Available online at www.sciencedirect.com
ScienceDirect
Journal of Computational and Applied Mathematics 221 (2008) 396–405
JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS
www.elsevier.com/locate/cam

Integral refinable operators exact on polynomials[☆]

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Received 11 December 2006; received in revised form 30 May 2007

Abstract

We study integral refinable operators of integral type exact on polynomials of even degree constructed by using refinable B-bases of GP type. We prove a general theorem of existence and uniqueness. Then we study the L^p-norm of these operators and we give error bounds in approximating functions and their derivatives belonging to suitable classes. Numerical results and comparisons with other quasi-interpolatory operators having the same order of exactness on polynomial reproduction are presented.

Keywords: Refinable functions; B-bases; Integral operators

1. Introduction

Recently refinable integral operators of Bernstein–Durrmeyer type have been studied either as regards L^2 convergence properties on the interval, or as regards their eigenstructure and spectral properties [2,3].

Such operators reproduce only the constant functions which correspond to degree $m = 0$ and then the approximations are often poor. In order to obtain operators involving integrals of functions f, with a higher order of accuracy then providing a better approximation, we require that the operators are endowed with the property of reproducing appropriate classes of polynomials.

In this paper, for a given positive integer n, using the GP scaling functions introduced in [1] and the relative B-bases on a given finite interval I (see [2,4,10]), we construct refinable integral operators that reproduce polynomials of degree $2m$ with $0 \le 2m \le n-2$, having assumed that the basis functions, utilized for constructing the operators, have order of polynomial reproducibility $n-2$.

These operators are linear combinations of refinable functions belonging to a fixed B-basis; the coefficients are suitable combinations of inner products involving the function f.

Our first objective is to examine, in Section 2, if, for any integer n, it is possible to construct an unique refinable integral operator reproducing polynomials of degree $2m$; in the same Section 2 we give the proof of existence and uniqueness. In Section 3 we prove some properties and give convergence results of the quoted operators. Finally,

[☆] Paper partially supported by MIUR.

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0377-0427/$ - see front matter
doi:10.1016/j.cam.2007.10.046

Section 4 is devoted to presenting some numerical results and comparisons between the operators considered here, the operators of Bernstein–Durrmeyer type studied in [2,3] and the quasi-interpolatory operators studied in [5,6] with the same reproducibility order of operators as presented here. We performed also a comparison with the integral operators constructed from uniformly partitioned spline bases studied in [9].

2. Construction of the operators $T_{j_0,m} f$

We consider any system of GP refinable functions [1]:

$$\Phi_{h,n} := \{\varphi_{h,n}(x-k), \forall k \in \mathbb{Z}\}, \tag{2.1}$$

where $\varphi_{h,n}(x)$ has support $[0, n+1]$ and satisfies the refinement equation:

$$\varphi_{h,n}(x) = \sum_{k\in\mathbb{Z}} a_{k,h,n}\varphi_{h,n}(2x-k), \quad h \geq n \geq 2 \tag{2.2}$$

where

$$a_{k,h,n} = g\widetilde{a}_{k,n+1} + (1-g)\widetilde{a}_{k-1,n-1}, \tag{2.3}$$

with $\widetilde{a}_{k,n} = \frac{1}{2^{n-1}}\binom{n}{k}$ and $g = \frac{1}{2^{h-n}} \in [0, 1]$; we recall that h is a real parameter. Here and in the rest of the paper $\binom{n}{k} = 0$ for $k < 0$ or $k > n$.

Such totally positive scaling functions are characterized, in particular, by the following properties: $\varphi_{h,n} \in C^{n-2}(\mathbb{R})$; $\varphi_{h,n}$ is centrally symmetric; $\sum_{k\in\mathbb{Z}} \varphi_{h,n}(x-k) = 1, \forall x \in \mathbb{R}$. Moreover, $\varphi_{h,n}$ generates a multiresolution analysis (MRA) of $L^2(\mathbb{R})$.

Since our interest is towards the construction of operators on finite interval I, having length $|I| \geq n+1$, we recall that, starting from any system (2.1) restricted to such an interval, it is possible to construct the corresponding B-basis $W_{h,n} = \{w_{i,n}(x), i = 0, 1, \ldots, N, N = |I| + n - 1\}$, on the same interval I [2,10]. The reason for introducing the B-basis $W_{h,n}$ is that the restriction of $\Phi_{h,n}$ to a finite interval presents some drawbacks; in particular it introduces some discontinuities at the end points because the edge functions are truncated. Using the basis functions in $W_{h,n}$ we overcome the drawbacks.

If we define

$$U_{h,n}^{(0)} = \{u_i^{(0)}(x) = \varphi_{h,n}(x+n-i), x \in I, i = 0, 1, \ldots, N\}, \tag{2.4}$$

the B-basis $W_{h,n}$ is characterized by

$$U_{h,n}^{(0)} = W_{h,n}A, \tag{2.5}$$

where the matrix A is given by

$$A = \left[\left(\prod_{k=0}^{n-2} A_k\right)\left(\prod_{k=0}^{n-2} L_k\right) D\right]^{-1}, \tag{2.6}$$

and it is stochastic and totally positive (TP); the matrices A_k and $L_k, k = 0, \ldots, n-2$, are suitable upper and lower triangular matrices respectively, with unit diagonal, and D is a diagonal matrix. We recall also that $w_{i,n} \in C^{n-2}, i = 0, 1, \ldots, N$; some interesting properties of $w_{i,n}$ have been proved in [10].

Now, let $I = [a, b]$ be a finite interval, with $a, b \in \mathbb{R}$ and let m be any integer satisfying $0 \leq 2m \leq n-2$; from now on let us denote by j_0 the first integer such that $b - a \geq 2^{-j_0}(n+2m+1)$; we shall set 0 as the index of all functions at level j_0. This condition guarantees that, in all B-bases that we are considering, there appears at least one function with support entirely contained in I.

Consider the system of functions

$$\Phi_{0,h,n} := \left\{\varphi_{0,k,n}(x) = 2^{j_0/2}\varphi_{h,n}(2^{j_0}x-k), 2^{j_0}a - n \leq k \leq 2^{j_0}b - 1\right\};$$

after having constructed the B-basis $W_{0,h,n} = \{w_{0,i,n}(x), i = 0, 1, \ldots, N_0\}$, with $N_0 = 2^{j_0}(b-a) + n - 1$, corresponding to $\Phi_{0,h,n}$, we introduce the class $X_{0,h,n}$ of functions $\chi_{0,i,n}(x)$ defined by

$$\chi_{0,i,n}(x) = \frac{w_{0,i,n}(x)}{\int_I w_{0,i,n}(x)\mathrm{d}x}, \tag{2.7}$$

and characterized by the property of having integral equal to 1 on interval I.

For any $f \in L^p(I)$, $1 \le p \le \infty$, with the usual interpretation for $p = \infty$, we define the integral operator

$$T_{0,m} f(x) := \sum_{i=0}^{N_0} \langle f, C_{0,m,i} \rangle w_{0,i,n}(x), \tag{2.8}$$

where $\langle f, g \rangle := \int_I f(x)g(x)\mathrm{d}x$ and each $C_{0,m,i}(x)$ is a linear combination of $2m+1$ functions defined in (2.7), i.e.,

$$C_{0,m,i}(x) = \sum_{k=-m}^{m} c_{0,i,k}\chi_{0,i+k,n}(x). \tag{2.9}$$

The m first and the m last combinations of functions $\chi_{0,i,n}$ are fixed as follows:

$$\text{for } 0 \le i \le m-1, \quad C_{0,m,i}(x) = \sum_{k=-m}^{m} c_{0,i,k}\chi_{0,m+k,n}(x); \tag{2.10}$$

$$\text{for } N_0 - m + 1 \le i \le N_0, \quad C_{0,m,i}(x) = \sum_{k=-m}^{m} c_{0,i,k}\chi_{0,N_0+k-m,n}(x). \tag{2.11}$$

The coefficients $c_{0,i,k}$, which depend on m and n, are determined in order that the operator $T_{0,m}$ be exact on the space of polynomials $\mathbb{P}_{2m}$ of degree $\le 2m$, i.e. satisfies $T_{0,m}p = p$, for all $p \in \mathbb{P}_{2m}$.

Operators of type (2.8) are extensions of integral operators studied in [2,3], which correspond to $m = 0$. They generalize to refinable functions some ideas developed in [8,9] for polynomials and splines, respectively.

We now recall some results on the moments of functions $\chi_{0,i,n}$ which have been obtained in [7] and which will be used later in the proof of Theorem 2.2 when we prove the existence and uniqueness of the operators $T_{0,m}$.

The moments of the functions $\chi_{0,i,n}$ are denoted by

$$\mu_{0,i}(l,n) = \int_I x^l \chi_{0,i,n}(x)\mathrm{d}x = \frac{\overline{\mu}_{0,i}(l,n)}{\int_I w_{0,i,n}(x)\mathrm{d}x}, \quad 0 \le l \le n-2, \tag{2.12}$$

where $\overline{\mu}_{0,i}(l,n) = \int_I x^l w_{0,i,n}(x)\mathrm{d}x$.

In particular $\mu_{0,i}(0,n) = 1$, according to the normalization of the functions $\chi_{0,i,n}$.

Property 2.1. *For all* $0 \le l \le n-2$ *one has*

$$\mu_{0,i+1}(l,n-1) - \mu_{0,i}(l,n-1) = l \cdot \mu_{0,i+1}(l-1,n) \int_I w_{0,i+1,n}(x)\,\mathrm{d}x, \tag{2.13}$$

$i = 0, 1, \ldots, N_0 - 2$ *(see Property* 3.3 *and Remark 3.1 in* [7]*).*

Let $B_{i,n}$ be the square matrix of order $2m+1$ whose entries, for $i = m, m+1, \ldots, N_0 - m$, are given by

$$B_{i,n}(l,s) = \mu_{0,i+s-m}(l,n), \quad 0 \le l, s \le 2m. \tag{2.14}$$

For $i = 0, \ldots, m-1$ and $i = N_0 + 1 - m, \ldots, N_0$, the matrices are defined by

$$B_{i,n}(l,s) = \mu_{0,s}(l,n), \quad 0 \le l, s \le 2m \tag{2.15}$$

and

$$B_{i,n}(l,s) = \mu_{0,N_0-2m+s}(l,n), \quad 0 \le l, s \le 2m \tag{2.16}$$

respectively. We define $D_{i,n}^{(2m+1)} = \det(B_{i,n})$.

Lemma 2.1. *Let $m \geq 1$ be such that $2m \leq n-2$; matrix $B_{i,n}$ is invertible, for all i, $0 \leq i \leq N_0$.*

Proof. Let us prove the lemma considering the matrices $B_{i,n}$ with $i = m, m+1, \ldots, N_0 - m$, that is,

$$B_{i,n} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \mu_{0,i-m}(1,n) & \mu_{0,i-m+1}(1,n) & \cdots & \mu_{0,i+m}(1,n) \\ \vdots & \vdots & & \vdots \\ \mu_{0,i-m}(2m,n) & \mu_{0,i-m+1}(2m,n) & \cdots & \mu_{0,i+m}(2m,n) \end{bmatrix}; \tag{2.17}$$

the claim can be proved for the first and last m matrices $B_{i,n}$, analogously.

Replacing in $B_{i,n}$ the column V_r with $V_r - V_{r-1}$, $r = i-m+1, \ldots, i+m$, leaving unchanged the first column V_{i-m}, by using Property 2.1, we obtain for the determinant of $B_{i,n}$ the relation

$$D_{i,n}^{(2m+1)} = (2m)! \left[\overline{\mu}_{0,i-m+1}(0,n+1) \ldots \overline{\mu}_{0,i+m}(0,n+1)\right] D_{i+1,n+1}^{(2m)}. \tag{2.18}$$

Proceeding iteratively and applying the same procedure to $D_{i+1,n+1}^{(2m)}, D_{i+2,n+2}^{(2m-1)}, \ldots, D_{i+2m-1,n+2m-1}^{(2)}$ we conclude the proof. $\square$

Now we can prove the following theorem for stating the existence and uniqueness of the operators (2.8).

Theorem 2.2. *For each integer m such that $0 \leq 2m \leq n-2$, there exists a unique operator $T_{0,m}f$ which is exact on $\mathbb{P}_{2m}$.*

Proof. The requirement $T_{0,m}p = p$, $\forall p \in \mathbb{P}_{2m}$, is satisfied if and only if $T_{0,m}x^r = x^r$, $0 \leq r \leq 2m$, i.e. if $\sum_{i=0}^{N_0} \langle x^r, C_{0,m,i} \rangle w_{0,i,n}(x) = x^r$. Considering that $x^r = \sum_{i=0}^{N_0} \eta_{0,i}^{(r)} w_{0,i,n}(x)$, for each r [4], we obtain a linear system of order $2m+1$ for the coefficients $c_{0,i,k}$:

$$\sum_{k=-m}^{m} c_{0,i,k} \int_I x^r \chi_{0,i+k,n}(x) \mathrm{d}x = \eta_{0,i}^{(r)} \quad r = 0, 1, \ldots, 2m. \tag{2.19}$$

Since the matrix of this linear system coincides with $B_{i,n}$, $i = 0, 1, \ldots, N_0$, from Lemma 2.1 the result follows since the sequences of coefficients $c_{0,i,k}$, $k = -m, \ldots, m$ are uniquely determined. $\square$

Remark 2.1. Assuming $r = 0$, on the basis of the results about the moments of functions $\chi_{0,i,n}$ obtained in [10], the relation (2.19) gives rise to the equation

$$\sum_{k=-m}^{m} c_{0,i,k} = 1. \tag{2.20}$$

If $m = 1$, taking into account that matrices $B_{i,n} \in \mathbb{R}^{3\times 3}$, we can perform the explicit formal computation of coefficients $c_{0,i,k}$ $-1 \leq k \leq 1$, by exploiting the procedure outlined in the proof of Lemma 2.1.

In fact, defining

$$\mathrm{inv}(B_{i,n}) = \frac{\left[B_{k,j}^*\right]}{D_{i,n}^{(3)}}, \tag{2.21}$$

with $B_{j,k}^*$ the cofactor of the element $b_{j,k}$ in $B_{i,n}$, the entries in the first and last rows of $\mathrm{inv}(B_{i,n})$ are characterized by having the moments of two successive functions $\chi_{0,i,n}, \chi_{0,i+1,n}$ and $\chi_{0,i-1,n}, \chi_{0,i,n}$ respectively. Therefore, by multiplying the rows $\left(\frac{B_{1,1}^*}{D_{i,n}^{(3)}} \frac{B_{2,1}^*}{D_{i,n}^{(3)}} \frac{B_{3,1}^*}{D_{i,n}^{(3)}}\right)$ and $\left(\frac{B_{1,3}^*}{D_{i,n}^{(3)}} \frac{B_{2,3}^*}{D_{i,n}^{(3)}} \frac{B_{3,3}^*}{D_{i,n}^{(3)}}\right)$ for the vector $[1 \eta_{0,i}^{(1)} \eta_{0,i}^{(2)}]^{\mathrm{T}}$ we obtain the values of $c_{0,i,-1}$ and $c_{0,i,+1}$ respectively; thus, using (2.20) we can evaluate $c_{0,i,0} = 1 - c_{0,i,-1} - c_{0,i,+1}$.

3. Norm of operator and convergence results

Consider now the refined system $\Phi_{j,h,n}$, $j \geq j_0$; by using such a system, we construct the jth corresponding normalized B-basis $W_{j,h,n} = \{w_{j,i,n}(x), i = 0, 1, \ldots, N_j\}$, with $N_j = 2^j(b-a) + n - 1$, and then we construct the operators $T_{j,m} f$, approximating $f \in L^p(I)$, $p \geq 1$. They have the form

$$T_{j,m} f(x) = \sum_{i=0}^{N_j} \left\langle f, \sum_{k=-m}^{m} c_{j,i,k} \chi_{j,i+k,n} \right\rangle w_{j,i,n}(x), \tag{3.1}$$

where $w_{j,i,n} \in W_{j,h,n}$, $\chi_{j,i,n}(x) = \frac{w_{j,i,n}(x)}{\int_I w_{j,i,n}(x)\mathrm{d}x} \in X_{j,h,n}$, $j \geq j_0$. The existence and uniqueness of $T_{j,m} f$ immediately follow considering that it is a refined operator.

In this section we shall analyze some properties of $T_{j,m}$ for $j \geq j_0$.

We denote by $\vartheta_{j,i,n}$ the quantity

$$\vartheta_{j,i,n} = \frac{1}{\int_I w_{j,i,n}(x)\mathrm{d}x}; \tag{3.2}$$

$\Gamma_{j,i,q}$ is defined by

$$\Gamma_{j,i,q} := \left[\sum_{k=-m}^{m} \left(\frac{\vartheta_{j,i+k,n}}{\vartheta_{j,i,n}} \right)^{q-1} \left| c_{j,i,k} \right|^q \right]^{\frac{1}{q}}, \tag{3.3}$$

$q \geq 1$, $0 \leq 2m \leq n-2$.
In (3.3) for $0 \leq i \leq m-1$ we assume $\vartheta_{j,i+k,n} = \vartheta_{j,m+k,n}$, while for $N_0 - m + i \leq i \leq N_0$ we assume $\vartheta_{j,i+k,n} = \vartheta_{j,N_0+k-m,n}$.
Then we have

$$\Gamma_{j,q} = \sup_i \Gamma_{j,i,q} < +\infty. \tag{3.4}$$

When $q = 1$, $\Gamma_{j,1} = \sum_{k=-m}^{m} \left| c_{j,i,k} \right|$ is simply the ℓ^1-norm of the vector $\left[c_{j,i,k} \right]_{k=-m,\ldots,m}$ in the $X_{j,h,n}$ basis.

We denote by $\mathcal{L}(L^p(I))$ the space of linear continuous operators on $L^p(I)$.

Theorem 3.1. *Let us consider $\Gamma_{j,q}$ as in (3.4). For all $0 \leq 2m \leq n-2$, $T_{j,m}$ is a bounded operator in $L^p(I)$, $1 \leq p, q \leq +\infty$ and $p^{-1} + q^{-1} = 1$. Moreover,*

$$\| T_{j,m} \|_{\mathcal{L}(L^p(I))} \leq \Gamma_{j,q} \, (n + 2 + 2m)^{\frac{1}{p}}. \tag{3.5}$$

Proof. We define

$$C_{j,m,i}(x) = \sum_{k=-m}^{m} c_{j,i,k} \chi_{j,i+k,n}(x) \tag{3.6}$$

and we set $y_{j,k} = \frac{k}{2^j} \in I$, considering also that some of these points can coincide with a or b when $k < 0$ or $k > N_j$.
There results

$$\sigma_{j,i} = \mathrm{supp}(C_{j,m,i}(x)) = [y_{j,i-m}, y_{j,i+n+1+m}]. \tag{3.7}$$

Using Hölder's inequality we get

$$\left| \langle f, C_{j,m,i} \rangle \right| \leq \| f \|_{p,\sigma_{j,i}} \cdot \left\| C_{j,m,i} \right\|_{q,\sigma_{j,i}}, \tag{3.8}$$

where $\|.\|_{s,V}$ denotes the norm in $L^s(V)$. Now, starting from the inequality

$$\left| T_{j,m} f(x) \right| \leq \sum_{i=0}^{N_j} \| f \|_{p,\sigma_{j,i}} \left\| \vartheta_{j,i,n}^{-\frac{1}{p}} C_{j,m,i} \right\|_{q,\sigma_{j,i}} \vartheta_{j,i,n}^{\frac{1}{p}} w_{j,i,n}(x), \tag{3.9}$$

we obtain

$$\left|\vartheta_{j,i,n}^{-\frac{1}{p}} C_{j,m,i}(x)\right|^q = \left|\vartheta_{j,i,n}^{\frac{1}{q}-1} C_{j,m,i}(x)\right|^q \tag{3.10}$$
$$\leq \left[\sum_{k=-m}^{m} \left|c_{j,i,k}\right| \vartheta_{j,i,n}^{\frac{1}{q}-1} \chi_{j,i+k,n}(x)\right]^q = \left[\sum_{k=-m}^{m} \left|c_{j,i,k}\right| \vartheta_{j,i,n}^{\frac{1}{q}-1} \vartheta_{j,i+k,n} w_{j,i+k,n}(x)\right]^q .$$

Applying Hölder's inequality, taking into account that $\sum_{k=-m}^{m} w_{j,i+k,n}(x) \leq 1$, $\vartheta_{j,i+k,n} w_{j,i+k,n}(x) = \chi_{j,i+k,n}(x)$ and considering that $w_{j,i+k,n}(x) = w_{j,i+k,n}^{\frac{1}{p}+\frac{1}{q}}(x)$, from (3.10) there results

$$\left|\vartheta_{j,i,n}^{-\frac{1}{p}} C_{j,m,i}(x)\right|^q \leq \left[\sum_{k=-m}^{m} \left|c_{j,i,k}\right|^q \vartheta_{j,i,n}^{1-q} \vartheta_{j,i+k,n}^{q} w_{j,i+k,n}(x)\right] \cdot \left[\sum_{k=-m}^{m} w_{j,i+k,n}(x)\right]^{\frac{q}{p}}$$
$$\leq \sum_{k=-m}^{m} \left(\frac{\vartheta_{j,i+k,n}}{\vartheta_{j,i,n}}\right)^{q-1} \left|c_{j,i,k}\right|^q \chi_{j,i+k,n}(x) .$$

Therefore

$$\left\|\vartheta_{j,i,n}^{-\frac{1}{p}} C_{j,m,i}\right\|_{q,\sigma_{j,i}} = \left[\int_{\sigma_{j,i}} \left|\vartheta_{j,i,n}^{-\frac{1}{p}} C_{j,m,i}(x)\right|^q \mathrm{d}x\right]^{\frac{1}{q}} \leq \Gamma_{j,i,q}, \tag{3.11}$$

and we can write

$$\left|T_{j,m} f(x)\right|^p \leq \left\{\left[\sum_{i=0}^{N_j} \|f\|_{p,\sigma_{j,i}} \Gamma_{j,i,q} \left(\vartheta_{j,i,n} w_{j,i,n}\right)^{\frac{1}{p}}\right] \cdot \left[\sum_{i=0}^{N_j} w_{j,i,n}\right]^{\frac{1}{q}}\right\}^p$$
$$\leq \sum_{i=0}^{N_j} \|f\|_{p,\sigma_{j,i}}^p \Gamma_{j,i,q}^p \chi_{j,i,n} \left(\sum_{i=0}^{N_j} w_{j,i,n}\right)^{\frac{p}{q}} \leq \Gamma_{j,q}^p \sum_{i=0}^{N_j} \|f\|_{p,\sigma_{j,i}}^p \chi_{j,i,n}. \tag{3.12}$$

We achieve the claim since

$$\int_I \left|T_{j,m} f(x)\right|^p \mathrm{d}x \leq \Gamma_{j,q}^p \sum_{i=0}^{N_j} \|f\|_{p,\sigma_{j,i}}^p \int_I \chi_{j,i,n}(x)\mathrm{d}x \leq \Gamma_{j,q}^p (n+2+2m) \|f\|_{p,I}^p . \tag{3.13}$$

In (3.13) we exploited Jensen's inequality (see e.g. [11]):

$$\sum_{i=0}^{N_j} \|f\|_{p,\sigma_{j,i}}^p \leq (n+2+2m) \|f\|_{p,I}^p$$

because $\sigma_{j,i}$ contains $n+2+2m$ intervals and then any piece of $[a,b]$ is added into the sum at most $(n+2+2m)$ times. □

We are now interested in estimating the error in approximating $D^k f$, the kth derivative of a smooth function f, by the kth derivative of $T_{j,m} f$, with $0 \leq k < s$, $1 \leq s \leq 2m+1$, exploiting the property that $T_{j,m}$ reproduces polynomials. We set

$$L_s^p(I) := \{f : D^{s-1} f \text{ is absolutely continuous on } I \text{ and } D^s f \in L^p(I)\}, \tag{3.14}$$

and we suppose $f \in L_s^p(I)$.

Let $x \in I$ and let ℓ be such that $x \in \left[y_{j,\ell}, y_{j,\ell+1}\right]$; for any polynomial $g \in \mathbb{P}_{s-1}$ and any f such that $D^k f$ exists, we write

$$E_{k,s}(x) = D^k R(x) - D^k T_{j,m} R(x), \quad 0 \leq k < s \leq 2m+1 \tag{3.15}$$

where $R(x) = f(x) - g(x)$.

Formula (3.15) reduces the problem of estimating $\left|E_{k,s}(x)\right|$ in obtaining estimates of $\left|D^k R(x)\right|$ and $\left|D^k T_{j,m} R(x)\right|$. We consider for g the Taylor expansion $g(x) = \sum_{r=0}^{s-1} \frac{D^r f(t_\ell)}{r!} (x - t_\ell)^r$ at $t_\ell \in \left[y_{j,\ell}, y_{j,\ell+1}\right]$ and we define

$$I_{j,\ell} = \bigcup_{i=\ell}^{\ell+n} \sigma_{j,i} \tag{3.16}$$

with $\sigma_{j,i}$ as in (3.7).

Lemma 3.2. *If $f \in L_s^p(I_{j,\ell})$, $1 \le p \le q \le \infty$, there results*

$$\left|D^k R(x)\right| \le C_{1,k} \left|x - y_{j,\ell}\right|^{s-k-1+\frac{1}{q}} \left\|D^s f\right\|_{p,I_{j,\ell}} \tag{3.17}$$

where $C_{1,k} = \frac{1}{(s-k-1)!((s-k-1)q+1)^{\frac{1}{q}}}$.

Proof. The rest $R(x)$ of the Taylor expansion $g(x)$ is given by

$$R(x) = \frac{1}{(s-1)!} \int_{t_\ell}^{x} (x - z)^{s-1} D^s f(z) \mathrm{d}z. \tag{3.18}$$

Therefore

$$D^k R(x) = \frac{1}{(s-k-1)!} \int_{t_\ell}^{x} (x - z)^{s-k-1} D^s f(z) \mathrm{d}z \tag{3.19}$$

and for Hölder's inequality

$$\left|D^k R(x)\right| \le \frac{1}{(s-k-1)!} \left[\int_{y_{j\ell}}^{x} |x - z|^{(s-k-1)q} \,\mathrm{d}z\right]^{\frac{1}{q}} \left\|D^s f\right\|_{p,I_{j,\ell}}. \tag{3.20}$$

Integrating, we get (3.17). $\square$

Lemma 3.3. *Given $x \in I$, let ℓ be such that $y_{j,\ell} \le x \le y_{j,\ell+1}$; if $f \in L_s^p\left(I_{j,\ell}\right)$ and R is as in Lemma 3.2, there results*

$$\left|D^k T_{j,m} R(x)\right| \le C_{2,k} 2^{-j(s-k)} \left\|D^s f\right\|_{p,I_{j,\ell}}, \tag{3.21}$$

where $C_{2,k}$ is any constant depending only on s, k, n, m.

Proof.

$$\begin{aligned}
\left|D^k T_{j,m} R(x)\right| &\le \sum_{i=0}^{N_j} \left|\left\langle R, \vartheta_{j,i,n}^{-\frac{1}{p}} C_{j,m,i}\right\rangle\right| \left|\vartheta_{j,i,n}^{\frac{1}{p}} \cdot D^k w_{j,i,n}(x)\right| \\
&= \sum_{i=\ell}^{\ell+n} \left|\left\langle R, \vartheta_{j,i,n}^{-\frac{1}{p}} C_{j,m,i}\right\rangle\right| \left|\vartheta_{j,i,n}^{\frac{1}{p}} \cdot D^k w_{j,i,n}(x)\right|.
\end{aligned}$$

Moreover, for Hölder's inequality, we have

$$\begin{aligned}
\left|\left\langle R, \vartheta_{j,i,n}^{-\frac{1}{p}} C_{j,m,i}\right\rangle\right| &= \left|\int_{\sigma_{j,i}} R(x) \vartheta_{j,i,n}^{-\frac{1}{p}} C_{j,m,i}(x) \mathrm{d}x\right| \\
&\le \left[\int_{\sigma_{j,i}} |R(x)|^p \,dx\right]^{\frac{1}{p}} \left[\int_{\sigma_{j,i}} \left|\vartheta_{j,i,n}^{-\frac{1}{p}} C_{j,m,i}(x)\right|^q \mathrm{d}x\right]^{\frac{1}{q}},
\end{aligned} \tag{3.22}$$

and using (3.11),

$$\left[\int_{\sigma_{j,i}} \left|\vartheta_{j,i,n}^{-\frac{1}{p}} C_{j,m,i}(x)\right|^q \mathrm{d}x\right]^{\frac{1}{q}} \le \Gamma_{j,i,q}, \quad \ell \le i \le \ell+n. \tag{3.23}$$

By (3.17) for $k=0$ we can write

$$|R(x)| \le \frac{1}{(s-1)!}\left[\frac{\left|x-y_{j\ell}\right|^{(s-1)+\frac{1}{q}}}{((s-1)\,q+1)^{\frac{1}{q}}}\right]\left\|D^s f\right\|_{p,I_{j,\ell}} \tag{3.24}$$

and then

$$\left[\int_{\sigma_{j,i}} |R\,(x)|^p \,\mathrm{d}x\right]^{\frac{1}{p}} \le \frac{1}{(s-1)!}\frac{2^{-js}\,(n+2m+1)^s\,\|D^s f\|_{p,I_{j,\ell}}}{[(s-1)\,q+1]^{\frac{1}{q}}\left[(s-1)\,p+\frac{p}{q}+1\right]^{\frac{1}{p}}}. \tag{3.25}$$

Therefore, recalling from [6] that

$$\left|D^k w_{j,i,n}(x)\right| \le C_k 2^{jk}, \tag{3.26}$$

we obtain

$$\left|D^k T_{j,m} R(x)\right| \le C_{2,k} 2^{-j(s-k)}\left\|D^s f\right\|_{p,I_{j,\ell}}. \tag{3.27}$$

In (3.27) we defined $C_{2,k} = \overline{C_{1,k}}\cdot\overline{C_2}$ with $\overline{C_{1,k}} = (n+1)\,C_k\cdot\overline{\theta}_j^{\frac{1}{p}}$, where $\overline{\theta}_j = \max\limits_{\ell\le i\le\ell+n}\left(\int_I w_{j,i,n}(x)\mathrm{d}x\right)^{-1}$ and $\overline{C_2} = \frac{(n+2m+1)^s\,\Gamma_{j,q}}{(s-1)![(s-1)q+1]^{\frac{1}{q}}\left[(s-1)p+\frac{p}{q}+1\right]^{\frac{1}{p}}}$. □

We are now ready to give a local error estimate. Integrating the inequalities (3.17) and (3.21) over the interval $\left[y_{j,\ell},\ y_{j,\ell+1}\right]$ we prove the following theorem.

Theorem 3.4. *Let $f\in L_s^p\left(I_{j\ell}\right)$, $1\le p\le q\le\infty$; then, for $0\le k<s\le 2m+1$, there results*

$$\left\|E_{k,s}\right\|_{q,\left[y_{j,\ell},\ y_{j,\ell+1}\right]} \le C_k^*\, 2^{-j\left(s-k+\frac{1}{q}-\frac{1}{p}\right)}\left\|D^s f\right\|_{p,I_{j\ell}} \tag{3.28}$$

where $C_k^* = \max\left\{C_{2,k},\ \frac{C_{1,k}}{[(s-k-1)q+2]^{\frac{1}{q}}}\right\}$.

This local error estimate leads immediately to the following global result:

Theorem 3.5. *Let $f\in L_s^p\,(I)$, $1\le p\le q\le\infty$; then, for $0\le k<s\le 2m+1$, there results*

$$\left\|E_{k,s}(x)\right\|_{q,I} \le 2\,(n+m+1)\,C_k^*\,2^{-j\left(s-k+\frac{1}{q}-\frac{1}{p}\right)}\left\|D^s f\right\|_{p,I}. \tag{3.29}$$

Proof. By Theorem 3.4, if $f\in L_s^p\,(I)$ raising the left and the right terms of (3.28) to the qth power and summing over i, $0\le i\le N_j$, leads to

$$\left\{\sum_i \left\|E_{k,s}(x)\right\|^q_{q,\left[y_{j,\ell},\ y_{j,\ell+1}\right]}\right\}^{\frac{1}{q}} \le C_k^*\,2^{-j\left(s-k+\frac{1}{q}-\frac{1}{p}\right)}\left\{\sum_i \left\|D^s f\right\|^q_{p,I_{j,\ell}}\right\}^{\frac{1}{q}}.$$

But, for $p\le q$, Jensen's inequality yields

$$\left\{\sum_i \left\|D^s f\right\|^q_{p,I_{j,\ell}}\right\}^{\frac{1}{q}} \le \left\{\sum_i \left\|D^s f\right\|^p_{p,I_{j,\ell}}\right\}^{\frac{1}{p}} \le 2\,(n+m+1)\left\|D^s f\right\|_{p,I}$$

since $I_{j,\ell}\subset\left[y_{j,\ell-m},\ y_{j,\ell+2n+m+1}\right]$, so any piece of I is added into the sum at most $2\,(n+m+1)$ times. □

4. Numerical results

We report, in this section, some numerical results showing the behavior of the integral refinable operators constructed here.

Tables 1–3 relate to the comparison of performances, at different levels j, between the operator constructed here denoted by Tf, the operator considered in [2,3] defined by $S_j f(x) = \sum_{i=0}^{N_j} \langle \chi_{j,i,n}, f \rangle w_{j,i,n}(x)$ and denoted here by Sf, and the quasi-interpolatory operator $Q_j f(x) = \sum_{i=0}^{N_j} (\lambda_{j,i,n} f) w_{j,i,n}(x)$, simply named Qf. That operator, with $\lambda_{j,i,n}$ suitable linear operators, has been studied in [5] and in [6] and has the same order of reproducibility as Tf.
In each table we report the maximum error in approximating the relative function and we specify the values of m and the values of parameters n and h identifying the basis used for constructing the operator.

Table 1
Absolute error for $f(x) = (25 - x^2)^{-\frac{1}{2}}$

$m=1, n=5, h=8$				$m=2, n=6, h=8$			
j	errSf	errQf	errTf	j	errSf	errQf	errTf
0	2.00 (−04)	2.76 (−06)	4.79 (−07)	0	1.10 (−04)	3.12 (−10)	1.19 (−10)
1	1.11 (−04)	4.12 (−07)	7.08 (−08)	1	5.78 (−05)	1.13 (−11)	4.33 (−12)
2	5.81 (−05)	5.64 (−08)	9.60 (−09)	2	2.97 (−05)	3.81 (−13)	1.45 (−13)
3	2.98 (−05)	7.38 (−09)	1.25 (−09)	3	1.51 (−05)	1.24 (−14)	4.75 (−15)
4	1.51 (−05)	9.43 (−10)	1.59 (−10)	4	7.56 (−06)	4.16 (−16)	5.27 (−16)

Table 2
Absolute error for $f(x) = (0.01 - x^2)^{-\frac{1}{2}}$

$m=1, n=5, h=8$				$m=2, n=6, h=8$			
j	errSf	errQf	errTf	j	errSf	errQf	errTf
0	9.29 (−01)	6.48 (−01)	2.04 (−01)	0	5.78 (−01)	1.24 (−01)	6.53 (−02)
1	5.79 (−01)	5.34 (−01)	1.36 (−01)	1	2.97 (−01)	9.16 (−03)	4.38 (−03)
2	2.77 (−01)	2.21 (−01)	4.42 (−02)	2	1.14 (−01)	1.91 (−03)	8.14 (−04)
3	9.58 (−02)	3.89 (−02)	6.75 (−03)	3	3.40 (−02)	1.00 (−04)	3.84 (−05)
4	2.69 (−02)	3.66 (−03)	6.01 (−04)	4	8.98 (−03)	2.23 (−06)	8.75 (−07)

Table 3
Absolute error for $f(x) = \sin(2\pi x)$

$m=1, n=5, h=8$				$m=2, n=6, h=8$			
j	errSf	errQf	errTf	j	errSf	errQf	errTf
0	1.80 (−01)	6.69 (−02)	2.55 (−02)	0	7.78 (−02)	4.50 (−04)	1.93 (−04)
1	7.88 (−02)	2.38 (−02)	4.48 (−03)	1	4.32 (−02)	2.70 (−05)	1.04 (−05)
2	4.35 (−02)	4.07 (−03)	7.01 (−04)	2	2.23 (−02)	9.65 (−07)	3.68 (−07)
3	2.24 (−02)	5.47 (−04)	9.28 (−05)	3	1.13 (−02)	3.11 (−08)	1.19 (−08)
4	1.13 (−02)	6.97 (−05)	1.18 (−05)	4	5.65 (−03)	9.80 (−10)	3.74 (−10)

The numerical results emphasize the influence of the higher reproducibility order of Tf and Qf with respect to Sf, in approximating functions; the last operator, however, enjoys very interesting spectral properties [3].

Moreover Tables 1–3 and many other experiments performed make evident a comparable and often better behavior of the operators presented here also with respect to the quasi-interpolatory operator Qf with the same degree of reproducibility.

We also compared the behavior, in approximating some test functions, of our refinable operator and the spline operator constructed in [9]. In Fig. 1 we report the behavior in $[-0.3, 0.3]$ of the above mentioned operators in approximating the function $f(x) = x^4 + |x|$; we worked with $n = 5, h = 8, m = 2$.

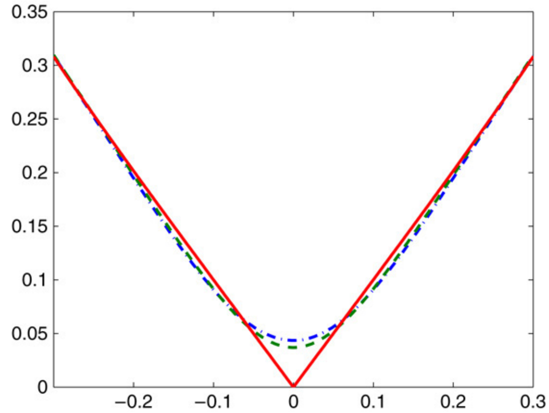

Fig. 1. f, solid line; operator Tf, dashed line; spline operator, dash–dotted line.

We remark that when the parameter h of the GP basis increases, the corresponding refinable function is better localized; this property is important in making the operator constructed here more effective than the operator constructed from B-splines.

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