Odd radical subgroups of some sporadic simple groups

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Abstract

The radical $p$-subgroups of the Monster, the Baby Monster and the Harada–Norton group $HN$ for $p = 3, 5, 7$ are determined up to conjugacy, which completes the classification problem of odd radical subgroups for all sporadic simple groups together with [M. Kitazume, S. Yoshiara, The radical subgroups of the Fischer simple groups, J. Algebra 255 (2002) 22–58] and the other works. © 2005 Elsevier Inc. All rights reserved.

1. Introduction

Let $p$ be a prime dividing the order of a finite group $G$. A nontrivial $p$-subgroup $R$ of $G$ is called radical if $R = \text{Op}(N_G(R))$, where $\text{Op}(X)$ denotes the largest normal $p$-subgroup of a group $X$. In this paper, the radical $p$-subgroups of the Monster, the Baby Monster and the Harada–Norton group are determined up to conjugacy, based on the classifications by R.A. Wilson and S.P. Norton of maximal odd local subgroups of these groups [13–15]. Together with previous works, this completes the classification of radical $p$-subgroups of all sporadic simple groups for all odd primes $p$. To emphasize the completeness, literatures are given to each sporadic simple group in Appendix A, though they may not be comprehensive.
See the introductions of [17] and [5] for the motivations and the strategy to determine the radical $p$-subgroups. We denote by $B_p(G)$ the set of all radical $p$-subgroups of a finite group $G$.

The main results are summarized in the form of tables in Theorems 2–5, 7–9. There, for a representative $R$ of each class of the radical 2-subgroups, we give a brief description of structure of $R$ as well as those of the center $Z(R)$ and the automizer $N_{G}(R)/R$. To describe the structure of a group, we follow the Atlas notation, in particular, $A.B$ (or $AB$) means an extension of $B$ by $A$ (a group with a normal subgroup isomorphic to $A$ and the quotient group by that group isomorphic to $B$), while $A : B$ means a split extension. Radical subgroups with the identical centers (up to conjugacy) are collected with the indication of fusion of the center, with class name following the Atlas notation. If fusion is not indicated, the center is identical with the group in the previous row.

Recall that a radical $p$-subgroup $R$ is called centric, if any $p$-element centralizing $R$ lies in $Z(R)$. For the importance of centric radical subgroups, see [7] and [9, Section 4]. We also refer to which representatives are centric, without giving any verification, as it is easy to see.

The symbols $S_n$ and $A_n$ are used to denote the symmetric and alternating groups of degree $n$. We also use the symbols $n$, $p^n$ and $2^{1+2n}$ to denote, respectively, the cyclic group of order $n$, the elementary abelian group of order $p^n$ and the extraspecial group of order $2^{1+2n}$ of type $(\epsilon)$ ($\epsilon = \pm 1$). Furthermore, the symbols $D_8$, $Q_8 \cong 2^{1+2}$ and $SD_{16}$ indicate the dihedral, quaternion and semidihedral group of order 8, 8 and 16, respectively.

For a conjugacy class $pX$ of an element of order $p$, an elementary abelian $p$-subgroup is called $pX$-pure if all its nontrivial elements lie in $pX$. A subgroup of order $p$ is called of type $pX$, if it is generated by an element in the class $pX$. I also call a subgroup $H$ of $G$ a $X$-subgroup, if $H \cong X$.

The next lemma is used in the final step of classification of $B_3(BM)$.

**Lemma 1.** Let $L$ be a finite group and $R$ be its nontrivial $p$-subgroup. Assume that $L$ has a normal abelian $p'$-subgroup $W$, so that $L/W$ acts on $W$ by conjugation. We set $U := C_W(R)$ (allowing $U = 1$) and let $C_{L/W}(U)$ be the centralizer of $U$ in the action of $L/W$ on $W$: that is, $C_{L/W}(U) = \{W \in L/W \mid l \in L, [lW, U] = 1\}$, which coincides with $C_U(U)/W$ by the commutativity of $W$.

Then $R$ is a radical $p$-subgroup of $L$ if and only if $RW/W$ is a radical $p$-subgroup of $C_{L/W}(U)$.

**Proof.** We set $D := C_L(U)$ and let $V := O_p(N_L(R)) \supseteq R$. As $U \subseteq C_W(R) = W \cap N_L(R)$ is normal in $N_L(R)$, the subgroup $N_D(R) = C_L(U) \cap N_L(R)$ is normal in $N_L(R)$. Thus $O_p(N_D(R)) \subseteq V$. On the other hand, as $U \subseteq C_L(U) \subseteq N_L(R)$ normalizes $V$ and $V \subseteq N_L(R)$ normalizes $N_L(R) \cap W = C_W(R) = U$, we have $[U, V] = U \cap V = 1$. Then $V \subseteq D$ and $V \subseteq O_p(N_D(R))$. Thus $V = O_p(N_D(R))$.

Assume that $RW/W$ is a radical subgroup of $C_{L/W}(U) = D/W$. By the above remark, $V$ is normal in $N_L(R) \cap D = N_D(R)$. As $R$ acts coprimely on $W$, we have $N_D(R)W/W = N_{D/W}(ZW/W)$. Then $VW/W$ is a normal $p$-subgroup of $N_{D/W}(ZW/W)$, and hence $RW/W = VW/W$ by the assumption. Then $R = V$ is a radical subgroup of $L$.  


Conversely, assume that $R$ is a radical subgroup of $L$. Then $R = V = O_p(N_D(R))$ by the above remark. Now let $T$ be a Sylow $p$-subgroup of the inverse image in $D$ of $O_p(N_{D/w}(R W/W))$. Then $T W/W$ is normal in $N_{D/w}(R W/W) = N_D(R) W/W$. Thus $T W$ is normal in $N_D(R) W$. As $N_D(R)$ contains a Sylow $p$-subgroup of $N_D(R) W$, there is some $w \in W$ such that $T w \leq N_D(R)$. Then $T w$ lies in $T W \cap N_D(R)$, which is normal in $N_D(R)$. As $T w W = T W$, we then have $T W \cap N_D(R) = T w (N_D(R) \cap D) = T w U$. Since $T w \leq D$, $T W \cap N_D(R) = T w \times U$. Thus $T w$ is a normal $p$-subgroup of $N_D(R)$, and $T w \trianglelefteq R$. Hence $R W/W \leq O_p(N_{D/w}(R W/W)) = T W/W = T w W/W \leq R W/W$, and then $R W/W = T W/W$ is a radical $p$-subgroup of $D/W$. 

2. The Baby Monster $BM$

2.1. Radical $3$-subgroups of $BM$

2.1.1. Maximal $3$-locals of $BM$

There are just two conjugacy classes of elements of order $3$ in $G := BM$, the Baby Monster, which are denoted $3A$ and $3B$, with normalizers $N_G(3A) \cong S_3 \times (Fi_{22} : 2)$ and $N_G(3B) \cong 3^{1+8} : (2^{1+6} \cdot (U_4(2) : 2))$. It follows from [13, Theorem 5.7], every $3$-local subgroup of $G$ is contained in one of the following groups up to conjugacy:

$L_1 := N_G(3B) \cong 3^{1+8} : (2^{1+6} \cdot (U_4(2) : 2)),$
$L_2 := N_G(3B^2) \cong 3^2 3^3 3^6(2_4 \times 2_4),$  
$L_3 := N_G(3B^3) \cong 3^3 3^6(L_3(3) \times D_6) < O_8^+(3) : S_4,$
$L_4 := N_G(3B^4) \cong 3^3 3^7(L_3(3) \times 2) = F_{23},$
$L_5 := N_G(3A) \cong S_3 \times (Fi_{22} : 2),$
$L_6 := N_G(3A^2) \cong (3^2 : D_8 \times U_4(3) : 2^2) \cdot 2,$  
$L_7 := N_G(3^6) \cong 3^6 : ((2 \times (L_4(3) : 2)) : 2) \cong O_8^+(3) : S_4.$

We set $V_i := O_3(L_i)$ ($i = 1, \ldots , 7$). Then $V_1 \cong 3^{1+8}$, $V_2 \cong 3^2 3^3 3^6$ with $Z(V_2)$ a $3B$-pure $3^2$-subgroup, $V_3 \cong 3^{3+6}$ with $Z(V_3)$ a $3B$-pure $3^3$-subgroup, $V_4 \cong 3^3 3^7$ with $Z(V_4)$ another $3B$-pure $3^3$-subgroup, $V_5$ is a $3A$-pure $3^2$-subgroup, and $V_7 \cong 3^6$.

Let $F$ be an $Fi_{23}$-subgroup of $G$. As $F$ contains a Sylow $3$-subgroup of $G$, we may assume $V_i \leq F$ ($i = 1, \ldots , 7$). The fusion of elements of order $3$ of $F$ in $G$ is easily determined, observing that the restriction of the irreducible character of $G$ of degree $4371$ to $F$ is the sum of irreducible characters of $F$ of degrees $1, 782$ and $3588$: the $F$-classes $3A, 3B, 3C$ and $3D$ lie in the $G$-classes $3A, 3B, 3A$ and $3B$, respectively.

We can verify that every $3^6$-subgroup of $G$ is conjugate to $V_7$, as follows. The exposition in the second paragraph of [13, Section 4] shows that every $3B$-pure elementary abelian $3$-subgroup $Y$ is contained in $X = O_3(C_G(x)) \cong 3^{1+8}$ for some $3B$-element $x$ (note that a $3B$-element in $C_G(x)$ outside $X$ generates with $x$ a $3^2$-subgroup of type $3B_3(h)$ or $3B_3(c)$ by [13, Theorem A.4]). In particular, any $3B$-pure elementary abelian subgroup is of order
at most $3^5$. Thus every $3^6$-subgroup of $G$ contains a $3A$-element, and hence it is conjugate to $V_7$ by [13, Proposition 3.2].

As $L_7$ is a maximal parabolic subgroup of an $O^+_6(3) : S_3$-overgroup, $V_7$ is a natural orthogonal module for $(L_7/V_7) \cong O^+_6(3)$. Then $V_7$ contains $130$ (respectively $117 \times 2$) subgroups of type $3B$ (respectively $3A$) corresponding to isotropic (respectively non-isotropic) points. As is described in [5, Lemma 19(2)(3)], we may assume that $Z(V_4)$ is a totally isotropic $3$-subspace of $V_7$ and that $Z(V_2) = Z(V_5) \cap Z(V_4)$.

2.1.2. Classification of $B$-conjugacy classes of radical $3$-subgroups of $L$

Step 2. Proof. Assume $R \neq V_6$. Then $R = V_6 \times U_F$ for some unipotent radical $U_F$ of $L_6^\infty \cong U_4(3)$ corresponding to a flag $F$ of totally isotropic subspaces of its natural module. If $F$ contains a point $p$, we have $Z(U_p) = R'$ or $R''$, as $U_p \cong 3^1 + 4$. Then $N_G(R) \leq L_1$ up to conjugacy, because $Z(U_p)$ is of type $3B$ by [13, Proposition 3.1]. If $F$ is a line, then $R \cong 3^6$, and hence $R$ is conjugate to $V_7$.

Step 1. If $N_G(R) \leq L_6$, then $R = V_6$, $V_7$ or $N_G(R) \leq L_1$ up to conjugacy.

Proof. Assume $R \neq V_6$. Then $R = V_6 \times U_F$ for some unipotent radical $U_F$ of $L_6^\infty \cong U_4(3)$ corresponding to a flag $F$ of totally isotropic subspaces of its natural module. If $F$ contains a point $p$, we have $Z(U_p) = R'$ or $R''$, as $U_p \cong 3^1 + 4$. Then $N_G(R) \leq L_1$ up to conjugacy, because $Z(U_p)$ is of type $3B$ by [13, Proposition 3.1]. If $F$ is a line, then $R \cong 3^6$, and hence $R$ is conjugate to $V_7$.

Step 2. If $N_G(R) \leq L_5$, then $R = V_j$ for $j = 5, 6, 7$ or $N_G(R) \leq L_1$ or $L_4$ up to conjugacy.

Proof. Assume $R \neq V_5$. Then $R = V_5 \times U$, where $U$ is one of the $7$ representatives of conjugacy classes of radical $3$-subgroups of $L_5^\infty \cong F_{I22}$ in [5, Theorem 18]. In the notation there, up to conjugacy under the direct factor $F_{I22}.2$ of $L_5$, we have $U = 'V_1'$ or 'V_3', or the normalizer of $U$ in $L_5^\infty$ is contained in that of 'V_2' or 'V_4'.

The fusion of elements of order $3$ of $L_5$ in $G$ is determined in [13, Proposition 3.1]: if $x$ lies in the class $3A$ (respectively $3B$, $3C$ or $3D$) of $L_5^\infty \cong F_{I22}$, then $x$ lies in the $G$-class $3A$ (respectively $3B$, $3A$ or $3B$) and $\langle V_5, x \rangle$ is a $3A$-pure (respectively of fusion types $3A^5B^1$, $3A^2B^2$ or $3A^1B^3$). Thus if $U = 'V_1'$, $R$ is a $3A$-pure $3^2$-subgroup, and hence $R$ is conjugate to $V_6$. If $U = 'V_3'$, then $R \cong 3^6$, and hence it is conjugate to $V_7$. As 'V_2' is an extraspecial group with center a subgroup of type $3B$ in $L_5^\infty \cong F_{I22}$, its center is a subgroup of type $3B$ in $G$. Thus if $N_G(U)$ is contained in the normalizer of 'V_2', then $N_G(R)$ normalizes $R'$, which is the center of 'V_2', and hence $N_G(R) \leq L_1$ up to conjugacy. In the remaining case where $N_G(U)$ is contained in the normalizer of 'V_4', we have $N_G(R) \leq L_4$, as the center $Z$ of $V_4$ is $3^3 + 3$ is the unique $3B$-pure $3^3$-subgroup of $\langle V_5, Z \rangle$.

Step 3. If $N_G(R) \leq L_7$, then $R = V_i$ or $N_G(R) \leq L_j$ for $j = 4, 2$ or $1$ up to conjugacy.

Proof. Assume $R \neq V_7$. Since $V_7$ is the natural orthogonal module for $L_7^\infty / V_7 \cong O^+_6(3)$, $Z(R) = C_{V_7}(R/V_7)$ is a totally isotropic $j$-subspace of $V_7$ for $j = 1, 2, 3$. As $3B$-elements of $V_7$ correspond to isotropic points, $Z(R)$ is conjugate to $V_1$, $V_2$ or $V_4$ according to $j = 1, 2$ or $4$. Then the claim follows.
Step 4. If $N_G(R) \leq L_i$ for $i = 3$ or $4$, then $R = V_i$ or $N_G(R) \leq L_1$ or $L_2$ up to conjugacy.

Proof. If $R \neq V_i$, $Z(R) = C_{Z(V_i)}(R/V_i)$ is a proper subspace of the natural module $Z(V_i) \cong 3^3$ for $L_i^\infty/V_i \cong L_1(3)$. As $3^2$-subgroups of $Z(V_3)$ and $Z(V_4)$ are conjugate to $V_2$, the claim follows. $\square$

Step 5. If $N_G(R) \leq L_2$, then $R = V_2$ or $V_2^{(1,0)} \cong V_2$ up to conjugacy, where the latter is a Sylow 3-subgroup of $C_G(V_2)$.

Proof. We have $R/V_2 = \hat{R}_1 \times \hat{R}_2$, where $\hat{R}_1$ is trivial or a radical 3-subgroup of $C_G(V_2)/V_2 \cong S_2$ and $\hat{R}_2$ is that of $2S_4 \cong GL_2(3)$ acting faithfully on $Z(V_2)$. If $\hat{R}_2 \neq 1$, then $Z(R) = C_{Z(V_2)}(R/V_2)$ is a subgroup of type $3B$, and hence $N_G(R) \leq L_1$. If $\hat{R}_2 = 1$, then $Z(R) = Z(V_2)$ and $N_G(R) \leq N_G(R)$. Thus it follows from [16, Lemma 1(3)] that there are just two classes of radical 3-subgroups of $G$ with centers $3B$-pure $3^2$-subgroups, with representatives $V_2$ and $V_2^{(1,0)}$. $\square$

Step 6. Now we are reduced to the case when $N_G(R) \leq L_1$. From [16, Lemma 1(4)] there is a bijective correspondence between the classes of radical 3-subgroups of $L_1/V_1 \cong 2^{1+6}U(2)$ including the trivial subgroup and those of $G$ with centers subgroups of type $3B$. The radical 3-subgroups of $L_1/V_1$ is classified as follows using Lemma 1.

As the center $Z$ of $O_2(L_1/V_1) \cong 2^{1+6}$ is the center of $L_1/V_1$, the radical 3-subgroups of $L_1/V_1$ bijectively correspond to those of $L := (L_1/V_1)/Z$, in which $W := O_2(L_1/V_1)/Z \cong 2^6$ is the natural module for $L/W \cong SO_6^-(2)$.

By Lemma 1, the radical 3-subgroups $R$ of $L$ with $C_W(R) = 1$ bijectively correspond to those of $L/W$. As $L/W \cong SO_5(3)$ is a group of Lie type of rank 2 in characteristic 3, there are exactly three classes of such radical 3-subgroups, indexed by nonempty flags of the natural module for $SO_5(3)$.

Next consider a radical 3-subgroup $R$ of $L$ with $C_W(R) \neq 1$. As the stabilizer of $SO_6(2)$ of an isotropic (respectively nonisotropic) point is $2^3 : SO_3^-(2)$ (respectively $2 \times Sp_4(2)$), the order of $R$ is at most 3 (respectively $3^2$) if $C_W(R)$ contains an isotropic point (respectively nonisotropic points only). Let $q(x) = x_1^2 + x_1x_2 + x_2^2 + x_3x_6 + x_4x_5$ be a quadratic form of minus type on $GF(2)^6$, identified with $W$, preserved by $SO_6^-(2) \cong L/W$. Then the following matrices $t, s_1, s_2$ with respect to the natural basis $e_i (i = 1, \ldots, 6)$ are elements of $L/W$ of order 3 such that $C_W(t) = \langle e_i \mid i = 3, \ldots, 6 \rangle$ and $C_W(s_1) = C_W(s_2) = \langle e_1, e_2 \rangle$:

$$t = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes I_4, \quad s_1 = I_2 \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad s_2 = I_2 \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$ 

Thus $\langle t \rangle$ (respectively $\langle s_1, s_2 \rangle \cong 3^2$) is a Sylow 3-subgroup of the stabilizer of an isotropic (respectively nonisotropic) point $\langle e_0 \rangle$ (respectively $\langle e_1 \rangle$). We may assume that $RW/W$ is $\langle t \rangle$, $\langle s_1 \rangle$ or $\langle s_1, s_2 \rangle$ up to conjugacy, as $\langle s_1 s_2^{-1} \rangle$ fixes an isotropic point and thus it is conjugate to $\langle t \rangle$.

If $RW/W = \langle t \rangle$, we have $U = C_W(R) = \langle e_i \mid i = 3, \ldots, 6 \rangle$. Moreover, $C_{L/W}(U) \cong S_3$ is generated by $t$ and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes I_4$. Thus $RW/W$ is a radical 3-subgroup of $C_{L/W}(U)$,
and hence $R$ is a radical 3-subgroup of $L$ by Lemma 1. If $RW/W = \langle s_1 \rangle$ or $\langle s_1, s_2 \rangle$, then $U = \langle e_i \mid i = 1, 2 \rangle$ and the group $C_{L/W}(U)$ is the subgroup of $SL_6(2)$ fixing $U$ and preserving the quadratic form $x_3x_6 + x_4x_5$ of plus type on the subspace $\langle e_i \mid i = 3, \ldots, 6 \rangle$ orthogonal to $U$. Hence $C_{L/W}(U) \cong SO_4^+(2) \cong S_3 \rtimes 2$. Thus $\langle s_1 \rangle$ is not radical but $\langle s_1, s_2 \rangle \in B_3(C_{L/W}(U))$. By Lemma 1, the $R$ corresponding to the latter is a radical 3-subgroup of $L$.

Hence we obtain two new radical subgroups $\tilde{V}_1^{(i)} \cong 3$ and $\tilde{V}_1^{(v)} \cong 3^2$ of $L = L_1/W_1$ with $C_{W}(\tilde{V}_1^{(i)}) \cong 2^4$ and $C_{W}(\tilde{V}_1^{(v)}) \cong 2^2$. Their normalizers act on both $\langle e_i \mid i = 1, 2 \rangle$ and $\langle e_i \mid i = 3, \ldots, 6 \rangle$, on which $L/W$ induces $S_3$ and $S_3 : 2$ respectively. Thus

$$\text{NG}(\tilde{V}_1^{(i)}) \cong 3^{1+8}2(S_3 \times (2^4(S_3 : 2))) \quad \text{and} \quad \text{NG}(\tilde{V}_1^{(v)}) \cong 3^{1+8}2((2^2S_3) \times (S_3 : 2)).$$

Then the corresponding radical 3-subgroups $V_1^{(x)}$ of $G$ to $\tilde{V}_1^{(x)}$ ($x = i$ or $v$) has the normalizer with the following structure:

$$\text{NG}(V_1^{(i)}) \cong 3^{1+8}2(S_3 \times (2^4(S_3 : 2))) \quad \text{and} \quad \text{NG}(V_1^{(v)}) \cong 3^{1+8}2((2^2S_3) \times (S_3 : 2)).$$

Summarizing, we established the following.

**Theorem 2.** There are just 13 classes of radical 3-subgroups of the Baby Monster with the representatives below, in which 11 except $V_6$ and $V_5$ are centric:

<table>
<thead>
<tr>
<th>$R$</th>
<th>$R \cong$</th>
<th>$Z(R)$</th>
<th>$N(R)/R$</th>
<th>$R$</th>
<th>$R \cong$</th>
<th>$Z(R)$</th>
<th>$N(R)/R$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V_6$</td>
<td>$3^3$</td>
<td>$A^4$</td>
<td>$(D_8 \times U_4(3) : 2)^2$</td>
<td>$V_4$</td>
<td>$3^{1+8}$</td>
<td>$3 = B^1$</td>
<td>$2^{1+6}U_4(2)2$</td>
</tr>
<tr>
<td>$V_5$</td>
<td>$3$</td>
<td>$A^4$</td>
<td>$2 \times (F_{22} : 2)$</td>
<td>$V_1^{(i)}$</td>
<td>$V_3$ 3</td>
<td>$3$</td>
<td>$2(2 \times 2^4S_3 : 2)$</td>
</tr>
<tr>
<td>$V_7$</td>
<td>$3^{36}$</td>
<td>$A^{2+3}B^{130}$</td>
<td>$(2 \times (L_4(3)^2) : 2)^2$</td>
<td>$V_1^{(v)}$</td>
<td>$V_3^{1+4}$</td>
<td>$3$</td>
<td>$(2^2S_3 \times D_8)$</td>
</tr>
<tr>
<td>$V_3$</td>
<td>$3^{3+6}$</td>
<td>$3^3 = B^{13}$</td>
<td>$L_3(3) \times D_8$</td>
<td>$V_1^{(i)}$</td>
<td>$V_3^{1+4}$</td>
<td>$3$</td>
<td>$(2A_2)$</td>
</tr>
<tr>
<td>$V_4$</td>
<td>$3^{3+7}$</td>
<td>$3^3 = B^{13}$</td>
<td>$L_3(3) \times 2$</td>
<td>$V_1^{(v)}$</td>
<td>$V_3^{3+4}$</td>
<td>$3$</td>
<td>$(2S_2)$</td>
</tr>
<tr>
<td>$V_2$</td>
<td>$3^{2+3}$</td>
<td>$3^2 = B^4$</td>
<td>$S_4 \times 2S_4$</td>
<td>$V_1^{(i)}$</td>
<td>$V_1^{[3^6]}$</td>
<td>$3$</td>
<td>$2^2$</td>
</tr>
<tr>
<td>$V_1^{[3,0]}$</td>
<td>$V_3$ 3</td>
<td>$3^2 \times 2$</td>
<td>$2 \times 2S_4$</td>
<td></td>
<td></td>
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</tr>
</tbody>
</table>

2.2. Radical 5-subgroups of BM

From [13, Theorem 6.4], any 5-local subgroup of $G = BM$, the Baby Monster, is contained in one of the following groups up to conjugacy:

$$L_1 := N(5A) \cong (5 : 4) \times (HS : 2),$$
$$L_2 := N(5B) \cong (5^{1+4} : 2^{1+4}A_5).4),$$
$$L_3 := N(5A^2) \cong (5^2 : 4S_4) \times S_5,$$
$$L_4 := N(5B^3) \cong 5^{3} \times L_3(5).$$

By maximality of $L_i$ as a 5-local subgroup, $V_i := O_{5}(L_i)$ is a radical 5-subgroup for every $i = 1, \ldots, 5$. Let $R$ be any radical 5-subgroup of $G$. Then $NG(R) \leq L_i$ for some $i = 1, \ldots, 5$ up to conjugacy.
Assume that $N_G(R) \leq L_1$. Then $R$ properly contains $V_1$, and $R = V_1 \times R_1$ for a radical $5$-subgroup $R_1$ of the $HS': 2$-factor of $C_G(V_1)$. As this factor contains $C_G(V_1)' \cong HS$ with index 2 which is coprime with 5, the group $R_1$ is in $B_2(HS)$. It is easy to classify $B_2(HS)$:

- they are subgroups of type $5B$ (in $HS$) or Sylow $5$-subgroups $5^{1+2}$ of $HS$. If $R_1$ is of the first type, then it follows from the fusion table in [13, Section 6] that $R$ is a 5$A$-pure (in $G$) $5^2$-subgroup and is conjugate to $V_3$. If $R_1$ is a Sylow $5$-subgroup of $HS$, then $R = V_1 \times R_1$ has the commutator subgroup $Z(R_1)$, which is of type $5A$ in $HS$ and hence type $5B$ in $G$. Thus $N_G(R) \leq N_G(R') \leq L_2$ up to conjugacy.

Assume that $N_G(R) \leq L_3$ but $R \neq V_3$. Then $R = V_3 \times R_3$ for some $R_3 = \langle y \rangle \cong 5$ contained in the $S_5$-factor of $L_3 \cong (S^2 : 4S_4) \times S_5$. In particular, $C_G(y)/\langle y \rangle$ contains a subgroup $5^2 : 4S_4$. If $y$ is a $5A$-element of $G$, then $C_G(y)/\langle y \rangle \cong HS : 2$ would contain $5^2 : 4S_4$. However, as one can verify easily that every $5^2$-subgroup of $HS$ contains a unique subgroup of type $5A$ in $HS$ and hence its normalizer in $HS : 2$ is contained in $N_{HS,2}(5A) \cong 5^4.2$. This contradiction shows that $y$ is a $5B$-element of $G$. As $V_3$ is 5$A$-pure, for every $1 \neq x \in V_3$ the element $y$ of $C_G(x)$ is a 5$A$ or 5$C$-element of $5^2$-complement permuting the 6 subgroups of type $5A$ of $V_3$, and therefore a subgroup of order at least $4S_4/6.4 = 4$ centralizes $x$ (and $y$ as well). This contradiction shows that $y$ is a $5A$-element of $C_G(x)/\langle x \rangle$. Hence $\langle y \rangle$ is a unique subgroup of $\langle x, y \rangle$ of type $5B$ by the fusion table in [13, Section 6]. As this holds for every $1 \neq x \in V_3$, the subgroup $\langle y \rangle$ is the unique subgroup of type $5B$ in $R = V_3 \times \langle y \rangle$. Thus $N_G(R)$ normalizes a subgroup $\langle y \rangle$ of type $5B$ and $N_G(R) \leq L_2$ up to conjugacy.

Next assume that $N_G(R) \leq L_4$ and $R \neq V_4$. Then $V_4 \cong R$ and $R/V_4$ is one of three unipotent radicals of $L_4/V_4 \cong L_3(5)$, a group of Lie type of Lie rank $2$ in characteristic $5$, acting faithfully on $V_4$. Note that $V_4$ is 5$B$-pure. If $R/V_4$ corresponds to a flag containing a $1$-subspace of $V_4$, then $C_{V_4}(R/V_4)$ is a subgroup of type $5B$ normalized by $N_G(R)$. Then $N_G(R) \leq L_2$ up to conjugacy. If $R/V_4$ corresponds to a 2-subspace of $V_4$, the centralizer $C_{V_4}(R/V_4) = E$ is a $5B$-pure $5^2$-subgroup normalized by $N_G(R)$. By [13, Theorem 6.4], every 5$B$-pure $5^2$-subgroup is conjugate to $E$ and $N_G(E) \cong 5^3 : GL_2(5)$ is contained in $N_G(V_4) = L_4$. Thus in this case, we have $N_G(R) \leq N_G(E) \leq L_4$, which implies that $R$ is in fact a new radical $5$-subgroup of $G$. We denote by $V_4^{(l)}$ this subgroup, the inverse image in $L_4$ of a unipotent radical of $L_4/V_4 \cong L_3(5)$ corresponding to a 2-subspace of $V_4$.

It remains the case when $N_G(R) \leq L_2$. As $L_2/V_2 \cong 5$, we have $R = V_2$ or $R$ is Sylow $5$-subgroup of $G$.

**Theorem 3.** There are just six classes of radical $5$-subgroups of the Baby Monster with the following representatives, among which four except $V_1$ and $V_3$ are centric:

<table>
<thead>
<tr>
<th>$R$</th>
<th>$R \cong Z(R)$</th>
<th>$N(R)/R$</th>
<th>$R$</th>
<th>$R \cong Z(R)$</th>
<th>$N(R)/R$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V_1$</td>
<td>$S_5$</td>
<td>$4 \times (HS : 2)$</td>
<td>$V_4^{(l)}$</td>
<td>$V_5^{2^3}$</td>
<td>$S^2 = B^6$</td>
</tr>
<tr>
<td>$V_3$</td>
<td>$S_5$</td>
<td>$4 \times (4S_4) \times S_5$</td>
<td>$V_2^{2^3}$</td>
<td>$5_{1+4}$</td>
<td>$5 = B^1$</td>
</tr>
<tr>
<td>$V_4$</td>
<td>$S_5$</td>
<td>$L_3(5)$</td>
<td>$V_2^{2^3}$</td>
<td>$V_5$</td>
<td>$5$</td>
</tr>
</tbody>
</table>

3. The Monster M

3.1. Radical 3-subgroups of the Monster

3.1.1. 3-local subgroups of M

There are three conjugacy classes of elements of order 3 in $G := M$, the Monster, called $3A$, $3B$ and $3C$, with normalizers $N_G(3A) \cong 3 \cdot {\text{Fi}}_{24}$, $N_G(3B) \cong 3^{1+12}(2\text{Suz} : 2)$ and $N_G(3C) \cong S_3 \times Th$. It follows from [14, Theorem 3] that every 3-local subgroup of $G$ is contained in one of the following groups up to conjugacy:

$L_1 := N_G(3A) \cong 3 \cdot {\text{Fi}}_{24}$,
$L_2 := N_G(3A^2) \cong \left((3^2 : 2) \times O_8^+(3)\right)S_4$,
$L_3 := N_G(3B) \cong 3^{1+12}(2\text{Suz} : 2)$,
$L_4 := N_G(3B^2) \cong 3^{2+15}(3) \times (M_{11} \times 2S_4)$,
$L_5 := N_G(3B^{-2}) \cong 3^3 \Sigma_6(3^2 \times L_3(3))$,
$L_6 := N_G(3^8) \cong 3^8 \cdot O_8^-(3)2$, and
$L_7 := N_G(3C) \cong S_3 \times Th$.

We set $O_3(L_i) := V_i$ ($i = 1, \ldots, 7$). From [14, Section 5], $V_6 \cong 3^8$ is the natural module for $L_6/V_6 \cong O_8^-(3)2$, in which singular (respectively nonsingular) points correspond to subgroups of type $3B$ (respectively $3A$). Then we may assume that $V_2 \cong 3^2$ is a 2-subspace of $V_6$ with no singular points. We also take $V_1 \leq V_2$.

We will show that up to conjugacy $Z(V_i)$ ($= 3, 4, 5$) are totally singular $(i - 2)$-subspaces of $V_6$.

Recall first that the $3B$ (respectively $3A$)-elements of $V_6$ are isotropic (respectively non-isotropic) vectors with respect to an orthogonal form on $V_6$ preserved by $L_6'/V_6 \cong O_8^-(3)$. It follows from [14, Section 4, Proposition 5.13] that there are just three classes of $3B$-pure $3^2$-subgroups of $G$, denoted by $3B_4(i)$, $3B_4(iii)$ (or $3a'$) and $3B_4(iii)$ (or $3b'$) in the notation there. Let $x$ be a $3B$-element of $G$. Then $X := O_3(C_G(x)) \cong 3^{1+12}$ and there exists a subgroup $C \cong 6\text{Suz}$ with $Z(C) = \langle x \rangle$, $C_G(x) = XC$ and $C_G(x)/X \cong 2\text{Suz}$. Let $y$ be an element of order 3 in $C$ whose image in $C/\langle x \rangle \cong 2\text{Suz}$ is in the class 3a. Then $V_5 = C_X(y) \langle y \rangle = O_3(C_G(x, y))$ up to conjugacy. In particular, $V_6$ contains a $3B$-pure $3^2$-subgroup of $X$, which is of type $3B_4(i)$ by [14, Section 4]. As $L_6/V_6 \cong O_8^-(3)2$ is transitive on the totally isotropic $i$-subspaces of $V_6$ for each $i = 1, 2, 3$, we conclude that all $3^2$-subgroups of $V_6$ are of type $3B_4(i)$, that is, they are conjugate to $Z(V_4)$.

Let $\pi$ be a $3^2$-subgroup of $V_6$ containing $x$. As every line of $\pi$ is of type $3B_4(i)$, it follows from [14, Section 4] that it lies in $X$, and hence $\pi \leq X$. For a line $l$ of $\pi$ containing $x$, we have $N_G(l) \cong L_4 \cong 3^2 \cdot 3^3 \cdot 3^{10}(M_{10} \times 2S_4)$. As explained in [14, Section 5], the normal subgroup $3^2 \cdot 3^5$ (in fact $3^7$) of $N_G(l)$ is contained in $X \cap N_G(l)$, in which $\pi/l$ is a 1-subspace of $3^2$ belonging to the $M_{11}$-orbit of length 55. The normalizer of a $3^2$-subgroup corresponding to any such 1-subspace of $3^5$ is $3^3 \Sigma_6(3^2 \times D_8)$, as shown in the proof of [14, Theorem 6.5]. Hence $\pi$ is conjugate to $Z(V_5)$. Clearly, $Z(V_3) \cong 3$ is conjugate to $\langle x \rangle$. 

3.1.2. Classification of $B_3(M)$

Now take a radical 3-subgroup $R = M$. Then $N_G(R) \leq L_i$ for some $i = 1, \ldots, 7$ up to conjugacy.

If $N_G(R) \leq L_7$ but $R \neq V_7$, then $R = V_7 \times U$ for a radical 3-subgroup $U$ of $L_7^i = \text{Th}$. Then $U$ is a unique maximal subgroup of $R$ without containing a 3C-elements, because every nontrivial 3-element $x$ of $L_7^i$ is not a 3C-element, but the diagonal elements $x \times (y) = V_7$ are 3C-elements (see the proof of [14, Proposition 2.1]). Hence $N_G(R) \leq N_G(U)$. By the arguments in [14], the latter normalizer is contained in $L_i$ for some $i = 1, \ldots, 6$ up to conjugacy.

If $N_G(R) \leq L_6$ but $R \neq V_6$, then $R = V_6 \times U$ for a unipotent radical $U$ of $L_6^\infty \cong O_8^+(3)$. As Aut$(O_8^+(3))$ is involved in $L_6^i$, there are just two classes of maximal parabolic subgroups of $L_6^\infty$ with $O_3$-parts $3_1^{+8}$ and $3^0$. The center of a $3_1^{+8}$-subgroup of $L_6^\infty$ is of type $3B$, and the direct product of $V_6$ with a $3^6$-subgroup of $L_6^\infty$ is conjugate to $V_6$. Thus $N_G(R) \leq N_G(3B) = L_3$ or $N_G(V_6) = L_6$ up to conjugacy.

If $N_G(R) \leq L_5$ but $R \neq V_5$, then $R/V_5$ is a radical 3-subgroup of $L_5^i/V_5 \cong F_{24}$. From [5, Theorem 20], $R/V_5$ is one of the subgroups given in the first column of the table below (with the notation in [5, Theorem 20]). In the second column the fusion types in $F_{24}'$ of their centers are given. The third column gives descriptions of the centers of those subgroups of $L_5^i/V_5$ in terms of radical subgroups $V_i$ of $G$ ($i = 1, \ldots, 7$). (See the discussion below.)

<table>
<thead>
<tr>
<th>In $F_{24}'$</th>
<th>Type in $F_{24}'$</th>
<th>In $G = M$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V_1 \cong 3$</td>
<td>$3A$</td>
<td>$V_2/V_1$</td>
</tr>
<tr>
<td>$V_6 \cong 3^2$</td>
<td>$3A^2E^2$</td>
<td>$V_7 V_1/V_1$</td>
</tr>
<tr>
<td>$V_2 \cong 3^7$</td>
<td>$V_6/V_1$</td>
<td></td>
</tr>
<tr>
<td>$V_5$</td>
<td>$3B^{13}$</td>
<td>$Z(V_5)/V_1$</td>
</tr>
<tr>
<td>$V_{4, V_4^{(1,0)}}$</td>
<td>$3B^4$</td>
<td>$Z(V_4)/V_1$</td>
</tr>
<tr>
<td>$V_{3, V_4^{(x)}}$</td>
<td>$3B^1$</td>
<td>$Z(V_4)/V_1$ for $x = 1, 2, 3, or s$</td>
</tr>
</tbody>
</table>

The fusion in $G$ of the inverse images of 3-elements of $F_{24}' \cong L_5^i/V_5$ in $L_5^i$ is determined in the paragraph after [12, Proposition 2.1]. From that table, we see that $V_i^j$ for $i = 1, 6$ and 2 respectively correspond to $V_2/V_1$, $V_7 V_1/V_1$ and $V_6/V_1$, and that the centers of $V_i$ for $j = 5, 4$ and 3 respectively correspond to $Z(V_5)/V_1$, $Z(V_4)/V_1$, and $Z(V_3)/V_1$. (Note that $Z(V_j)$ is the largest $3B$-pure subgroup of $Z(V_j)/V_1$ for $j = 5, 4, 3$.) Thus in this case we conclude that $R = V_i$ ($i = 1, 2, 6, 7$) or $N_G(R) \leq L_j$ ($j = 5, 4, 3$).

Next consider the case when $N_G(R) \leq L_6$ but $R \neq V_6$. Then $R/V_5$ is a unipotent radical of $L_6^i/V_6 \cong O_8^+(3)$, which acts naturally on $V_6 \cong 3\infty$. Thus $Z(R) = CV_6(R/V_5)$ is a totally isotropic $i$-subspace of $V_6$ (that is, a $3B$-pure $3^i$-subgroup) for some $i = 1, 2, 3$, and hence $N_G(R) \leq L_{i+2}$ up to conjugacy.

If $N_G(R) \leq L_5$ but $R \neq V_5$, then $R/V_5$ is a unipotent radical of $L_5/C_{L_5}(Z(V_5)) \cong L_3(3)$ acting naturally on $Z(L_3) \cong 3\infty$. Then $Z(R)$ is a $3B$-pure $3^i$-subgroup of $Z(V_5)$ for $i = 1, 2$, and hence $N_G(R) \leq L_{2+i}$.
If \( N_G(R) \leq L_4 \), then \( R/V_i = \tilde{R}_1 \times \tilde{R}_2 \), where \( \tilde{R}_1 \) is a radical 3-subgroup of \( M_{11} \cong C_G(Z(V_i))/V_i \) and \( \tilde{R}_2 \) is a radical 3-subgroup of \( 2S_4 \cong GL_2(3) \cong L_4/C_G(Z(V_i)) \) which acts faithfully on \( Z(V_i) \cong \mathbb{Z}^3 \), allowing the trivial group as either factor. Thus if \( \tilde{R}_2 \neq 1 \), then \( N_G(R) \leq L_3 \) up to conjugacy. As \( Z(R) = C_Z(V_i)(\tilde{R}_2) \), we have \( \tilde{R}_2 \neq 1 \) iff \( Z(R) \) is of type 3B. As Sylow 3-subgroups are the only radical 3-subgroups of \( M_{11} \), it follows from [16, Lemma 1(3)] that there are exactly two classes of radical 3-subgroups of \( G \) with centers conjugate to \( Z(V_i) \), with representatives \( V_4 \) and a Sylow 3-subgroup \( V_4^{(1,0)} \) of \( C_G(Z(V_i)) \).

Now we are reduced to the case when \( N_G(R) \leq L_3 \). As \( V_3 \cong \mathbb{Z}_3^{1+12} \), there is a bijective correspondence of radical 3-subgroups of \( L_3/V_3 \cong 2.Suz.2 \) (allowing the trivial group) with those of \( G \) with centers subgroups of type 3B. From [18, Proposition 6], there are just 5 classes of radical 3-subgroups of \( 2.Suz.2 \) with representatives \( 'V_i' (i = 1, \ldots, 4) \) and a Sylow 3-subgroup \( S \). Hence we established:

**Theorem 4.** There are just 13 classes of radical 3-subgroups of the Monster with the following representatives, among which 10 except \( V_7, V_2, V_1 \) are centric:

<table>
<thead>
<tr>
<th>( R )</th>
<th>( R \cong )</th>
<th>( Z(R) )</th>
<th>( N(R)/R )</th>
<th>( R )</th>
<th>( R \cong )</th>
<th>( Z(R) )</th>
<th>( N(R)/R )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( V_7 )</td>
<td>3</td>
<td>( C^1 )</td>
<td>( 2 \times Th )</td>
<td>( V_3 )</td>
<td>3( ^2 + 12 )</td>
<td>( 3 = B^1 )</td>
<td>( 2.Suz : 2 )</td>
</tr>
<tr>
<td>( V_2 )</td>
<td>3( ^2 )</td>
<td>( A^4 )</td>
<td>( (2 \times O_9^+(3))S_4 )</td>
<td>( V_3^{(1)} )</td>
<td>( V_3^3 )</td>
<td>3</td>
<td>( U_4(3)2 )</td>
</tr>
<tr>
<td>( V_1 )</td>
<td>3</td>
<td>( A^1 )</td>
<td>( F_{24} )</td>
<td>( V_3^{(2)} )</td>
<td>( V_3^{32+4} )</td>
<td>3</td>
<td>( 2(A_4 \times 2^3)2 )</td>
</tr>
<tr>
<td>( V_6 )</td>
<td>3( ^6 )</td>
<td>( A^{214}B_{1066} )</td>
<td>( O_8^-(3)/2 )</td>
<td>( V_3^{(3)} )</td>
<td>( V_3^{35} )</td>
<td>3</td>
<td>( M_{11} )</td>
</tr>
<tr>
<td>( V_5 )</td>
<td>3( ^3 )</td>
<td>( B^{13} )</td>
<td>( (D_8 \times L_3(3)) )</td>
<td>( V_3^{(4)} )</td>
<td>( V_3^{32} )</td>
<td>3</td>
<td>( (4 \times A_6)2 )</td>
</tr>
<tr>
<td>( V_4 )</td>
<td>3( ^2 )</td>
<td>( B^{14} )</td>
<td>( M_{11} \times 2S_4 )</td>
<td>( V_3^{(5)} )</td>
<td>( V_3^{3532} )</td>
<td>3</td>
<td>( Q_8 )</td>
</tr>
</tbody>
</table>

### 3.2. Radical 5-subgroups of \( M \)

By [14, Theorem 5], any 5-local subgroup of the Monster \( G = M \) is contained in exactly one of the following groups up to conjugacy:

- \( L_1 = N_G(5A) \cong (D_{10} \times F_5) : 2 \),
- \( L_2 = N_G(5A^2) \cong (5^2 : 4 \cdot 2^2 \times U_3(5)) : S_3 \),
- \( L_3 = N_G(5B) \cong 5^{1+6} : 2J_4 : 4 \),
- \( L_4 = N_G(5B^2) \cong 5^2.5^2.5^4 : (S_3 \times GL_2(5)) \),
- \( L_5 = N_G(5B^3) \cong 5^{3+3}(2 \times L_3(5)) \), and
- \( L_6 = N_G(5B^4) \cong 5^4 : (3 \times 2 \cdot L_2(25)) : 2 \).

We set \( V_i := O_5(L_i) \) (i = 1, \ldots, 6). By maximality of \( L_i \) as a 5-local subgroup, \( V_i \) is a radical 5-subgroup for every \( i = 1, \ldots, 6 \). Let \( R \) be any radical 5-subgroup of \( G \). Then \( N_G(R) \leq L_i \) for some \( i = 1, \ldots, 6 \) up to conjugacy.
Assume first that $N_G(R) \leq L_1$ but $R \notin V_1$. Then $R = V_1 \times R_1$, where $R_1$ is a radical 5-subgroup of $L_1/V_1 \cong F_5.4$. As $L_1/V_1$ contains the $F_5$-factor $C_G(V_1)'$ of $C_G(V_1) \cong 5 \times F_5$ with index coprime to 5, the group $R_1$ is a radical 5-subgroup of $C_G(V_1)' \cong F_5$. In view of the list of $B_5(F_5)$ (see Theorem 9), $R_1$ is one of the following subgroups of $F_5$ with fusion type in $F_5$:

- a subgroup of type $5A$,
- a subgroup $5^2S^{1+2}$ with center a $5B$-pure $5^2$-subgroup,
- $5^1\times 4$ with center of type $5B$, or a Sylow 5-subgroup of order 5.

In the first case, it follows from the first table in [14, Section 9] that $R = V_1 \times R_1$ is a $5A$-pure $5^2$-subgroup in $G$, and hence it coincides with $V_2$ up to conjugacy. In the second case, the same table shows that $Z(R) = V_1 \times Z(R_1)$ has a unique 5$B$-pure $5^2$-subgroup $Z(R_1)$. As $N_G(Z) \cap L_1$ contains $(D_{10} \times 5^2.5^{1+2}.A_5)2$, the group $Z(R_1)$ is of type $5B_6(i)$ in the notation of the table in [14, Section 9], and hence $Z(R_1) = V_4$ up to conjugacy. In the last two cases, $Z(R) = V_1 \times Z(R_1)$ has a unique subgroup of type $5B$, and then $N_G(R) \leq N_G(5B) = L_3$ up to conjugacy.

Assume next that $N_G(R) \leq L_2$ but $R \notin V_2$. Then $R = V_2 \times R_2$ for $R_2 \in B_5(U_3(5))$ by the reasoning similar to that in the previous paragraph. As $U_3(5)$ is a group of Lie type of Lie rank 1 in characteristic 5, we then have $R_2 \cong 5^{1+2}$. As $Z(R_2)$ is a subgroup of type $5B$ in $F_5 \cong C_5(x)'$ for $x \in V_2^5$, we see that $Z(R_2)$ is a unique subgroup of $Z(R) = V_2 \times Z(R_2)$ of type $5B$ in $G$. Then we have $N_G(R) \leq N_G(Z(R_2)) = L_3$ up to conjugacy.

Assume that $N_G(R) \leq L_6$ but $R \neq V_6$. Then $R/V_6 \in B_5(L_2(25))$, and hence $R/V_6 \cong 5^2$ and $C_{V_6}(R/V_6)$ is a $5^2$-subgroup of $V_6$. This $5^2$-subgroup is of type $5B_6(i)$ in the notation of [14, Section 9], in view of its normalizer in $L_6$. Thus it is $V_4$ and $N_G(R) \leq N_G(C_{V_6}(R/V_6)) = L_4$ up to conjugacy.

Assume that $N_G(R) \leq L_5$ but $R \neq V_5$. Then $R/V_5 \in B_5(L_3(5))$, and hence $R/V_5$ is a unipotent radical $U_F$ of $L_3(5)$ corresponding to a flag $F$ of the projective plane associated with $Z(V_5) \cong S^3$, on which $L_5/V_5.2$ acts faithfully. If the flag $F$ contains a point, then $N_G(R)$ stabilizes the corresponding subgroup $C_{Z(V_5)}(R/V_5)$ of type $5B$, and hence $N_G(R) \leq L_3$ up to conjugacy. If $F$ corresponds to a line, the corresponding subgroup $C_{Z(V_5)}(R/V_5)$ is a $5B$-pure $5^2$-subgroup, which is of type $5B_6(i)$ in view of its normalizer in $L_5$. Thus $N_G(R) \leq L_4$ up to conjugacy.

When $N_G(R) \leq L_4$ but $R \neq V_4$, we have $R/V_4 \in B_5(GL_2(5))$. Then $R/V_4 \cong 5$ and $C_{Z(V_4)}(R/V_4)$ is a subgroup of type $5B$. Hence $N_G(R) \leq N_G(5B) = L_3$.

Finally, when $N_G(R) \leq L_3$, such $R$’s bijectively correspond to the radical 5-subgroups of $L_3/V_3 \cong 2J_24$, as $V_3 \cong S_4^{1+6}$ is extraspecial. As $|J_2|_5 = 5^2$, in view of [3] it is easy to see that $B_5(J_2)$ has two classes with representatives a subgroup of type $5A$ (in $J_2$) and a Sylow 5-subgroup (with their normalizers $(5.2) \times A_5$ and $5^2(2 \times S_3)$ in $J_2$). Hence we proved the following.

**Theorem 5.** The radical 5-subgroups of the Monster split into exactly eight conjugacy classes with representatives below, in which six except $V_1$ and $V_2$ are centric:
3.3. Radical 7-subgroups of $M$

3.3.1. The radical 7-subgroups of the Held simple group

The Held simple group $H := He$ has three classes of subgroups of order 7, of type $7AB$, $7C$ and $7DE$, with the following normalizers:

\[
\begin{align*}
N_H(7AB) &\approx (7 : 3) \times L_2(7), \\
N_H(7C) &\approx 7^{1+2} : (S_3 \times 3), \\
N_H(7DE) &\approx (7 \times D_{14})^3.
\end{align*}
\]

Under the action of $N_H(7C)$, the eight $7^2$-subgroups of $O_7(N_H(7C))$ split into three (respectively two and three) subgroups of type $7C^8$ (respectively $7C^1(AB)^7$ and $7C^1(DE)^3$). In the latter two cases, the normalizer of the $7^2$-subgroup is contained in the normalizer of its unique subgroup of type $7C$. In the first case, the normalizer is calculated as $N_G(7C^2) \approx 7^2 : SL_2(7)$.

Based on the information above, we now determine $B_7(He)$. If a radical 7-subgroup $R$ is of order 7, then $R$ is of type $7AB$. If $R$ is of order $7^2$, then $R$ is conjugate to one of the eight $7^2$-subgroups contained in a Sylow 7-subgroup $O_7(N_H(7C)) \approx 7^{1+2}$ of $H$.

The remark above shows that then either $N_G(R) \leq N_G(7C)$ or $R$ is $7C$-pure. In the first case, $R$ should be $O_7(N_H(7C))$, which is against $|R| = 7^2$. In the latter case, $R$ is in fact a radical subgroup. Thus, including a Sylow 7-subgroup, we obtain:

**Theorem 6.** The simple Held group has just three classes of radical 7-subgroups with the following representatives, in which two except $V_1$ are centric:

<table>
<thead>
<tr>
<th>$R$</th>
<th>$R \cong$</th>
<th>$Z(R)$</th>
<th>$(N(R)/R)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V_1$</td>
<td>7</td>
<td>$=(AB)^1$</td>
<td>$3 \times L_2(7)$</td>
</tr>
<tr>
<td>$V_2$</td>
<td>$7^2$</td>
<td>$C^8$</td>
<td>$2L_2(7)$</td>
</tr>
<tr>
<td>$V_3$</td>
<td>$7^{1+2}$</td>
<td>$C^1$</td>
<td>$S_3 \times 3$</td>
</tr>
</tbody>
</table>

3.3.2. Radical 7-subgroups of $M$

By [14, Theorem 7], every 7-local subgroup of the Monster $G = M$ is contained in one of the following subgroups up to conjugacy:

\[
\begin{align*}
L_1 &:= N_G(7A) \cong (7 : 3) \times He), \\
L_2 &:= N_G(7A^2) \cong ((7^2 : (3 \times 2A_4) \times L_2(7)) : 2, \\
L_3 &:= N_G(7B) \cong 7^{1+4} : (3 \times 2S_7).
\end{align*}
\]
Let $V_i := \text{Or}(L_i)$ $(i = 1, \ldots, 5)$. By maximality of $L_i$ as a 7-local subgroup, $V_i$ is a radical 7-subgroup for every $i = 1, \ldots, 5$. Let $R$ be any radical 7-subgroup of $G$. Then $NG(R) \leq L_i$ for some $i = 1, \ldots, 5$ up to conjugacy.

Assume first that $NG(R) \leq L_1$ but $R \neq V_1$. Then $R/V_1$ is a radical 7-subgroup of $L_1/V_1 \cong (3 \times \text{He}) : 2$. As $L_1/V_1$ contains $L_1^2V_1/V_1 \cong \text{He}$ with index coprime with 7, and the extension $L_1^4V_1/V_1$ splits, we have $R = V_1 \times \bar{R}_1$ for some radical 7-subgroup $\bar{R}_1$ of $L_1'' \cong \text{He}$. Then the classification of $B_7(\text{He})$ in Theorem 6 implies that $R_1$ is one of the following subgroups in $\text{He}$: a subgroup of type $7AB$, a $7C$-pure $7^2$-subgroup, or a Sylow 7-subgroup $7^{1+2}$. In the last case, we have $R' = Z(R_1)$ is a subgroup of type 7C in $\text{He}$, and thus of type 7B in $G$ (see the first table in [14, Section 10]). Then $NG(R) \leq NG(R') = NG(7B) = L_3$ up to conjugacy. In the first case, the same table shows that $R = V_1 \times R_1$ is a $7A$-pure (in $G$) $7^2$-subgroup, which is conjugate to $V_2$. In the second case, $R_1$ is a $7B$-pure $7^2$-subgroup of $G$ and $C_G(R_1) \cap L_1$ contains $(7 : 3) \times 7^2$. For a $7B$-element $x \in R_1$, this implies that $R_1$ lies in $O_7(C_G(x)) \cong 7^{1+2}$, because any element of order 7 in $C_G(x) \setminus O_7(C_G(x))$ is contained in a $7^2$-subgroup of type $7A^4B^4$ or $7B^8$, and the latter subgroup is conjugate to $V_3$ with $C_G(V_3) \cong V_3$ (see the exposition in [14, Section 10]). Then $R_1$ is conjugate to $V_3$, and $R_1$ is a unique maximal $7B$-pure subgroup of $R = V_1 \times R_1$ in view of the first table of [14, Section 10]. Thus $NG(R) \leq NG(R_1) = L_4$ up to conjugacy.

Next assume that $NG(R) \leq L_2$ but $R \neq V_2$. Then $R = V_2 \times R_2$, where $R_2$ is a Sylow 7-subgroup of the $L_2(7)$-factor of $C_G(V_2) \cong 7^2 \times L_2(7)$. Write $V_2 = \langle x, y \rangle$ for 7A-elements $x, y$ of $G$. Then $\langle y \rangle \times R_2$ lies in the $\text{He}$-factor of $C_G(x) \cong 7 \times \text{He}$, in which $\langle y \rangle$ and $R_2$ are subgroups of type $7AB$ and $7C$ respectively. Then it follows from the first table of [14, Section 10] that every subgroup of $R = V_2 \times R_2$ of order 7 other than $R_2$ is of type $7A$ in $G$. Thus $NG(R)$ normalizes the unique subgroup $R_2$ of type $7B$ in $G$, and hence $NG(R) \leq NG(R_2) = L_3$ up to conjugacy.

Assume that $NG(R) \leq L_1$ but $R \neq V_i$ for $i = 4$ or 5. Then $R/V_i$ is a Sylow 7-subgroup of $L_i/V_i \cong GL_2(7)$ or $SL_2(7)$, which acts faithfully on $V_i \cong 7^2$. Thus $C_{V_i}(R/V_i) = Z(R)$ is a 1-subspace of $V_i$ invariant under $NG(R)$. As $V_i$ is $7B$-pure, this implies that $NG(R) \leq NG(Z(R)) = L_3$ up to conjugacy.

Finally, when $NG(R) \leq L_3$, either $R = V_3$ or $R$ is a Sylow 7-subgroup of $G$. This completed the classification of $B_7(G)$.

**Theorem 7.** The radical 7-subgroups of the Monster split into just six conjugacy classes with representatives below, among which four except $V_1$ and $V_2$ are centric:

<table>
<thead>
<tr>
<th>$R$</th>
<th>$R \cong$</th>
<th>$Z(R) \cap R$</th>
<th>$NG(R)/R$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V_1$</td>
<td>$7$</td>
<td>$A^1 \times (3 \times \text{He})/2$</td>
<td></td>
</tr>
<tr>
<td>$V_2$</td>
<td>$7^2$</td>
<td>$A^8 \times (3 \times 4A_4) \times L_2(7)/2$</td>
<td></td>
</tr>
<tr>
<td>$V_3$</td>
<td>$7^2$</td>
<td>$SL_2(7)$</td>
<td></td>
</tr>
<tr>
<td>$V_4$</td>
<td>$7^2 \times 4 + 2$</td>
<td>$7^2 = B^8 \times GL_2(7)$</td>
<td></td>
</tr>
<tr>
<td>$V_5$</td>
<td>$7^2 + 4$</td>
<td>$7 = B_4 \times 3 \times 2S_3$</td>
<td></td>
</tr>
<tr>
<td>$V_6$</td>
<td>$7^2 + 4$</td>
<td>$7 = B_4 \times 3 \times 2S_3$</td>
<td></td>
</tr>
</tbody>
</table>

where $V_1^{(1)} = V_5\bar{7}$.
4. The Harada–Norton group $F_5$

4.1. The radical 3-subgroups of $F_5$

In $G := F_5$, the Harada–Norton simple group, there are exactly two classes of elements of order 3, called $3A$ and $3B$. In [15, Section 3.2], it is shown that every 3-local subgroup is conjugate to a subgroup of one of the following groups:

$$L_1 := N_G(3A) \cong (3 \times A_9) : 2,$$
$$L_2 := N_G(3A^2) \cong ((3^2 : 4) \times A_6)2_2,$$
$$L_3 := N_G(3B) \cong 3^3.A_5,$$
$$L_4 := N_G(3^4) \cong (2 \cdot (A_4 \times A_4))4.

Then $V_i := O_3(L_i)$ are radical 3-subgroups for all $i = 1, 2, 3$.

To analyze the fusion of 3-elements in $L_1$ and $L_2$, observe that $L_1$ is contained in a subgroup $A$ of $G$ isomorphic to $A_{12}$. There are four classes of elements of order 3 in $A$ of cycle types $3^1$, $3^2$, $3^3$ and $3^4$. The centralizer in $A$ of an element of cycle type $3^1$ or $3^2$ contains $A_6$, and hence it is a $3A$-element in $G$. An element of cycle type $3^4$ centralizes an involution of cycle type $2^6$, which is a $2A$-element in $G$ [15, Section 3.1]. As $3B$-elements do not centralize $2A$-elements [15, Section 5.1], elements of cycle type $3^4$ are $3A$-elements in $G$. It is shown in [15, Section 3.2] that $G$ has one class of $3^4$-subgroups (of fusion type $3A2^4B^{16}$). As $A \cong A_{12}$ contains a $3^4$-subgroup, elements of $A$ of cycle type $3^3$ are $3B$-elements in $G$.

Let $x$ be an element of $A$ of cycle type 3. Then $C_G(x) = \langle x \rangle \times C$ with $C \cong A_0$, which is a subgroup of $A$. Thus, for an element $y$ of order 3 in $C$, we may determine the fusion type of $\langle x, y \rangle$ based on the above remark: $\langle x, y \rangle$ is of type $3A^4$ (respectively $3A^2B^2$) in which $\langle x \rangle$ and $\langle y \rangle$ are of type $3A$, and $3A^3B^1$ in which $\langle y \rangle$ is the unique subgroup of type $3B$) if $y$ is of cycle type $3^1$ (respectively $3^2$ and $3^3$).

Now let $R$ be a radical 3-subgroup of $G$. Then $N_G(R) \leq L_i$ for some $i = 1, 2, 3$ up to conjugacy.

Assume first that $N_G(R) \leq L_1$ but $R \neq V_1$. Then $R = V_1 \times R_1$, where $R_1$ is a subgroup of $C_G(V_1) \cong A_9$ which corresponds to a radical 3-subgroup of $L_1/V_1 \cong S_9$. Remark that we may take $V_1$ and $R_1$ as subgroups of $A \cong A_{12}$ with $V_1$ of cycle type $3^1$. The fundamental result of Alperin and Fou [1] shows that a radical $p$-subgroup of the symmetric group $S_n$ is the direct product of basic subgroups $P^{s_1} \times P^{s_2} \times \cdots \times P^{s_m}$ with $p^{s_i}$ acting regularly on a subset of the $n$ letters, for $1 \leq c_1 \leq \cdots \leq c_m \leq n$ with $c_1 + \cdots + c_m \leq \log_p(n)$. In our case $n = 9$, there are just three classes for basic subgroups corresponding to $(c_1) = (1), (c_1) = (2)$ and $(c_1, c_2) = (1, 1)$. Thus $R_1$ is either the direct product of $t$ basic subgroups of cycle type $3^1$ for some $t = 1, 2, 3$, or $R_1$ itself is a basic subgroup isomorphic to $3^2$ or $3^3$.

In the former case, $R = V_1 \times R_1$ is generated by $3A$-elements. Thus if $t = 1$ (respectively $3$), $R$ is a $3A$-pure $3^3$-subgroup (respectively a $3^4$-subgroup), and hence $R = V_2$ (respectively $V_4$) up to conjugacy. If $t = 2$, we have $C_G(R) = V_1 \times C_{L_1}(R_1) \cong 3^4$, and hence $N_G(R) \leq L_4$ up to conjugacy. When $R_1$ is a regular $3^2$-subgroup, all nonidentity elements of $R_1$ are of cycle type $3^3$. Hence the above remark implies that $R_1$ is the unique $3B$-
pure $3^2$-subgroup of $R = V_1 \times R_1$. Thus $NG(R) \leq NG(R_1)$, which is contained in $L_3 = NG(3B)$ or $L_4 = NG(3^3)$ up to conjugacy in view of the arguments in [15, Section 3.2]. In the last case when $R_1 \cong 3_3 : 3 \cong 3^3 : 3$, $Z(R_1) \cong 3$ is generated by an element of cycle type $3^3$. Then it follows from the remark above that $Z(R_1)$ is a unique subgroup of $Z(R) = V_1 \times Z(R_1)$ of type $3B$. Thus $NG(R) \leq NG(Z(R)) \leq NG(Z(R_1)) = L_3$ up to conjugacy.

Assume next that $NG(R) \leq L_2$ but $R \neq V_2$. Then $R = V_2 \times R_2$ for some radical $3$-subgroup $R_2$ of $L_2 \cong A_6$. As the centralizer of an element of order 3 in $A_6$ contains a unique Sylow 3-subgroup of $A_6$, there is no radical subgroup of order 3 in $A_6$. Thus $R_2 \cong 3^2$ and $R = V_2 \times R_2 \cong 3^3$, which is conjugate to $V_4$.

Assume that $NG(R) \leq L_4$. Then $R/V_4$ is a radical $3$-subgroup of $L_4/V_4 \cong GO_4^+(3).2$, which is of Lie rank 2 in characteristic 3. Thus $R/V_4$ is a unipotent radical $U_F$ corresponding to a flag $F$ of totally isotropic subspaces of $V_4$, regarded as the natural module for $L_4/V_4$. Now $V_4$ contains 16 subgroups of type $3B$, and hence they correspond to isotropic points. If the flag $F$ contains a point, $Z(R) = CV_4(R/V_4)$ is a 1-subspace of $V_4$ corresponding to that point. Then $NG(R) \leq NG(Z(R)) = L_3$ up to conjugacy. If $F$ is an isotropic line, then $Z(R) = GV_4(R/V_4)$ is a $3B$-pure $3^2$-subgroup of $V_4$. Conversely, if we take $V_4^{(l)}$ as the inverse image in $L_4$ of a unipotent radical $U_l \cong 3$ of $GO_4^+(3)$ corresponding to an isotropic line $l$, then it follows from [15, Section 3.2] that

$$NG(V_4^{(l)}) \leq NG(Z(V_4^{(l)})) \leq NG(V_4) = L_4.$$ 

Thus $V_4^{(l)} (\cong V_4 3)$ is a new radical $3$-subgroup with center a $3B$-pure $3^2$-subgroup.

Finally, if $NG(R) \leq L_3$, then $R = V_3$ or $V_3 3$, a Sylow $3$-subgroup of $G$.

**Theorem 8.** There are exactly six classes of radical $3$-subgroups of the Harada–Norton simple group with the following representatives, in which four except $V_1$ and $V_2$ are centric:

<table>
<thead>
<tr>
<th>Name</th>
<th>$R$</th>
<th>$Z(R)$</th>
<th>$NG(R)/R$</th>
<th>Name</th>
<th>$R$</th>
<th>$Z(R)$</th>
<th>$NG(R)/R$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V_1$</td>
<td>3</td>
<td>$A^4_1$</td>
<td>$S_9$</td>
<td>$V_4^{(l)}$</td>
<td>$V_4 3$</td>
<td>$3^2 = B^4_1$</td>
<td>$2S_4$</td>
</tr>
<tr>
<td>$V_2$</td>
<td>$3^2$</td>
<td>$A^4_4$</td>
<td>$(4 \times A_6) 2.2$</td>
<td>$V_3$</td>
<td>$3_1^{1+4}$</td>
<td>$3 = B^1_1$</td>
<td>$4A_5$</td>
</tr>
<tr>
<td>$V_4$</td>
<td>$3^4$</td>
<td>3</td>
<td>$2(A_4 \times A_4 4)$</td>
<td>$V_3^{(l)}$</td>
<td>$V_3 3$</td>
<td>3</td>
<td>4.2</td>
</tr>
</tbody>
</table>

### 4.2. The radical $5$-subgroups of $F_5$

There are exactly four classes of subgroups of $G := F_5$ of order 5 in $G$, of type $5A$, $5B$, $5CD$ and $5E$, with the following normalizers:

- $NG(5A) \cong (D_{10} \times U_3(5))^2$,
- $NG(5B) \cong 5^{1+4} : (2^{1+4} : 5 : 4)$,
- $NG(5CD) \cong 5^3 : 4A_5$,
- $NG(5E) \cong (5 \times 5^{1+2} : 2^2 4)$. 
In [15, Section 3.3], it is shown that any 5-local subgroup of \( G \) is contained in one of the following groups up to conjugacy:

\[
\begin{align*}
L_1 &:= N_G(5A) \cong (D_{10} \times U_3(5))_2, \\
L_2 &:= N_G(5B) \cong 5^{1+4} : (2^{1+4} : 5 : 4), \\
L_3 &:= N_G(5B^2) \cong 5^2.5^{1+2} : 4A_5.
\end{align*}
\]

Then \( V_i := O_5(L_i) \) are radical 5-subgroups for all \( i = 1, 2, 3 \).

We now remark that there is no subgroup isomorphic to \( 5^{1+2} \) with center a subgroup of type \( 5A, 5CD \) or \( 5E \); this is evident for type \( 5A \), as \( C_G(5A) \cong 5 \times U_3(5) \). For type \( 5CD \) or \( 5E \), the extraspecial group \( V_2 \) contains a subgroup \( X \cong 5 \) of that type [15, Section 3.3], and hence \( C_{V_2}(X) \cong X \times 5^{1+2} \), which is a Sylow 5-subgroup of \( C_G(X) \) in view of \( |C_G(X)| \). This shows the claim for type \( 5CD \) or \( 5E \).

Let \( R \) be a radical 5-subgroup of \( G \). Then \( N_G(R) \leq L_i \) for some \( i = 1, 2, 3 \). If \( N_G(R) \leq L_1 \) but \( R \neq V_1 \), then \( R = V_1 \times R_1 \) for some \( R_1 \in B_5(U_3(5)) \). As \( U_3(5) \) is of Lie rank 1 in characteristic 5, \( R_1 \) is a Sylow 5-subgroup of \( U_3(5) \) which is isomorphic to \( 5^{1+2} \). The centralizer of \( Z(R_1) \) in \( L_1 \) is \( D_{10} \times 5^{1+2} \), in which \( Z(R_1) \) corresponds to the center of \( 5^{1+2} \). The remark above implies that \( Z(R_1) = R_1' = R' \) is of type \( 5B \), and hence \( N_G(R) \leq N_G(5B) = L_2 \).

If \( N_G(R) \leq L_i \) for \( i = 2 \) or 3, then \( R = V_i \) or \( R \) is a Sylow 5-subgroup of \( G \), as \( |L_i|_5 = |G|/5 \). Hence we obtain:

**Theorem 9.** There are just four classes of radical 5-subgroups of the Harada–Norton simple group with the following representatives, in which three except \( V_1 \) are centric:

<table>
<thead>
<tr>
<th>Name</th>
<th>( R )</th>
<th>( Z(R) )</th>
<th>( N_G(R)/R )</th>
<th>Name</th>
<th>( R )</th>
<th>( Z(R) )</th>
<th>( N_G(R)/R )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( V_1 )</td>
<td>5</td>
<td>( A^1 )</td>
<td>(2 \times U_3(5))_2</td>
<td>( V_2 )</td>
<td>( 5^{1+4} )</td>
<td>( 5 = B_1 )</td>
<td>2^{1+4}(5 : 4)</td>
</tr>
<tr>
<td>( V_3 )</td>
<td>( 5^2.5^{1+2} )</td>
<td>( 5^2 = B^6 )</td>
<td>4A_5</td>
<td>( V_2^{(s)} )</td>
<td>( V_2 5 )</td>
<td>5</td>
<td>2.4</td>
</tr>
</tbody>
</table>

**Appendix A**

In Table 1, we give the number of the classes of radical \( p \)-subgroups for each sporadic simple group \( X \) and each prime \( p \) with \( |X|_p \geq p^2 \). I also provide one published reference when \( |X|_p \geq p^4 \). Besides the references given here, the papers and manuscripts by An, O’Brien and others (see http://www.math.auckland.ac.nz/) can be quote, although they classify the radical subgroups with the aid of GAP. The references are omitted when \( |X|_p \leq p^3 \), as it is not difficult to determine radical \( p \)-subgroups in this case. I ignore the case \( |X|_p = p \), as the class number in question is 1 in this case.

Note that there is no odd prime \( p \) with \( |X|_p \geq p^2 \) for \( J_1 \), and hence \( J_1 \) does not appear in Table 1.
Table 1
Radical $p$-subgroups of the sporadic simple groups for $p$ with $|X|_p \geq p^2$

| $X$   | $|X|_p$ | Number and reference |
|-------|---------|----------------------|
| $M_{11}$ | 3$^2$ | 1 |
| $M_{12}$ | 3$^3$ | 4 |
| $M_{22}$ | 3$^2$ | 2 |
| $M_{23}$ | 3$^2$ | 2 |
| $M_{24}$ | 3$^3$ | 4 |
| $J_2$ | 3$^3/5^2$ | 3/2 |
| $Suz$ | 3$^2/5^2$ | 5 [18, Proposition 6]/3 |
| $M_{23}$ | 3$^3/5^3$ | 3/2 |
| $HS$ | 3$^2/5^3$ | 2/2 |
| $Co_3$ | 3$^3/5^3$ | 4/2 |
| $Co_2$ | 3$^3/5^3$ | 4/2 |
| $Co_1$ | 3$^3/5^2/7^2$ | 12 [6, Main results]/7/3 |
| $F_{22}$ | 3$^3/5^2$ | 10 [5, Theorem 23]/2 |
| $F_{23}$ | 3$^3/5^2$ | 10 [2]/2 |
| $F_{24}'$ | 3$^6/5^2/7^3$ | 11 [5, Theorem 20]/2/3 |
| $He$ | 3$^3/5^2/7^3$ | 4/2/3 |
| $HN$ | 3$^6/5^6$ | 6 [Theorem 8]/4 [Theorem 9] |
| $Th$ | 3$^{10}/5^3/7^2$ | 5 [10, Section 6.1]/2/2 |
| $BM$ | 3$^{13}/5^6/7^2$ | 13 [Theorem 2]/6 [Theorem 3]/2 |
| $M$ | 3$^{20}/5^3/7^6/11^2/13^3$ | 13 [Theorem 4]/8 [Theorem 5]/6 [Theorem 7]/2/3 |
| $J_3$ | 3$^{5}$ | 2 [4, Proposition 3.1.4] |
| $Ru$ | 3$^3/5^3$ | 3/3 |
| $O'N$ | 3$^5/7^3$ | 2 [11, Section 6.1]/3 |
| $Ly$ | 3$^1/5^6$ | 4 [8, Section 4.1]/4 [8, Section 5.1] |
| $J_4$ | 3$^1/11^3$ | 3/1 |

References

