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Vacuum Einstein metrics with bidimensional Killing leaves. I. Local aspects ☆

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Abstract

The solutions of vacuum Einstein's field equations, for the class of Riemannian metrics admitting a non-Abelian bidimensional Lie algebra of Killing fields, are explicitly described. They are parametrized either by solutions of a transcendental equation (the *tortoise equation*), or by solutions of a linear second order differential equation in two independent variables. Metrics, corresponding to solutions of the tortoise equation, are characterized as those that admit a 3-dimensional Lie algebra of Killing fields with bidimensional leaves.

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1. Introduction

In this paper we describe in an exact form local solutions (metrics) of the vacuum Einstein equations assuming that they admit a Lie algebra $\mathcal{G}$ of Killing vector fields such that:

I. the distribution $\mathcal{D}$, generated by the vector fields belonging to $\mathcal{G}$, is bidimensional,
II. the distribution $\mathcal{D}^{\perp}$, orthogonal to $\mathcal{D}$, is completely integrable and transversal to $\mathcal{D}$.

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Global, in a sense, solutions of the Einstein equations constructed on the basis of the local solutions found in this paper are discussed in the subsequent one. There can occur two qualitatively different cases according to whether the dimension of $\mathcal{G}$ is 2 or 3. Both of them, however, have an important feature in common, which makes reasonable to study them together. Namely, all manifolds satisfying the assumptions I and II are in a sense fibered over ζ-complex curves (see Section 7 and [10,11]).

$\dim \mathcal{G} = 2$ Recall that, up to isomorphisms, there are two bidimensional Lie algebras: Abelian and non-Abelian, which in what follows will be denoted by $\mathcal{A}_2$ and $\mathcal{G}_2$ respectively.
A metric g satisfying the assumptions I and II, with $\mathcal{G} = \mathcal{A}_2$ or $\mathcal{G}_2$, will be called *$\mathcal{G}$-integrable*.
The study of $\mathcal{A}_2$-integrable metrics were started by Belinsky, Geroch, Khalatnikov, Zakharov and others [3,4,7]. Some remarkable properties of the reduced, according to the above symmetry assumptions, vacuum Einstein equations were discovered in 1978. In particular, a suitable generalization of the Inverse Scattering Transform, allowed to integrate the equations and to obtain solitary wave solutions [4]. Some physical consequences of these reduced equations were analyzed in a number of works (see for instance [2,5]). This paper will be devoted to the analysis of $\mathcal{G}_2$-integrable metrics, for which some partial results can be found in [1,6,8].
In this case, the Killing fields "interact" non-trivially one another (for instance, $[X, Y] = Y$, for a suitable choice of the basis vectors in $\mathcal{G}$), while in the Abelian case these fields are absolutely free (i.e., $[X, Y] = 0$). Hence, it is natural to expect that the former case is more rigid, with respect to the latter, and, as such, it allows a more complete analysis. It occurs to be the case, namely, metrics in question are parametrized by solutions of a linear equation in two independent variables, which, in its turn, depends linearly on a choice of a ζ-harmonic function. Thus, this class of solutions has a "bilinear structure" and, hence, is subjected to two superposition laws.

$\dim \mathcal{G} = 3$ In this case, assumption II follows automatically from I and the local structure of this class of Einstein metrics can be explicitly described. Some well known exact solutions [9], such as, for instance, that of Schwarzschild, belong to this class.

Geometrical properties of solutions described in the paper will be discussed with more details separately.

In the paper, as it is usual, everything is assumed to be of C^∞ class and the following terminological and notational convention are adopted.

- manifolds are assumed to be connected and C^∞,
- *metric* refers to a non-degenerate symmetric $(0, 2)$ tensor field,
- *k-metric* refers to a metric on a k-dimensional manifold,
- the Lie algebra of all Killing fields of a metric g is denoted by $\mathcal{K}il(g)$ while the term *Killing algebra* refers to a subalgebra of $\mathcal{K}il(g)$,
- *integral submanifolds* of the distribution, generated by vector fields of a Killing algebra $\mathcal{G}$, are called *Killing leaves*,
- $\mathcal{A}_2$ stands for a bidimensional Abelian Lie algebra, while $\mathcal{G}_2$ for a non-Abelian one,
- a *$\mathcal{G}$-integrable* metric is a metric satisfying the assumptions I and II, with $\mathcal{G} = \mathcal{A}_2$ or $\mathcal{G}_2$,
- the elements of a matrix will be denoted with the corresponding lower case letter, for instance $\mathbf{A} = (a_{ij})$.

2. Metrics admitting a bidimensional Lie algebra $\mathcal{G}_2$ of Killing fields

For a given $s \in \mathbb{R}, s \neq 0$, we fix a basis $\{e, \varepsilon\}$ in $\mathcal{G}_2$ such that $[e, \varepsilon] = s\varepsilon$. It is defined uniquely up to transformations of the form

$$e \mapsto \lambda e + \mu\varepsilon, \varepsilon \mapsto \lambda^{-1}\varepsilon, \quad \lambda, \mu \in \mathbb{R},\ \lambda \neq 0.$$

The parameter s is introduced in order to include, into our subsequent analysis, the Abelian case $(s = 0)$ as well.

In what follows, it will be useful the following general fact.

Lemma 1. *Let g be a metric on a differential manifold M. If $X \neq 0$ and fX, $f \in C^\infty(M)$, are two of its Killing fields, then f is constant.*

Proof. The proof results from the formula

$$L_{fX}(g) = f L_X(g) + i_X(g)\, df, \tag{1}$$

where the second term in the right hand side is the *symmetric product* of two differential 1-forms, and $i_X(g)$ the natural insertion of X in g. Indeed, $L_X(g) = 0$ and $L_{fX}(g) = 0$ imply, in view of relation (1), $i_X(g)\, df = 0$. This shows that df vanishes at those points where $i_X(g) \neq 0$. Since g is non-degenerate, $i_X(g)$ vanishes exactly at the same points where X does. Therefore, $df = 0$, on $\operatorname{supp} X = \overline{\{a \in M \mid X_a \neq 0\}}$. On the other hand, if a Killing field vanishes on an open subset of M, then, obviously, it vanishes everywhere on M. For this reason $\operatorname{supp} X$ coincides with M and, so, $df = 0$ on M. □

Let g be a metric on a manifold M admitting $\mathcal{G}_2$ as a Killing algebra. Then, for the Killing vector fields X and Y corresponding, respectively, to e and ε, one has

$$[X, Y] = sY. \tag{2}$$

Denote by $\mathcal{D}$ the Frobenius distribution, possibly with singularities, generated by X and Y.

Proposition 2. *The distribution $\mathcal{D}$ is bidimensional and in a neighborhood of a non-singular point of $\mathcal{D}$ there exists a local chart (x_α) in M such that*

$$X = \partial_{n-1}, \qquad Y = e^{sx_{n-1}}\partial_n.$$

Proof. First of all, show that $\dim \mathcal{D} = 2$. Indeed, in view of the above lemma if locally $X = \phi Y$, then ϕ is constant and X and Y commute, in contradiction with Eq. (2). Thus, the vector Y_a and X_a are independent for almost all points $a \in M$, i.e., in an everywhere dense open subset M_0 of M. Choose now a function ϕ such that the fields X and ϕY commute. In view of Eq. (2), this is equivalent to $X(\phi) + s\phi = 0$. This equation admits, obviously, a solution in a neighborhood of any point $a \in M_0$.

In a local chart (y_μ) in which $X = \frac{\partial}{\partial y_{n-1}}$, $\phi Y = \frac{\partial}{\partial y_n}$, the equality $X(\phi) + s\phi = 0$ looks as $\frac{\partial \phi}{\partial y_{n-1}} + s\phi = 0$ and hence, $\phi = e^{-sy_{n-1}+\lambda}$ where the function λ does not depend on y_{n-1}. By passing now to coordinates (x_α) with $x_\alpha = y_\alpha$, $\alpha < n$, and $x_n = \beta(y_1, \ldots, y_{n-2}, y_n)$ one finds the desired result with β such that $\frac{\partial \beta}{\partial y_n} = e^{-\lambda}$. Indeed, since λ does not depend on y_{n-1}, the last equation admits a solution not depending on y_{n-1}. □

Definition 1. A chart of the kind introduced in the above proposition will be called *semi-adapted* (with respect to X, Y).

All metrics g admitting the $\{X, Y\}$ Killing algebra, i.e., such that $L_Y(g) = L_X(g) = 0$, are characterized by the following proposition.

Proposition 3. *An n-metric g admits the vector fields X and Y as Killing fields iff in a semi-adapted chart it has the following block matrix form*

$$M_C(g) = \begin{pmatrix} (g_{ij}) & (sm_i x_n + l_i) & (-m_i) \\ (sm_i x_n + l_i)^T & s^2\lambda x_n^2 - 2s\mu x_n + \nu & -s\lambda x_n + \mu \\ (-m_i)^T & -s\lambda x_n + \mu & \lambda \end{pmatrix}$$

where $C = \{dx_\mu\}$, and $g_{ij}, m_i, l_i, \lambda, \mu, \nu$, are functions of x_l, $1 \leqslant l \leqslant n-2$.

Proof. Indeed, the invariance with respect to X shows that the components of the metric do not depend on x_{n-1} while the invariance with respect to Y is equivalent to

$$\partial_n g_{ij} = 0, \quad \forall i, j \leqslant n-2, \tag{3}$$

$$\partial_n g_{n-1n-1} + s g_{nn-1} = 0, \tag{4}$$

$$\partial_n g_{n-1n} + s g_{nn} = 0, \tag{5}$$

$$\partial_n g_{nn} = 0, \tag{6}$$

$$\partial_n g_{in-1} + s g_{in} = 0, \tag{7}$$

$$\partial_n g_{in} = 0. \tag{8}$$

Eq. (3) tells that, for $i, j < n-1$, the components g_{ij} do not depend also on x_n, while Eqs. (4), (5) and (6), imply that, for $a, b = n-1, n$

$$(g_{ab}) = \begin{pmatrix} s^2\lambda x_n^2 - 2s\mu x_n + \nu & -s\lambda x_n + \mu \\ -s\lambda x_n + \mu & \lambda \end{pmatrix}, \tag{9}$$

where λ, μ and ν depend only on the coordinates x_i.

Eqs. (7) and (8) have the solution

$$(g_{in-1}, g_{in}) = \big(sm_i x_n + l_i(x_j), -m_i(x_j)\big),$$

where l_i and m_i are arbitrary functions. □

For further computations it is more convenient to work with a basis, say $\{e_i\}$, of vector fields invariant with respect to the Killing algebra. It is easy to see that all such fields are linear combinations of

$$e_i = \partial_i, \qquad e_{n-1} = \partial_{n-1} + s x_n \partial_n, \qquad e_n = -\partial_n \tag{10}$$

whose coefficients are $\mathcal{G}_2$-invariant functions, i.e., not depending on x_{n-1}, x_n. So, the set (10) can be taken as such a basis. Obviously, the basis of differential 1-forms $\Theta = \{\vartheta^i\}$ dual to $\{e_i\}$

$$\vartheta^i = dx_i, \qquad \vartheta^{n-1} = dx_{n-1}, \qquad \vartheta^n = s x_n dx_{n-1} - dx_n \tag{11}$$

is also $\mathcal{G}_2$-invariant. The bases (10), (11) are "slightly" non-holonomic because in the relations

$$[e_\mu, e_\nu] = C^\alpha_{\mu\nu} e_\alpha, \qquad d\vartheta^\alpha = -\frac{1}{2} C^\alpha_{\mu\nu} \vartheta^\mu \wedge \vartheta^\nu,$$

all the structure constants $C^\alpha_{\mu\nu}$ are vanishing, except C^n_{n-1n}, which equals $-s$. They will be called *non-holonomic semi-adapted*.

The expression of the metric of Proposition 3 in terms of the basis (11) is

$$g = g_{ij}\vartheta^i\vartheta^j + \lambda\vartheta^n\vartheta^n + \nu\,\vartheta^{n-1}\vartheta^{n-1} - 2\mu\vartheta^{n-1}\vartheta^n + 2l_i\vartheta^i\vartheta^{n-1} + 2m_i\vartheta^i\vartheta^n.$$

Corollary 4. *An n-metric g admits the vector fields X and Y as Killing fields iff its components, in a semi-adapted non-holonomic basis Θ, do not depend on x_{n-1} and x_n. The matrix of g with respect to the basis Θ is*

$$\mathbf{M}_\Theta(g) = \begin{pmatrix} (g_{ij}) & (l_i) & (m_i) \\ (l_i)^T & \nu & -\mu \\ (m_i)^T & -\mu & \lambda \end{pmatrix}.$$

3. Killing leaves

The assumption II of the introduction imposed on the metrics g considered in this paper allows, obviously, to construct semi-adapted charts, $\{x_i\}$, such that the fields $e_i = \frac{\partial}{\partial x_i}$, $i = 1, \dots, n-2$, belong to $\mathcal{D}^\perp$. In such a chart, called from now on, *adapted*, the components l_i's and m_i's vanish. The corresponding non-holonomic semi-adapted bases will be called *non-holonomic adapted*.

We will call *orthogonal leaf* an integral (bidimensional) submanifold of $\mathcal{D}^\perp$. Since $\mathcal{D}^\perp$ is assumed to be transversal to $\mathcal{D}$, the restriction of g to any Killing leaf, say S, is non-degenerate. So, $(S, g|_S)$ is a homogeneous bidimensional Riemannian manifold. In particular, the Gauss curvature $K = K(S)$ of the Killing leaves is constant. It can be easily computed by noticing that the matrix of the components of $g|_S$ with respect to the chart $\tilde{x} = x_{n-1}|_S$, $\tilde{y} = x_n|_S$ is

$$\mathbf{M}_{(d\tilde{x}, d\tilde{y})}(g|_S) = \begin{pmatrix} s^2\tilde{\lambda}\tilde{y}^2 - 2s\tilde{\mu}\tilde{y} + \tilde{\nu} & -s\tilde{\lambda}\tilde{y} + \tilde{\mu} \\ -s\tilde{\lambda}\tilde{y} + \tilde{\mu} & \tilde{\lambda} \end{pmatrix},$$

where the symbol "*tilde*" refers to the restriction to S and $\tilde{\lambda}$, $\tilde{\mu}$, and $\tilde{\nu}$ are constants according to Proposition 3. The result is

$$K(S) = \frac{\tilde{\lambda}s^2}{\tilde{\mu}^2 - \tilde{\lambda}\tilde{\nu}}, \qquad \tilde{\lambda}\tilde{\nu} - \tilde{\mu}^2 = \det \mathbf{M}_{(d\tilde{x}, d\tilde{y})}(g|_S).$$

This shows that the following cases can occur for $(S, g|_S)$.

1. $\tilde{\lambda} > 0$, $\tilde{\lambda}\tilde{\nu} - \tilde{\mu}^2 > 0$: $(S, g|_S)$ is a non-Euclidean plane, i.e., a bidimensional Riemannian manifold of negative constant Gauss curvature.
2. $\tilde{\lambda} < 0$, $\tilde{\lambda}\tilde{\nu} - \tilde{\mu}^2 > 0$: $(S, g|_S)$ is an "anti" non-Euclidean plane, i.e., is endowed with the metric of the previous case multiplied by -1.
3. $\tilde{\lambda}\tilde{\nu} - \tilde{\mu}^2 < 0$: $(S, g|_S)$ is any indefinite bidimensional metric of constant Gauss curvature.

Since the Killing leaves are parametrized by x_1, x_2, the function

$$K = K(x_1, \dots, x_{n-2}) = \frac{\lambda s^2}{\mu^2 - \lambda\nu}$$

describes the behavior of the Gauss curvature when passing from one Killing leave to another.

It is worth to note that the Killing algebra $\mathcal{G}_2$ is a subalgebra of the algebra $\mathcal{K}il(g_0)$, g_0 being a bidimensional metric of constant curvature (for instance, $g_0 = g|_S$).

If g_0 is positive (respectively, negative) definite and of positive (respectively, negative) Gauss curvature, then $\mathcal{K}il(g_0)$ is isomorphic to $so(3)$. But $so(3)$ does not admit bidimensional subalgebras at all. This explains why $g|_S$ cannot be a positively (respectively, negative) curved metric in the case (1) (respectively, (2)).

Similarly, if g_0 is a positive or negative definite flat metric, then $\mathcal{K}il(g_0)$ admits only Abelian bidimensional subalgebras. This explains why both positive and negative definite flat metrics are absent in the above list for $g|_S$.

In all other cases, the algebra $\mathcal{K}il(g_0)$ admits bidimensional non-Abelian subalgebras.

More exactly, if g_0 is not flat, then $\mathcal{K}il(g_0)$ is isomorphic to $so(2,1)$. Let $\mathbf{g}$ be the Killing form of $so(2,1)$. Then, the tangent planes to the isotropic cone of $\mathbf{g}$ exhaust the bidimensional non-Abelian Lie subalgebras of $so(2,1)$. If g_0 is flat and, thus, indefinite, then any bidimensional subspace of the algebra $\mathcal{K}il(g_0)$ different from its *commutator*, which is Abelian, is a non-Abelian subalgebra.

It is not difficult to describe the algebra $\mathcal{K}il(g|_S)$ in the semi-adapted coordinates $(\tilde{x}, \tilde{y})$. A direct computation shows that $\mathcal{K}il(g_0)$ has the following basis:

$$\widetilde{X} = \partial_{\tilde{x}}, \qquad \widetilde{Y} = e^{s\tilde{x}}\partial_{\tilde{y}}, \qquad \widetilde{Z} = e^{-s\tilde{x}}\big[2(s\tilde{\lambda}\tilde{y} - \tilde{\mu})\partial_{\tilde{x}} + \big(s^2\tilde{\lambda}\tilde{y}^2 - 2s\tilde{\mu}\tilde{y} + \tilde{\nu}\big)\partial_{\tilde{y}}\big],$$
$$\big[\widetilde{X}, \widetilde{Y}\big] = s\widetilde{Y}, \qquad \big[\widetilde{X}, \widetilde{Z}\big] = -s\widetilde{Z}, \qquad \big[\widetilde{Y}, \widetilde{Z}\big] = 2s\tilde{\lambda}\widetilde{X}.$$

In the case $\lambda = 0$, the metric $g|_S$ is flat indefinite and it is convenient to identify $(S, g|_S)$ with the standard plane $(\mathbb{R}^2, d\xi^2 - d\eta^2)$, $\mathbb{R}^2 = \{(\xi, \eta)\}$. To do that it is necessary to choose a bidimensional non-commutative subalgebra in $\mathcal{K}il(d\xi^2 - d\eta^2)$ (they are all equivalent). For instance, by choosing $Y_0 = \partial_\xi + \partial_\eta$, $X_0 = -\eta\partial_\xi - \xi\partial_\eta$, we have $[X_0, Y_0] = Y_0$, $X_0, Y_0 \in \mathcal{K}il(d\xi^2 - d\eta^2)$ and, for $s \neq 0$, one can identify the quadruple $(S,\ 2(d\tilde{x}\, d\tilde{y} - \tilde{y}\, d\tilde{x}^2),\ X|_S,\ Y|_S)$ with $(\mathbb{R}^2,\ d\xi^2 - d\eta^2,\ X_0, Y_0)$.

The simply connected Lie group G corresponding to $\mathcal{G}$ is isomorphic to the group of affine transformations of $\mathbb{R}$. Then, both S and $\mathbb{R}^2$ are diffeomorphic to G as homogeneous G-spaces and the above identification of them is an equivalence of G-spaces.

The Killing form of $\mathcal{G}$ determines naturally a symmetric covariant tensor field on the G-space G which is identified with $d\tilde{x}^2$ on S and with $(\frac{d\xi - d\eta}{\xi - \eta})^2$ on $\mathbb{R}^2$. We will continue to call it *Killing form*. Thus, in the above identification the metric $g|_S$ for $\lambda = 0$ and $s = 0$ corresponds to

$$\tilde{\mu}\big(d\xi^2 - d\eta^2\big) + \tilde{\nu}\left(\frac{d\xi - d\eta}{\xi - \eta}\right)^2. \tag{12}$$

This representation of the metric $g|_S$ will be used to describe global solutions of the Einstein equations in Section 5.

4. The Ricci tensor field

In the following we will consider 4-dimensional manifolds and will use the following convention for the indices: Greek letters take values from 1 to 4; the first Latin letters take values from 3 to 4, while i, j from 1 to 2.

Let g be a $\mathcal{G}_2$-integrable 4-metric. The results of the previous sections allow to choose a non-holonomic adapted basis Θ such that the matrix $\mathbf{M}_\Theta(g)$ associated to g is of the form

$$\mathbf{M}_\Theta(g) = \begin{pmatrix} \mathbf{F} & \mathbf{0} \\ \mathbf{0} & \mathbf{H} \end{pmatrix} \tag{13}$$

where $\mathbf{F}$ and $\mathbf{H}$ are 2×2 matrices whose elements depend only on x_1 and x_2. We will distinguish two cases according to whether $\mathbf{F}$, i.e., the matrix associated to the metric restricted to $\mathcal{D}^\perp$, has negative or positive determinant.

- $\det \mathbf{F} < 0$. In this case, owing to the bidimensionality of $\mathcal{D}^\perp$, and the independence of $\mathbf{F}$ on x_3 and x_4, the coordinates x_1 and x_2, can be further specified to be characteristic coordinates on any integral submanifold of $\mathcal{D}^\perp$, so that, without changing the properties of $\mathbf{M}_\Theta(g)$ in (13), $\mathbf{F}$ takes the following form
$$\mathbf{F} = \begin{pmatrix} 0 & f \\ f & 0 \end{pmatrix}.$$
- $\det \mathbf{F} > 0$. Similarly, in this case, in some isothermal coordinates, the matrix $\mathbf{F}$ gets the form
$$\mathbf{F} = \begin{pmatrix} f & 0 \\ 0 & f \end{pmatrix}.$$

Thus, we have:

Proposition 5. *A 4-metric g, is $\mathcal{G}_2$-integrable iff there exists a non-holonomic adapted basis Θ such that the matrix $M_\Theta(g)$ of g takes one of the following block forms, according to whether* $\det \mathbf{F} < 0$ *or* $\det \mathbf{F} > 0$.

$$\mathbf{M}_\Theta(g) = \begin{pmatrix} \begin{matrix} 0 & f \\ f & 0 \end{matrix} & \mathbf{0} \\ \mathbf{0} & \mathbf{H} \end{pmatrix}, \qquad \mathbf{M}_\Theta(g) = \begin{pmatrix} \begin{matrix} f & 0 \\ 0 & f \end{matrix} & \mathbf{0} \\ \mathbf{0} & \mathbf{H} \end{pmatrix}, \qquad \mathbf{H} = \begin{pmatrix} \nu & -\mu \\ -\mu & \lambda \end{pmatrix}$$

λ, μ, ν being arbitrary functions of x_i. In the corresponding adapted holonomic basis $C = \{dx^\mu\}$ we have

$$\mathbf{M}_C(g) = \begin{pmatrix} \begin{matrix} 0 & f \\ f & 0 \end{matrix} & \mathbf{0} \\ \mathbf{0} & \overline{\mathbf{H}} \end{pmatrix}, \qquad \mathbf{M}_C(g) = \begin{pmatrix} \begin{matrix} f & 0 \\ 0 & f \end{matrix} & \mathbf{0} \\ \mathbf{0} & \overline{\mathbf{H}} \end{pmatrix},$$

where

$$\overline{\mathbf{H}} = \begin{pmatrix} s^2\lambda x_4^2 - 2s\mu x_4 + \nu & -s\lambda x_4 + \mu \\ -s\lambda x_4 + \mu & \lambda \end{pmatrix}.$$

It is worth to observe that $\det \overline{\mathbf{H}} = \det \mathbf{H} = \lambda\nu - \mu^2$ is a functions of x_i's only.

In the following sections the explicit expressions of the components $R_{\mu\nu}$ of the Ricci tensor field in terms of the function f and of the elements h_{ab} of the matrix $\mathbf{H}$ in the adapted non-holonomic basis of Proposition 5 are found.

Recall that

$$R_{\mu\nu} = R^{\beta}_{\mu\nu\beta} = e_{[\nu}\big(\gamma^{\beta}_{\beta]\mu}\big) + \gamma^{\beta}_{[\nu\rho}\gamma^{\rho}_{\beta]\mu} - C^{\rho}_{\nu\beta}\gamma^{\beta}_{\rho\mu}$$

with the Christoffel symbols

$$\gamma^{\alpha}_{\mu\nu} = \frac{1}{2}g^{\alpha\sigma}\big(-e_\sigma(g_{\mu\nu}) + e_\mu(g_{\sigma\nu}) + e_\nu(g_{\sigma\mu})\big) - \frac{1}{2}\big(C^{\alpha}_{\nu\mu} + g^{\rho\alpha}g_{\sigma\mu}C^{\sigma}_{\nu\rho} + g^{\rho\alpha}g_{\sigma\nu}C^{\sigma}_{\mu\rho}\big).$$

It is easy too see that the $\gamma^{\alpha}_{\mu\nu}$'s and $R_{\mu\nu}$'s are first order polynomials in s and it is convenient to single out their constant terms $\Gamma^{\alpha}_{\mu\nu}$ and $S_{\mu\nu}$, respectively. More exactly, one has:

$$\gamma_\mu = \Gamma_\mu + \Lambda_\mu = \frac{1}{2}g^{-1}G_\mu + \Lambda_\mu$$

where $\gamma_\mu, \Gamma_\mu, \Lambda_\mu, G_\mu$ are matrices whose elements $\gamma^{\alpha}_{\mu\nu}, \Gamma^{\alpha}_{\mu\nu}, \Lambda^{\alpha}_{\mu\nu}, G_{\mu\alpha\nu}$, are defined by

$$\begin{aligned}
\Gamma^{\alpha}_{\mu\nu} &= \frac{1}{2}g^{\alpha\sigma}\big(-e_\sigma(g_{\mu\nu}) + e_\mu(g_{\sigma\nu}) + e_\nu(g_{\sigma\mu})\big),\\
\Lambda^{\alpha}_{\mu\nu} &= -\frac{1}{2}\big(C^{\alpha}_{\nu\mu} + g^{\rho\alpha}g_{\sigma\mu}C^{\sigma}_{\nu\rho} + g^{\rho\alpha}g_{\sigma\nu}C^{\sigma}_{\mu\rho}\big),\\
G_{\mu\sigma\nu} &= -e_\sigma(g_{\mu\nu}) + e_\mu(g_{\sigma\nu}) + e_\nu(g_{\sigma\nu}),\\
R_{\mu\nu} &= S_{\mu\nu} + T_{\mu\nu},
\end{aligned} \tag{14}$$

where

$$\begin{aligned}
S_{\mu\nu} &= e_{[\nu}\Gamma^{\beta}_{\beta]\mu} + \Gamma^{\beta}_{[\nu\rho}\Gamma^{\rho}_{\beta]\mu},\\
T_{\mu\nu} &= e_{[\nu}\Lambda^{\beta}_{\beta]\mu} + (\Gamma_{[\nu}\Lambda_{\beta]})^{\beta}_{\mu} + (\Lambda_{[\nu}\Gamma_{\beta]})^{\beta}_{\mu} + (\Lambda_{[\nu}\Lambda_{\beta]})^{\beta}_{\mu} - C^{\rho}_{\nu\beta}\gamma^{\beta}_{\rho\mu}.
\end{aligned}$$

Now we pass to the calculation of the Ricci tensor.

4.1. *The Ricci tensor in the case det* $\mathbf{F} < 0$

Note that for $s = 0$ the adapted non-holonomic basis becomes holonomic and coincides with the one used in [4]. This is why the expressions for $S_{\mu\nu}$ given below coincide with the expressions for the components of the Ricci tensor found in [4]. Observe also that only the fields e_1 and e_2 give nontrivial contributions to expressions (14) for the $\Gamma^{\alpha}_{\mu\nu}$'s and all components $\Lambda^{\alpha}_{\mu\nu}$, except possibly Λ^{c}_{ba}, vanish.

- *Components* R_{ij}:
  Let us note that

  $$T_{ij} = \Lambda^{\beta}_{\beta\rho}\Gamma^{\rho}_{ji} - C^{\rho}_{j\beta}\gamma^{\beta}_{\rho i} = 0,$$

  this is due to the fact that $C^{\rho}_{j\beta} = 0$, the components $\Lambda^{\alpha}_{\mu\nu}$ with an index equal to 1 or 2 vanish and $\Gamma^{a}_{ij} = 0$. So,

  $$R_{ij} = S_{ij} = e_{[j}\big(\Gamma^{\beta}_{\beta]i}\big) + \Gamma^{\beta}_{[j\rho}\Gamma^{\rho}_{\beta]i}.$$

The first term of this expression gives,

$$e_{[j}\big(\Gamma^{\beta}_{\beta]i}\big) = \partial_j\partial_i(\ln|f|) - \delta_{ij}\partial_i^2(\ln|f|) + \partial_j\partial_i(\ln\alpha),$$

where $\alpha = \sqrt{|\det \mathbf{H}|}$ and the second term gives

$$\begin{aligned}\Gamma^{\beta}_{[j\rho}\Gamma^{\rho}_{\beta]i} &= \mathrm{tr}(\Gamma_j\Gamma_i) - (\Gamma_\beta\Gamma_j)^\beta_i\\ &= \frac{1}{4}\mathrm{tr}\big[\mathbf{H}^{-1}\partial_j(\mathbf{H})\mathbf{H}^{-1}\partial_i(\mathbf{H})\big] + \frac{(\partial_i f)^2}{f^2}\delta_{ij} + \delta_{ij}\frac{(\partial_i f)^2}{f^2} + \delta_{ij}\partial_i(\ln|f|)\partial_i(\ln\alpha).\end{aligned}$$

Finally, one has

$$\begin{aligned}R_{ij} &= \partial_j\partial_i(\ln|f|) - \delta_{ij}\partial_i^2(\ln|f|) + \partial_j\partial_i(\ln\alpha) + \frac{1}{4}\mathrm{tr}\big[\mathbf{H}^{-1}\partial_j(\mathbf{H})\mathbf{H}^{-1}\partial_i(\mathbf{H})\big]\\ &\quad - \delta_{ij}\partial_i(\ln|f|)\partial_i(\ln\alpha).\end{aligned}$$

- *Components* $R_{ab} = S_{ab} + T_{ab}$:
  For what concerns S_{ab}, it is more convenient to use the following expression

$$S_{ab} = \frac{1}{\sqrt{|\det g|}}\partial_\rho\Big(\sqrt{|\det g|}\,\Gamma^{\rho}_{ab}\Big) - \partial_a\partial_b\Big(\ln\sqrt{|\det g|}\Big) - \Gamma^{\beta}_{\rho a}\Gamma^{\rho}_{\beta b}$$

taking into account that $|\det g| \equiv |\det \mathbf{F}||\det \mathbf{H}| = f^2\alpha^2$ and $\alpha = \sqrt{|\det \mathbf{H}|}$.
The result is

$$(S_{ab}) = \frac{1}{2f\alpha}\mathbf{H}\Big[\big(\alpha\mathbf{H}^{-1}\partial_1(\mathbf{H})\big)_{,2} + \big(\alpha\mathbf{H}^{-1}\partial_2(\mathbf{H})\big)_{,1}\Big].$$

For T_{ab} one finds

$$T_{ab} = e_{[b}\big(\Lambda^{\beta}_{\beta]a}\big) + (\Gamma_{[b}\Lambda_{\beta]})^\beta_a + (\Lambda_{[b}\Gamma_{\beta]})^\beta_a + (\Lambda_{[b}\Lambda_{\beta]})^\beta_a - C^{\rho}_{b\beta}\gamma^{\beta}_{\rho a} = -C^{\rho}_{b\beta}\gamma^{\beta}_{\rho a},$$

so that

$$(T_{ab}) = s^2 h_{22}\big(\det \mathbf{H}^{-1}\big)\mathbf{H}$$

and

$$(R_{ab}) = \frac{1}{2f\alpha}\mathbf{H}\left[\big(\alpha\mathbf{H}^{-1}\partial_2(\mathbf{H})\big)_{,1} + \big(\alpha\mathbf{H}^{-1}\partial_1(\mathbf{H})\big)_{,2} + \frac{2s^2}{\alpha}f h_{22}\mathbf{1}_2\right],$$

where $\mathbf{1}_2$ stands for the unit (2×2)-matrix.
- *Components* R_{ai}:
  In this case,

$$S_{ai} = e_{[i}\big(\Gamma^{\beta}_{\beta]a}\big) + \Gamma^{\beta}_{[i\rho}\Gamma^{\rho}_{\beta]a} = 0.$$

Indeed, the first term vanishes since Γ_i's are diagonal and Γ_a are anti-diagonal. The second term also vanishes since the matrices $\Gamma_i\Gamma_j$ are diagonal while $\Gamma_i\Gamma_b$ or $\Gamma_b\Gamma_i$ anti-diagonal. Thus,

$$R_{ai} = T_{ai}$$

and

$$T_{ai} = e_i\big(\Lambda^{b}_{ba}\big) + (\Gamma_{[i}\Lambda_{\beta]})^\beta_a + (\Lambda_{[i}\Gamma_{\beta]})^\beta_a + (\Lambda_{[b}\Lambda_{\beta]})^\beta_a - C^{\rho}_{i\beta}\gamma^{\beta}_{\rho a} = (\Gamma_i\Lambda_b)^b_a - (\Lambda_b\Gamma_i)^b_a$$

or, equivalently,

$$\begin{pmatrix} T_{3i} \\ T_{4i} \end{pmatrix} = s \begin{pmatrix} (\mathbf{H}^{-1}\partial_i(\mathbf{H}))_2^2 - (\mathbf{H}^{-1}\partial_i(\mathbf{H}))_1^1 \\ -2(\mathbf{H}^{-1}\partial_i(\mathbf{H}))_2^1 \end{pmatrix}.$$

So, the final result is

$$(R_{i3}, R_{i4}) = s\big(\big(\mathbf{H}^{-1}\partial_i(\mathbf{H})\big)_2^2 - \big(\mathbf{H}^{-1}\partial_i(\mathbf{H})\big)_1^1, -2\big(\mathbf{H}^{-1}\partial_i(\mathbf{H})\big)_2^1\big).$$

The above calculations are summarized in the following proposition

Proposition 6. *Let g be a $\mathcal{G}_2$-integrable 4-metric. If $\det \mathbf{F} < 0$, then the components of the Ricci tensor in a non-holonomic adapted basis are*

$$(R_{ab}) = \frac{\mathbf{H}}{2f\alpha}\left[\big(\alpha\mathbf{H}^{-1}\partial_1(\mathbf{H})\big)_{,2} + \big(\alpha\mathbf{H}^{-1}\partial_2(\mathbf{H})\big)_{,1} + \frac{2s^2}{\alpha} f h_{22}\mathbf{1}_2\right],$$

$$R_{12} = \partial_1\partial_2(\ln|f| + \ln\alpha) + \frac{1}{4}\operatorname{tr}\big[\mathbf{H}^{-1}\partial_1(\mathbf{H})\mathbf{H}^{-1}\partial_2(\mathbf{H})\big],$$

$$R_{ii} = -\partial_i(\ln\alpha)\partial_i(\ln|f|) + \partial_i^2(\ln\alpha) + \frac{1}{4}\operatorname{tr}\big[\mathbf{H}^{-1}\partial_i(\mathbf{H})\mathbf{H}^{-1}\partial_i(\mathbf{H})\big],$$

$$\begin{pmatrix} R_{i3} \\ R_{i4} \end{pmatrix} = s \begin{pmatrix} (\mathbf{H}^{-1}\partial_1(\mathbf{H}))_2^2 - (\mathbf{H}^{-1}\partial_1(\mathbf{H}))_1^1 & -2(\mathbf{H}^{-1}\partial_1(\mathbf{H}))_2^1 \\ (\mathbf{H}^{-1}\partial_2(\mathbf{H}))_2^2 - (\mathbf{H}^{-1}\partial_2(\mathbf{H}))_1^1 & -2(\mathbf{H}^{-1}\partial_2(\mathbf{H}))_2^1 \end{pmatrix}$$

with $\alpha = \sqrt{|\det \mathbf{H}|}$.

Remark 1. Note that for $s = 0$ the above expressions for the components of the Ricci tensor field coincide with the ones given in [4]. In particular, the components R_{ai} vanish identically.

4.2. *The Ricci tensor field in the case* $\mathbf{F} > 0$

We use again the adapted non-holonomic basis Θ described in Proposition 2, so that the matrix of g is

$$\mathbf{M}_\Theta(g) = \begin{pmatrix} 2f & 0 & \mathbf{0} \\ 0 & 2f & \\ \mathbf{0} & & \mathbf{H} \end{pmatrix} = \begin{pmatrix} \mathbf{F} & \mathbf{0} \\ \mathbf{0} & \mathbf{H} \end{pmatrix}.$$

In this case essentially the same computation as before gives the following result.

Proposition 7. *Let g be a $\mathcal{G}_2$-integrable 4-metric. If $\det \mathbf{F} > 0$, then the components of the Ricci tensor in a non-holonomic adapted basis are*

$$(R_{ia}) = s \begin{pmatrix} (\mathbf{H}^{-1}\partial_1(\mathbf{H}))_2^2 - (\mathbf{H}^{-1}\partial_1(\mathbf{H}))_1^1 & -2(\mathbf{H}^{-1}\partial_1(\mathbf{H}))_2^1 \\ (\mathbf{H}^{-1}\partial_2(\mathbf{H}))_2^2 - (\mathbf{H}^{-1}\partial_2(\mathbf{H}))_1^1 & -2(\mathbf{H}^{-1}\partial_2(\mathbf{H}))_2^1 \end{pmatrix};$$

$$(R_{ab}) = \frac{\mathbf{H}}{2f\alpha}\left[\frac{1}{2}\big[\big(\alpha\mathbf{H}^{-1}\partial_1(\mathbf{H})\big)_{,1} + \big(\alpha\mathbf{H}^{-1}\partial_2(\mathbf{H})\big)_{,2}\big] + \frac{2s^2}{\alpha} f h_{22}\mathbf{1}_2\right];$$

$$R_{11} = \frac{1}{2}\left[\Delta(\ln\alpha\ln|f|) + \frac{1}{2}\operatorname{tr}\bigl(\mathbf{H}^{-1}\partial_1\mathbf{H}\bigr)^2 - \frac{\alpha_{,1}}{\alpha}\partial_1(\ln|f|)\right] + \frac{1}{2}\left[\frac{\alpha_{,2}}{\alpha}\partial_2(\ln|f|) + \partial_1\left(\frac{\alpha_{,1}}{\alpha}\right) - \partial_2\left(\frac{\alpha_{,2}}{\alpha}\right)\right];$$

$$R_{22} = \frac{1}{2}\left[\Delta(\ln\alpha\ln|f|) + \frac{1}{2}\operatorname{tr}\bigl(\mathbf{H}^{-1}\partial_2\mathbf{H}\bigr)^2 + \frac{\alpha_{,1}}{\alpha}\partial_1(\ln|f|)\right] - \frac{1}{2}\left[\frac{\alpha_{,2}}{\alpha}\partial_2(\ln|f|) - \partial_1\left(\frac{\alpha_{,1}}{\alpha}\right) + \partial_2\left(\frac{\alpha_{,2}}{\alpha}\right)\right];$$

$$R_{12} = \frac{1}{2}\left[-\frac{\alpha_{,1}}{\alpha}\partial_2(\ln|f|) - \frac{\alpha_{,2}}{\alpha}\partial_1(\ln|f|) + 2\partial_1\partial_2(\ln\alpha)\right] + \frac{1}{4}\operatorname{tr}\bigl[\mathbf{H}^{-1}\partial_1(\mathbf{H})\mathbf{H}^{-1}\partial_2(\mathbf{H})\bigr];$$

with

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}.$$

Remark 2. Also in this case the components R_{ai} vanish identically for $s = 0$.

5. Solutions of vacuum Einstein field equations

In this section we will limit ourselves to discuss only the general form of local solutions of vacuum Einstein equations

$$R_{\mu\nu} = 0$$

for $\mathcal{G}_2$-integrable *normal* (see after) metrics.

Let us consider separately the cases characterized by $\det\mathbf{F} < 0$ and $\det\mathbf{F} > 0$.

5.1. *Solutions of Einstein equations in the case* $\det\mathbf{F} < 0$

Note that, for $s = 0$ (Abelian case) the equations $R_{ai} = 0$ become identities, while for $s \neq 0$ they impose the following strong conditions on the metric:

$$\begin{cases} (\mathbf{H}^{-1}\partial_i(\mathbf{H}))^2_2 = (\mathbf{H}^{-1}\partial_i(\mathbf{H}))^1_1, \\ (\mathbf{H}^{-1}\partial_i(\mathbf{H}))^1_2 = 0. \end{cases} \tag{15}$$

The two cases $h_{22} \neq 0$ and $h_{22} = 0$ are qualitatively different and will be discussed separately.

5.1.1. *The case* $h_{22} \neq 0$

In this case Eqs. (15) imply that $(\mathbf{H}^{-1}\partial_i(\mathbf{H}))^2_1 = 0$ for any symmetric (2×2)-matrix $\mathbf{H}$. This means that $\mathbf{H}^{-1}\partial_i(\mathbf{H})$ is a scalar matrix, i.e., $\partial_1(\mathbf{H}) = \varphi\mathbf{H}$, $\partial_2(\mathbf{H}) = \psi\mathbf{H}$ for some functions $\varphi = \varphi(x_i)$, $\psi = \psi(x_i)$.

The compatibility condition $\partial_2(\varphi) = \partial_1(\psi)$ for the above system, implies the existence (locally) of a function $\gamma(x_i)$ such that $\varphi = \partial_1(\gamma)$, $\psi = \partial_2(\gamma)$. The function γ can be chosen in such a way that $\mathbf{H} = e^{\gamma}\mathbf{M}$, $\mathbf{M}$ being a constant symmetric (2×2)-matrix such that $\det\mathbf{M} = \pm 1$. Thus,

$$\alpha = e^{\gamma}.$$

Then the equations $R_{ab} = 0$ can be written as

$$\alpha,_{12} + s^2 f m_{22} = 0, \tag{16}$$

or

$$f = c\alpha,_{12},$$

$\alpha,_i \equiv \partial_i(\alpha)$, $\alpha,_{ij} \equiv \partial_i \partial_j(\alpha)$, and

$$c = -\frac{1}{s^2 m_{22}}.$$

This brings Einstein equations to the form

$$\mathbf{H} = e^{\gamma}\mathbf{M} = \alpha\mathbf{M}, \tag{17}$$

$$f = c\alpha,_{12} \tag{18}$$

$$\partial_i(\ln|f|) = \partial_i\left(\ln\frac{|\alpha,_i|}{\sqrt{\alpha}}\right), \tag{19}$$

$$\partial_1\partial_2(\ln|f|) = -\frac{1}{\alpha}\alpha,_{12} + \frac{1}{2\alpha^2}\alpha,_1\alpha,_2. \tag{20}$$

For the two possible values of the index i Eq. (19) gives

$$f = H(x_2)\partial_1\big(\sqrt{\alpha}\big) \tag{21}$$

$$= K(x_1)\partial_2\big(\sqrt{\alpha}\big), \tag{22}$$

where H and K are arbitrary functions, or, equivalently,

$$H\partial_1\alpha = K\partial_2\alpha. \tag{23}$$

From Eq. (18) one gets

$$\alpha,_1 = \frac{1}{c}K\big(\sqrt{\alpha} - A\big), \qquad \alpha,_2 = \frac{1}{c}H\big(\sqrt{\alpha} - A\big),$$

where A is a constant, or, equivalently,

$$d\alpha = \frac{1}{c}\big(\sqrt{\alpha} - A\big)(K\,dx_1 + H\,dx_2).$$

By setting $\beta^2 = \alpha$ the above equation integrates to the equality

$$\beta + A\ln|\beta - A| = F(x_1) + G(x_2),$$

with $F(x_1) \equiv \frac{1}{2c}\int K\,dx_1$, $G(x_2) \equiv \frac{1}{2c}\int H\,dx_2$. The above equation will be called the *tortoise equation*. Finally, the remaining Einstein equations show Eq. (20) to be an identity.

By summing up we give the components of the metric in the basis $C = \{dz_1, dz_2, dx, dy\}$ with $z_1 = \frac{1}{2}(x_1 + x_2)$, $z_2 = \frac{1}{2}(x_1 - x_2)$, $x = x_3$, $y = x_4$, where the x_μ's are the adapted coordinates mentioned in Proposition 5.

Proposition 8. *Any $\mathcal{G}_2$-integrable 4-metric g satisfying the vacuum Einstein equations, and such that $\det\mathbf{F} < 0$ and $h_{22} \neq 0$, has in the adapted coordinate (z_1, z_2, x, y) the following matrix form*

$$\mathbf{M}_C(g) = \begin{pmatrix} \begin{matrix} 2f & 0 \\ 0 & -2f \end{matrix} & \mathbf{0} \\ \mathbf{0} & \beta^2 \begin{pmatrix} s^2ky^2 - 2sly + m & -sky + l \\ -sky + l & k \end{pmatrix} \end{pmatrix}$$

where

- *k, l, m, are arbitrary constants such that $km - l^2 = \pm 1$, $k \neq 0$,*
- $$f = -\frac{1}{4s^2k}\left(\frac{\partial^2}{\partial z_1^2} - \frac{\partial^2}{\partial z_2^2}\right)\beta^2, \tag{24}$$
- *β is a solution of the tortoise equation*

  $$\beta + A\ln|\beta - A| = F(z_1 + z_2) + G(z_1 - z_2), \tag{25}$$

  A, F, G being an arbitrary constant and arbitrary functions respectively.

Remark 3. As it will be clarified in [10,11], the *tortoise equation* (25) leads to a deeper understanding of the so called Regge–Wheeler tortoise coordinate, which, apart from constant terms, is defined as its left hand side.

Remark 4. Concerning the signature of the metric and the character of the Killing fields, we observe that:

If $\det\mathbf{M} = 1$ (see Eq. (17)), then $\mathbf{H}$ is either positive or negative definite according to the sign of k and $g(Y, Y)$, $g(X, X)$ have the same sign as k. The signature of g is equal to ± 2, so that these metrics are of interest for general relativity;

If $\det\mathbf{M} = -1$, then $\mathbf{H}$ is indefinite, $g(Y, Y)$ has again the same sign as k while the sign of $g(X, X)$ varies depending on the values of y. The signature of g in this case is equal to 0.

By using the results of Section 3, we have:

Corollary 9. *The metric g of the above proposition admits an additional Killing field*

$$Z = e^{-sx}\left[2(sky - l)\partial_x + \left(s^2ky^2 - 2sly + m\right)\partial_y\right],$$

which generates together with $X = \partial_x$ and $Y = e^{sx}\partial_y$ a 3-dimensional Lie algebra isomorphic to $so(2,1)$ (assuming that $s \neq 0$):

$$[X, Y] = sY, \qquad [X, Z] = -sZ, \qquad [Y, Z] = 2skX.$$

5.1.2. The case $h_{22} = 0$

Now, Eqs. (15) are identically satisfied, while the remaining Einstein equations become

$$\left(\alpha\mathbf{H}^{-1}\partial_1\mathbf{H}\right)_{,2} + \left(\alpha\mathbf{H}^{-1}\partial_2\mathbf{H}\right)_{,1} = 0, \tag{26}$$

$$\partial_1\partial_2(\ln|f| + \ln\alpha) + \frac{1}{4}\operatorname{tr}\big[\mathbf{H}^{-1}\partial_1(\mathbf{H})\mathbf{H}^{-1}\partial_2(\mathbf{H})\big] = 0, \tag{27}$$

$$-\partial_i \ln|\alpha|\partial_i \ln|f| + \partial_i^2 \ln\alpha + \frac{1}{4}\operatorname{tr}\big[\mathbf{H}^{-1}\partial_i(\mathbf{H})\mathbf{H}^{-1}\partial_i(\mathbf{H})\big] = 0. \tag{28}$$

In terms of the components μ and ν of $\mathbf{H}$ they reduce to

$$\alpha,_{12} = 0, \tag{29a}$$

$$(\alpha w_{,1})_{,2} + (\alpha w_{,2})_{,1} = 0, \tag{29b}$$

$$\partial_1\partial_2(\ln|f|) = \frac{\alpha,_2\,\alpha,_1}{2\alpha^2}, \tag{29c}$$

$$\alpha,_i\,\partial_i(\ln|f|) = \alpha,_{ii} - \frac{\alpha,_i^2}{2\alpha}, \tag{29d}$$

with $\alpha = \sqrt{|\det\mathbf{H}|} = |\mu|$ and $w = \nu/\alpha$.

The general solution of Eq. (29a) is

$$\alpha = F(x_1) + G(x_2),$$

F and G being arbitrary functions such that α is positive.

The general solution of Eq. (29c) is

$$f = \pm\alpha^{-\frac{1}{2}} e^{P(x_1)+Q(x_2)},$$

where P and Q are arbitrary functions.

Now Eq. (29d) takes the form

$$P'(x_1)\alpha,_1 = \alpha,_{11},$$

$$Q'(x_2)\alpha,_2 = \alpha,_{22}$$

and are resolved as

$$F = C_1\int e^P\,dx_1 + D_1, \qquad G = C_2\int e^Q\,dx_2 + D_2.$$

Thus as the final result we see that the general solution of the differential system (29a), (29c), (29d) is given by

$$\alpha = C_1\int e^P\,dx_1 + C_2\int e^Q\,dx_2 + C,$$

$$f = \pm\alpha^{-\frac{1}{2}} e^{P(x_1)+Q(x_2)},$$

where C, C_1, C_2, are arbitrary constants such that α is positive.

Eq. (29b) is a *linear second order partial differential equation* and can be studied by standard methods. We postpone this problem to a further publication.

As in Proposition 8 we summarize the obtained results by giving the components of g in the frame $C = \{dz_1, dz_2, dx, dy\}$ where $z_1 = \frac{1}{2}(x_1 + x_2)$, $z_2 = \frac{1}{2}(x_1 - x_2)$, $x = x_3$, $y = x_4$, and x_μ's are the adapted coordinates introduced in Proposition 5.

Proposition 10. *Any $\mathcal{G}_2$-integrable 4-metric g satisfying the vacuum Einstein equations and such that* $\det \mathbf{F} < 0$ *and* $h_{22} = 0$, *has the following matrix form in the adapted coordinates* (z_1, z_2, x, y),

$$\mathbf{M}_C(g) = \begin{pmatrix} \begin{matrix} 2f & 0 \\ 0 & -2f \end{matrix} & \mathbf{0} \\ \mathbf{0} & \mu \begin{pmatrix} -2sy + w & 1 \\ 1 & 0 \end{pmatrix} \end{pmatrix}$$

where

-
$$\mu = C_1 F(z_1 + z_2) + C_2 G(z_1 - z_2) + C, \tag{30}$$
$$f = |\mu|^{-\frac{1}{2}} F' G', \tag{31}$$
F, G and C, C_1, C_2 being arbitrary functions and arbitrary constants respectively, such that μ and f are everywhere nonvanishing;
- *w is an arbitrary solution of the equation*
$$\mu\left(\frac{\partial^2}{\partial z_1^2} - \frac{\partial^2}{\partial z_2^2}\right) w + \frac{\partial \mu}{\partial z_1}\frac{\partial w}{\partial z_1} - \frac{\partial \mu}{\partial z_2}\frac{\partial w}{\partial z_2} = 0.$$

In this case, $\det \mathbf{H} < 0$ and the metric g has signature equal to 0. The Killing field Y is isotropic, while the sign of $g(X, X)$ varies as a function of y. The curvature K of the Killing leaves vanishes.

Remark 5. In contrast with the case $h_{22} \neq 0$ (see Section 5.1.1) an additional Killing field, say Z, tangent to the Killing leaves and independent on X and Y exists only if w is a constant, say w_0. In such a case

$$Z = e^{-sx}\left[-2\partial_x + (-2sy + w_0)\partial_y\right],$$

and generates together with $X = \partial_x$ and $Y = e^{sx}\partial_y$ a 3-dimensional Lie algebra isomorphic to $\mathcal{K}il(dx^2 - dy^2)$:

$$[X, Y] = sY, \qquad [X, Z] = -sZ, \qquad [Y, Z] = 0.$$

A canonical form for Eq. (29b) may be obtained by passing to coordinates

$$\xi = F(x_1), \qquad \eta = G(x_2)$$

in which Eq. (29b) becomes

$$2(\xi + \eta)\frac{\partial^2 \widetilde{w}}{\partial \xi \partial \eta} + \frac{\partial \widetilde{w}}{\partial \xi} + \frac{\partial \widetilde{w}}{\partial \eta} = 0,$$

with $\widetilde{w}(\xi, \eta) \equiv w(F^{-1}(\xi), G^{-1}(\eta))$, or, alternatively,

$$\frac{\partial^2 Z}{\partial \xi \partial \eta} + \frac{1}{4(\xi + \eta)^2} Z = 0, \qquad Z = \sqrt{\xi + \eta}\, \widetilde{w}.$$

Its geometrical interpretation is given in [10,11].

5.2. *Solutions of Einstein equations in the case* $\det \mathbf{F} > 0$

As before, the equations $R_{ai} = 0$ are satisfied trivially if $s = 0$ while for $s \neq 0$ they coincide with (15):

$$\begin{cases} (\mathbf{H}^{-1}\partial_i(\mathbf{H}))_2^2 = (\mathbf{H}^{-1}\partial_i(\mathbf{H}))_1^1, \\ (\mathbf{H}^{-1}\partial_i(\mathbf{H}))_3^2 = 0. \end{cases} \tag{32}$$

Again it is convenient to treat separately the cases $h_{22} \neq 0$ and $h_{22} = 0$.

5.2.1. *The case* $h_{22} \neq 0$

As in Section 5.1.1, equations $R_{ia} = 0$ are solved as

$$\mathbf{H} = e^{\gamma}\mathbf{M}.$$

$\mathbf{M}$ being a constant symmetric (2×2)-matrix such that $\det M = \pm 1$ and $\alpha = e^{\gamma}$. Because of the non-degeneracy of g the first derivatives of α are non-vanishing, so that Einstein equations can be brought to the following form

$$\mathbf{H} = \alpha\mathbf{M}, \tag{33}$$

$$\left(\frac{\triangle(\alpha)}{4f} + s^2 m_{22}\right)\mathbf{M} = \mathbf{0}, \tag{34}$$

$$\Delta(\ln\alpha|f|) - \frac{1}{\alpha f}(\alpha,_1 f,_1 - \alpha,_2 f,_2) + \frac{(\alpha,_2)^2}{\alpha^2} + \frac{\alpha,_{11} - \alpha,_{22}}{\alpha} = 0, \tag{35}$$

$$\Delta(\ln\alpha|f|) + \frac{1}{\alpha f}(\alpha,_1 f,_1 - \alpha,_2 f,_2) + \frac{(\alpha,_1)^2}{\alpha^2} - \frac{\alpha,_{11} - \alpha,_{22}}{\alpha} = 0, \tag{36}$$

$$\frac{1}{2\alpha f}(\alpha,_1 f,_2 + \alpha,_2 f,_1) + \frac{\alpha,_2\alpha,_1}{2\alpha^2} - \frac{\alpha,_{12}}{\alpha} = 0. \tag{37}$$

In its turn the last system is equivalent to

$$\mathbf{H} = \alpha\mathbf{M},$$
$$f = \frac{c}{4}\Delta\alpha,$$
$$\partial_1\left[\ln|f| - \frac{1}{2}\left(\ln\alpha + \ln\frac{|\nabla(\alpha)|^2}{\alpha^2}\right)\right] = -\vartheta_2,$$
$$\partial_2\left[\ln|f| - \frac{1}{2}\left(\ln\alpha + \ln\frac{|\nabla(\alpha)|^2}{\alpha^2}\right)\right] = \vartheta_1$$

where $c = -\frac{1}{s^2 m_{22}}$ and ϑ_1 and ϑ_2 are the partial derivatives of

$$\vartheta = \arctan\frac{\alpha,_2}{\alpha,_1}.$$

These equations show that ϑ and $\ln\frac{\sqrt{\alpha}|f|}{|\nabla(\alpha)|}$ are conjugated harmonic functions so that the above system can be brought to the form:

$$\Delta(\vartheta) = 0,$$

$$\frac{\alpha_{,2}}{\alpha_{,1}} = \tan\vartheta, \tag{38}$$

$$\ln\frac{\sqrt{\alpha}|\Delta(\alpha)|}{|\nabla(\alpha)|} = \Phi, \tag{39}$$

$$f = \frac{c}{4}\Delta(\alpha), \tag{40}$$

$$\mathbf{H} = \alpha\mathbf{M},$$

where Φ is a harmonic function conjugated to ϑ, that is a primitive of the exact differential 1-form $\omega = \vartheta_1\,dx_2 - \vartheta_2\,dx_1$. Now one can easily check that the above system is reduced to the *tortoise* equation (see Section 5.1.1)

$$\beta + A\ln|\beta - A| = \Psi,$$

where $\beta^2 = \alpha$, Ψ is an arbitrary harmonic function and A is an arbitrary constant. The functions ϑ and Φ are given, respectively, by

$$\vartheta = \arctan\frac{\Psi_{,2}}{\Psi_{,1}}$$

$$\Phi = \ln|\nabla(\Psi)|.$$

By summing up we give the components of the metric in terms of the adapted holonomic frame $C = \{dx_1, dx_2, dx, dy\}$ with $x = x_3$, $y = x_4$, the x_μ's being the adapted coordinates introduced in Proposition 5.

Proposition 11. *Any $\mathcal{G}_2$-integrable 4-metric g satisfying the vacuum Einstein equations, and such that $\det\mathbf{F} > 0$ and $h_{22} \neq 0$, has the following matrix form in the adapted coordinates (x_μ)*

$$\mathbf{M}_C(g) = \begin{pmatrix} \begin{matrix} 2f & 0 \\ 0 & 2f \end{matrix} & \mathbf{0} \\ \mathbf{0} & \beta^2\begin{pmatrix} s^2ky^2 - 2sly + m & -sky + l \\ -sky + l & k \end{pmatrix} \end{pmatrix}$$

where

- *k, l, m, are arbitrary constants such that $km - l^2 = \pm 1$, $k \neq 0$,*
- $$f = -\frac{1}{4s^2k}\Delta(\beta^2), \tag{41}$$
- *β is a solution of the tortoise equation*

  $$\beta + A\ln|\beta - A| = \Psi, \tag{42}$$

  such that $\Delta\beta^2 \equiv (\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2})\beta^2$ is everywhere nonvanishing, A and Ψ being an arbitrary constant and an arbitrary harmonic function.

Remark 6. Concerning the signature of g and the character of the Killing fields, we remark that:

If $\det \mathbf{M} = 1$ (see Eq. (33)), then $\mathbf{H}$ is either positive or negative definite according to the sign of k as well as $g(Y, Y)$, and $g(X, X)$. Since the sign of the constant c is opposite to the one of k, the signature of g is always equal to 0.

If $\det \mathbf{M} = -1$, then $\mathbf{H}$ is indefinite, $g(Y, Y)$ has the same sign as k while the sign of $g(X, X)$ varies with as a function of y. The signature of g is equal to ± 2, so that these metrics are of interest for General Relativity.

Moreover, as in Section 5.1.1 we have:

Corollary 12. *The metric of the above proposition admits a third Killing field*

$$Z = \alpha e^{-sx_2}\big[(m - sky)\partial_x + \big(s^2ky^2 - 2smy + l\big)\partial_y\big],$$

which together with X and Y generate a 3-dimensional Lie algebra isomorphic to $so(2, 1)$

$$[X, Y] = sZ, \quad [X, Z] = -sZ, \quad [Y, Z] = -2skX.$$

5.2.2. The case $h_{22} = 0$

In this case the equations $R_{ia} = 0$ are satisfied automatically while the matrix $\mathbf{H}$ has the form

$$\mathbf{H} = \begin{pmatrix} \nu & \mu \\ \mu & 0 \end{pmatrix},$$

and $\alpha = |\mu|$. The remaining Einstein equations reduce now to

$$\Delta(\alpha) = 0, \tag{43}$$

$$(\alpha\partial_1 w)_{,1} + (\alpha\partial_2 w)_{,2} = 0, \tag{44}$$

$$\Delta(\ln|f|) = \frac{1}{2}\left[\left(\frac{\alpha_{,1}}{\alpha}\right)^2 + \left(\frac{\alpha_{,2}}{\alpha}\right)^2\right], \tag{45}$$

$$\alpha_{,1}\,\partial_1(\ln|f|) - \alpha_{,2}\,\partial_2(\ln|f|) = \alpha_{,11} - \alpha_{,22} - \frac{\alpha_{,1}^2 - \alpha_{,2}^2}{2\alpha}, \tag{46}$$

$$\alpha_{,2}\,\partial_1(\ln|f|) + \alpha_{,1}\,\partial_2(\ln|f|) = 2\alpha_{,12} - \frac{\alpha_{,2}\,\alpha_{,1}}{\alpha}, \tag{47}$$

where $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$ and $w = \frac{\nu}{\alpha}$. If α is a solution of Eq. (43), i.e., a harmonic function, then the general solution of Eq. (45) is

$$f = \pm\alpha^{-\frac{1}{2}}e^{\psi}$$

ψ being a harmonic function. Substituting this expression in Eqs. (46), (47) one gets

$$\alpha_{,1}\,\psi_{,1} - \alpha_{,2}\,\psi_{,2} = 2\alpha_{,11},$$
$$\alpha_{,2}\,\psi_{,1} + \alpha_{,1}\,\psi_{,2} = 2\alpha_{,12},$$

the last relations are locally equivalent to

$$|\nabla(\alpha)|^2 = ce^{\psi}$$

c being a constant. Therefore,

$$f = \pm \frac{|\nabla(\phi)|^2}{\sqrt{|D\phi + B|}}, \tag{48}$$

where $\alpha = |\mu| = |D\phi + B|$, A and B are constants and ϕ a harmonic function such that α is nonvanishing. Eq. (44) is a *linear second order partial differential equation* and can be analyzed with standard methods.

Thus, as the final result we have:

Proposition 13. *Any $\mathcal{G}_2$-integrable 4-metric g satisfying the vacuum Einstein equations, and such that* $\det \mathbf{F} > 0$ *and* $h_{22} = 0$, *has the following matrix form in the adapted coordinates* $(x_1, x_2, x.y)$

$$\mathbf{M}_C(g) = \begin{pmatrix} \varepsilon \frac{|\nabla(\phi)|^2}{\sqrt{|D\phi+B|}} \mathbf{1}_2 & \mathbf{0} \\ \mathbf{0} & (D\phi + B)\begin{pmatrix} -2sy + w & 1 \\ 1 & 0 \end{pmatrix} \end{pmatrix}$$

where $\varepsilon = \pm 1$, ϕ is a harmonic function, D and B are constants such that $\mu = D\phi + B$ is everywhere nonvanishing and w is a solution of the equation

$$(\mu w_{,1})_{,1} + (\mu w_{,2})_{,2} = 0.$$

In the considered case $\det \mathbf{H}$ is negative and the signature of g is equal to ± 2. The Killing vector field Y is isotropic while the sign of $g(X, X)$ varies as a function of y. The Gauss curvature K of the Killing leaves vanishes.

Remark 7. According to Section 3, an additional Killing field, say Z, tangent to the Killing leaves and independent of X and Y, exists iff w is a constant, say w_0. In such a case it is given by

$$Z = e^{-sx}[-2\partial_x + (-2sy + w_0)\partial_y],$$

which generates together with $X = \partial_x$ and $Y = e^{sx}\partial_y$ a 3-*dimensional Lie algebra* isomorphic to $\mathcal{K}il(dx^2 - dy^2)$:

$$[X, Y] = sY, \qquad [X, Z] = -sZ, \qquad [Y, Z] = 0.$$

A canonical form for Eq. (44) can be found by introducing new coordinates, namely ξ and η, by

$$\xi = \alpha + \tilde{\alpha}, \qquad \eta = \alpha - \tilde{\alpha}$$

in which Eq. (44) becomes

$$(\xi + \eta)\left(\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2}\right)(\widetilde{w}) + \frac{\partial \widetilde{w}}{\partial \xi} + \frac{\partial \widetilde{w}}{\partial \eta} = 0,$$

with $\widetilde{w}(\xi, \eta) \equiv w(x_1(\xi, \eta), x_2(\xi, \eta))$, or, alternatively,

$$\left(\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2}\right)(Z) + \frac{1}{2(\xi + \eta)^2} Z = 0$$

with

$$Z = \sqrt{\xi + \eta}\, \widetilde{w}.$$

For its geometrical meaning see [10,11].

6. The Abelian limit $(s=0)$

The solutions of the Einstein equations found in the previous section allow one to get exact solutions of the Belinsky–Zahkarov case just by passing to the "Abelian limit" $s=0$. Since the Abelian case was extensively studied (see, for instance, [3,4,7]) we shall limit ourself here simply to describe these solutions. In what follows we use the adapted coordinates to which the propositions refer and consider separately the cases $h_{22}\neq 0$ and $h_{22}=0$.

The case $h_{22}\neq 0$. With this assumption Eqs. (15) and, which is the same (32) play the role of an "ansatz" when passing to the Abelian limit: So, in that case as in Sections 5.1.1 and 5.1.2 one sees that $\mathbf{H}=\alpha\mathbf{M}$, $\mathbf{M}$ being a constant unimodular matrix.

- If $\det\mathbf{F}<0$ then Eq. (16) becomes

  $$\alpha_{,12}=0$$

  and the remaining Einstein equations coincide with Eqs. (29c) and (29d) as they appeared in the analysis of the non-Abelian case with $h_{22}=0$ and $\det F<0$ (see Section 5.1.2). Thus, the same procedure leads us to the following result:

  $$\mathbf{M}_C(g)=\begin{pmatrix} \begin{matrix} 2f & 0 \\ 0 & -2f \end{matrix} & \mathbf{0} \\ \mathbf{0} & \alpha\mathbf{M} \end{pmatrix}, \tag{49}$$

  where α and f are given by

  $$\alpha = C_1F(z_1+z_2)+C_2G(z_1-z_2)+C, \tag{50}$$
  $$f=\frac{F'G'}{\sqrt{|\alpha|}}, \tag{51}$$

  F and G being arbitrary functions, C, C_1, C_2, arbitrary constants such that α and f are everywhere nonvanishing.
- If $\det\mathbf{F}>0$, then by referring to Eqs. (43)–(47) one finds that

  $$\mathbf{M}_C(g)=\begin{pmatrix} \varepsilon\frac{|\nabla(\phi)|^2}{\sqrt{|D\phi+B|}}\mathbf{1}_2 & \mathbf{0} \\ \mathbf{0} & (D\phi+B)\mathbf{M} \end{pmatrix}.$$

  where $\varepsilon=\pm 1$, ϕ is a harmonic function, and D and B are constants such that $D\phi+B$ is everywhere nonvanishing and $\mathbf{M}$ is as above.

The case $h_{22}=0$. With this assumption the Abelian limit is, obviously, obtained from the corresponding non-Abelian result (Propositions 10 and 13) just by putting $s=0$. Namely:

- If $\det\mathbf{F}<0$, then (Proposition 10)

  $$\mathbf{M}_C(g)=\begin{pmatrix} \begin{matrix} -2f & 0 \\ 0 & 2f \end{matrix} & \mathbf{0} \\ \mathbf{0} & \mu\begin{pmatrix} w & 1 \\ 1 & 0 \end{pmatrix} \end{pmatrix}, \tag{52}$$

where

$$\mu = C_1 F(z_1 + z_2) + C_2 G(z_1 - z_2) + C, \tag{53}$$

$$f = |\mu|^{-\frac{1}{2}} F' G', \tag{54}$$

F, G and C, C_1, C_2, being arbitrary functions and constants, respectively, such that μ and f be everywhere nonvanishing while w is an arbitrary solution of the equation

$$(\mu w_{,1})_{,2} + (\mu w_{,2})_{,1} = 0.$$

- If $\det \mathbf{F} > 0$, then (Proposition 13)

$$\mathbf{M}_C(g) = \begin{pmatrix} \varepsilon \frac{|\nabla(\phi)|^2}{\sqrt{|D\phi+B|}} \mathbf{1}_2 & \mathbf{0} \\ \mathbf{0} & (D\phi + B) \begin{pmatrix} w & 1 \\ 1 & 0 \end{pmatrix} \end{pmatrix}, \tag{55}$$

where $\varepsilon = \pm 1$, ϕ is a harmonic function, D and B are arbitrary constants such that $\mu = D\phi + B$ is everywhere nonvanishing and w is an arbitrary solution of the equation

$$(\mu w_{,1})_{,1} + (\mu w_{,2})_{,2} = 0.$$

Remark 8. It is worth to note that in the Abelian case the Gauss curvature of the Killing leaves is equal to zero.

7. Ricci-flat metrics admitting a 3-dimensional Killing algebra with bidimensional leaves

Let g be a metric and $\mathcal{G}$ be one of its Killing algebras. In what follows, the Killing algebra $\mathcal{G}$ will be called *normal* if the restrictions of g to its Killing leaves are non-degenerate.

Obviously, a normal Killing algebra $\mathcal{G}$ is isomorphic to a subalgebra of $\mathcal{K}il(g|_S)$ where S is a generic Killing leaf of $\mathcal{G}$. Thus, when $\dim \mathcal{G} = 3$ and the Killing leaves are bidimensional, $\mathcal{G} = \mathcal{K}il(g|_S)$. As it is easy to see, in this situation there are exactly five options for $\mathcal{K}il(g|_S)$ and, therefore, for $\mathcal{G}$. Namely, they are:

$$so(2,1), \qquad \mathcal{K}il\big(dx^2 - dy^2\big), \qquad so(3), \qquad \mathcal{K}il\big(dx^2 + dy^2\big), \qquad \mathcal{A}_3, \tag{56}$$

where $\mathcal{A}_3$ is a 3-dimensional Abelian Lie algebra. Since the Lie algebra $\mathcal{A}_3$ belongs to the case treated in [4] it will not be considered in the following.

Only two of these algebras, namely $so(2,1)$ and $\mathcal{K}il(dx^2 - dy^2)$, possess a non-commutative bidimensional subalgebra. Thus, one may expect that the corresponding Ricci flat 4-metrics are among the solutions described in Section 5. It will be shown below that this is in fact true and that they belong to one of the cases $h_{22} \neq 0$, or $h_{22} = 0$ with w fixed to be constant (see Section 5).

As for the algebra $\mathcal{K}il(dx^2 + dy^2)$, it has only a bidimensional commutative subalgebra and we shall see that the corresponding Ricci-flat 4-metrics are among the solutions described in the previous Section 6 (the Abelian limit with $h_{22} \neq 0$).

The following assertion generalizes Lemma 1 (Section 2).

Lemma 14. *Let X_1, X_2 and $f_1X_1 + f_2X_2$, $f_1, f_2 \in C^\infty(M)$, be Killing fields of a metric g. Then, supposing that X_1 and X_2 are independent, either f_1 and f_2 are functionally independent, or f_1 and f_2 are constant.*

Proof. It results from relation (1) taking into account $L_{X_1}(g) = L_{X_2}(g) = 0$ that

$$0 = L_{f_1X_1+f_2X_2}(g) = i_{X_1}(g)\,df_1 + i_{X_2}(g)\,df_2. \tag{57}$$

Assuming, say, that $f_2 = \varphi(f_1)$ we see that

$$0 = L_{f_1X_1+f_2X_2}(g) = \bigl(i_{X_1}(g) + \varphi' i_{X_2}(g)\bigr)\,df_1 = i_{X_1+\varphi' X_2}(g)\,df_1.$$

If $df_1 \neq 0$, then the last equality implies, obviously, $i_{X_1+\varphi' X_2}(g) = 0$. In that case, $X_1 + \varphi' X_2 = 0$ due to the non-degeneracy of g in contradiction with the assumed independence of X_1 and X_2. If on the contrary $df_1 = 0$, then $df_2 = 0$ and the second alternative takes place.

Note that it cannot happen that on a connected manifold M the first alternative takes place in $U_1 \subset M$ and the second one in $U_2 \subset M$ if $\bigcap_i U_i \neq \emptyset$. It results from the fact that if a Killing field vanishes on an open subset of M, then it vanishes everywhere. □

Corollary 15. *If $\overline{\mathcal{G}}$ is a 3-dimensional Killing algebra having bidimensional Killing leaves and the fields X_1, X_2, X_3 generate it as a linear space, then almost everywhere $X_3 = f_1X_1 + f_2X_2$ and f_1 and f_2 are functionally independent.*

Proof. The fields X_1 and X_2 are independent according to Lemma 1. So they generate almost everywhere, say in U, the tangent spaces to the Killing leaves. Thus, $X_3 = f_1X_1 + f_2X_2$, $f_i \in C^\infty(U)$. The possibility that f_1 and f_2 be constant offered by Lemma 1 cannot occur in this context since X_1, X_2 and X_3 are supposed to be linearly independent. □

Proposition 16. *Any Killing algebra from the list* (56) *having bidimensional Killing leaves is normal. Moreover, the distribution $\mathcal{D}^\perp$ orthogonal to its Killing leaves is integrable.*

Proof. Below the notation of Corollary 15 is used. Since df_1 and df_2 are almost everywhere point-wise independent and one can deduce easily from (57) that

$$i_{X_1}(g) = \lambda\,df_2, \qquad i_{X_2}(g) = -\lambda\,df_1, \tag{58}$$

being g nondegenerate, λ is almost everywhere non-vanishing.

Let now Y be an almost everywhere nonvanishing vector field. Then the equality

$$i_Y\bigl(i_{X_1}(g)\,df_1 + i_{X_2}(g)\,df_2\bigr) = 0,$$

which is an obvious consequence of (57), is equivalent to

$$g(X_1, Y)\,df_1 + g(X_2, Y)\,df_2 = -Y(f_1)i_{X_1}(g) - Y(f_2)i_{X_2}(g).$$

In view of (58) it gives

$$g(X_1, Y)\,df_1 + g(X_2, Y)\,df_2 = -\lambda Y(f_1)\,df_2 + \lambda Y(f_2)\,df_1,$$

so that

$$g(X_1, Y) = \lambda Y(f_2), \qquad g(X_2, Y) = -\lambda Y(f_1).$$

Hence $Y(f_1) = Y(f_2) = 0$ iff $g(X_1, Y) = g(X_2, Y) = 0$, i.e., such fields Y are orthogonal to the Killing leaves and *vice versa*. If Y is tangent to the Killing leaves, then

$$Y(f_1) = Y(f_2) = 0 \iff Y = 0,$$

since by the above corollary applied to the case $M = S$, $df_i|_S$ is nondegenerate for a generic Killing leaf S. This proves that the fields Y such that $Y(f_1) = Y(f_2) = 0$ are transversal to the Killing leaves and that $g|_S$ is non-degenerate for a generic Killing leaf S. Thus $\mathcal{G}$ is normal.

Finally note that the distribution $\widetilde{\mathcal{D}}$ spanned by the vector fields Y such that $Y(f_1) = Y(f_2) = 0$ is of co-dimension 2 since df_1 and df_2 are independent almost everywhere. Being both transversal and orthogonal to the Killing leaves, $\widetilde{\mathcal{D}}$ coincides with $\mathcal{D}^\perp$ by a dimension argument. □

Corollary 17. *The solutions found in Section* 5 *exhaust all local Ricci-flat* 4*-metrics admitting a Killing algebra isomorphic to* $so(2,1)$ *or to* $\mathcal{K}il(dx^2 - dy^2)$.

Proof. As we already noticed, the first two algebras possess non-Abelian bidimensional subalgebras and according to the previous proposition the distribution $\mathcal{D}^\perp$ orthogonal to Killing leaves is transversal to them and integrable. □

7.1. $\mathcal{K}il(dx^2 + dy^2)$-invariant Ricci-flat metrics

As it has been already noticed, the algebra $\mathcal{K}il(dx^2 + dy^2)$ has a bidimensional commutative subalgebra. We shall see that the corresponding Ricci-flat 4-metrics are among the solutions of previous Section 6 (the Abelian limit with $h_{22} \neq 0$).

First, let $\mathcal{G}$ be a Killing algebra isomorphic to $\mathcal{K}il(dx^2 + dy^2)$ and let X_i, $i = 1, 2, 3$, be its standard basis, i.e.,

$$[X_1, X_2] = 0, \qquad [X_1, X_3] = X_2, \qquad [X_2, X_3] = -X_1.$$

With the notation of Corollary 15, let $X_3 = f_1 X_1 + f_2 X_2$. Then

$$X_2 = [X_1, X_3] = [X_1, f_1 X_1 + f_2 X_2] = X_1(f_1) X_1 + X_1(f_2) X_2$$

and

$$X_1 = [X_3, X_2] = [f_1 X_1 + f_2 X_2, X_2] = -X_2(f_1) X_1 - X_2(f_2) X_2,$$

so that, for the independence (Section 2, Lemma 1) of X_1 and X_2, implies that we have

$$X_1(f_1) = 0, \qquad X_1(f_2) = 1,$$
$$X_2(f_1) = -1, \qquad X_2(f_2) = 0.$$

Joining to f_1, f_2 a couple of independent functions z_1, z_2 such that $X_i(z_j) = 0$, $\forall i, j$, one gets a local chart on M. Taking into account the above relations and passing to the standard coordinate notation $x = f_1$, $y = f_2$, we see that in the chart (x, y, z_1, z_2)

$$X_1 = \partial_y, \qquad X_2 = -\partial_x, \qquad X_3 = x\partial_y - y\partial_x.$$

Introducing on S *polar coordinates* (r, φ), i.e., $x = r\cos\varphi$, $y = r\sin\varphi$, the above fields read as

$$X_1 = \sin\varphi\,\partial_r + \frac{\cos\varphi}{r}\partial_\varphi, \qquad X_2 = \cos\varphi\,\partial_r + \frac{\sin\varphi}{r}\partial_\varphi, \qquad X_3 = \partial_\varphi.$$

Then, in view of Proposition 16, a direct computation similar to the one of Section 2 shows that any $\mathcal{G}$-invariant metric has in the adapted local chart (z_1, z_2, r, φ) the form

$$g = 2f\left(dz_1^2 + \varepsilon\, dz_2^2\right) + \mu(z_1, z_2)\left[dr^2 + r^2\, d\varphi^2\right],$$

and, therefore, belongs to the class of metrics considered in Section 6 with definite $\mathbf{H}$ and $h_{22} \neq 0$.

Thus, we have:

Corollary 18. *The solutions found in Section* 6 *exhaust all local Ricci-flat* 4*-metrics admitting a Killing algebra isomorphic to* $\mathcal{K}il(dx^2 + dy^2)$.

7.2. *so*(3)*-invariant Ricci-flat metrics*

The above results lead to expect that Ricci-flat 4-metrics admitting a Killing algebra isomorphic to *so*(3) with 2-dimensional leaves can be described essentially in the same way as it was done in Section 5 with respect to those admitting a Killing algebra isomorphic to *so*(2, 1). The details are as follows.

First, let $\mathcal{G}$ be a Killing algebra isomorphic to *so*(3) and let $X_i,\ i = 1, 2, 3$, be its standard basis, i.e.,

$$[X_1, X_2] = X_3, \qquad [X_2, X_3] = X_1, \qquad [X_3, X_1] = X_2.$$

In the notation of Corollary 15 let $X_3 = f_1X_1 + f_2X_2$. Then

$$\begin{aligned} X_1 &= [X_2, X_3] = [X_2, f_1X_1 + f_2X_2] \\ &= X_2(f_1)X_1 + f_1[X_2, X_1] + X_2(f_2)X_2 \\ &= \left(X_2(f_1) - f_1^2\right)X_1 + \left(X_2(f_2) - f_1f_2\right)X_2. \end{aligned}$$

Since X_1 and X_2 are independent (Lemma 1)

$$X_2(f_1) - f_1^2 = 1, \qquad X_2(f_2) - f_1f_2 = 0. \tag{59}$$

Similarly, from the relation $[X_3, X_1] = X_2$ one finds

$$X_1(f_1) + f_1f_2 = 0, \qquad X_1(f_2) + f_2^2 = -1. \tag{60}$$

Joining to f_1, f_2 a couple of independent functions z_1, z_2 such that $X_i(z_j) = 0,\ \forall i, j$, one gets a local chart on M. Taking into account relations (59) and (60) and passing to the standard coordinate notation $x = f_1$, $y = f_2$, we see that in the chart (x, y, z_1, z_2)

$$X_1 = -xy\partial_x - \left(1 + y^2\right)\partial_y, \qquad X_2 = \left(x^2 + 1\right)\partial_x + xy\partial_y, \qquad X_3 = y\partial_x - x\partial_y.$$

In the *geographic coordinates* (r, φ), i.e., $x = \tan\vartheta\cos\varphi,\ y = \tan\vartheta\sin\varphi$, the above fields read as

$$X_1 = -\frac{\cos\varphi}{\tan\vartheta}\partial_\varphi - \sin\varphi\,\partial_\vartheta, \qquad X_2 = -\frac{\sin\varphi}{\tan\vartheta}\partial_\varphi + \cos\varphi\,\partial_\vartheta, \qquad X_3 = -\partial_\varphi.$$

Then, in view of Proposition 16, a direct computation similar to the one of Section 2 shows that any $\mathcal{G}$-invariant metric has in the adapted local chart $(z_1, z_2, \vartheta, \varphi)$ the form

$$g = f\left(dz_1^2 + \varepsilon\, dz_2^2\right) + \alpha(z_1, z_2)\left[d\vartheta^2 + \sin^2\vartheta\, d\varphi^2\right]. \tag{61}$$

The Ricci tensor of the above metric can be easily computed as in Section 4 and the corresponding Einstein equations lead to the same equations for f and $\alpha \equiv r^2$ as already found in Section 5 in the case

$h_{22} \neq 0$. Namely,

$$f = -\frac{1}{2}\left(\frac{\partial^2}{\partial z_1^2} + \varepsilon\frac{\partial^2}{\partial z_2^2}\right)(r^2), \tag{62}$$

$$r + A\ln|r - A| = u, \tag{63}$$

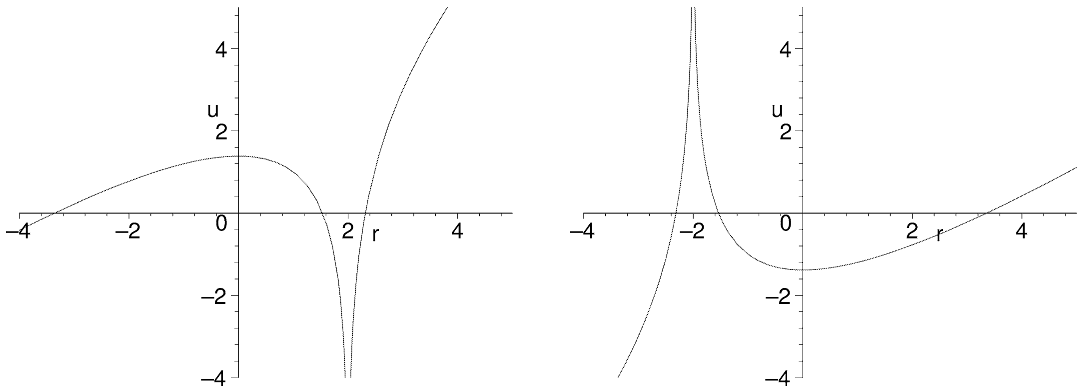
with $\varepsilon = \pm 1$, A being an arbitrary constant and u being an arbitrary function satisfying the equation

$$\left(\frac{\partial^2}{\partial z_1^2} + \varepsilon\frac{\partial^2}{\partial z_2^2}\right)(u) = 0.$$

Additionally, f is required to be nonvanishing.

Remark 9. In the case $\varepsilon = -1$, these solutions are locally diffeomorphic to the Schwarzschild solution. This will be discussed in [10,11].

Below, the graph of the left hand side of Eq. (63) is reported for the values $A = 2$ and $A = -2$.



$u = r + 2\ln|r - 2|$ $\quad\quad$ $u = r - 2\ln|r + 2|$

One can see that for $A \neq 0$ there exactly three possibilities for $r = r(u)$ that correspond to the intervals of monotonicity of $u(r)$. For instance, for $A > 0$ these are $]-\infty, 0[$, $]0, A[$, and $]A, \infty[$. In these regions the corresponding metric (61) is regular and has some singularities along the curves $r = 0$ and $r - A = 0$.

Some geometrical peculiarities of the obtained *local* solutions show how to match them together in order to get *global nonextendible* Einstein metrics. To this purpose, in [10,11] a formalism is developed which allows to construct, starting from known solutions, "new" global ones and to describe their singularities as well. For instance, by extracting the *square root* of the Schwarzschild solution, one easily finds an Einstein metric which describes *parallel universes*. Other examples which illustrate some aspects of our approach can be found in [10,11]. We stress that it generalizes naturally to some other situations as, for instance, *cosmological Einstein metrics* satisfying assumptions I and II (work in progress).

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