, V_\mu] + [Y_{m-2}, V_{\mu+1}] + \cdots + [Y_1, V_{\mu+m-2}] = 0,$$

namely, from Remark 3.2,

$$(Y_1, Y_2, ..., Y_{m-1}) \in \operatorname{Ker} L_1^{(m-1)}.$$

Thus

$$W_{\mu+m} = V_{\mu+m} + [Y, V]_{\mu+m}.$$

Note that

$$[Y, V]_{\mu+m} = [Y_1, V_{\mu+m-1}] + \cdots + [Y_m, V_1],$$

This means $[Y, V]_{\mu+m} \in \operatorname{Im} L_1^{(m)}$.

On the other hand, $V_{\mu+m}$ and $W_{\mu+m}$ are both in the same complementary space $N_{\mu+m}^{(m)}$ to $\operatorname{Im} L_1^{(m)}$. Therefore

$$W_{\mu+m} - V_{\mu+m} = [Y, V]_{\mu+m} \in N_{\mu+m}^{(m)} \cap \operatorname{Im} L_1^{(m)} = \{0\},$$

and hence $W_{\mu+m} = V_{\mu+m}$. This contradiction shows that the conclusion of the theorem is true. ▮

COROLLARY 4.3. *If*

$$V^{(N)} = V_\mu^{(0)} + V_{\mu+1}^{(1)} + \cdots + V_{\mu+N}^{(N)} + V_{\mu+N+1}^{(N)} + \cdots$$

and

$$W^{(N)} = V_\mu^{(0)} + W_{\mu+1}^{(1)} + \cdots + W_{\mu+N}^{(N)} + W_{\mu+N+1}^{(N)} + \cdots$$

are both Nth order normal form of (5), *then*

$$V_{\mu+k}^{(k)} = W_{\mu+k}^{(k)}, \qquad k = 1, ..., N.$$

5. A SPECIAL CASE

In this section we assume that all linear operators $L_k^{(m)}$ are based on the same sequence of polynomials $V_\mu, V_{\mu+1}, \ldots$.

PROPOSITION 5.1. *If there exists an $N \in \mathbb{N}$ such that*

$$\operatorname{Ker} L_k^{(N+1)} = \{0\} \times \operatorname{Ker} L_{k+1}^{(N)}, \qquad \forall k \in \mathbb{N}$$

holds, then

$$\operatorname{Ker} L_k^{(N+m)} = \underbrace{\{0\} \times \cdots \times \{0\}}_{m} \times \operatorname{Ker} L_{k+m}^{(N)}, \qquad \forall k, m \in \mathbb{N} \tag{14}$$

and

$$\operatorname{Im} L_k^{(N+m+1)} = \operatorname{Im} L_{k+m}^{(N+1)}, \qquad \forall k \in \mathbb{N}. \tag{15}$$

Proof. By assumption, (14) with $m = 1$ apparently holds. Suppose we have $X_{\mu+k+N+1} \in \operatorname{Im} L_k^{(N+2)}$, namely, there exists $(Y_k, ..., Y_{k+N+1})$ satisfying

$$(Y_k, ..., Y_{k+N}) \in \operatorname{Ker} L_k^{(N+1)}, \qquad Y_{k+N+1} \in H_{k+N+1},$$

$$[Y_k, V_{\mu+N+1}] + \cdots + [Y_{k+N+1}, V_\mu] = X_{\mu+k+N+1}.$$

From assumption, $Y_k=0$ and $(Y_{k+1}, ..., Y_{k+N}) \in \operatorname{Ker} L^{(N)}_{k+N}$. Hence

$$[Y_{k+1}, V_{\mu+N}] + \cdots + [Y_{k+N+1}, V_{\mu}] = X_{\mu+k+N+1}.$$

This implies that $X_{\mu+k+N+1} \in \operatorname{Im} L^{(N+1)}_{k+1}$ and hence $\operatorname{Im} L^{(N+2)}_{k} \subset \operatorname{Im} L^{(N+1)}_{k+1}$.

Conversely, if we assume $X_{\mu+k+N+1} \in \operatorname{Im} L^{(N+1)}_{k+1}$, namely, there exists $(Y_{k+1}, ..., Y_{k+N+1})$ satisfying

$$(Y_{k+1}, ..., Y_{k+N}) \in \operatorname{Ker} L^{(N)}_{k+1}, \qquad Y_{k+N+1} \in H_{k+N+1},$$

$$[Y_{k+1}, V_{\mu+N}] + \cdots + [Y_{k+N+1}, V_{\mu}] = X_{\mu+k+N+1}.$$

Then taking $Y_k=0$, it holds that $(Y_k, ..., Y_{k+N+1}) \in \operatorname{Ker} L^{(N+1)}_{k}$ and apparently

$$[Y_k, V_{\mu+N+1}] + [Y_{k+1}, V_{\mu+N}] + \cdots + [Y_{k+N+1}, V_{\mu}] = X_{\mu+k+N+1}.$$

This implies $X_{\mu+k+N+1} \in \operatorname{Im} L^{(N+1)}_{k}$, and hence $\operatorname{Im} L^{(N+1)}_{k+1} \subset \operatorname{Im} L^{(N+2)}_{k}$. Therefore

$$\operatorname{Im} L^{(N+2)}_{k} = \operatorname{Im} L^{(N+1)}_{k+1},$$

namely (15) holds for $m=1$.

Suppose (14) and (15) hold for a fixed $m \geqslant 1$. Let $(Y_k, ..., Y_{k+N+m}) \in \operatorname{Ker} L^{(N+m+1)}_{k}$. Then

$$[Y_k, V_{\mu+N+m}] + \cdots + [Y_{k+N+m}, V_{\mu}] = 0. \tag{16}$$

Note that $(Y_k, ..., Y_{k+N+m-1}) \in \operatorname{Ker} L^{(N+m)}_{k}$. By induction hypothesis, $Y_k=0, ..., Y_{k+m-1}=0$ and $(Y_{k+m}, ..., Y_{k+m+N-1}) \in \operatorname{Ker} L^{(N)}_{k+m}$. Hence from (16),

$$L^{(N+1)}_{k+m}(Y_{k+m}, ..., Y_{k+N+m}) = 0,$$

or in other words,

$$(Y_{k+m}, ..., Y_{k+N+m}) \in \operatorname{Ker} L^{(N+1)}_{k+m}.$$

By assumption, $Y_{k+m}=0$ and $(Y_{k+m+1}, ..., Y_{k+m+N}) \in \operatorname{Ker} L^{(N)}_{k+m+1}$. Hence

$$\operatorname{Ker} L^{(N+m+1)}_{k} \subset \underbrace{\{0\} \times \cdots \times \{0\}}_{m+1} \times \operatorname{Ker} L^{(N)}_{k+m+1}, \qquad \forall k \in \mathbb{N}.$$

Conversely, take $Y_k=0, ..., Y_{k+m}=0$ and $(Y_{k+m+1}, ..., Y_{k+m+N}) \in \operatorname{Ker} L^{(N)}_{k+m+1}$. Then

$$\begin{aligned} L_k^{(N+m+1)}&(Y_k, ..., Y_{k+m+N}) \\ &= [Y_{k+m+1}, V_{\mu+N-1}] + \cdots + [Y_{k+m+N}, V_\mu] \\ &= 0, \end{aligned}$$

namely,

$$\underbrace{\{0\} \times \cdots \times \{0\}}_{m+1} \times \operatorname{Ker} L^{(N)}_{k+m+1} \subset \operatorname{Ker} L_k^{(N+m+1)}.$$

Therefore (14) holds for $m+1$ and for any $k \in \mathbb{N}$. In a similar way, (15) can be proved as in the case $m=1$. ∎

Corollary 5.2. *If there exists an* N *such that*

$$\operatorname{Ker} L_k^{(N+1)} = \{0\} \times \operatorname{Ker} L^{(N)}_{k+1}, \qquad \forall k \in \mathbb{N},$$

then an $(N+1)$*th order normal form must be an infinite order normal form.*

Proof. From Proposition 5.1, we have

$$\operatorname{Im} L_k^{(N+m+1)} = \operatorname{Im} L^{(N+1)}_{k+m}, \qquad \forall k, m \in \mathbb{N}.$$

Hence we may set

$$N^{(N+m+1)}_{\mu+k+N+m} = N^{(N+1)}_{\mu+k+N+m}$$

as complementary subspaces to $\operatorname{Im} L_k^{(N+m+1)}$ for $\forall k, m \in \mathbb{N}$. Thus, for any $m > N+1$,

$$N^{(m)}_{\mu+m} = N^{(N+(m-N-1)+1)}_{\mu+1+N+(m-N-1)} = N^{(N+1)}_{\mu+1+N+(m-N-1)} = N^{(N+1)}_{\mu+m}$$

which implies that, if $V^{(N+1)}_{\mu+m} \in N^{(N+1)}_{\mu+m}$, then $V^{(N+1)}_{\mu+m} \in N^{(m)}_{\mu+m}$ for any $m \geqslant N+1$. The conclusion thus follows. ∎

Corollary 5.3. *If there exists an* $N \in \mathbb{N}$ *such that* $\operatorname{Im} L_k^{(N+m)} = \operatorname{Im} L^{(N)}_{k+m}$ *for any* $k, m \in \mathbb{N}$, *then the* N*th order normal form is an infinite order normal form.*

Example 5.4. If $\operatorname{Ker} L_k^{(1)} = \{0\}$, $\forall k \in \mathbb{N}$, then a first order normal form is also an infinite order normal form, and hence it is unique normal form of the original equation.

6. THE BOGDANOV–TAKENS NORMAL FORM: THE CASE $\mu = 2$, $v = 1$

Baider and Sanders [BS2] gave unique normal forms for cases $\mu < 2v$ and $\mu > 2v$ of Bogdanov–Takens singularities. But the case $\mu = 2v$ is still unsolved. In this section we consider a special case, i.e., $\mu = 2$, $v = 1$. By using our method introduced above we give the unique normal form for this case.

We consider the following equation:

$$\begin{aligned} \dot{x} &= y + a_{11}xy + a_{02}y^2 + O(3), \\ \dot{y} &= \alpha xy + \beta x^3 + b_{02}y^2 + O(3), \end{aligned} \tag{17}$$

where $\alpha, \beta \neq 0$.

Define $\delta: D_2 \to \mathbb{Z}$ by

$$\delta\begin{pmatrix} x^m y^n \\ 0 \end{pmatrix} = m + 2n - 1, \qquad \delta\begin{pmatrix} 0 \\ x^m y^n \end{pmatrix} = m + 2n - 2.$$

Then δ is a linear grading function with

$$\delta\begin{pmatrix} y \\ 0 \end{pmatrix} = \delta\begin{pmatrix} 0 \\ xy \end{pmatrix} = \delta\begin{pmatrix} 0 \\ x^3 \end{pmatrix} = 1, \text{ and } \delta\begin{pmatrix} x^2 \\ 0 \end{pmatrix} = 1.$$

Let

$$V_1^{(0)} = \begin{pmatrix} y \\ \alpha xy + \beta x^3 \end{pmatrix}.$$

Then the equation (17) can be written as

$$V^{(0)} = V_1^{(0)} + V_2^{(0)} + \cdots + V_m^{(0)} + \cdots \tag{18}$$

where $V_m^{(0)} \in H_m$, $m = 1, 2, \ldots$.

LEMMA 6.1. *The following vectors form a basis of the space H_m: For $m = 2k + 1$,*

$$\begin{pmatrix} 0 \\ x^{2k+3} \end{pmatrix}, \begin{pmatrix} 0 \\ x^{2k+1}y \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ x^3 y^k \end{pmatrix}, \begin{pmatrix} 0 \\ xy^{k+1} \end{pmatrix},$$

$$\begin{pmatrix} x^{2k+2} \\ 0 \end{pmatrix}, \begin{pmatrix} x^{2k}y \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} x^2 y^k \\ 0 \end{pmatrix}, \begin{pmatrix} y^{k+1} \\ 0 \end{pmatrix};$$

For $m=2k+2$,

$$\begin{pmatrix} 0 \\ x^{2k+4} \end{pmatrix}, \begin{pmatrix} 0 \\ x^{2k+2}y \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ x^{2}y^{k+1} \end{pmatrix}, \begin{pmatrix} 0 \\ y^{k+2} \end{pmatrix},$$

$$\begin{pmatrix} x^{2k+3}y \\ 0 \end{pmatrix}, \begin{pmatrix} x^{2k+1}y \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} x^{3}y^{k} \\ 0 \end{pmatrix}, \begin{pmatrix} xy^{k+1} \\ 0 \end{pmatrix}.$$

In particular, $\dim H_m = m+3$.

LEMMA 6.2.

$$\left[\begin{pmatrix} 0 \\ x^m y^n \end{pmatrix}, V_1^{(0)}\right] = \begin{pmatrix} -x^m y^n \\ mx^{m-1}y^{n+1} + (n-1)\,\alpha x^{m+1}y^n + n\beta x^{m+3}y^{n-1} \end{pmatrix},$$

$$\left[\begin{pmatrix} x^m y^n \\ 0 \end{pmatrix}, V_1^{(0)}\right] = \begin{pmatrix} mx^{m-1}y^{n+1} + n\alpha x^{m+1}y^n + n\beta x^{m+3}y^{n-1} \\ -\alpha x^m y^{n+1} - 3\beta x^{m+2}y^n \end{pmatrix}.$$

LEMMA 6.3. *Let*

$$Y_{2k+1} = \sum_{i=0}^{k+1} a_i \begin{pmatrix} 0 \\ x^{2k+3-2i}y^i \end{pmatrix} + \sum_{i=0}^{k+1} b_i \begin{pmatrix} x^{2k+2-2i}y^i \\ 0 \end{pmatrix}.$$

Then

$$\begin{aligned}[Y_{2k+1}, V_1^{(0)}] = \sum_{i=-1}^{k+1} & \begin{pmatrix} 0 \\ x^{2k+3-2i}y^{i+1} \end{pmatrix} \{(2k+3-2i)\,a_i + ia_{i+1}\alpha \\ & + (i+2)\,a_{i+2}\,\beta - b_i\alpha - 3b_{i+1}\,\beta\} \\ & + \sum_{i=-1}^{k} \begin{pmatrix} x^{2k+1-2i}y^{i+1} \\ 0 \end{pmatrix} \{(2k+2-2i)\,b_i + (i+1)\,b_{i+1}\alpha \\ & + (i+2)\,b_{i+2}\,\beta - a_{i+1}\},\end{aligned}$$

where $a_i = 0$, $b_j = 0$ *if* $i, j < 0$ *or* $i, j > k+2$.

LEMMA 6.4. *Let*

$$Y_{2k+2} = \sum_{i=0}^{k+2} a_i \begin{pmatrix} 0 \\ x^{2k+4-2i}y^i \end{pmatrix} + \sum_{i=0}^{k+1} b_i \begin{pmatrix} x^{2k+3-2i}y^i \\ 0 \end{pmatrix},$$

Then

$$[Y_{2k+2}, V_1^{(0)}] = \sum_{i=-1}^{k+1} \begin{pmatrix} 0 \\ x^{2k+3-2i}y^{i+1} \end{pmatrix} \{(2k+4-2i)\, a_i + ia_{i+1}\alpha$$
$$+ (i+2)\, a_{i+2}\beta - b_i\alpha - 3b_{i+1}\beta\}$$
$$+ \sum_{i=-1}^{k+1} \begin{pmatrix} x^{2k+2-2i}y^{i+1} \\ 0 \end{pmatrix} \{(2k+3-2i)\, b_i + (i+1)\, b_{i+1}\alpha$$
$$+ (i+2)\, b_{i+2}\beta - a_{i+1}\},$$

where $a_i = 0,\ b_j = 0$ *if* $i, j < 0$ *or* $i > k+2,\ j > k+1$.

Using these results, we have a matrix representation for the adjoint operator $\mathrm{ad}(V_1^{(0)})$, i.e., $L_m^{(1)}$. Note that $L_m^{(1)}: H_m \to H_{m+1};\ Y_m \mapsto [Y_m, V_1^{(0)}]$. Hence the matrix representation L of $L_m^{(1)}$ is given by an $(m+4)\times(m+3)$-matrix. Let

$$L = \begin{pmatrix} L_1 & L_2 \\ L_3 & L_4 \end{pmatrix},$$

where the submatrix L_3 is such that $-L_3$ is the identity matrix of the size $l = [m/2] + 2$. Here $[q]$ stands for the integer part of q. The other three submatrices are (almost) tri-diagonal matrices. More specifically, they are given as follows:

For $m = 2k+1$,

$$L_1 = \begin{pmatrix} -\alpha & \beta & 0 & & & \\ 2k+3 & 0 & 2\beta & & & \mathbf{0} \\ & 2k+1 & \alpha & 3\beta & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & \ddots & (k+1)\beta \\ & & & & 3 & k\alpha \\ \mathbf{0} & & & & & 1 \end{pmatrix};$$

$$L_2 = \begin{pmatrix} -3\beta & & & & & \\ -\alpha & -3\beta & & & & \mathbf{0} \\ & -\alpha & -3\beta & & & \\ & & \ddots & \ddots & & \\ & & & \ddots & \ddots & \\ & & & & -\alpha & -3\beta \\ \mathbf{0} & & & & & -\alpha \end{pmatrix};$$

$$L_4 = \begin{pmatrix} 0 & \beta & 0 & & & \\ 2k+2 & \alpha & 2\beta & & \mathbf{0} & \\ & 2k & 2\alpha & 3\beta & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & \ddots & (k+1)\beta \\ \mathbf{0} & & & & 2 & (k+1)\alpha \end{pmatrix}.$$

For $m = 2k+2$,

$$L_1 = \begin{pmatrix} \alpha & \beta & 0 & & & \\ 2k+4 & 0 & 2\beta & & \mathbf{0} & \\ & 2k+2 & \alpha & 3\beta & & \\ & & 2k & 2\alpha & \ddots & \\ & & & \ddots & \ddots & (k+2)\beta \\ \mathbf{0} & & & & 2 & (k+1)\alpha \end{pmatrix};$$

$$L_2 = \begin{pmatrix} -3\beta & & & & & \\ -\alpha & -3\beta & & & \mathbf{0} & \\ & -\alpha & -3\beta & & & \\ & & \ddots & \ddots & & \\ & & & \ddots & \ddots & \\ & & & & -\alpha & -3\beta \\ \mathbf{0} & & & & & -\alpha \end{pmatrix};$$

$$L_4 = \begin{pmatrix} 0 & \beta & 0 & & & \\ 2k+3 & \alpha & 2\beta & & \mathbf{0} & \\ & 2k+1 & 2\alpha & 3\beta & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & \ddots & (k+1)\beta \\ & & & & & (k+1)\alpha \\ \mathbf{0} & & & & & 1 \end{pmatrix}.$$

Remark 6.5. In order to simplify the expression, we may assume that $\alpha = 1$ since we may make a suitable linear change of variables x, y, t in the equation (17) such that the coefficient of xy is changed to 1 and the coefficient of x^3 is changed to β/α^2 accordingly.

LEMMA 6.6. *For both $m = 2k+1$ and $m = 2k+2$, the first $k+3$ rows of matrix L can be reduced to the form*

$$(0 \quad \tilde{M})$$

by a suitable row transformation. Here the matrix

$$\tilde{M} = (M_{ij}) \qquad (-1 \leqslant i \leqslant k+1;\ 0 \leqslant j \leqslant k+1)$$

be given, using $\alpha = 1$, *as follows*:

Case I. *For* $m = 2k+1$,

$$\begin{aligned}
M_{i,\,i-1} &= (2k+3-2i)(2k+4-2i) && (i = 1, ..., k+1)\\
M_{i,\,i} &= (4k+5-4i)\, i - 1 && (i = 0, ..., k+1)\\
M_{i,\,i+1} &= i(i+1) + \{(4i+6)\, k - (4i+3)\, i\}\, \beta && (i = -1, ..., k)\\
M_{i,\,i+2} &= 2(i+1)(i+2)\, \beta && (i = -1, ..., k-1)\\
M_{i,\,i+3} &= (i+2)(i+3)\, \beta^2 && (i = -1, ..., k-2)
\end{aligned}$$

and the other entries are all zero.

Case II. *For* $m = 2k+2$,

$$\begin{aligned}
M_{i,\,i-1} &= (2k+4-2i)(2k+5-2i) && (i = 1, ..., k+1)\\
M_{i,\,i} &= (4k+7-4i)\, i - 1 && (i = 0, ..., k+1)\\
M_{i,\,i+1} &= i(i+1) + \{(4i+2)\, k - (4i^2+3i-9)\}\, \beta && (i = -1, ..., k)\\
M_{i,\,i+2} &= 2(i+1)(i+2)\, \beta && (i = -1, ..., k-1)\\
M_{i,\,i+3} &= (i+2)(i+3)\, \beta^2 && (i = -1, ..., k-2)
\end{aligned}$$

and the other entries are all zero.

For convenience, we denote $M_{i,\,i-1} = a_i$, $M_{i,\,i} = b_i$, $M_{i,\,i+1} = c_i + d_i \beta$, $M_{i,\,i+2} = e_i \beta$, $M_{i,\,i+3} = f_i \beta^2$ for both cases.

LEMMA 6.7. *If* β *is not an algebraic number, then*

$$\operatorname{Ker} L_m^{(1)} = \{0\}, \qquad \forall m \in \mathbb{N}.$$

To show the lemma, it is sufficient to show that

$$\det M \neq 0$$

where $M = (M_{ij})_{0 \leqslant i, j \leqslant k+1}$ is a submatrix of $\tilde{M}$. Since $\det M$ is a polynomial of β, we only need to show that $\det M$ is not identically equal to zero, because β is not an algebraic number.

First we consider case II.

LEMMA 6.8. *In case II, we have*

$$\det M|_{\beta=0} \neq 0$$

Proof. Let D_l be the following subdeterminant:

$$D_l = \det(M_{ij}|_{\beta=0})_{1 \leqslant i,j \leqslant l}.$$

Then it is easy to see that

$$\det(M|_{\beta=0}) = (-1) \cdot D_{k+1}.$$

By induction we can show

$$D_l = \frac{(l+1)!\,(2k+1)!!}{(2k+1-2l)!!}.$$

In fact, it is true for $l=1$ and 2. Since $M|_{\beta=0}$ takes a tri-diagonal form, we have

$$D_{l+1} = b_{l+1} \cdot D_l - c_l a_{l+1} \cdot D_{l-1}.$$

Therefore, using

$$\begin{aligned} a_{l+1} &= (2k+2-2l)(2k+3-2l), \\ b_{l+1} &= (4k+3-4l)(l+1)-1, \\ c_l &= l(l+1), \end{aligned}$$

we have

$$\begin{aligned} D_{l+1} &= \{(4k+3-4l)(l+1)-1\} \frac{(l+1)!\,(2k+1)!!}{(2k+1-2l)!!} \\ &\quad - l(l+1)(2k+2-2l)(2k+3-2l) \frac{l!\,(2k+1)!!}{(2k+3-2l)!!} \\ &= \frac{(l+1)!\,(2k+1)!!}{(2k+1-2l)!!} \{(4k+3-4l)\,l + (4k+2-4l) - l(2k+2-2l)\} \\ &= \frac{(l+1)!\,(2k+1)!!}{(2k+1-2l)!!} \cdot (l+2)(2k+1-2l) \\ &= \frac{\{(l+1)+1\}!\,(2k+1)!!}{\{2k+1-2(l+1)\}!!}. \end{aligned}$$

The conclusion is thus obtained.

It follows that

$$\det(M|_{\beta=0}) = -D_{k+1} = -(k+2)!\,(2k+1)!! \neq 0,$$

and hence the lemma is proved. ∎

Next we consider case I. We introduce the same subdeterminant

$$D_l = \det(M_{ij}|_{\beta=0})_{1 \leqslant i,j \leqslant l}$$

for this case as well. Again, by induction, we can show

LEMMA 6.9. *For case I, we have*

$$D_l = \frac{(l+1)!\,(2k)!!}{(2k-2l)!!}.$$

From the lemma, we have

$$\det(M|_{\beta=0}) = -D_{k+1} = 0,$$

and hence we cannot conclude $\det M \neq 0$ immediately. We therefore differentiate $\det M$ with respect to β and will show that

$$\frac{\partial}{\partial\beta} \det M \bigg|_{\beta=0} \neq 0.$$

Let $M^{(l)}$ be the matrix given by differentiating the lth column of the matrix M with respect to β. It thus takes the following form:

$$M^{(0)}|_{\beta=0} = \begin{pmatrix} 0 & 6k & 4 & 0 & \dots & 0 \\ a_1 & b_1 & c_1 & & & \\ & \ddots & \ddots & \ddots & & \mathbf{0} \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & \ddots & c_k \\ & \mathbf{0} & & & a_{k+1} & b_{k+1} \end{pmatrix};$$

$$M^{(l)}|_{\beta=0} = \begin{pmatrix} -1 & 0 & \dots & \dots & \dots & \dots & \dots & & 0 \\ a_1 & b_1 & c_1 & & & & & & \\ & \ddots & \ddots & \ddots & & & & \mathbf{0} & \\ & & \ddots & \ddots & \ddots & & & & \\ & & & a_{l-1} & b_{l-1} & c_{l-1} & & & \\ & & & & 0 & 0 & d_l & e_l & \\ & & & & & a_{l+1} & b_{l+1} & c_{l+1} & \\ & & & & & & \ddots & \ddots & \ddots \\ & & & & & & & \ddots & \ddots & c_k \\ & \mathbf{0} & & & & & & & a_{k+1} & b_{k+1} \end{pmatrix};$$

$$M^{(k+1)}|_{\beta=0} = \begin{pmatrix} -1 & 0 & \dots & \dots & \dots & 0 \\ a_1 & b_1 & c_1 & & & \\ & \ddots & \ddots & \ddots & & \mathbf{0} \\ & & \ddots & \ddots & \ddots & \\ & \mathbf{0} & & a_k & b_k & c_k \\ 0 & \dots & \dots & \dots & 0 & 0 \end{pmatrix}.$$

The determinants of these matrices are given as follows:

$$\det(M^{(0)}|_{\beta=0}) = (-a_1)\{6k\,\delta_k - 4a_2\delta_{k-1}\},$$
$$\det(M^{(l)}|_{\beta=0}) = a_{l+1}D_{l-1}\{d_l\delta_{k-l} - a_{l+2}e_l\delta_{k-l-1}\}, \qquad 1 \leqslant l \leqslant k+1,$$

where

$$\delta_m = \det(M_{ij}|_{\beta=0})_{k+2-m \leqslant i,j \leqslant k+1}, \qquad 1 \leqslant m \leqslant k-1,$$

and

$$\delta_0 = \delta_{-1} = \delta_{-2} = 1.$$

By induction, we can show:

LEMMA 6.10.

$$\delta_m = \frac{(2m-1)!!\,k!}{(k-m)!}, \qquad 1 \leqslant m \leqslant k.$$

From these formulas, we shall compute

$$\frac{\partial}{\partial\beta}\det M\Big|_{\beta=0} = (-a_1)\{6k\delta_k - 4a_2\delta_{k-1}\}$$
$$+ \sum_{l=1}^{k-1}(a_{l+1}D_{l-1})\{d_l\delta_{k-l} - a_{l+2}e_l\delta_{k-l-1}\}$$
$$+ (a_{k+1}D_{k-1})\,d_k.$$

First we compute the second term. Since

$$D_{l-1} = \frac{l!\,(2k)!!}{(2k+2-2l)!!} = \frac{l!\,2^k\,k!}{2^{k+1-l}(k+1-l)!} = \frac{2^{l-1}\,k!\,l!}{(k+1-l)!},$$

$$\delta_{k-l} = \frac{(2k-1-2l)!!\,k!}{l!} = \frac{(2k-1-2l)!\,k!}{l!\,2^{k-1-l}(k-1-l)!} = \frac{(2k-2l)!\,k!}{2^{k-l}l!\,(k-l)!},$$

$$\delta_{k-l-1} = \frac{(2k-3-2l)!!\,k!}{(l+1)!} = \frac{(2k-3-2l)!\,k!}{(l+1)!\,2^{k-2-l}(k-2-l)!}$$
$$= \frac{(2k-2-2l)!\,k!}{2^{k-1-l}(l+1)!\,(k-1-l)!},$$

we have

$$D_{l-1}(d_l\delta_{k-l} - a_{l+2}e_l\delta_{k-l-1}) = -(2k-5l)\,\frac{2^{2l-1}(k!)^2\,(2k-2l)!}{(k+1-l)!\,(k-l)!\,2^k}.$$

Hence

$$a_{l+1}D_{l-1}(d_l\delta_{k-l} - a_{l+2}e_l\delta_{k-l-1}) = -(2k-5l)\,\frac{2^{2l-1}(k!)^2\,(2k+2-2l)!}{(k+1-l)!\,(k-l)!\,2^k}.$$

We need to compute

$$\sum_{l=1}^{k-1}(2k-5l)\,\frac{2^{2l-1}(2k+2-2l)!}{(k+1-l)!\,(k-l)!}\cdot\frac{(k!)^2}{2^k}.$$

Let $i = k+1-l$ and

$$A_p(i) = \frac{(2i)!}{i!\,i!}\,i^p\cdot 4^{k-i}, \qquad p = 1, 2, 3,$$

for simplicity of notation. Then $l = k+1-i$ and

$$(2k-5l)\,\frac{2^{2l-1}(2k+2-2l)!}{(k+1-l)!\,(k-l)!}\cdot\frac{(k!)^2}{2^k} = -\frac{(3k+5)(k!)^2}{2^{k-1}}\,A_1(i) + \frac{5(k!)^2}{2^{k-1}}\,A_2(i).$$

Therefore we need to compute

$$\sum_{i=2}^{k} A_p(i) \qquad \text{for} \quad p = 1, 2.$$

LEMMA 6.11. (1) $A_1(i+1) - A_1(i) = \frac{1}{2}A_0(i)$

(2) $A_2(i+1) - A_2(i) = \frac{3}{2}A_1(i) + \frac{1}{2}A_0(i)$

(3) $A_3(i+1) - A_3(i) = \frac{5}{2}A_2(i) + 2A_1(i) + \frac{1}{2}A_0(i)$

Proof. A simple computation shows

$$A_p(i+1) - A_p(i) = \frac{(2i)!}{i!\,i!}\,4^{k-i}\cdot\left\{\left(i+\frac{1}{2}\right)(i+1)^{p-1} - i^p\right\}.$$

For $p=1, 2, 3$, we get

$$\left(i+\frac{1}{2}\right)(i+1)^{1-1}-i^{1}=\frac{1}{2}i^{0},$$

$$\left(i+\frac{1}{2}\right)(i+1)^{2-1}-i^{2}=\frac{3}{2}i^{1}+\frac{1}{2}i^{0},$$

$$\left(i+\frac{1}{2}\right)(i+1)^{3-1}-i^{2}=\frac{5}{2}i^{2}+2i^{1}+\frac{1}{2}i^{0},$$

and hence the lemma follows. ∎

Summing them up from $i=2$ to k, we have

$$\sum_{i=2}^{k} A_1(i)=\frac{2}{3}\{A_2(k+1)-A_2(2)-A_1(k+1)+A_1(2)\}$$
$$=\frac{2}{3}\left\{\frac{(2k+2)!}{(k+1)!\,(k+1)!}\cdot\frac{k(k+1)}{4}-3\cdot 4^{k-1}\right\};$$

$$\sum_{i=2}^{k} A_2(i)=\frac{2}{5}\left\{A_3(k+1)-\frac{4}{3}A_2(k+1)+\frac{1}{3}A_1(k+1)-A_3(2)\right\}$$
$$+\frac{2}{5}\left\{\frac{4}{3}A_2(2)-\frac{1}{3}A_1(2)\right\}$$
$$=\frac{2}{5}\left\{\frac{(2k+2)!}{(k+1)!\,(k+1)!}\cdot\frac{k(k+1)(k+(2/3))}{4}-5\cdot 4^{k-1}\right\}.$$

From these expressions, we have

$$-\frac{(3k+5)(k!)^2}{2^{k-1}}\sum_{i=2}^{k} A_1(i)+\frac{5(k!)^2}{2^{k-1}}\sum_{i=2}^{k} A_2(i)$$
$$=\frac{(2k+2)!}{2^{k-1}}\cdot\left\{-\frac{(3k+5)\,k}{6(k+1)}+\frac{k(3k+2)}{6(k+1)}\right\}$$
$$-\frac{(k!)^2\cdot 4^{k-1}}{2^{k-1}}\cdot\{-2(3k+5)+10\}$$
$$=\frac{(2k+1)!}{2^{k-1}}\cdot(-k)+2^{k}\cdot(k!)^2\cdot 3k.$$

Therefore we finally obtain

$$\frac{\partial}{\partial\beta}\det M\Big|_{\beta=0}$$
$$= -(2k+1)(2k+2)\cdot\{6k\cdot(2k-1)!!\,k! - 4(2k-1)\cdot 2k\cdot(2k-3)!!\,k!\}$$
$$+k\,\frac{(2k+1)!}{2^{k-1}} - 2^k(k!)^2\cdot 3k + k!\,(2k)!!\cdot 3k$$
$$= (2k+1)(2k+2)\cdot 2k\cdot(2k-1)!!\,k! + k\,\frac{(2k+1)!}{2^{k-1}}$$
$$- 3k\cdot 2^k(k!)^2 + 3k\cdot k!\,(2k)!!$$
$$= \frac{(2k+2)!}{2^{k-1}(k-1)!}\,k! + \frac{(2k+1)!\,k}{2^{k-1}} - 3k\cdot 2^k(k!)^2 + 3k\cdot k!\cdot 2^k\cdot k!$$
$$= 2k\cdot k!\,(2k+3)!!,$$

and hence we conclude $\det M \neq 0$.

THEOREM 6.12. *If β/α^2 is not an algebraic number, then the first order normal form of Eq.* (17) *with respect to the grading function δ is unique, which is*

$$\dot{x} = y, \qquad \dot{y} = \alpha xy + \beta x^3 + \sum_{m=4}^{\infty} a_m x^m. \tag{19}$$

Proof. The uniqueness of the first order normal form follows from Corollary 5.2, Example 5.4 and Lemma 6.7. A simple calculation shows that $\text{span}\langle \binom{0}{x^{m+3}} \rangle$ is a complement to $\text{Im}\, L_m^{(1)}$ for each $m \in \mathbb{N}$. Therefore the first order normal form (19) is obtained. ∎

Remark 6.13. F. Dumortier pointed out to us that bifurcations from the singularity treated in this paper was studied by [DRS] almost completely. They used different normal forms using notion of equivalence, not conjugacy, for which the change of time variable is permitted as well as changes of space variables. However, Eq. (19) can also be obtained under equivalence, without assuming any condition on α and β. Therefore our results indicates that, if we assume non-algebraicity of β/α^2, the normal form obtained in [DRS] under equivalence can be obtained only under conjugacy.

Remark 6.14. By a similar argument, we can show that

$$\det \bar{M} \neq 0,$$

if β is not an algebraic number, where $\bar{M}$ is the matrix which is made from $\tilde{M}$ by removing the second row. Hence the unique normal form can be also taken as

$$\dot{x} = y, \qquad \dot{y} = \alpha xy + \beta x^3 + \sum_{m=2}^{\infty} b_m x^m y.$$

More generally, we can take the following form as a unique normal form:

$$\dot{x} = y, \qquad \dot{y} = \alpha xy + \beta x^3 + \sum_{m=2}^{\infty} c_m X_m,$$

where X_m can be either x^{m+2} or $x^m y$.

ACKNOWLEDGMENTS

This work was done while D. Wang was visiting Ryukoku University in 1994 and while H. Kokubu and H. Oka were visiting Tsinghua University in 1995. We are grateful for the hospitality of both universities and the support which made these visits possible. We also thank Freddy Dumortier for several useful remarks, including Remark 6.13.

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