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Lie algebras with few centralizer dimensions

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Abstract

It is known that a finite group with just two different sizes of conjugacy classes must be nilpotent and it has recently been shown that its nilpotence class is at most 3. In this paper we study the analogs of these results for Lie algebras and some related questions.

1. Introduction

Several authors have investigated finite groups that have just two different conjugacy class sizes. For example, in [3], N. Ito showed that such a group must be nilpotent. (And in fact, once we know that the group is nilpotent, it is easy to see that it must be the direct product of an abelian group and a p-group for some prime p.) Recently, a dramatic improvement of Ito's result was obtained by K. Ishikawa, who showed in [2] that a finite group with just two class sizes must have nilpotence class at most 3. (The second author of this paper was able to simplify Ishikawa's proof somewhat, and he circulated his argument privately. We include a slightly improved version of it here as Appendix A.)

Of course, the number of different conjugacy class sizes in a finite group G is equal to the number of different orders of centralizers of elements of G. This observation allows us to consider possible analogs of Ito's and Ishikawa's theorems for Lie algebras: What can be said about a finite-dimensional Lie algebra L if the centralizer subalgebras $\mathbf{C}_L(x)$ have just two different dimensions as x runs over the elements of L?

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We consider only finite-dimensional Lie algebras in this paper. Also, although some of our arguments would work more generally, we limit ourselves to Lie algebras over the complex numbers except in Theorem B, which we state and prove over an arbitrary field. (We will usually omit explicit mention of the modifiers “finite-dimensional” and “complex”, however.) For general background on Lie algebras, we refer the reader to the books [1,4] by N. Jacobson and J.E. Humphreys, respectively.

Results that hold for finite groups and that make sense for Lie algebras are often valid for such algebras. In fact, the Lie-algebra version of such a result is usually somewhat less complicated to state and easier to prove than is the corresponding theorem about groups. Nevertheless, the Lie algebra analog of Ito’s theorem is not valid: A Lie algebra with just two dimensions of element centralizers need not be nilpotent or even solvable. It is easy to see, for example, that in the three-dimensional simple Lie algebra $\mathfrak{sl}_2(\mathbb{C})$, the centralizer of each nonzero element has dimension 1. Also, the unique non-abelian two-dimensional Lie algebra shows that even if we assume that the algebra is solvable, it need not be nilpotent.

Nevertheless, we do obtain some results in the direction of Ito’s theorem for Lie algebras. We show, in fact, that if a Lie algebra L has just two centralizer dimensions, then either it is nilpotent or else its center $\mathbf{Z}(L)$ has codimension at most 3. (It is not hard to see, however, that if L is nilpotent and has just two centralizer dimensions, then the codimension of the center can be unboundedly large.)

Theorem A. *Suppose that L is a nonnilpotent finite-dimensional complex Lie algebra and that the subalgebras $\mathbf{C}_L(t)$ have just two different dimensions as t runs over the elements of L. Then $\dim(L/\mathbf{Z}(L)) \leqslant 3$ and one of the following possibilities occurs.*

(1) *$L/\mathbf{Z}(L)$ is isomorphic to the unique non-abelian 2-dimensional Lie algebra.*
(2) *$L/\mathbf{Z}(L)$ is isomorphic to $\mathfrak{sl}_2(\mathbb{C})$.*
(3) *$L/\mathbf{Z}(L)$ is isomorphic to the Lie algebra with basis $\{a, x, y\}$ and relations $[a, x] = x$, $[a, y] = -y$, and $[x, y] = 0$.*

We shall see that all three possibilities in Theorem A can occur, and we observe that L is nonsolvable only in Case (2). Also, we remark that in the first two cases, the subalgebra $\mathbf{Z}(L)$ is necessarily a direct summand of L. (This is fairly easy to see in Case (1) and it follows from Levi’s theorem [4, p. 91] in Case (2).) The center cannot be a direct summand in Case (3) however, because the Lie algebra $L/\mathbf{Z}(L)$ has more than two different centralizer dimensions.

Unlike the situation for Ito’s theorem, the analog of Ishikawa’s theorem is valid for nilpotent Lie algebras, and this works over any field.

Theorem B. *Let L be a finite-dimensional nilpotent Lie algebra over an arbitrary field and assume that the subalgebras $\mathbf{C}_L(t)$ have at most two different dimensions as t runs over the elements of L. Then the nilpotence class of L is at most 3.*

We also obtain results for Lie algebras in which there are more than just two centralizer dimensions. To state these, we consider the set of dimensions (over $\mathbb{C}$) of centralizers of noncentral elements of a Lie algebra L and we write $\mathrm{cd}(L)$ to denote this set of “nontrivial” centralizer dimensions. (Thus L is abelian if and only if $\mathrm{cd}(L) = \emptyset$ and there are just two centralizer dimensions if and only if $|\mathrm{cd}(L)| = 1$.)

We show that if $|\mathrm{cd}(L)|$ is given, then the Lie algebra L is "almost" solvable. To state our result, we recall that the radical of a Lie algebra is its unique largest solvable ideal and that the (uniquely determined) rank of a Lie algebra is the dimension of an arbitrary Cartan subalgebra.

Theorem C. *Let R be the radical of a finite-dimensional complex Lie algebra L. Then the rank of L/R is at most $|\mathrm{cd}(L)|$.*

Recall that if R is the radical of a Lie algebra L, then L/R is a direct sum of simple ideals. If $|\mathrm{cd}(L)| = 1$, it follows from Theorem C that L/R has rank at most 1, and thus if $R < L$, the only possibility is that $L/R \cong \mathfrak{sl}_2(\mathbb{C})$. In this case, R has codimension 3 and we know by Theorem A that R must actually be the center $\mathbf{Z}(L)$. In fact, the case of Theorem C where $|\mathrm{cd}(L)| = 1$ is used in the proof of Theorem A.

It is natural to ask whether or not Theorems A and B have analogs for Lie algebras L for which $|\mathrm{cd}(L)| > 1$. It seems that the relevant parameter here is not $|\mathrm{cd}(L)|$, but instead it is the related quantity $\max(\mathrm{cd}(L)) - \min(\mathrm{cd}(L))$, which we denote $\Delta(L)$. Of course, we have defined $\Delta(L)$ only when L is non-abelian, and so we set $\Delta(L) = -1$ if L is abelian. Note that $|\mathrm{cd}(L)| \leqslant \Delta(L) + 1$ and that $|\mathrm{cd}(L)| = 1$ if and only if $\Delta(L) = 0$.

It follows easily from Theorem A that if L is solvable and $\Delta(L) = 0$, then L has a nilpotent ideal with codimension at most 1. (This is because each of the two solvable possibilities for $L/\mathbf{Z}(L)$ in Theorem A has an abelian ideal with codimension 1.) The following theorem is a generalization of this fact, and indeed, we use the case $\Delta(L) = 0$ of this result in our proof of Theorem A. We recall that the nilradical of a Lie algebra L is the unique largest nilpotent ideal of L.

Theorem D. *Let L be a solvable finite-dimensional complex Lie algebra with nilradical N. Then $\Delta(N) \leqslant \Delta(L)$ and the codimension of N in L is at most $\Delta(L) + 1$.*

We mention that an argument similar to that in our proof of Theorem D can be used to show that $\Delta(H) \leqslant \Delta(L)$, where H is a Cartan subalgebra of an arbitrary finite-dimensional complex Lie algebra L. We do not present that proof here, however.

The assertion of Theorem B is that if L is a nilpotent Lie algebra and $\Delta(L) = 0$, then the nilpotence class of L is at most 3. We do not know if this result can be generalized to cases where $\Delta(L) > 0$, but it seems reasonable to conjecture that there does exist such a generalization.

Conjecture E. *Let L be a nilpotent Lie algebra. Then the nilpotence class of L is bounded in terms of $\Delta(L)$.*

In general, the nilpotence class of a nilpotent Lie algebra L is definitely not bounded in terms of $|\mathrm{cd}(L)|$. In fact, even when $|\mathrm{cd}(L)| = 2$, the nilpotence class can be arbitrarily large. To see this, let L be the semidirect product of an abelian algebra A with basis $\{a_1, a_2, \ldots, a_n\}$, acted on by a 1-dimensional algebra with generator x acting according to the formula $[a_i, x] = a_{i+1}$ for $1 \leqslant i < n$ and $[a_n, x] = 0$. (It is easy to check that L exists for every positive integer n and that L is nilpotent of class n.) The center of L is the 1-dimensional subspace spanned by a_n; every noncentral element of L in A has centralizer A, and all elements of L outside of A have centralizers of dimension 2. If $n > 2$, therefore, $\mathrm{cd}(L) = \{2, n\}$ has cardinality 2, and yet the nilpotence class n is unbounded.

If Conjecture E is true, then it would follow by Theorem D that the nilpotence class of the nilradical N of a solvable Lie algebra L is bounded in terms of $\Delta(L)$. Since L/N is abelian, we would have the following as a consequence.

Conjecture F. *Let L be a solvable Lie algebra. Then the derived length of L is bounded in terms of $\Delta(L)$.*

Finally, we return to finite groups. It is known that there is no bound on the nilpotence class of a finite p-group if there are exactly three conjugacy class sizes, and so Ishikawa's theorem does not generalize in that direction. For example, if P is the wreath product of a cyclic group of order p^n with a cyclic group of order p, it is not hard to see that the class sizes of P are 1, p, and $p^{(p-1)n}$, but that the nilpotence class is unbounded in terms of n.

In this wreath product example, however, the analog of our parameter Δ is unboundedly large. If P is a non-abelian p-group, we define $\Delta(P) = e - f$, where p^e is the size of the largest conjugacy class in P and p^f is the size of the smallest noncentral class in P. (Also, we set $\Delta(P) = -1$ if P is abelian.) Ishikawa's theorem asserts that if $\Delta(P) = 0$, then the nilpotence class of P is at most 3, and we have the following conjecture.

Conjecture G. *Let P be a finite p-group. Then the nilpotence class of P is bounded in terms of $\Delta(P)$.*

2. Semisimple rank and the number of centralizer dimensions

In this section we prove Theorem C. The key idea is nothing but a bit of elementary linear algebra, which we state below as a lemma. We need to establish some notation.

Let F^m be the m-dimensional row vector space over a field F. If $v \in F^m$ and $1 \leqslant i \leqslant m$, we write $v(i) \in F$ to denote the ith coordinate of v; we define $\mathrm{supp}(v) = \{i \mid v(i) \neq 0\}$, the *support* of v, and we set $s(v) = |\mathrm{supp}(v)|$. If $U \subseteq F^m$ is a subspace, we define $S(U) = \{s(u) \mid 0 \neq u \in U\}$.

Lemma 2.1. *Suppose that F is an infinite field and that $U \subseteq F^m$ is a subspace, then* $\dim(U) \leqslant |S(U)|$.

Proof. If $S(U)$ is empty, then U contains no nonzero vectors, and hence $\dim(U) = 0$, as required. We can assume, therefore, that $S(U)$ is nonempty, and we work by induction on $|S(U)|$.

Let $x, y \in U$ and $\alpha \in F$, so that we have $\mathrm{supp}(x - \alpha y) \subseteq \mathrm{supp}(x) \cup \mathrm{supp}(y)$. If $i \in \mathrm{supp}(x) \cup \mathrm{supp}(y)$ but i is not in $\mathrm{supp}(x - \alpha y)$, then $x(i) = \alpha y(i)$, and we see that there is most one possibility for α. Since F is infinite, it follows that we can choose $\alpha \in F$ so that all of these “bad” elements are avoided, and it follows that $\operatorname{supp}(x) \cup \operatorname{supp}(y) = \operatorname{supp}(z)$ for some element $z \in U$.

Now let $k = \max(S(U))$ and let $x \in U$ with $s(x) = k$. If $y \in U$, we know that $\operatorname{supp}(x) \cup \operatorname{supp}(y) = \operatorname{supp}(z)$ for some element $z \in U$ and it follows from the maximality of k that $\operatorname{supp}(y) \subseteq \operatorname{supp}(x)$. Fix $j \in \operatorname{supp}(x)$ and let $\pi : U \to F$ be the linear functional defined by $u \mapsto u(j)$. Since $j \in \operatorname{supp}(x)$, we see that $x \notin \ker(\pi)$ and so $\dim(\ker(\pi)) = \dim(U) - 1$. Also, if $y \in \ker(\pi)$, then $\operatorname{supp}(y) < \operatorname{supp}(x)$, and hence $s(y) < k$ and $k \notin S(\ker(\pi))$. Thus $S(\ker(\pi))$ is a proper subset of $S(U)$ and, by the inductive hypothesis, we see that $\dim(\ker(\pi)) \leqslant |S(\ker(\pi))| < |S(U)|$. It follows that $\dim(U) = 1 + \dim(\ker(\pi)) \leqslant |S(U)|$, as required. □

Proof of Theorem C. Let R be the radical of L. We know by Levi's theorem [4, p. 91] that there is a subalgebra $S \subseteq L$ such that $S \cong L/R$ and, in particular, S is semisimple and has an abelian Cartan subalgebra T. We need to show that $\dim(T) \leqslant |\operatorname{cd}(L)|$.

Since $S \cap \mathbf{Z}(L) = 0$, we see that S acts faithfully on L, and thus by [1, Theorem 6.4], we know that if $s \in S$ is an element whose adjoint action on S is semisimple, then its adjoint action on L is also semisimple. Since S is semisimple, its Cartan subalgebra T is toral, and this means that the adjoint action of each element of T on S is semisimple. We conclude, therefore, that the adjoint action of each element of T on L is semisimple.

Since T is abelian and each of its elements acts semisimply on L, it follows that there is a basis for L such that the adjoint representation of T on L is via $m \times m$ diagonal matrices, where $m = \dim(L)$. In particular, we see that if $t \in T$, then $\dim(\mathbf{C}_L(t))$ is exactly the number of zero entries on the diagonal of the diagonal matrix $\rho(t)$ representing t. Also, since $T \cap \mathbf{Z}(L) = 0$, we see that ρ is a vector-space isomorphism from T into the space of diagonal $m \times m$ matrices over $\mathbb{C}$.

We can identify the space of all diagonal $m \times m$ matrices over $\mathbb{C}$ with the row space $\mathbb{C}^m$. Under our identifications, therefore, T is a subspace of this space of row vectors. Furthermore, if $t \in T$, we see that $\dim(\mathbf{C}_L(t)) = m - |\operatorname{supp}(t)|$, and so there are at most $|\operatorname{cd}(L)|$ different numbers that can occur as $|\operatorname{supp}(t)|$ for nonzero elements $t \in T$. It follows by Lemma 2.1 that $\dim(T) \leqslant |\operatorname{cd}(L)|$, as required. □

3. Dimensions of centralizers in the nilradical

In this section we prove Theorem D. We begin by recalling some basic facts about the Zariski topology on a Lie algebra L. If we fix a basis, we can identify L with the set of n-tuples of elements of $\mathbb{C}$, where $n = \dim(L)$. If I is an ideal in the ring R of polynomials in n indeterminates over $\mathbb{C}$, then the corresponding variety $\mathcal{V}(I)$ is the subset of L consisting of those n-tuples that are simultaneous zeros for all of the polynomials in the ideal I. The closed sets in the Zariski topology on L are exactly the varieties of the various ideals of R. (It is not hard to check that this is a topology and that it is independent of the initial choice of the basis for L.) The key fact that we will use is that two nonempty Zariski-open sets cannot be disjoint. (This is equivalent to the assertion that L is not the union of two proper varieties $\mathcal{V}(I)$ and $\mathcal{V}(J)$. But $\mathcal{V}(I) \cup \mathcal{V}(J) = \mathcal{V}(IJ)$ and $\mathcal{V}(IJ) < L$ since $IJ \neq 0$ and the field $\mathbb{C}$ is infinite.)

Lemma 3.1. *Let $m = \min(\mathrm{cd}(L))$, where L is a non-abelian Lie algebra. Then $\{x \in L \mid \dim(\mathbf{C}_L(x)) = m\}$ is a nonempty Zariski-open subset of L.*

Proof. Our set is certainly nonempty and we let M be its complement in L. Then $M = \{x \in L \mid \dim(\mathbf{C}_L(x)) > m\}$ and we need to show that M is Zariski-closed.

Choose a basis for L and view the adjoint representation of L as a map ρ from L into the space of $n \times n$ matrices over $\mathbb{C}$. If $x \in L$, we see that $\dim(\mathbf{C}_L(x)) = n - \mathrm{rank}(\rho(x))$. Our set M, therefore, is exactly the set of elements $x \in L$ such that $\mathrm{rank}(\rho(x)) < k$, where we have written $k = n - m$. Since these are exactly the elements of L for which every $k \times k$ submatrix of $\rho(x)$ has determinant 0, it follows that x lies in M if and only if the coefficients of x with respect to the specified basis for L are simultaneous solutions of certain polynomial equations. This completes the proof. □

Next, we need some general (and presumably well known) results about nilpotent Lie algebras.

Lemma 3.2. *Let N be a nilpotent Lie algebra and suppose that $H \subseteq N$ is a maximal proper subalgebra. Then H is an ideal of N.*

Proof. We work by induction on $\dim(N)$. Let $Z = \mathbf{Z}(N)$, the center, and note that $Z > 0$ since N is nontrivial and nilpotent. If $Z \nsubseteq H$ then since $H + Z$ is clearly a subalgebra, we have $H + Z = N$. Thus $[N, H] = [H + Z, H] = [H, H] \subseteq H$, as required. We can assume, therefore, that $Z \subseteq H$, and thus H/Z is a maximal proper subalgebra of the nilpotent Lie algebra N/Z. Since $\dim(N/Z) < \dim(N)$, we conclude by the inductive hypothesis that H/Z is an ideal of N/Z. The result now follows. □

Corollary 3.3. *Let N be a nilpotent Lie algebra and suppose that $H \subseteq N$ is a subalgebra such that $H + N' = N$, where N' is the derived subalgebra of N. Then $H = N$.*

Proof. If $H < N$, we can replace H by a maximal proper subalgebra, and thus by Lemma 3.2, we can assume that H is an ideal of N. Then N/H is a Lie algebra having no nonzero proper subalgebras, and we deduce that $\dim(N/H) = 1$, and thus N/H is abelian. But then $N' \subseteq H$ and $N = H + N' = H < N$. This is a contradiction and the proof is complete. □

Lemma 3.4. *Let L be a Lie algebra and suppose that $N \subseteq L$ is a nilpotent ideal. If L/N' is nilpotent, then L is nilpotent.*

Proof. First, note that the derived subalgebra N' is an ideal of L, and so the hypothesis makes sense. By Engel's theorem, it suffices to show that $\mathrm{ad}\, x$ is nilpotent for an arbitrary element $x \in L$. Given x, we consider the Fitting decomposition of L with respect to $\mathrm{ad}\, x$. In other words, we write $L = L_0 + L_1$, where $\mathrm{ad}\, x$ is nilpotent on L_0 and $\mathrm{ad}\, x$ is invertible on L_1. Also, we recall that L_0 is a subalgebra of L (see [4, Proposition III.2]).

Since L/N' is nilpotent, we see that the linear transformation of L/N' induced by $\mathrm{ad}\, x$ is nilpotent, and thus $L_1 \subseteq N'$. It follows that $L = N' + L_0$, and hence $N = N' + (L_0 \cap N)$.

By Corollary 3.3, we conclude that $L_0 \cap N = N$, and thus $N \subseteq L_0$. It follows that $L_1 \subseteq N' \subseteq N \subseteq L_0$, and we conclude that $L_1 = 0$, and thus $L_0 = L$. In other words, $\operatorname{ad} x$ is nilpotent, as required. □

Lemma 3.5. *Let N be the nilradical of a solvable Lie algebra L and let $x \in L$. If the linear transformation of N/N' induced by the action of* $\operatorname{ad} x$ *is nilpotent, then $x \in N$.*

Proof. Since L is solvable, we know that L/N is abelian, and thus the subalgebra $H = N + \mathbb{C}x$ is actually an ideal of L, and it suffices to show that this ideal is nilpotent. By Lemma 3.4, therefore, it suffices to show that the Lie algebra H/N' is nilpotent.

Since the action of $\operatorname{ad} x$ on N/N' is nilpotent and for each element $n \in N$, the action of $\operatorname{ad} n$ on N/N' is trivial, we see that the action of $\operatorname{ad} h$ on N/N' is also nilpotent, where h is an arbitrary element of H. Also, if $h \in H$ is arbitrary, then since $[H, h] \subseteq N$, it follows that $\operatorname{ad} h$ induces a nilpotent linear transformation on H/N'. We conclude that each element of the Lie algebra H/N' is ad-nilpotent, and thus H/N' is nilpotent by Engel's theorem. This completes the proof. □

Theorem 3.6. *Let N be the nilradical of a solvable Lie algebra L. Then the set $\{x \in N \mid \mathbf{C}_L(x) \subseteq N\}$ contains a nonempty Zariski-open subset of N.*

Proof. Note that L/N is abelian since L is solvable. The vector space N/N' is a module for L/N, and thus we can decompose N/N' as a finite direct sum of nonzero L-invariant weight spaces W_α, where each weight α is a function from L/N into the field $\mathbb{C}$. If we view α as a function defined on L, we can say that for each element $x \in L$, the linear transformation of W_α induced by $\operatorname{ad} x - \alpha(x) \cdot 1$ is nilpotent.

Let A be the set of elements $x \in N$ such that the image $\bar{x}$ of x in N/N' has a nonzero component in each weight space W_α. It is clear that A is nonempty and Zariski-open in N, and so it suffices to show that $\mathbf{C}_L(a) \subseteq N$ for all elements $a \in A$. By Lemma 3.5, therefore, it suffices to show that if $x \in L$ centralizes $a \in A$, then the induced action of $\operatorname{ad} x$ on N/N' is nilpotent. Finally, since the action of $\operatorname{ad} x - \alpha(x) \cdot 1$ on the weight space W_α is nilpotent, it suffices to show that $\alpha(x) = 0$ for each of the weights α.

Since $[a, x] = 0$ and a has a nonzero component in each weight space W_α, we see that the linear transformation of W_α induced by $\operatorname{ad} x$ annihilates some nonzero vector, and thus has 0 as an eigenvalue. But since $\operatorname{ad} x - \alpha(x) \cdot 1$ is nilpotent on W_α, we see that $\alpha(x)$ is the unique eigenvalue of the action of $\operatorname{ad} x$ on W_α. It follows that $\alpha(x) = 0$, as required. □

We are now ready to prove part of Theorem D of the introduction.

Theorem 3.7. *Let L be a solvable Lie algebra with nilradical N. Then $\Delta(N) \leqslant \Delta(L)$. Also, if N is non-abelian, then* $\min(\operatorname{cd}(N)) \in \operatorname{cd}(L)$.

Proof. We can certainly assume that $N < L$. If N is abelian, then $\Delta(N) = -1 \leqslant \Delta(L)$, and there is nothing further to prove. We can assume, therefore, that N is non-abelian.

Let $m = \min(\operatorname{cd}(N))$ and $M = \max(\operatorname{cd}(N))$, so that $M - m = \Delta(N)$. By Lemma 3.1 and Theorem 3.6, together with the fact that nonempty Zariski-open subsets of N cannot be disjoint, we can choose $x \in N$ such that $\dim(\mathbf{C}_N(x)) = m$ and $\mathbf{C}_L(x) \subseteq N$. Then $\dim(\mathbf{C}_L(x)) = \dim(\mathbf{C}_N(x)) = m$ and $m \in \mathrm{cd}(L)$, as required. In particular, we have $\min(\mathrm{cd}(L)) \leqslant m$.

Now let $y \in N$ with $\dim(\mathbf{C}_N(y)) = M$. Then $y \notin \mathbf{Z}(N)$, and so $y \notin \mathbf{Z}(L)$, and we have $M = \dim(\mathbf{C}_N(y)) \leqslant \dim(\mathbf{C}_L(y)) \in \mathrm{cd}(L)$. Thus $\max(\mathrm{cd}(L)) \geqslant M$ and, since $\min(\mathrm{cd}(L)) \leqslant m$, we see that $\Delta(L) \geqslant \Delta(N)$, and the proof is complete. □

To complete the proof of Theorem D, we need to review a bit of the theory of Cartan subalgebras. We recall that if L is any Lie algebra, then L has a Cartan subalgebra H, which, by definition, is nilpotent. It follows that L decomposes as a finite direct sum of nonzero weight spaces L_α with respect to the adjoint action of H on L, where each weight α is some function from H into the field $\mathbb{C}$. Also, $H \subseteq L_0$, and in fact we must have equality here since otherwise H would annihilate a nonzero element of the vector space L_0/H, and this contradicts the fact that the Cartan subalgebra H is its own normalizer in L. Finally, if α is any nonzero weight, we have $[L_\alpha, H] = L_\alpha$ and thus $L_\alpha \subseteq L'$. It follows from all of this that $L = H + L'$.

The following result contains the part of Theorem D that we have not yet proved.

Theorem 3.8. *Let L be a solvable Lie algebra and suppose that N is its nilradical. Then* $\dim(L/N) \leqslant 1 + \Delta(L)$.

Proof. Since, by definition, $\Delta(L) \geqslant -1$, there is nothing to prove if $N = L$, and so we can suppose that L is not nilpotent. Since L is solvable, however, we know that L' is nilpotent, and hence $L' \subseteq N$.

Let H be a Cartan subalgebra of L and observe that $H + N \supseteq H + L' = L$, and thus $H + N = L$. Also, since $H < L$, there exists some nonzero weight for the action of H on L and we let $W = L_\alpha$ be the corresponding weight space, so that $W = [W, H] \subseteq L' \subseteq N$.

Since H is solvable, there exists a nonzero element $w \in W$ such that $[w, H] \subseteq \mathbb{C}w$, and thus the codimension in H of $C \cap H$ is at most 1, where $C = \mathbf{C}_L(w)$. Also, since $\alpha \neq 0$, we have $[w, H] \neq 0$, and so w is not central in L and $\dim(C) \in \mathrm{cd}(L)$. In particular, $\dim(C) \leqslant \max(\mathrm{cd}(L))$.

We claim that $\dim(C \cap N) \geqslant \min(\mathrm{cd}(L))$. First, we see that if N is non-abelian, we have $\dim(C \cap N) = \dim(\mathbf{C}_N(w)) \geqslant \min(\mathrm{cd}(N)) \in \mathrm{cd}(L)$ by Theorem 3.7, and thus $\dim(C \cap N) \geqslant \min(\mathrm{cd}(L))$, as required. If N is abelian, on the other hand, then $N \subseteq C$ since $w \in N$. Also, by Theorem 3.6 we know that N is the centralizer in L of one of its elements, and thus $\dim(C \cap N) = \dim(N) \in \mathrm{cd}(L)$. In this case too, we have $\dim(C \cap N) \geqslant \min(\mathrm{cd}(L))$, as claimed.

Since $H + N = L$ and $C \cap H$ has codimension at most 1 in H, we see that $C + N$ has codimension at most 1 in L, and thus

$$\begin{aligned}\dim(L/N) &\leqslant 1 + \dim\big((C+N)/N\big) = 1 + \dim(C) - \dim(C \cap N) \\ &\leqslant 1 + \max\big(\mathrm{cd}(L)\big) - \min\big(\mathrm{cd}(L)\big) = 1 + \Delta(L),\end{aligned}$$

as required. □

Corollary 3.9. *Let N be the nilradical of a solvable Lie algebra L and suppose that $|\mathrm{cd}(L)| = 1$. Then $\dim(L/N) \leqslant 1$ and, if N is non-abelian, then $\mathrm{cd}(N) = \mathrm{cd}(L)$.*

Proof. For any Lie algebra X, we know that $|\mathrm{cd}(X)| = 1$ if and only if $\Delta(X) = 0$. The result is now immediate from Theorems 3.8 and 3.7. □

4. One centralizer dimension in nonsolvable algebras

Our goal in this section is to prove the following theorem.

Theorem 4.1. *Let L be a Lie algebra and assume that $|\mathrm{cd}(L)| = 1$. If L is not solvable, then L is the direct sum of its center $\mathbf{Z}(L)$ and a copy of $\mathfrak{sl}_2(\mathbb{C})$ and, in particular, $\dim(L/\mathbf{Z}(L)) = 3$.*

We suppose throughout this section that L is a nonsolvable Lie algebra such that $\mathrm{cd}(L) = \{n\}$. Let R denote the radical of L and note that the semisimple algebra L/R has rank 1 by Theorem C, and thus $L/R \cong \mathfrak{sl}_2(\mathbb{C})$ and, in particular, R has codimension 3 in L. By Levi's theorem, there is a subalgebra $S \subseteq L$ such that $S \cong \mathfrak{sl}_2(\mathbb{C})$, and we see that $R \cap S = 0$. To complete the proof of Theorem 4.1, therefore, it suffices to show that R is central in L.

Now R is a module for $S \cong \mathfrak{sl}_2(\mathbb{C})$, and since S is semisimple, we know by Weyl's theorem that R is a direct sum of simple S-modules (see [1, Theorem 6.3]). Furthermore, according to Section 7 of [1], the isomorphism types of the simple S-modules are comparatively easy to describe.

Fix a basis $\{x, h, y\}$ for S, where $[h, x] = 2x$, $[h, y] = -2y$, and $[x, y] = h$. Then for each integer $m \geqslant 0$, there is exactly one isomorphism type of simple S-module $M = M_m$ of dimension $m + 1$. This module has a basis $\{v_i\}$, where $0 \leqslant i \leqslant m$, and where the action of S is given as follows. Each basis vector v_i is an eigenvector for h and we have $h \cdot v_i = (m - 2i)v_i$ for $0 \leqslant i \leqslant m$. The element $y \in S$ acts according to the formula $y \cdot v_i = (i + m)v_{i+1}$ for $0 \leqslant i < m$ and $y \cdot v_m = 0$. Finally, the action of the x is given by $x \cdot v_0 = 0$ and $x \cdot v_i = (m - i + 1)v_{i-1}$ for $i > 0$.

In particular, we have the following lemma.

Lemma 4.2. *Let M be a simple S module of dimension d, where $S \cong \mathfrak{sl}_2(\mathbb{C})$, and let $\{x, h, y\}$ be the basis of S as in the previous discussion. Then $\dim(\mathbf{C}_M(x)) = 1 = \dim(\mathbf{C}_M(y))$ and $\dim(\mathbf{C}_M(h))$ is 1 or 0, according to whether d is odd or even.*

Corollary 4.3. *Let L, n, R, and S be as before, and write $R = \sum M_i$, a direct sum of simple S-modules. Then there are exactly $n - 1$ summands M_i and each of them has odd dimension.*

Proof. Let $x, h \in S$ be as before, and note that $L = \sum M_i + S$ is a direct sum of subspaces, each of which is invariant under both x and h. Also, $\dim(\mathbf{C}_S(x)) = 1 = \dim(\mathbf{C}_S(h))$ and, in particular, neither x nor h is central in L. Since we are assuming that $\mathrm{cd}(L) = \{n\}$, we have $\dim(\mathbf{C}_L(x)) = n = \dim(\mathbf{C}_L(h))$, and so $\sum \dim(\mathbf{C}_{M_i}(x)) = n - 1 = \sum \dim(\mathbf{C}_{M_i}(h))$. In each direct summand M_i, however, we see by Lemma 4.2 that $\dim(\mathbf{C}_{M_i}(x)) \geqslant \dim(\mathbf{C}_{M_i}(h)) = 1$, and thus there are exactly $n - 1$ summands, as required. Also, we see that equality must hold for each of these simple S-modules M_i, and it follows by Lemma 4.2 that M_i must have odd dimension. □

Proof of Theorem 4.1. Let R, n, S, and the basis $\{x, h, y, \}$ of S be as in the previous discussion, and let N be the nilradical of R. Our first step is to show that $\dim(R/N) \leqslant 1$; then we show that R is nilpotent, and finally, to complete the proof, we show that R is central in L.

Let $R \subseteq B \subseteq L$, where B is a subalgebra and $\dim(B/R) = 2$. (For example, we could take B to be $R + \mathbb{C}h + \mathbb{C}x$ or $R + \mathbb{C}h + \mathbb{C}y$.) Then B/R is solvable, and hence B is solvable, and we let P be the nilradical of B. Since B/R is isomorphic to a 2-dimensional subalgebra of $\mathfrak{sl}_2(\mathbb{C})$, it is not nilpotent, and it follows that $P + R < B$, and thus $\dim(R/(R \cap P)) < \dim(B/P)$.

If b is a noncentral element of B, then b is not central in L, and hence since we are assuming that $\mathrm{cd}(L) = \{n\}$, it follows that $\dim(\mathbf{C}_L(b)) = n$. Since $\mathbf{C}_B(b) = \mathbf{C}_L(b) \cap B$ and B has codimension 1 in L, we see that the only possibilities for $\dim(\mathbf{C}_B(b))$ are n and $n - 1$. Thus $\mathrm{cd}(B) \subseteq \{n-1, n\}$, and so $\Delta(B) \leqslant 1$. By Theorem 3.8, therefore, we conclude that $\dim(B/P) \leqslant 2$, and thus $\dim(R/(R \cap P)) \leqslant 1$. But $R \cap P$ is a nilpotent ideal of R, and so if R is not nilpotent, we have $R \cap P = N$, and thus $\dim(R/N) \leqslant 1$, as required.

Suppose now that R is not nilpotent, so that $\dim(R/N) = 1$ and $N = R \cap P$ is an ideal of B. Since we can choose B to contain the elements h and x or the elements h and y, and in either case, N is an ideal of B, it follows that N is actually an ideal of L.

By Weyl's theorem, R is completely reducible as an S-module, and we know that N is a submodule of codimension 1. It follows that we can write $R = N + A$, where $\dim(A) = 1$ and A is an S-module. But S acts trivially on its unique (up to isomorphism) module of dimension 1, and hence if we choose $0 \neq a \in A$, we can write $S \subseteq \mathbf{C}_L(a)$.

Now let $R = R_0(a) + R_1(a) = U + V$ be the Fitting decomposition of R with respect to $\mathrm{ad}\, a$, so that the action of a is nilpotent on U and is invertible on V. Also, since S centralizes a, we observe that U and V are S-submodules of R. Since $a \notin N$, we see by Lemma 3.5 that the action of $\mathrm{ad}\, a$ on N is not nilpotent, and thus $N \nsubseteq U$ and, in particular, $V > 0$.

Note that the action of the element $x \in S$ on every S-module is nilpotent and, in particular, $\dim(\mathbf{C}_V(x)) > 0$. Since a acts invertibly on V and x and a commute, we deduce that the action of $x + a$ on V is invertible, and hence $\dim(\mathbf{C}_V(x + a)) = 0$.

Since S, U, and V are all invariant under both x and $x + a$, it follows that

$$\dim\bigl(\mathbf{C}_L(x)\bigr) = \dim\bigl(\mathbf{C}_U(x)\bigr) + \dim\bigl(\mathbf{C}_V(x)\bigr) + \dim\bigl(\mathbf{C}_S(x)\bigr) \quad \text{and}$$
$$\dim\bigl(\mathbf{C}_L(x+a)\bigr) = \dim\bigl(\mathbf{C}_U(x+a)\bigr) + \dim\bigl(\mathbf{C}_V(x+a)\bigr) + \dim\bigl(\mathbf{C}_S(x+a)\bigr).$$

Note that $\dim(\mathbf{C}_S(x)) = 1 = \dim(\mathbf{C}_S(x + a))$ and, in particular, neither of these elements is central in L, so that we have $\dim(\mathbf{C}_L(x)) = n = \dim(\mathbf{C}_L(x + a))$. Since we have seen that $\dim(\mathbf{C}_V(x + a)) < \dim(\mathbf{C}_V(x))$, it follows that $\dim(\mathbf{C}_U(x)) < \dim(\mathbf{C}_U(x + a))$.

The action of a on U is nilpotent, and so we can define a series of subspaces $U = U_0 > U_1 > \cdots > U_m = 0$ by setting $U_{i+1} = [U_i, a]$. We observe that since S centralizes a, each of the subspaces U_i is an S-submodule, and hence by Weyl's theorem, U is isomorphic as an S-module to the direct sum of the factors U_i/U_{i+1} for $0 \leqslant i < m$. Since $x \in S$, it follows that $\dim(\mathbf{C}_U(x))$ is exactly the sum of the dimensions of the centralizers of the action of x on each factor. But a acts trivially on each factor, and thus the actions of x and $x + a$ on the factors are identical. Clearly, however, $\dim(\mathbf{C}_U(x + a))$ is at most equal to the sum of the dimensions of the centralizers of $x + a$ on the factors, and thus $\dim(\mathbf{C}_U(x + a)) \leqslant \dim(\mathbf{C}_U(x))$. This contradicts the inequality that was established in the previous paragraph, and so we conclude that R is nilpotent.

We can now begin our proof that R is central in L. We know that $R = \sum M_i$ is a direct sum of simple S-modules, and by Lemma 4.2, each summand has odd dimension. On each such module $M = M_i$, the action of the element $h \in S$ is diagonal and its eigenvalues are all of the even integers between $-2k$ and $2k$, inclusive, where $\dim(M) = 2k + 1$. Also, we recall that the element $x \in S$ centralizes the h-eigenvectors in M corresponding to the maximum eigenvalue $2k$.

Given any integer k, write W_k to denote the (possibly zero) h-eigensubspace of R corresponding to the eigenvalue k. Let $m \geqslant 0$ be the maximum (necessarily even) integer such that $W_m > 0$ and note that $W_m \subseteq \mathbf{C}_L(x)$. Also, by Lemma 4.2, the number of S-simple direct summands of R is exactly $n - 1$, where $\mathrm{cd}(L) = \{n\}$, and thus $\dim W_0 = n - 1$. Finally, we remark that for each integer k, it is easy to show that $[W_k, W_0] \subseteq W_k$.

Let $R = R^1 > R^2 > \cdots > 0$ be the lower central series for R. Since $0 < W_m = W_m \cap R^1$, there is some maximum positive integer t that such that $W_m \cap R^t > 0$, and we fix a nonzero element $a \in W_m \cap R^t$. If $w \in W_0$ is arbitrary, we have $[a, w] \in W_m$ and $[a, w] \in [R^t, R] = R^{t+1}$, and thus, by the choice of t, we see that $[a, w] = 0$. We have shown, therefore, that $W_0 \subseteq \mathbf{C}_L(a)$.

If $m > 0$, then $a \notin W_0$, and hence $\dim(\mathbf{C}_R(a)) \geqslant 1 + \dim(W_0) = n$. But also $x \in S$ centralizes a since $a \in W_m$. Thus $\dim(\mathbf{C}_L(a)) > n$, and we conclude that $a \in \mathbf{Z}(L)$. But $[h, a] = ma \neq 0$, and this contradiction shows that $m = 0$. We conclude that all of the S-simple direct summands of R have dimension 1, and thus $[R, S] = 0$.

Now let $r \in R$. Since r centralizes S, we see that $\mathbf{C}_S(r + x) = \mathbb{C}x = \mathbf{C}_S(x)$ and, in particular, x and $r + x$ are not central in L. Thus $\dim(\mathbf{C}_L(r + x)) = n = \dim(\mathbf{C}_L(x))$, and since both $r + x$ and x stabilize both R and S, it follows that $\dim(\mathbf{C}_R(r + x)) = \dim(\mathbf{C}_R(x))$. But we know that x centralizes R, and it follows that $r + x$ also centralizes R, and thus r centralizes R. Since $r \in R$ was arbitrary, we conclude that R is abelian, and thus $R = \mathbf{Z}(L)$, as desired. □

5. One centralizer dimension in nonnilpotent algebras

We continue to assume that L is a Lie algebra such that $|\mathrm{cd}(L)| = 1$. In Section 4, we showed that if L is nonsolvable, then $L/\mathbf{Z}(L) \cong \mathfrak{sl}_2(\mathbb{C})$ and, in particular, $\dim(L/\mathbf{Z}(L)) = 3$. Here, we complete the proof of Theorem A by determining all possibilities (up to isomorphism) for $L/\mathbf{Z}(L)$ if L is solvable but not nilpotent.

Theorem 5.1. *Let L be a solvable nonnilpotent Lie algebra and assume that $|\mathrm{cd}(L)| = 1$. Then $L/\mathbf{Z}(L)$ is isomorphic either to the unique non-abelian 2-dimensional algebra or to the 3-dimensional algebra with basis $\{x, y, a\}$ and relations $[a, x] = x$, $[a, y] = -y$, and $[x, y] = 0$.*

We remark that in the nonsolvable case discussed in Section 4, the Lie algebra L splits over $\mathbf{Z}(L)$, and thus L is the direct sum of $\mathfrak{sl}_2(\mathbb{C})$ with an abelian algebra, and furthermore, every such direct sum has the property that $|\mathrm{cd}(L)| = 1$. If $L/\mathbf{Z}(L)$ is the non-abelian 2-dimensional algebra, one can show that in this situation too, L must split over $\mathbf{Z}(L)$, and thus L is the direct sum of the non-abelian 2-dimensional Lie algebra with an abelian algebra. Here too, it is easy to see that every such direct sum has the desired property on centralizer dimensions.

In the remaining case, where L is solvable and $\dim(L/\mathbf{Z}(L)) = 3$, we see that L cannot split over its center because the algebra $L/\mathbf{Z}(L)$ described in Theorem 5.1 does not satisfy the condition on centralizer dimensions. As we shall see, this case does occur, and there is a certain 4-dimensional algebra S that satisfies the conditions. It is not hard to see that in general, L must be the direct sum of S and an abelian algebra, and that every such direct sum satisfies the condition on centralizer dimensions.

Proof of Theorem 5.1. Let N be the nilradical of L. Since $N < L$, it follows by Corollary 3.9 that $\dim(L/N) = 1$. Also, since L is not nilpotent, we can choose $a \in L$ such that $\mathrm{ad}\, a$ is not nilpotent, and we write $C = \mathbf{C}_L(a)$. Then $a \notin N$ and, in particular, a is not central in L and $\dim(C)$ is the unique member of $\mathrm{cd}(L)$.

Suppose first that N is abelian. Certainly, N is not central in L, and thus N is the full centralizer in L of one of its elements and $\dim(N) \in \mathrm{cd}(L)$. Thus $\dim(C) = \dim(N) = \dim(L) - 1$, and it follows that $N \cap C$ has codimension at most 2 in L. But $\mathbf{C}_L(C \cap N) \supseteq N + \mathbb{C}a = L$, and thus $C \cap N \subseteq \mathbf{Z}(L)$ and the result follows in this case.

We can now suppose that N is non-abelian. If $z \in N \cap C$, then $a \in \mathbf{C}_L(z)$, and so $\mathbf{C}_L(z) > \mathbf{C}_N(z)$. Since $\mathrm{cd}(N) = \mathrm{cd}(L)$ by Corollary 3.9, it follows that $z \in \mathbf{Z}(L)$, and this shows that $N \cap C = \mathbf{Z}(L)$. Also, since $\dim(C) = 1 + \dim(C \cap N) = 1 + \dim(\mathbf{Z}(L))$ is the unique member of $\mathrm{cd}(L)$, it follows that for every noncentral element $b \in L$, we have $\mathbf{C}_L(b) = \mathbf{Z}(L) + \mathbb{C}b$.

Next, we decompose L as a direct sum of nonzero weight spaces $L_\lambda = L_\lambda(a)$ as λ runs over the set Λ of weights for a. (These, of course, are just the eigenvalues of $\mathrm{ad}\, a$.) Note that $0 \in \Lambda$ and also there is at least one nonzero weight since $\mathrm{ad}\, a$ is not nilpotent. We know that $[L_\lambda, L_\mu] \subseteq L_{\lambda+\mu}$, for all choices of eigenvalues λ and μ. (Note that if $\lambda + \mu \notin \Lambda$, then $L_{\lambda+\mu} = 0$.) In particular, we have $[L_0, L_\lambda] \subseteq L_\lambda$, and hence L_0 is a subalgebra of L. Also, $L_0 \supseteq \mathbf{C}_L(a) = \mathbf{Z}(L) + \mathbb{C}a$ and $L_0 \cap L_\lambda = 0$ if $\lambda \neq 0$.

We show next that $L_0 = \mathbf{Z}(L) + \mathbb{C}a$. Let λ be a nonzero eigenvalue of $\mathrm{ad}\, a$ and note that L_λ is a nonzero module for the solvable Lie algebra L_0. There must exist, therefore, a nonzero element $t \in L_\lambda$ such that $[L_0, t] \subseteq \mathbb{C}t$, and it follows that $L_0 \cap \mathbf{C}_L(t)$ has codimension at most 1 in L_0. Since $0 \neq t \in L_\lambda$, we see that $t \notin L_0$, and thus t is noncentral and $\mathbf{C}_L(t) = \mathbf{Z}(L) + \mathbb{C}t$. It follows that $L_0 \cap \mathbf{C}_L(t) = \mathbf{Z}(L)$, and since this intersection has codimension at most 1 in L_0, we conclude that $L_0 = \mathbf{Z}(L) + \mathbb{C}a$.

Again let $0 \neq \lambda \in \Lambda$ and suppose that there exists $\mu \in \Lambda$ such that μ is not an integer multiple of λ. If $\mu + \lambda \in \Lambda$, we can replace μ by $\mu + \lambda$, which is also not a multiple of λ. Since Λ is finite, we can repeat this process until we have $\mu + \lambda \notin \Lambda$. Then $[L_\mu, L_\lambda] = 0$, and thus if $0 \neq t \in L_\lambda$, we have $0 < L_\mu \subseteq \mathbf{C}_L(t) = \mathbf{Z}(L) + \mathbb{C}t \subseteq L_0 + L_\lambda$. This is impossible, however, since μ is different from 0 and λ and the sum of the weight spaces is direct. We conclude from this contradiction that every member of Λ is an integer multiple of λ. It follows that if there exist two distinct nonzero members of Λ, then each of them must be an integer multiple of the other, and thus each is the negative of the other. We conclude that either $\Lambda = \{0, \lambda\}$ or $\Lambda = \{0, \lambda, -\lambda\}$.

We claim now that if $0 \neq \lambda \in \Lambda$, then $\dim(L_\lambda) = 1$. To see why this is true, let $0 \neq t \in L_\lambda$ and note that t is not central in L. Observe that $2\lambda \notin \Lambda$, and thus $[L_\lambda, L_\lambda] = 0$ and we have $L_\lambda \subseteq \mathbf{C}_L(t)$. Since $\mathbf{Z}(L)$ has codimension 1 in this space and $\mathbf{Z}(L) \cap L_\lambda = 0$, it follows that $\dim(L_\lambda) = 1$, as desired. Of course, if $-\lambda \in \Lambda$, then similar reasoning shows that $\dim(L_{-\lambda}) = 1$, and so in any case we have $\dim(L_{-\lambda}) \leqslant 1$.

Since $L = L_0 + L_\lambda + L_{-\lambda} = \mathbf{Z}(L) + \mathbb{C}a + L_\lambda + L_{-\lambda}$, we see that the codimension of $\mathbf{Z}(L)$ in L is at most 3, as required. Also, if $\dim(L/\mathbf{Z}(L)) = 3$, then there is a basis for $L/\mathbf{Z}(L)$ of the form $\{\bar{a}, \bar{x}, \bar{y}\}$, where $x \in L_\lambda$ and $y \in L_{-\lambda}$. If we replace a by a suitable scalar multiple, we can assume that $\lambda = 1$, and it is easy to see that $L/\mathbf{Z}(L)$ has the required form. □

In order to see that it really is possible to have $\dim(L/\mathbf{Z}(L)) = 3$ in Theorem 5.1, we construct a 4-dimensional Lie algebra S as follows. First, we let N be the unique non-abelian nilpotent 3-dimensional nilpotent Lie algebra with basis $\{x, y, z\}$, where z is central and $[x, y] = z$. If we let $\varphi : N \to N$ be the linear map defined by $\varphi(x) = x$, $\varphi(y) = -y$, and $\varphi(z) = 0$, it is routine to check that φ is a derivation of N. We can then define S to be the semidirect product $\mathbb{C}a + N$, where a acts on N according to the derivation φ. Thus S has the basis $\{a, x, y, z\}$ and we see that z is central in S and that $[x, y] = z$, $[a, x] = x$, and $[a, y] = -y$.

Theorem 5.2. *Let S be the 4-dimensional algebra defined above. Then S is solvable and nonnilpotent. Also,* $\mathrm{cd}(S) = \{2\}$ *and* $\mathbf{Z}(S) = \mathbb{C}z$.

Proof. Since N is nilpotent and S/N is abelian, it is clear that S is solvable. Write $Z = \mathbb{C}z$, and note that $Z \subseteq \mathbf{Z}(S)$. To show that $Z = \mathbf{Z}(S)$ and that S is not nilpotent, it suffices to check that $\mathbf{Z}(S/Z) = 0$. But $\{\bar{a}, \bar{x}, \bar{y}\}$ is a basis for $\overline{S} = S/Z$, and we have $[\bar{a}, \bar{x}] = \bar{x}$ and $[\bar{a}, \bar{y}] = -\bar{y}$, and from this information, it is trivial to check that $\mathbf{Z}(\overline{S}) = 0$, as required.

Now let t be a noncentral element of S and write $t = \alpha a + \beta x + \gamma y + \delta z$, where the coefficients are complex numbers and $(\alpha, \beta, \gamma) \neq (0, 0, 0)$. We want to show that $\dim(\mathbf{C}_S(t)) = 2$, and for this purpose we let $c \in \mathbf{C}_S(t)$ have the form $c = \lambda a + \mu x + \nu y$ and show that (λ, μ, ν) is a scalar multiple of (α, β, γ). We compute that

$$0 = [t, c] = (\alpha\mu - \beta\lambda)x + (\gamma\lambda - \alpha\nu)y + (\beta\nu - \gamma\mu)z.$$

It follows that

$$0 = \begin{vmatrix} \alpha & \beta \\ \lambda & \mu \end{vmatrix} = \begin{vmatrix} \alpha & \gamma \\ \lambda & \nu \end{vmatrix} = \begin{vmatrix} \beta & \gamma \\ \mu & \nu \end{vmatrix},$$

and so the rank of the matrix

$$\begin{bmatrix} \alpha & \beta & \gamma \\ \lambda & \mu & \nu \end{bmatrix}$$

is 1. This completes the proof. □

6. Nilpotent algebras

In this section we prove Theorem B. In fact, the following result is somewhat stronger than the theorem stated in the introduction.

Theorem 6.1. *Let L be a finite-dimensional nilpotent Lie algebra generated by elements t such that* $\dim(\mathbf{C}_L(t)) = \max(\mathrm{cd}(L))$. *Then the nilpotence class of L is at most* 3.

Proof. Let $L = L^1 > L^2 > \cdots > L^m > L^{m+1} = 0$ be the lower central series of L, where m is the nilpotence class, and assume that $m > 3$. We have $[L^{m-2}, L, L] = L^m > 0$, and thus $[L^{m-2}, L] \not\subseteq \mathbf{Z}(L)$. Since the set $\{t \in L \mid [L^{m-2}, t] \subseteq \mathbf{Z}(L)\}$ is a proper subalgebra of L, there must exist some element $v \in L$ that does not lie in this set and such that $\dim(\mathbf{C}_L(v)) = n$, where $n = \max(\mathrm{cd}(L))$. Choose $u \in L^{m-2}$ such that $[u, v] \notin \mathbf{Z}(L)$ and write $x = [u, v]$, so that $x \in L^{m-1}$. Recall that $[L^i, L^j] \subseteq L^{i+j}$ for all superscripts $i, j \geqslant 1$, and thus $[u, x] \in [L^{m-2}, L^{m-1}] \subseteq L^{2m-3}$. We are assuming that $m > 3$, and so we have $2m - 3 > m$, and thus $[u, x] \in L^{m+1} = 0$. We want to obtain a contradiction.

Let $S = [u, [v, L]]$ and $T = [v, L^{m-1}]$. We have

$$[L, x] = \big[L, [u, v]\big] \subseteq \big[u, [v, L]\big] + \big[v, [L, u]\big] \subseteq S + \big[v, L^{m-1}\big] = S + T,$$

and thus $\dim(S) + \dim(T) \geqslant \dim([L, x]) = \dim(L) - r$, where $r = \dim(\mathbf{C}_L(x)) \in \mathrm{cd}(L)$, and so $r \leqslant n$.

Since $S = (\mathrm{ad}\, u)(\mathrm{ad}\, v)L$, we can choose a subspace $A \subseteq L$ such that $(\mathrm{ad}\, u)(\mathrm{ad}\, v)$ maps A injectively onto S and, in particular, we have $\dim(A) = \dim(S)$. Similarly, $T = (\mathrm{ad}\, v)L^{m-1}$, and so we choose a subspace $B \subseteq L^{m-1}$ such that $(\mathrm{ad}\, v)$ maps B injectively onto T and $\dim(B) = \dim(T)$. Finally, let $C = \mathbf{C}_L(v)$ and recall that $\dim(C) = n$.

We claim that the sum $A + B + C$ is direct. First, observe that $C = \ker(\mathrm{ad}\, v)$ and that B was chosen so that $B \cap \ker(\mathrm{ad}\, v) = 0$. It follows that $B \cap C = 0$ and it suffices now to check that $A \cap (B + C) = 0$. Since $B \subseteq L^{m-1}$, we have $(\mathrm{ad}\, u)(\mathrm{ad}\, v)B = L^{m+1} = 0$ and B is contained in $\ker((\mathrm{ad}\, u)(\mathrm{ad}\, v))$. Also, $C = \ker(\mathrm{ad}\, v) \subseteq \ker((\mathrm{ad}\, u)(\mathrm{ad}\, v))$, and thus $B + C \subseteq \ker((\mathrm{ad}\, u)(\mathrm{ad}\, v))$. By the choice of A, however, we know that $A \cap \ker((\mathrm{ad}\, u)(\mathrm{ad}\, v)) = 0$, and thus the sum $A + B + C$ is direct, as claimed.

We now have

$$\begin{aligned} \dim(A + B + C) &= \dim(A) + \dim(B) + \dim(C) = \dim(S) + \dim(T) + n \\ &\geqslant (\dim(L) - r) + n \geqslant \dim(L), \end{aligned}$$

and thus $A + B + C = L$ and we can write $u = a + b + c$ with the obvious notation. Since we have seen that $B + C \subseteq \ker((\mathrm{ad}\, u)(\mathrm{ad}\, v))$, it follows that $(\mathrm{ad}\, u)(\mathrm{ad}\, v)a = (\mathrm{ad}\, u)(\mathrm{ad}\, v)u =$

$[u,[v,u]] = -[u,x] = 0$, where we recall that the last equality is a consequence of our assumption that $m > 3$. Since $(\mathrm{ad}\,u)(\mathrm{ad}\,v)$ is injective on A, we see that $a = 0$, and thus $u \in B + C$. But then $x = [u,v] \in [B,v] \subseteq [L^{m-1}, v] \subseteq L^m \subseteq \mathbf{Z}(L)$, and this is a contradiction. □

Appendix A

We present here a simplified proof of a somewhat strengthened form of Ishikawa's theorem for nilpotent groups. Recall that Ishikawa showed that if all noncentral classes of a finite nilpotent group G have equal sizes, then the nilpotence class of G is at most 3. It was pointed out by A. Mann that Ishikawa's argument could be modified to yield the same conclusion, that G has class at most 3, under the weaker hypothesis that G is generated by all of its noncentral elements that are in classes of the smallest possible size. (This, of course, is the group-theory analog of our Theorem 6.1, which was motivated by Mann's observation.)

As we mentioned in the introduction, a simplified proof of Ishikawa's theorem was circulated privately by the second author, and we present a version of it here. The argument is exactly parallel to the proof of Theorem 6.1 for Lie algebras, but as might be expected, the proof for groups is a bit more technical. Mann's stronger form of Ishikawa's theorem follows with no extra effort, and so we have included it the following theorem.

Theorem (Ishikawa). *Let G be a finite non-abelian nilpotent group. Let n be the size of the smallest noncentral conjugacy class of G and assume that G is generated by elements in classes of size n. Then the nilpotence class of G is at most* 3.

We begin with a brief review of some notation and basic facts. If G is any group, we define the lower central series by writing $G^1 = G$ and $G^r = [G^{r-1}, G]$ for $r > 1$, and we recall that $[G^i, G^j] \subseteq G^{i+j}$ for all $i, j \geqslant 1$. Now suppose that G is nilpotent of class m, which means that $G = G^1 > G^2 > \cdots > G^m > G^{m+1} = 1$. In this case, if $u \in G^i$, $v \in G^j$, $w \in G^k$, and $i + j + k = m$, then $[u,v,w] \in G^m \subseteq \mathbf{Z}(G)$. (Recall that we left associate in multiple commutators, so that by definition, $[u,v,w] = [[u,v],w]$.) Continuing with the assumption that $i + j + k = m$, we recall the Witt identity $[u,v,w][v,w,u][w,u,v] = 1$, which, of course, plays the role of the Jacobi identity in Lie algebras.

Proof of Ishikawa's theorem. Let m be the nilpotence class of G and assume that $m > 3$. Then $[G^{m-2}, G, G] = G^m > 1$, and hence $[G^{m-2}, G] \nsubseteq \mathbf{Z}(G)$. Since the centralizer modulo $\mathbf{Z}(G)$ of G^{m-2} is a proper subgroup of G, it must fail to contain some element v of G that lies in a class of size n. We have $[G^{m-2}, v] \nsubseteq \mathbf{Z}(G)$, and we can choose $u \in G^{m-2}$ such that the element $x = [u,v]$ is noncentral. But $x \in G^{m-1}$, and thus $[x,u] \in [G^{m-1}, G^{m-2}] \subseteq G^{2m-3}$. Since we are assuming that $m > 3$, we have $2m - 3 > m$, and thus $[x,u] \in G^{m+1} = 1$. We want to obtain a contradiction.

Now let $y \in G$ be arbitrary. We have $u \in G^{m-2}$ and, of course, $v, y \in G^1$, and so the Witt identity applies and we have $[v,u,y][y,v,u][u,y,v] = 1$. Since $[u,v] = x$, it follows that

$$[y,v,u][u,y,v] = [v,u,y]^{-1} = \left[x^{-1}, y\right]^{-1} = [x,y],$$

where the last equality holds because $[x, y] \in G^m \subseteq \mathbf{Z}(G)$.

Next, we define maps $V : G \to G$ and $U : G \to G$ by $gV = [g, v]$ and $gU = [g, u]$ for all $g \in G$. Then $[y, v, u] \in (G)VU$ and also, $[u, y, v] \in (G^{m-1})V$ since $u \in G^{m-2}$, and hence $[u, y] \in G^{m-1}$. If we set $S = (G)VU$ and $T = (G^{m-1})V$, it follows that the subset ST contains every commutator $[x, y] = x^{-1}x^y$ for $y \in G$. The number of such commutators is, of course, the size of the conjugacy class of the noncentral element x, and hence is at least n. It follows, therefore, that $|ST| \geqslant n$.

Now choose subsets $A \subseteq G$ and $B \subseteq G^{m-1}$ so that VU maps A injectively onto $(G)VU = S$ and V maps B injectively onto $(G^{m-1})V = T$. Also, assume, as we can, that $1 \in A$. We have $|A| = |S|$ and $|B| = |T|$, and thus $|A||B| \geqslant n$. Let $C = \mathbf{C}_G(v)$ and recall that v lies in a class of size n, so that $n = |G : C|$ and we have $|A||B||C| \geqslant n|C| = |G|$.

Now let $a \in A$, $b \in B$, and $c \in C$. We will show that

$$(cba)VU = (a)VU, \qquad (cb)V = (b)V \tag{$*$}$$

and it will follow from the fact that VU is injective on A and V is injective on B that the element cba uniquely determines the factors a, b, and c. Since v centralizes c, we have $[cb, v] = [b, v]$ and the second assertion of $(*)$ is immediate. Also $(cba)V = [cba, v] = [ba, v] = [b, v]^a[a, v]$. But $b \in G^{m-1}$, which yields $[b, v] \in G^m \subseteq \mathbf{Z}(G)$ and we have $[b, v]^a = [b, v]$. Thus $(cba)V = [b, v][a, v] = (bV)(aV)$. Again using the fact that $bV = [b, v]$ is central, we conclude that $(cba)VU = [(bV)(aV), u] = [(aV), u] = (a)VU$, as desired.

As we have remarked, it follows from $(*)$ that the element cba uniquely determines $a \in A$, $b \in B$, and $c \in C$, and thus $|CBA| = |C||B||A| \geqslant |G|$. We conclude that $CBA = G$ and, in particular, we can choose a, b, and c such that $u = cba$. As we have seen, the assumption that $m > 3$ yields $1 = [x, u] = [u, v, u] = (u)VU = (a)VU$, where the last equality follows from the first part of $(*)$. But also $(1)VU = 1$ and $1 \in A$, and since VU is injective on A, we conclude that $a = 1$ and $u = cb$. But then $x = [u, v] = (cb)V = bV$, which is central in G. This is the desired contradiction, and the proof is complete. □

References

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