ELSEVIER

Twisted loop groups and their affine flag varieties

G. Pappas [a,*,1], M. Rapoport [b]

[a] *Department of Mathematics, Michigan State University, E. Lansing, MI 48824-1027, USA*
[b] *Mathematisches Institut der Universität Bonn, Beringstrasse 1, 53115 Bonn, Germany*

Abstract

We develop a theory of affine flag varieties and of their Schubert varieties for reductive groups over a Laurent power series local field $k((t))$ with k a perfect field. This can be viewed as a generalization of the theory of affine flag varieties for loop groups to a "twisted case"; a consequence of our results is that our construction also includes the flag varieties for Kac–Moody Lie algebras of affine type. We also give a coherence conjecture on the dimensions of the spaces of global sections of the natural ample line bundles on the partial flag varieties attached to a fixed group over $k((t))$ and some applications to local models of Shimura varieties.
© 2008 Elsevier Inc. All rights reserved.

Keywords: Loop group; Affine Kac–Moody algebra; Schubert variety; Shimura variety

0. Introduction

Loop groups are familiar objects in several branches of mathematics. Let us mention here three variants.

The first variant is differential-geometric in nature. One starts with a Lie group G (e.g., a compact Lie group or its complexification). The associated loop group is then the group of (C^0-, or C^1-, or C^∞-) maps of S^1 into G, cf. [41] and the literature cited there. A twisted version arises

---

* Corresponding author.
  E-mail addresses: pappas@math.msu.edu (G. Pappas), rapoport@math.uni-bonn.de (M. Rapoport).

from an automorphism α of G. The associated twisted loop group is the group of maps $\gamma : \mathbf{R} \to G$ such that

$$\gamma(\theta + 2\pi) = \alpha\big(\gamma(\theta)\big).$$

The second variant is algebraic and arises in the context of Kac–Moody algebras. Here one constructs an infinite-dimensional algebraic group variety with Lie algebra equal or closely related to a given Kac–Moody algebra. (This statement is an oversimplification and the situation is in fact more complicated: there exist various constructions at a formal, a minimal, and a maximal level which produce infinite-dimensional groups with Lie algebras closely related to the given Kac–Moody Lie algebra, see [34,17], also [49,50] and the literature cited there.)

The third variant is algebraic-geometric in nature and is our main concern in this paper. Let us recall the basic definitions in the untwisted case. Let k be a field and let G_0 be an algebraic group over $\mathrm{Spec}(k)$. We consider the functor LG_0 on the category of k-algebras,

$$R \mapsto LG_0(R) = G_0\big(R((t))\big).$$

Here $R((t)) = R[[t]][1/t]$ is the ring of Laurent series with coefficients in R. One proves that this functor is representable by an ind-scheme (= inductive limit of k-schemes), called *the algebraic loop group* associated to G_0. At the same time, one considers the associated flag varieties, mostly the affine Grassmannian $\mathcal{G}$ and the affine flag variety $\mathcal{F}$, the *fpqc*-sheaves associated to the functors

$$R \mapsto G_0\big(R((t))\big)/G_0\big(R[[t]]\big), \quad \text{resp. } R \mapsto G_0\big(R((t))\big)/\mathcal{B}(R).$$

Here $\mathcal{B}(R)$ is the inverse image of a Borel subgroup under the reduction map $G_0(R[[t]]) \to G_0(R)$. These sheaves are also representable by ind-schemes. Important results on the structure of algebraic loop groups and their associated flag varieties are due to Beauville, Laszlo, Sorger, Faltings, Beilinson, Drinfeld and Gaitsgory, cf. [2,3,30,13,4,15]. These results have applications in the theory of vector bundles on algebraic curves [2,30,13], in geometric Langlands theory [4,15], and to local models of Shimura varieties [18,19,38].

The aim of the present paper is to develop a similar theory of twisted algebraic loop groups and of their associated flag varieties. As it turns out, this theory gives a geometric interpretation of many aspects of Bruhat–Tits theory in the equal characteristic case.

The basic definition is very simple. Let G be a linear algebraic group over $K = k((t))$ (we always assume that G is connected reductive). Then G defines its associated algebraic loop group LG, which is the ind-scheme representing the functor

$$R \mapsto LG(R) = G\big(R((t))\big).$$

Note that, since $R((t))$ is a $k((t))$-algebra, the right-hand side makes sense. When $G = G_0 \otimes_k K$, one recovers the definition in the untwisted case. To any facet $\underline{a}$ in the Bruhat–Tits building of G over K there is associated a unique smooth group scheme with connected fibers $P_{\underline{a}}$ over $k[[t]]$ such that $P_{\underline{a}}(k[[t]])$ is equal to the parahoric subgroup of $G(K)$ attached to $\underline{a}$. To $P_{\underline{a}}$ corresponds an infinite-dimensional affine group scheme $L^+ P_{\underline{a}}$ over $\mathrm{Spec}(k)$ with

$$L^+ P_{\underline{a}}(R) = P_{\underline{a}}\big(R[[t]]\big).$$

The quotient $\mathcal{F}_{\underline{a}} = LG/L^+P_{\underline{a}}$ (in the sense of *fpqc*-sheaves) is representable by an ind-scheme over k which we call the (partial) affine flag variety associated to G and $\underline{a}$. In the untwisted case, the affine Grassmannian is associated to the special vertex arising from an isomorphism $G \simeq G_0 \otimes_k k((t))$, and the affine flag variety is associated to an alcove in the Bruhat–Tits building containing this vertex. Note that even in the untwisted case, the definition of $\mathcal{F}_{\underline{a}}$ is useful, as it re-establishes the natural symmetry attached to the K-group scheme $G_0 \otimes_k \bar{K}$ which has no preferred special vertex.

Our main results may now be formulated as follows.

Let $\pi_1(G)$ denote the algebraic fundamental group of G (see [5]). The group $\pi_1(G)$ can be defined as the quotient $P^\vee(G)/Q^\vee(G)$ of the coweight lattice by the coroot lattice of $G(K^{\mathrm{sep}})$. If k is algebraically closed, let $I = \mathrm{Gal}(\bar{K}/K)$ denote the inertia group. Let $\pi_1(G)_I$ be the co-invariants under I.

Theorem 0.1. *Let k be algebraically closed. The Kottwitz homomorphism (*[27]*, see* 2.a.2*) induces bijections on the set of connected components,*

$$\pi_0(LG) = \pi_1(G)_I, \qquad \pi_0(\mathcal{F}_{\underline{a}}) = \pi_1(G)_I.$$

Note that in the untwisted case $G = G_0 \otimes_k K$, the action of I is trivial so that $\pi_1(G)_I = \pi_1(G)$ and we recover the result of Beauville, Laszlo and Sorger [3, Lemma 1.2], resp. of Beilinson and Drinfeld [4, Prop. 4.5.4], which is analogous to the corresponding result in topology.

Theorem 0.2. *Suppose that G is semi-simple and splits over a tamely ramified extension of K and that the order of the fundamental group of the derived group $\pi_1(G_{\mathrm{der}})$ is prime to the characteristic of k. Then the ind-schemes LG and $\mathcal{F}_{\underline{a}}$ are reduced.*

In the case of SL_n this is due to Beauville and Laszlo [2], and in the case of a general split group and when k has characteristic 0, to Beilinson and Drinfeld [4, Thm. 4.5.1]. In characteristic $p > 0$, this is due to Laszlo and Sorger [30], and to Faltings [13], when G is split and simply connected.

We next consider the Schubert varieties contained in a partial affine flag variety $\mathcal{F}_{\underline{a}}$. By definition, these are the reduced closures of orbits of $L^+P_{\underline{a}}$. They are finite-dimensional projective varieties.

Theorem 0.3. *Suppose that G splits over a tamely ramified extension of K and that the order of the fundamental group of the derived group $\pi_1(G_{\mathrm{der}})$ is prime to the characteristic of k. Then all Schubert varieties in $\mathcal{F}_{\underline{a}}$ are normal and have only rational singularities. In positive characteristic, they are Frobenius-split.*

In characteristic zero, normality of Schubert varieties in the context of Kac–Moody partial affine flag varieties is due to Kumar [28], Littelmann [31] and Mathieu [33]. By an argument of Faltings these coincide with the Schubert varieties in this paper. In positive characteristic the above theorem is due to Faltings [13] in the split simply connected case. (The normality of Schubert varieties in the case of SL_n is proved in [37].) Normality of Schubert varieties in characteristic p has also been shown by Mathieu [33] and Littelmann [31] (in the context of Kac–Moody theory). Their definition of Schubert varieties is a priori different. They are given using an embedding into the infinite-dimensional projective space associated to a highest-weight

representation of the Kac–Moody algebra. In contrast, we are essentially defining Schubert varieties as varieties of certain lattices and so [33] and [31] does not imply our result. Nevertheless, it is a corollary of Theorem 0.3 combined with the results of Mathieu and Littelmann that the two notions of Schubert varieties coincide (see Section 9.h). From this, it also follows a posteriori that (for an absolutely simple, simply connected G that splits over a tame extension) our partial affine flag varieties are isomorphic to the ones defined in the Kac–Moody theory.

Let us now comment on our proofs of the above theorems. The proofs of Theorems 0.2 and 0.3 are closely intertwined. The proofs go along the lines of the proof of the theorem corresponding to Theorem 0.3 of Faltings [13] (for more details on this proof, cf. [19]). The main method is to lift the situation in positive characteristic to characteristic zero and compare the lifted situation with the Kac–Moody setting. It is therefore somewhat reminiscent of the method introduced by Kazhdan to relate the representation theory of a group over a local field of characteristic p to that of a group over a local field of unequal characteristic. This also explains why we have to exclude groups which split over a wild extension (although we strongly believe that appropriate versions of Theorems 0.2 and 0.3 also hold in this case). We note that in the previous proofs of Theorem 0.2 (in [2] for the case of SL_n, in [30] for a split semi-simple, simply connected group in characteristic 0, in [13,19] for a semi-simple split simply connected group in characteristic p) all use the big cell. Our argument, which we learned in Faltings' course [14] is different. Finally, the proof of Theorem 0.1 generalizes the proof of the corresponding theorem in [4], using the construction of the Kottwitz map [27].

We would like to stress that interesting problems arise when one tries to generalize to the setting of twisted loop groups classical results in the untwisted case. The theory of perverse sheaves on their partial flag varieties has still to be worked out. Also, we expect a theory of bundles on algebraic curves for corresponding non-constant algebraic groups, of their Picard groups and their theta functions.

Our main motivation for developing the theory of twisted loop groups is its application to the theory of local models of Shimura varieties. These are schemes defined in linear algebra terms that describe the étale local structure of integral models for Shimura varieties and other moduli spaces [42]. Consider the group G of unitary similitudes corresponding to a ramified quadratic extension of K. Then G is not of the form $G = G_0 \otimes_k K$. Generalizing the method of Görtz [18], we wish to embed the special fiber of the naive local model associated to a ramified group of unitary similitudes over a p-adic local field into the affine flag variety $\mathcal{F}$ of G. At this point we require knowledge of the structure of $\mathcal{F}$ which we then can in turn use to deduce structure results on the local model. This is the way in which we will use the results of this paper. To give the taste of this application we explain in the last section this method in a representative simple case.

This method hinges on a coherence conjecture on the dimensions of the spaces of global sections of the natural ample invertible sheaves on partial flag varieties attached to a fixed group G over K. This conjecture, which may be stated in purely combinatorial terms, seems to us of independent interest. We show that in the cases of SL_n and Sp_{2n} it follows from previous results [18,19,38,39] on local models. The main point of the application we have in mind here is to turn the logic around and deduce facts on local models from this coherence conjecture.

We now explain the lay-out of the paper. In Section 1 we give the construction of twisted loop groups and their quotients. In Section 2 we recall the definition of parahoric subgroups and introduce the (generalized) affine flag varieties. In Section 3 we make these notions explicit in the case of tori. In Section 4 we discuss in some detail the case of a unitary group and relate their partial affine flag varieties to the lattice model. As explained above, this example plays a central role

in the applications of our results. Section 5 is devoted to the proof of Theorem 0.1. In Section 6 we discuss Theorem 0.2 and reduce its proof to the case of a simple simply connected group. In Section 7 we explain how to lift the group G and its partial flag varieties to mixed characteristic, i.e., from $k((t))$ to $W((t))$, where W denotes the ring of Witt vectors of k. In Section 8 we introduce the Schubert varieties in twisted affine flag varieties and their Demazure resolutions. The proofs of Theorems 0.2 and 0.3 are given in Section 9. In Section 10 we state and discuss the coherence conjecture mentioned above. In the final section, we discuss how we apply our results to the local models of Shimura varieties associated to ramified unitary groups. As mentioned above, these applications depend on a positive solution to the coherence conjecture.

1. General constructions

Let k be a field. We set $K = k((t))$ for the field of Laurent power series with indeterminate t and coefficients in k. Let $V = k[[t]] \subset K$ be the discretely valued ring of power series with coefficients in k.

1.a. Let X be a scheme over $K = k((t))$. We denote by LX the functor from the category of k-algebras to that of sets given by

$$R \mapsto LX(R) := X\big(\operatorname{Spec}\big(R((t))\big)\big).$$

If $\mathcal{X}$ is a scheme over V, we denote by $L^+\mathcal{X}$ the functor from the category of k-algebras to that of sets given by

$$R \mapsto L^+\mathcal{X}(R) = \mathcal{X}\big(\operatorname{Spec}\big(R[[t]]\big)\big).$$

The functors LX, $L^+\mathcal{X}$ give sheaves of sets for the fpqc topology on k-algebras. In what follows, for simplicity, we will call such functors "k-spaces." Recall that a k-space is called an ind-scheme (resp. a strict ind-scheme) if it is the inductive limit of the functors associated to a directed family of k-schemes (resp. via transition morphisms which are closed embeddings). A group ind-scheme is an ind-scheme which is a group object in the category of ind-schemes.

If $\mathcal{X} = \operatorname{Spec}(A)$ is affine of finite type, $L^+\mathcal{X}$ is represented by an affine scheme. If $\mathcal{X} = \mathbf{A}_V^r$ is the affine space of dimension r over $\operatorname{Spec}(V)$, then $L^+\mathcal{X}$ is the infinite-dimensional affine space $L^+\mathcal{X} = \prod_{i=0}^{\infty}(\mathbf{A}^r)$, via

$$L^+\mathcal{X}(R) = \operatorname{Hom}_{k[[t]]}\big(k[[t]][T_1, \ldots, T_r], R[[t]]\big) = R[[t]]^r = \prod_{i=0}^{\infty} R^r = \prod_{i=0}^{\infty} \mathbf{A}^r(R).$$

Let $\mathcal{X}$ be the closed subscheme of $\mathbf{A}_V^r$ defined by the vanishing of a polynomial f in $k[[t]][T_1, \ldots, T_r]$. Then $L^+\mathcal{X}(R)$ is the subset of $L^+\mathbf{A}^r(R)$ of $k[[t]]$-algebra homomorphisms $k[[t]][T_1, \ldots, T_r] \to R[[t]]$ which factor through $k[[t]][T_1, \ldots, T_r]/(f)$. Write $f = \sum a_{\underline{n}}(t) \cdot T^{\underline{n}}$. Then $L^+\mathcal{X}$ is defined by the vanishing of the infinite set of polynomials in the variables $X_{1,\bullet}, X_{2,\bullet}, \ldots, X_{r,\bullet}$ with $\bullet = 0, 1, 2, \ldots$,

$$\sum_{\underline{n}} C_i^{(\underline{n})}(a_{\underline{n},\bullet}; X_{1,\bullet}, \ldots, X_{r,\bullet}) = 0, \quad i = 0, 1, \ldots.$$

Here the polynomials $C_i^{(n_1,...,n_r)}$ in the indeterminates $a_0, \ldots, a_i, X_{1,0}, \ldots, X_{1,i}, \ldots,$ $X_{r,0}, \ldots, X_{r,i}$ are defined by the identity

$$\left(\sum_{i=0}^{\infty} a_i \cdot t^i\right) \cdot \left(\sum_{i=0}^{\infty} X_{1,i} \cdot t^i\right)^{n_1} \cdots \left(\sum_{i=0}^{\infty} X_{r,i} \cdot t^i\right)^{n_r} = \sum_{i=0}^{\infty} C_i^{(n_1,...,n_r)} \cdot t^i.$$

Note that if $\mathcal{X}$ is an affine k-scheme, which we regard as an affine scheme over V via $V \to k$ which sends t to 0, then $L^+\mathcal{X} = \emptyset$. If X is an affine K-scheme, then $L^+X = \emptyset$. On the other hand, in this case LX is represented by a strict ind-scheme. Finally, we mention that $L(X \times_{\mathrm{Spec}\,K} Y) = LX \times_{\mathrm{Spec}\,k} LY$, and similarly for L^+.

Definition 1.1. Let G be a linear algebraic group over K. The *loop group* associated to G is the ind-group scheme LG over $\mathrm{Spec}(k)$.

In the case $G = GL_n$, the fact that LG is an ind-group scheme is well known, comp. [2, Prop. 1.2]. In fact, if G is of the form $G = G_0 \times_{\mathrm{Spec}(k)} \mathrm{Spec}(K)$ for a linear algebraic group G_0 over k, then the notion of a loop group is in [4,2,13].

1.a.1. We now list some easy functoriality properties of this construction.

(a) If k' is a k-field extension then we have an isomorphism of ind-schemes over k'

$$LG \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k') \simeq L\big(G \times_{\mathrm{Spec}(k((t)))} \mathrm{Spec}\big(k'((t))\big)\big). \tag{1.1}$$

(b) Assume that K'/K is a finite extension of K; then a choice of uniformizer u for K' allows us to write $K' = k'((u))$. If $G = \mathrm{Res}_{K'/K} H$ for some linear algebraic group H over K', then the choice of u gives an isomorphism of ind-schemes over k',

$$LG \simeq \mathrm{Res}_{k'/k}(LH). \tag{1.2}$$

In particular, if $k' = k$ then we have

$$L(\mathrm{Res}_{K'/K} H) \simeq LH. \tag{1.3}$$

Indeed, for a k-algebra R, we have

$$(LG)(R) = G\big(R((t))\big) = H\big(R((t)) \otimes_{k((t))} k'((u))\big) = H\big((R \otimes_k k')((u))\big)$$
$$= LH(R \otimes_k k') = \mathrm{Res}_{k'/k}(LH)(R).$$

(Note that $R((t)) \otimes_{k((t))} k'((u)) = (R \otimes_k k')((u))$ uses that $k'((u))/k((t))$ is finite.)

1.a.2. We will denote by $Lie(LG)$ the Lie algebra functor of the group functor of LG over k (see for example [10, II, §4, 1]). By definition

$$Lie(LG)(R) = \ker\big(G\big(R[\epsilon]((t))\big) \to G\big(R((t))\big)\big). \tag{1.4}$$

The points $Lie(LG)(k)$ form a k-vector space. If R is a k-algebra which is finite-dimensional over k and $J \subset R$ is an ideal of square 0 then there is an exact sequence

$$0 \to Lie(LG)(k) \otimes_k J \to LG(R) \to LG(R/J). \tag{1.5}$$

1.a.3. Suppose that k is algebraically closed and G is a connected reductive group over $K = k((t))$. By a theorem of Steinberg ([46], extended by Borel and Springer to the case of a non-perfect field, cf. [45, III, 2.3, Remark 1 after Theorem 1′]), every G-torsor over K is trivial. Let R be an Artinian k-algebra with algebraically closed residue field k'; then $R((t))$ is a complete local ring with residue field $k'((t))$. Hence, $R((t))$ is henselian and by the above, every G-torsor over $R((t))$ is trivial. It follows that if

$$1 \to G' \to G \to G'' \to 1$$

is an exact sequence of connected reductive groups over K and R is as above, the sequence

$$1 \to LG'(R) \to LG(R) \to LG''(R) \to 1$$

is exact.

1.b. Now let P be a flat affine group scheme of finite type over $\mathrm{Spec}(V)$, with associated affine group scheme L^+P over k. We wish to form the quotient of the generic fiber LP_η by L^+P.

Proposition 1.2. *Suppose that $Q \subset G$ are both flat affine group schemes of finite type over* $\mathrm{Spec}(V)$ *and that Q is a proper closed subgroup scheme of G. Then there exist a finitely generated free V-module $\Lambda \simeq V^n$ and a direct summand $\Lambda' \subset \Lambda$ of Λ of V-rank 1 such that:*

(a) *there is a representation $\rho : G \to \mathrm{GL}(\Lambda)$ that identifies G with a closed subgroup scheme of* $\mathrm{GL}(\Lambda)$,
(b) *the representation ρ identifies Q with the closed subgroup scheme of G that normalizes the V-line Λ'.*

Proof. This follows the argument of the proof of [10, II §2, 3.3]. Let $A(G)$, $A(Q)$ be the V-algebras of regular functions on G and Q respectively. Since Q and G are V-flat, these are V-torsion free. Denote by I the ideal of definition of Q in G so that $A(Q) = A(G)/I$. The group scheme G acts on $A(G)$ via the "regular representation." Using [44, §1, Prop. 2], and the fact that $A(G)$ is V-flat, we see that there is a finitely generated free V-submodule L of $A(G)$ with the following properties:

(i) G acts on L,
(ii) $A(G)/L$ is V-torsion free,
(iii) L generates the V-algebra $A(G)$ and is such that $I \cap L$ generates the ideal I.

Note that since Q is flat, I is also V-cotorsion free in $A(G)$. Set $L' = I \cap L$. Then we can see by following the proof of [10, II §2, 3.3], that $\Lambda = \bigwedge^{\mathrm{rank}(L')} L$ and $\Lambda' = \bigwedge^{\mathrm{rank}(L')} L' = \det(L')$ satisfy the requirements of the proposition. $\quad\square$

Proposition 1.3. *Suppose that P is a flat affine group scheme of finite type over* $\mathrm{Spec}(V)$. *Then*

(a) *P is linear, i.e., it is a closed subgroup scheme of the group scheme* GL_n *over* $\mathrm{Spec}(V)$, *for some n.*
(b) *There exists a group scheme closed immersion $P \subset \mathrm{GL}_n \times \mathbf{G}_m$ such that the quotient fppf sheaf $(\mathrm{GL}_n \times \mathbf{G}_m)/P$ is representable by a quasi-affine scheme over* $\mathrm{Spec}(V)$.

Proof. To obtain (a) we apply Proposition 1.2 to $Q = \{1\}$ and $G = P$. Now let us discuss the proof of (b). We apply Proposition 1.2 to $Q = P$ and $G = \mathrm{GL}_n$. Let $\chi : P \to \mathbf{G}_m = \mathrm{GL}(\Lambda')$ be the character of P giving the action on the V-line Λ' of Proposition 1.2. Now consider the embedding $i : P \to G' := \mathrm{GL}_n \times \mathbf{G}_m$ given by $p \mapsto (p, \chi^{-1}(p))$ and the representation $\rho' : G' \to \mathrm{GL}(\Lambda \otimes_V \Lambda')$ given by $\rho'((g, \lambda)) = \rho(g) \otimes \lambda$. Let $v \in \Lambda'$ be a generator of the V-line Λ', i.e., such that $\Lambda' = V \cdot v$. Then the subgroup scheme P is identified with the stabilizer of $v \in \Lambda$. Hence, the quotient $(\mathrm{GL}_n \times \mathbf{G}_m)/P$ (which is representable by a V-scheme by a general result of [1, Thm. 4.C]) is identified with the $\mathrm{GL}_n \times \mathbf{G}_m$-orbit of $v \in \Lambda$. A standard argument (cf. [10, proof of II §5, Prop. 3.1]) shows that this orbit is open in its Zariski closure and hence it is quasi-affine. $\quad\square$

1.c. Let P be a flat affine group scheme of finite type over V. Let P_η denote the generic fiber of P (a group scheme over K). We consider the quotient fpqc sheaf over $\mathrm{Spec}(k)$

$$\mathcal{F}_P := LP_\eta/L^+P. \tag{1.6}$$

By definition, this is the fpqc sheaf associated to the presheaf which to a k-algebra R associates the quotient $P(R((t)))/P(R[[t]])$.

1.c.1. When $P = \mathrm{GL}_n$, the fpqc sheaf $\mathcal{F}_P$ is (represented by) the affine Grassmannian of lattices in the vector space K^n. More precisely, for any k-algebra R, a lattice in $R((t))^n$ is a sub-$R[[t]]$-module $\mathcal{L}$ of $R((t))^n$ which is projective of rank n and is such that $\mathcal{L} \otimes_{R[[t]]} R((t)) = R((t))^n$. Consider the functor $\mathcal{F}_n$ which to a k-algebra R associates the set of lattices in $R((t))^n$. Then there is a natural isomorphism $\mathcal{F}_{\mathrm{GL}_n} \simeq \mathcal{F}_n$. It is well known [2, Prop. 2.3], that $\mathcal{F}_n$ is represented by an ind-k-scheme which is ind-proper over k (i.e., given as an increasing union of proper schemes over k). In this case ($P = \mathrm{GL}_n$), the quotient morphism

$$L\mathrm{GL}_n \to L\mathrm{GL}_n/L^+\mathrm{GL}_n = \mathcal{F}_{\mathrm{GL}_n}$$

admits sections locally for the Zariski topology, cf. [2, Thm. 2.5]. In the general case we have the following statement.

Theorem 1.4. *Let P be an affine group scheme which is smooth over V. Then the fpqc sheaf $\mathcal{F}_P$ is represented by an ind-k-scheme of ind-finite type over k. The quotient morphism $LP_\eta \to \mathcal{F}_P$ admits sections locally for the étale topology (i.e., L^+P-equivariant isomorphisms $\mathrm{Spec}(R) \times_{\mathcal{F}_P} LP_\eta \simeq \mathrm{Spec}(R) \times_{\mathrm{Spec}(k)} L^+P$ for each point of $\mathcal{F}_P$ with values in a strictly henselian ring R).*

Proof. The result is shown in [4, Theorem 4.5.1], or [15, Appendix], in the case when P is obtained by base change $P = P_0 \otimes_k V$, with P_0 an affine smooth group scheme over k. In fact, in these references it is shown that if P_0 is reductive, then $\mathcal{F}_P$ (which is then the affine

Grassmannian for P_0) is ind-proper over k. Here, we will also follow the argument in the proof of [4, Theorem 4.5.1]. By Proposition 1.3, there is an embedding $P \to G = \mathrm{GL}_n \times \mathbf{G}_m$ such that the quotient $U := G/P$ (over $\mathrm{Spec}(V)$) is quasi-affine, i.e., U is an open subscheme of an affine V-scheme Z. We will consider the functors LU, LZ, L^+U, L^+Z, defined as above. Then L^+U, L^+Z are schemes over k and L^+U is an open subscheme of L^+Z. Note that L^+U represents the fpqc quotient L^+G/L^+P: Actually, if R is strictly henselian so is $R[[t]]$ and since P is smooth, we have $U(R[[t]]) = G(R[[t]])/P(R[[t]])$. We can see that, since Z is affine over V, L^+Z is a closed subscheme of the ind-scheme LZ. Therefore, L^+U is a locally closed sub-ind-scheme of LZ. Now consider the natural morphism $f : LG \to LZ$ given by the quotient $G \to U = G/P$ followed by the open immersion $U \to Z$. Let $Y := f^{-1}(L^+U)$ be the inverse image of L^+U; it is a locally closed sub-ind-scheme of LG. Suppose that we have $g \in Y(R) \subset LG(R) = G(R((t)))$. We have $LG(R)/LP_\eta(R) \subset LU(R)$ and after an étale cover $R \to R'$, $f(g) \in L^+U(R)$ comes from an element of $L^+G(R')$ modulo $L^+P(R')$. Therefore, locally for the étale topology, we can write g as a product of an element from LP_η with one from L^+G. The ind-scheme Y is invariant under right translation by L^+G. Hence, and since $LG \to \mathcal{F}_G$ admits a section locally for the Zariski topology, Y is the preimage of a locally closed sub-ind-scheme $Y' \subset LG/L^+G = \mathcal{F}_G$; in fact, if the quotient G/P is affine, then $U = Z$ and Y' is closed in $\mathcal{F}_G$. In any case, Y' is of ind-finite type. Now observe that LP_η is contained in Y, since it maps to the trivial coset under f. This gives a natural morphism

$$\pi : LP_\eta \to Y'. \tag{1.7}$$

We claim that π identifies Y' with the fpqc quotient $\mathcal{F}_P = LP_\eta/L^+P$. It is clear that π induces a map of fpqc sheaves $LP_\eta/L^+P \to Y'$. To see that this gives an isomorphism note that $Y \to Y'$ is an L^+G-torsor, that the morphism $f : Y \to L^+U$ is L^+P-equivariant and that $LP_\eta = f^{-1}(e \bmod L^+P)$. This combined with the above gives the result. In fact, the proof shows that π admits a section locally for the étale topology. $\square$

2. Affine flag varieties

2.a.   Let G be a (connected) reductive group over the local field K. (Recall, that then, by [11, XXIII 5.6], G splits over the separable closure K^{sep} of K.) Our main examples of group schemes P to which we want to apply Theorem 1.4 are given by the group schemes associated to the parahoric subgroups of $G(K)$ by Bruhat and Tits [7,8]. These group schemes are affine smooth over $\mathrm{Spec}(V)$ and their generic fibers coincide with G. Since these are basic for what follows we take some space to properly set the definition and notation by tracing down the corresponding sections in [8]. The corresponding quotients $\mathcal{F}_P$ are the partial affine flag varieties and will be our main concern.

2.a.1.   Set $K^\flat = \bar{k}((t))$ with $\bar{k}$ the algebraic closure of k. Let $\mathcal{B} = \mathcal{B}(G(K))$ be the Bruhat–Tits building of $G(K)$ [7, 7.4]. Suppose that $\underline{a}$ is a facet in $\mathcal{B}$. This corresponds to a facet $\underline{a}^\flat$ of the building $\mathcal{B}(G(K^\flat))$ which is fixed under $\Gamma := \mathrm{Gal}(K^\flat/K) = \mathrm{Gal}(\bar{k}/k)$ [8, Thm. 5.1.25]. Now Bruhat and Tits associate to $\underline{a}^\flat$ the "connected stabilizer" group scheme $P_{\underline{a}}^\flat$ (see [8, 4.6]), comp. also [51, Section 7]. This comes with Γ-descent data (see [8, 5.1.9]) which give the group scheme $P_{\underline{a}}$. The V-valued points $P_{\underline{a}}(V)$ give the corresponding parahoric subgroup of $G(K)$ (see [8, p. 165]), and $P_{\underline{a}}$ is the unique extension to $\mathrm{Spec}(V)$ of G as a smooth affine group scheme with this property, cf. [8, 1.7].

Note that by construction $P_{\underline{a}}(V) = P_{\underline{a}^\flat}(V^\flat) \cap G(K)$.

2.a.2. Assume now that k is algebraically closed in which case $K^\flat = K$. Then we can also obtain the parahoric subgroup $P_{\underline{a}}(V)$ as follows [23, Prop. 3] (see also [42]): Consider the corresponding facet $\underline{a}'$ in the Bruhat–Tits building of the adjoint group $\mathcal{B}(G_{\mathrm{ad}}(K))$. Then $P_{\underline{a}}(V)$ is the intersection of the stabilizer of $\underline{a}'$ in $G(K)$ with the kernel of the Kottwitz homomorphism

$$\kappa_G : LG(k) = G(K) \to \pi_1(G)_I. \tag{2.1}$$

Here again we denote by $\pi_1(G)$ the $\mathrm{Gal}(K^{\mathrm{sep}}/K)$-module given by the algebraic fundamental group [5]. Recall $\pi_1(G)$ is the quotient $P^\vee(G)/Q^\vee(G)$ of the coweight lattice by the coroot lattice of $G(K^{\mathrm{sep}})$. The inertia group $I = \mathrm{Gal}(K^{\mathrm{sep}}/K)$ acts on $\pi_1(G)$ and we denote by $\pi_1(G)_I$ the coinvariants under this action.

Let us quickly recall the construction of the Kottwitz homomorphism in this case (cf. [27, §7], where the case is considered, in which $k((t))$ is replaced by the completion of the maximal unramified extension of a local p-adic field). It proceeds in four steps:

Step 1. The case that $G = \mathbf{G}_m$. Then $\pi_1(G) = \mathbf{Z}$ with trivial I-action. Write $f \in LG(k) = k((t))^\times$ in the form $f = t^k \cdot u$, $u \in k[[t]]^\times$. Then set $\kappa(f) = k$. This definition extends in the obvious way to the case when $G = \mathrm{Res}_{K'/K}(\mathbf{G}_m)$ and more generally to the case of an induced torus $G = \prod_i \mathrm{Res}_{K_i/K}(\mathbf{G}_m^{n_i})$.

Step 2. The case that $G = T$ is a torus. Note that in this case we can identify $\pi_1(T)$ with the cocharacter lattice $X_*(T)$. As in [27] one uses an exact sequence

$$R \xrightarrow{f} S \xrightarrow{g} T \to 1,$$

where R and S are induced tori and $\ker(f)$, $\ker(g)$ are tori. This induces an exact sequence

$$\pi_1(R)_I \to \pi_1(S)_I \to \pi_1(T)_I \to 1$$

and this together with 1.a.3 and Step 1 suffices to construct κ_T.

Step 3. The case where the derived group G_{der} of G is simply connected. In this case, $\pi_1(G) = \pi_1(D) = X_*(D)$ where $D = G/G_{\mathrm{der}}$ and we define κ_G as the composition

$$LG(k) \to LD(k) \to \pi_1(D)_I.$$

Step 4. The general case. One uses a z-extension

$$1 \to S \to G' \to G \to 1,$$

i.e. a central extension with G'_{der} simply connected and S an induced torus [36, Prop. 3.1], to reduce to the previous case, cf. [7].

By [7] the Kottwitz homomorphism κ_G is surjective. We will give a geometric interpretation of the Kottwitz homomorphism and additional properties of it in the following sections.

The parahoric subgroup scheme $P_{\underline{a}}$ leads by the procedure of the previous section to the fpqc sheaf $\mathcal{F}_{P_{\underline{a}}} = LG/L^+P_{\underline{a}}$ which in the sequel we will often denote simply by $\mathcal{F}_{\underline{a}}$; we will call the corresponding ind-schemes (cf. Theorem 1.4) *generalized affine flag varieties*.

Remark 2.1. More generally, one can also consider the group schemes P corresponding to the Moy–Prasad type subgroups of $G(K)$ which are constructed by Yu in [51]. These are also smooth and affine over $\mathrm{Spec}(V)$ with generic fiber G. As an example, consider the case $G = \mathrm{GL}_n$. Take the group scheme P_m given in [51] with V-valued points

$$P_m(V) = \left\{ g \in \mathrm{GL}_n\big(k[\![t]\!]\big) \;\big|\; g \equiv \mathrm{Id}_n \bmod t^m \right\}. \tag{2.2}$$

Then the fpqc-quotient $\mathcal{F}_{P_m}$ is a $\mathrm{Res}_{k[\![t]\!]/(t^m)/k}\,\mathrm{GL}_n$-bundle over the affine Grassmannian $\mathcal{F}_{P_0}$. The morphism $\mathcal{F}_{P_m} \to \mathcal{F}_{P_0}$ represents the forgetful functor

$$\big(\mathcal{L}, \sigma : \mathcal{L}/t^m\mathcal{L} \xrightarrow{\sim} \big(R[\![t]\!]/(t^m)\big)^n\big) \mapsto \mathcal{L} \tag{2.3}$$

where $\mathcal{L}$ denotes a lattice as in 1.c.1.

3. Tori and groups of multiplicative type

In this section we present various examples.

3.a. (a) For $P = \mathbf{G}_m$, we have $L^+P = R[\![t]\!]^\times$, $LP_\eta = R(\!(t)\!)^\times$. For

$$a = \sum_{i \gg -\infty}^{+\infty} a_i t^i \in R(\!(t)\!)^\times$$

denote by $\mathrm{ord}(a)$ the smallest integer $n \in \mathbf{Z}$ for which a_n is invertible in R. (We are assuming that $\mathrm{Spec}(R)$ is connected.) Then for all $m < n$, a_m is nilpotent. When R is reduced we can write $a = t^{\mathrm{ord}(a)} \cdot u$ with $u \in R[\![t]\!]^\times$. The map ord gives a Zariski locally constant map

$$\mathrm{ord} : \mathcal{F}_{\mathbf{G}_m} \to \mathbf{Z}.$$

Consider the ind-scheme $L^{--}\mathbf{G}_m$ over k which represents the functor which to a k-algebra R associates

$$L^{--}\mathbf{G}_m(R) = \big(1 + t^{-1}R[t^{-1}]\big)^\times, \tag{3.4}$$

i.e., the elements of $L^{--}\mathbf{G}_m(R)$ are polynomials $1 + a_1 t^{-1} + \cdots + a_i t^{-i}$ with a_i nilpotent in R. Then it is easily seen that $L^{--}\mathbf{G}_m$ can be identified with the open and closed sub-ind-scheme of $\mathcal{F}_{\mathbf{G}_m}$ for which $\mathrm{ord} = 0$. From this it follows that the fibers of ord are the connected components of $\mathcal{F}_{\mathbf{G}_m}$. In fact, the topological space underlying the ind-scheme $\mathcal{F}_{\mathbf{G}_m}$ is $\mathbf{Z}$ and ord gives a homeomorphism. The connected components of $\mathcal{F}_{\mathbf{G}_m}$ are infinitesimal ind-schemes; they are all isomorphic to each other via morphisms given by multiplication by the appropriate power of t.

(b) Consider $K' = k(\!(u)\!)$ with $u^2 = t$ as a degree 2 extension of K, and suppose $\mathrm{char}(k) \neq 2$. Let P be the group scheme over $V = k[\![t]\!]$ which is given as the kernel of the norm

$$P = \ker\big(\mathrm{Res}_{k[\![u]\!]/k[\![t]\!]}(\mathbf{G}_m) \xrightarrow{\mathrm{N}} \mathbf{G}_m\big).$$

Note that if R is reduced, then any $a \in R((u))^\times$ is of the form $u^m \cdot v$ with $v \in R[[u]]^\times$; the equation $N(a) = 1$ then gives $m = 0$ and $N(v) = 1$. This shows $\mathcal{F}_P(R) = \{1\}$ for all reduced R and therefore the corresponding topological space is a point. On the other hand, if $R = \mathrm{Spec}(k[x]/(x^2))$ then $a = xu^{-m} + 1$, for any odd $m > 0$, is a non-trivial element of $\mathcal{F}_P(R)$. We conclude that $\mathcal{F}_P$ is not reduced and is an infinitesimal ind-scheme.

(c) Let $P = \mu_n$. Then $LP_\eta(R) = \mu_n(R((t)))$, $L^+P(R) = \mu_n(R[[t]])$. If the characteristic of k does not divide n, then $\mu_n(R((t))) = \mu_n(R[[t]]) = \mu_n(R)$ and $\mathcal{F}_P = \mathrm{Spec}(k)$. However, this is not true in general. In fact, if $n = p = \mathrm{char}(k)$, then we can see that $\mathcal{F}_{\mu_p}$ is represented by the ind-scheme

$$\prod_{i=-1}^{-\infty} \mathrm{Spec}\big(k[x_i]/(x_i^p)\big). \tag{3.5}$$

3.b. Let $G = T$ be a torus over K. In this case, the connected Neron model of T over V (also called the Neron–Raynaud model in [51]) gives the (unique) parahoric subgroup scheme of T [8, 4.4]. Let us briefly recall this construction below. Let K' denote a splitting field for T, so that $T_{K'}$ is a split torus. We can assume that K'/K is finite and separable. The natural homomorphism

$$T \to \mathrm{Res}_{K'/K} T_{K'}$$

realizes T as a subgroup scheme of an induced torus. Denote by $\mathcal{T}_{V'}$ the natural extension of the split torus $T_{K'}$ to a torus over V' (if $T_{K'} \simeq (\mathbf{G}_m^h)_{K'}$, then $\mathcal{T}_{V'} \simeq (\mathbf{G}_m^h)_{V'}$). Consider the Zariski closure $\mathcal{T}$ of T in $\mathrm{Res}_{V'/V}(\mathcal{T}_{V'})$; this is an affine flat group scheme of finite type over V' which is independent of all choices (see [8, 4.4.6]). One can construct its smoothening ("lissification") $\mathcal{T}^R$ following Raynaud (see [8, 4.4.12]). This is a (commutative) group scheme affine and smooth over V with generic fiber T; its special fiber is not necessarily connected. (Let us remark here that if K'/K is tamely ramified, then it is known by [12] that $\mathcal{T}$ is smooth over V and so $\mathcal{T} = \mathcal{T}^R$.) We have [8, 4.4.12]

$$\mathcal{T}^R\big(V^\flat\big) = T_b\big(K^\flat\big) := \big\{t \in T\big(K^\flat\big) \,\big|\, \mathrm{val}\big(\chi(t)\big) = 0, \text{ for all } \chi \in X_{K^\flat}^*(T)\big\}. \tag{3.6}$$

The connected Neron model $\mathcal{T}^0$ of T is the connected component of the smooth group scheme $\mathcal{T}^R$. This agrees with the connected component of the locally finite type (lft) Neron model of the torus T (see [6, §10]), so there is no confusion in our terminology. Furthermore, $\mathcal{T}^R$ is the maximal subgroup scheme of the lft Neron model of finite type over V, comp. [42, Prop. on p. 314].

If T is an induced torus, i.e., of the form $T \simeq \prod_i \mathrm{Res}_{K_i/K}(\mathbf{G}_m^{n_i})_{K_i}$, then by [8, 4.4] we have $\mathcal{T}^0 = \mathcal{T}^R = \mathcal{T}$. Then by (1.2) and the above construction, there is a finite extension k' of k such that

$$LT \times_k k' \simeq (L\mathbf{G}_m)^n, \qquad L^+\mathcal{T} \times_k k' \simeq \big(L^+\mathbf{G}_m\big)^n, \qquad \mathcal{F}_\mathcal{T} \times_k k' \simeq (\mathcal{F}_{\mathbf{G}_m})^n, \tag{3.7}$$

with $n = [k':k] \cdot \sum_i n_i$.

Let us consider $\mathcal{F}_{\mathcal{T}^0}$ and $\mathcal{F}_{\mathcal{T}^R}$ for a general torus T. By Theorem 1.4, the $\bar{k}$-valued points of these ind-schemes are dense and we have

$$\mathcal{F}_{\mathcal{T}^R}(\bar{k}) = T\big(\bar{k}((t))\big)\big/\mathcal{T}^R\big(\bar{k}[[t]]\big), \qquad \mathcal{F}_{\mathcal{T}^0}(\bar{k}) = T\big(\bar{k}((t))\big)\big/\mathcal{T}^0\big(\bar{k}[[t]]\big). \tag{3.8}$$

By (3.6), $\mathcal{F}_{\mathcal{T}^R}(\bar{k}) = T(K^\flat)/T_b(K^\flat) \subset \mathrm{Hom}(X^*_{K^\flat}(T), \mathbf{Z})$. We have an exact sequence

$$0 \to \mathcal{T}^R\big(\bar{k}[\![t]\!]\big)/\mathcal{T}^0\big(\bar{k}[\![t]\!]\big) \to \mathcal{F}_{\mathcal{T}^0}(\bar{k}) \to \mathcal{F}_{\mathcal{T}^R}(\bar{k}) \to 0. \tag{3.9}$$

It will follow from our discussion of the Kottwitz homomorphism (Section 5, Corollary 5.2) that this exact sequence can be identified with the exact sequence

$$0 \to \big(X_*(T)_I\big)_{\mathrm{tor}} \to X_*(T)_I \to \big(X_*(T)_I\big)_{\mathrm{cotor}} \to 0. \tag{3.10}$$

(Here I is the inertia subgroup of $\mathrm{Gal}(K^{\mathrm{sep}}/K)$.)

3.b.1. Consider $K' = k((u))$ with $u^2 = t$ as a degree 2 extension of K and suppose that $\mathrm{char}(k) \neq 2$. Let T be the torus over K given as the kernel of the norm,

$$T := \ker\big(\mathrm{Res}_{K'/K}(\mathbf{G}_m) \xrightarrow{N} \mathbf{G}_m\big).$$

In this case, the torus $\mathcal{T}$ above is the group scheme over $k[\![t]\!]$ which is also given as the kernel of the norm

$$\mathcal{T} = \ker\big(\mathrm{Res}_{k[\![u]\!]/k[\![t]\!]}(\mathbf{G}_m) \xrightarrow{N} \mathbf{G}_m\big). \tag{3.11}$$

Notice that $\mathcal{T}$ is the fixed point scheme of an involution on $\mathrm{Res}_{k[\![u]\!]/k[\![t]\!]}(\mathbf{G}_m)$. Hence, it follows by [12, Prop. 3.5], that $\mathcal{T}$ is smooth over V and so $\mathcal{T} = \mathcal{T}^R$. However, $\mathcal{T}$ is not connected; the special fiber has two connected components corresponding to units $a \in k[\![u]\!]^\times$ with $a \cdot \bar{a} = 1$ having residue class 1, resp. -1 modulo u. Hilbert's theorem 90 implies that there is an exact sequence

$$1 \to \mathbf{G}_m \to \mathrm{Res}_{k[\![u]\!]/k[\![t]\!]} \mathbf{G}_m \xrightarrow{b \mapsto \bar{b} \cdot b^{-1}} \mathcal{T}^0 \to 1 \tag{3.12}$$

over $k[\![t]\!]$. In this case, the Kottwitz homomorphism

$$\kappa_T : T\big(\bar{k}((t))\big) \to \{\pm 1\} \tag{3.13}$$

is given as follows: If $a \in \bar{k}((u))^\times$ with $a \cdot \bar{a} = 1$, then write $a = \bar{b} \cdot b^{-1}$, $b \in \bar{k}((u))^\times$ and we have $\kappa_T(a) = (-1)^{\mathrm{ord}_u(b)}$. Note that for $a \in \bar{k}[\![u]\!]^\times$ we have $a \equiv \kappa_T(a) \bmod (u)$ and so we have $\mathcal{T}^0(\bar{k}[\![t]\!]) = \mathcal{T}(\bar{k}[\![t]\!]) \cap \ker(\kappa_T)$, see also Section 5 below.

4. Affine flag varieties for quasi-split unitary groups

We continue with an example which is essential for our applications.

4.a. Let $\mathrm{char}(k) \neq 2$. Let K'/K be a ramified quadratic extension and fix a uniformizer u of K' with image under the Galois involution $\bar{u} = -u$, $u^2 = t$. Let W be a K'-vector space of dimension $n \geqslant 2$ and let

$$\phi : W \times W \to K'$$

be a K'/K-hermitian form. Let

$$U(W, \phi) = \left\{ g \in GL_{K'}(W) \mid \phi(gv, gw) = \phi(v, w), \ \forall v, w \in W \right\}$$

and consider also $SU(W, \phi) = \{g \in U(W, \phi) \mid \det(g) = 1\}$ (a semi-simple simply connected group over K). In these cases, the group is not of the form $G_0 \times_{\mathrm{Spec}(k)} \mathrm{Spec}(K)$. Let us consider the case when the form ϕ is split. This means that there exists a basis $e_1, \ldots, e_n$ of W such that

$$\phi(e_i, e_j) = \delta_{i, n+1-j}, \quad \forall i, j = 1, \ldots, n. \tag{4.1}$$

In what follows, we fix a choice of such a form and denote the groups by U_n, SU_n. Since SU_n is semi-simple simply connected, the Kottwitz homomorphism for U_n is the composition

$$\kappa_{U_n} : U_n\left(\bar{k}((t))\right) \xrightarrow{\det} T\left(\bar{k}((t))\right) \xrightarrow{\kappa_T} \{\pm 1\}. \tag{4.2}$$

Here T is the torus considered in 3.b.1. Note that there are two associated $\mathcal{O}_K$-bilinear forms

$$(x, y) = \mathrm{Tr}_{K'/K}\left(\phi(x, y)\right), \qquad \langle x, y \rangle = \mathrm{Tr}_{K'/K}\left(u^{-1}\phi(x, y)\right). \tag{4.3}$$

The form $(,)$ is symmetric while $\langle, \rangle$ is alternating. They satisfy the identities

$$(x, uy) = -(ux, y), \qquad \langle x, uy \rangle = -\langle ux, y \rangle. \tag{4.4}$$

For any $\mathcal{O}_{K'}$-lattice Λ in W, we set

$$\hat{\Lambda} = \left\{ w \in W \mid \phi(w, \Lambda) \subset \mathcal{O}_{K'} \right\} = \left\{ v \in W \mid \langle v, \Lambda \rangle \subset \mathcal{O}_K \right\}.$$

Similarly, we set

$$\hat{\Lambda}^s = \left\{ w \in W \mid (w, \Lambda) \subset \mathcal{O}_K \right\},$$

so that $\hat{\Lambda}^s = u^{-1}\hat{\Lambda}$. For $i = 0, \ldots, n-1$, let us set

$$\lambda_i := \mathrm{span}_{\mathcal{O}_{K'}}\left\{ u^{-1}e_1, \ldots, u^{-1}e_i, e_{i+1}, \ldots, e_n \right\}.$$

The lattice λ_0 is self-dual for the alternating form $\langle, \rangle$. We now distinguish two cases:

(a) $n = 2m + 1 \geqslant 3$ is odd. [Then SU_n is quasi-split of type C-BC$_m$; see [48, p. 60].] We have $\hat{\lambda}_m^s = \lambda_{m+1}$. Let I be a non-empty subset of $\{0, \ldots, m\}$ and consider the lattice chain consisting of λ_i, $i \in I$. In fact, we can extend λ_i, $i \in I$, to a periodic self-dual lattice chain by first including the duals $\hat{\lambda}_i^s = \lambda_{n-i}$ for $i \neq 0$, and then all the u-multiples of our lattices: For $j \in \mathbf{Z}$ of the form $j = k \cdot n + i$ with $0 \leqslant i < n$ we put

$$\lambda_j = u^{-k} \cdot \lambda_i.$$

Then $\{\lambda_j\}_j$ forms a periodic lattice chain λ_I ($u \cdot \lambda_j = \lambda_{j-n}$) which satisfies $\hat{\lambda}_j = \lambda_{-j}$. Now consider the subgroups

$$P_I = \{g \in U_n \mid g\lambda_i = \lambda_i, \ \forall i \in I\}, \qquad P_I' = \left\{ g \in P_I \mid \det(g) = 1 \right\}$$

of $U_n(K)$ and $SU_n(K)$ respectively. We then have the following statements:

The subgroup P'_I is a parahoric subgroup of $SU_n(K)$. Any parahoric subgroup of $SU_n(K)$ is conjugate to a subgroup P'_I for a unique subset I. The sets $I = \{0\}$ and $I = \{m\}$ correspond to the special maximal parahoric subgroups.

We indicate briefly how these assertions can be proved by following the method of Kaiser [25, §5.2] (in [25] the symplectic group is considered). (Another possible reference is [48, 1.15 and 3.11], or the work of Gan and Yu [16].) We consider the embedding of the Bruhat–Tits building $\mathcal{B}$ of $SU_n(K)$ into the Bruhat–Tits building $\mathcal{B}'$ of $SL_n(K')$. Denote by $z \mapsto \hat{z}$ the involution of $\mathcal{B}'$ induced by the map which to a lattice Λ associates $\hat{\Lambda}$. The following points can be checked by imitating Kaiser's reasoning in the case at hand.

(a) Every maximal parahoric subgroup P of $SU_n(K)$ fixes a vertex z in $\mathcal{B}'$ such that z and $\hat{z}$ are neighbouring vertices (or coincide), and, conversely, the unordered pair $(z, \hat{z})$ determines P uniquely.
(b) Conversely, the stabilizer of a vertex z of $\mathcal{B}'$ of the kind described above is a maximal parahoric subgroup.
(c) To a vertex z as above we associate as follows a flat group scheme $\mathcal{G}$ over $k[\![t]\!]$. Let z be the homothety class of a lattice Z with $uZ \subset \hat{Z} \subset Z$. Let $\mathcal{G}$ be the Zariski closure of SU_n in $GL(Z)$. Then $\mathcal{G}$ is a smooth group scheme whose special fiber has as its maximal reductive quotient the group $\bar{\mathcal{G}} = Sp(Z/\hat{Z}) \times SO(\hat{Z}/uZ)$. The map $P \to \bar{\mathcal{G}}(k)$ is surjective. Furthermore, the link of the vertex in $\mathcal{B}$ corresponding to P can be identified with the Tits building of $\bar{\mathcal{G}}$.
(d) It follows that the maximal parahoric subgroup P is special if and only if either $Sp(Z/\hat{Z})$ or $SO(\hat{Z}/uZ)$ is trivial, i.e., if either $Z = \hat{Z}$ or $\dim \hat{Z}/uZ = 1$.
(e) Fix a lattice Z with $\hat{Z} = Z$ and a standard basis $e_1, \ldots, e_n$ of it satisfying (4.1). Then any simplex in $\mathcal{B}$ is conjugate under $SU_n(K)$ to a simplex in the Tits building of the corresponding group $\bar{\mathcal{G}}$ which is unique up to conjugation; and hence is conjugate to a unique simplex in the fundamental chamber of the corresponding apartment of the Tits building of $SO(Z/uZ)$. It follows that any parahoric subgroup of $SU_n(K)$ is conjugate to P'_I for a unique subset I.

Note that the subgroups P_I are *never* parahoric subgroups of $U_n(K)$. One can see this by observing that P_I always contains the unitary transformation $e_m \mapsto -e_m$, $e_i \mapsto e_i$ for $i \neq m$; the Kottwitz invariant of this element of $U_n(K)$ is non-trivial and so the special fiber of P_I is not connected. To obtain the corresponding parahoric subgroups of $U_n(K)$ we need to intersect with the kernel of the Kottwitz homomorphism, i.e., consider

$$P_I^0 = \{g \in P_I \mid \kappa_{U_n}(g) = 1\}.$$

(b) $n = 2m$. [We will always assume $m \geqslant 2$, in which case SU_n is quasi-split with local Dynkin diagram of type B-C$_m$ if $m \geqslant 3$, and C-B$_2$ if $m = 2$. If $m = 1$ then $SU_n \simeq SL_2$ and is split.] In this case, we also introduce

$$\lambda_{m'} = \mathrm{span}_{\mathcal{O}_{K'}}\{u^{-1}e_1, \ldots, u^{-1}e_{m-1}, e_m, u^{-1}e_{m+1}, e_{m+2}, \ldots, e_n\}.$$

Then both λ_m and $\lambda_{m'}$ are self-dual for the symmetric form $(,)$. Now consider non-empty subsets $I \subset \{0, \ldots, m-2, m, m'\}$ (we view m' as an additional symbol) and as above consider the subgroups

$$P_I = \{g \in U_n(K) \mid g\lambda_i = \lambda_i, \; \forall i \in I\}, \qquad P'_I = \{g \in P_I \mid \det(g) = 1\}.$$

The subgroup P'_I is a parahoric subgroup of $SU_n(K)$. Any parahoric subgroup of $SU_n(K)$ is conjugate to a subgroup P'_I for a unique subset I. The sets $I = \{m\}$ and $I = \{m'\}$ correspond to the special maximal parahoric subgroups.

The way to see these facts is similar to the case of odd n considered above. But there are differences. In (b) above, there is the exceptional case when z is represented by a lattice Z with $uZ \subset \hat{Z} \subset Z$, where $\dim \hat{Z}/uZ = 2$. The action of the stabilizer of z on the link of z factors through $\bar{\mathcal{G}}$ and fixes in this case the two isotropic lines in the 2-dimensional quadratic space $\hat{Z}/uZ$. Hence this stabilizer is not a maximal parahoric subgroup, but is the intersection of the two maximal parahoric subgroups which fix the lattices corresponding to the isotropic lines. Therefore we have to exclude this kind of pair $(z, \hat{z})$ in the correspondence described in (a). Similarly, in (d), to identify the special maximal parahoric subgroups, the condition in this case becomes that $SO(\hat{Z}/uZ)$ is trivial, which now means that $\hat{Z} = uZ$. But other than these minor differences the argument is the same.

Out of the lattices λ_i, $i \in I$, we can create a periodic self-dual lattice chain whose stabilizer in $SU_n(K)$ is equal to P'_I in the following manner. We first introduce a set $I^{\sharp} \subset \{0, \ldots, m-1, m\}$ that depends on I and is defined as follows: If both m and m' belong to I we set $I^{\sharp}$ to be the set $(I - \{m'\}) \cup \{m-1\}$ which is obtained from I by replacing m' by $m-1$. We can see that we have $P'_I = P'_{I^{\sharp}}$, $P_I = P_{I^{\sharp}}$. In all other cases, i.e., when not both m and m' are in I, set $I^{\sharp} = I$. Now we can saturate λ_i, $i \in I^{\sharp}$, as above to create a periodic self-dual lattice chain λ_I.

Note again that the subgroups P_I are not parahoric subgroups of $U_n(K)$ unless the set I contains either m or m'. Indeed, suppose that I contains neither m nor m'. Then the unitary transformation $e_m \mapsto e_{m+1}$, $e_{m+1} \mapsto e_m$, $e_i \mapsto e_i$ for $i \neq m, m+1$, belongs to P_I and has non-trivial Kottwitz invariant. Conversely, assume that I contains for example m so that $P_I \subset P_{\{m\}}$. Let $g \in P_{\{m\}}$ and consider the action of g on $\lambda_m/u\lambda_m$. We can see that g respects the perfect alternating form on $\lambda_m/u\lambda_m$ given by $s(\bar{x}, \bar{y}) = (x, u^{-1} \cdot y)$. Therefore the determinant of g is 1 modulo u; by 3.b.1 the Kottwitz invariant of any such g is trivial. This is enough to imply that P_I is parahoric. When I contains neither m nor m', to obtain the corresponding parahoric subgroup of $U_n(K)$ we need again to consider the kernel of the Kottwitz homomorphism, i.e.,

$$P_I^0 = \{g \in P_I \mid \kappa_{U_n}(g) = 1\}.$$

In both cases, we will also denote by P_I, P'_I the affine group schemes over $k[[t]]$ whose S-valued points for a $k[[t]]$-algebra S are the $k[[u]] \otimes_{k[[t]]} S$-automorphisms of the (polarized) chain $\lambda_I \otimes_{k[[t]]} S$, that preserve the form $\phi \otimes_{k[[t]]} S$ (and in the case of P'_I have $k[[u]] \otimes_{k[[t]]} S$-determinant 1). By [43, App. to Ch. 3], the group scheme P_I is smooth over $k[[t]]$ (this is a variant of Case III of [43], with $\mathcal{O}_F$ replaced by the d.v.r. $k[[u]]$ and $\mathcal{O}_{F_0}$ by $k[[t]]$; actually [43] would only allow S for which t is locally nilpotent, but this is enough to imply our result). Therefore, since, as we will see below in Theorem 5.1, the Kottwitz homomorphism is locally constant, the group scheme P_I^0 is also smooth and affine. A similar argument shows that P'_I is also smooth

over $k[[t]]$; therefore, by appealing to [8, 1.7.], P'_I coincides with the corresponding parahoric group scheme constructed in [8]. We have an exact sequence of group schemes over $k[[t]]$

$$1 \to P'_I \to P_I \xrightarrow{\det} \mathcal{T}, \tag{4.5}$$

where $\mathcal{T}$ is the torus of (3.11). The neutral component P^0_I is the preimage of the neutral component $\mathcal{T}^0$ of $\mathcal{T}$. This fact, together with the description of the Kottwitz invariant in this case, implies that the $\bar{k}[[t]]$-valued points of the neutral component P^0_I of P_I are given by the $\bar{k}[[t]]$-valued points of P_I that have trivial Kottwitz invariant (so our notation is consistent). This in turn implies that P^0_I coincides with the corresponding parahoric group scheme that is constructed in [8]. We have a short exact sequence of (connected, smooth) group schemes over $k[[t]]$,

$$1 \to P'_I \to P^0_I \xrightarrow{\det} \mathcal{T}^0 \to 1. \tag{4.6}$$

4.b. Recall the definition above of $I^\sharp$ when n is even. We set $I^\sharp = I$ if n is odd. In both cases $I^\sharp$ is totally ordered, if in the case that $n = 2m$ is even, we consider $\{0, 1, \ldots, m, m'\} = \{0, 1, \ldots, m\} \cup \{m'\}$ with the partial order which extends the standard order and for which $m' > j$ for $j = 0, \ldots, m - 1$. Let us now write $I^\sharp = \{i_0 < i_1 < \cdots < i_k\}$. Then we can consider the "standard" lattice chain

$$\lambda_{i_0} \subset \lambda_{i_1} \subset \cdots \subset \lambda_{i_k} \subset u^{-1}\lambda_{i_0}. \tag{4.7}$$

We can consider the functor $\mathcal{F}_{I^\sharp}$ which to a k-algebra R associates the set of $R[[u]]$-lattice chains

$$L_{i_0} \subset L_{i_1} \subset \cdots \subset L_{i_k} \subset u^{-1}L_{i_0} \tag{4.8}$$

in $W \hat{\otimes}_{K'} R = R((u))^n$ which satisfy the following conditions:

(a) For any $q \in \{0, \ldots, k\}$, we have

$$L_{i_q} \subset u^{-1}\hat{L}_{i_q} \subset u^{-1}L_{i_q}. \tag{4.9}$$

(b) The quotients $L_{i_{q+1}}/L_{i_q}, u^{-1}\hat{L}_{i_q}/L_{i_q}, u^{-1}L_{i_0}/L_{i_{q+1}}$ are projective R-modules of rank equal to the rank of the corresponding quotients for the standard chain (4.7) (when $q = k$, these conditions have to be interpreted in the obvious way).

The ind-group scheme LU_n over k given by $LU_n(R) = U_n(R((t)))$ acts naturally on $\mathcal{F}_{I^\sharp}$.

Theorem 4.1. *There is an LU_n-equivariant isomorphism $LU_n/L^+P_I \simeq \mathcal{F}_{I^\sharp}$ of sheaves for the fpqc topology.*

Proof. To prove the theorem it suffices to check the following two statements:

(i) For any R, the subgroup of $LU_n(R)$ consisting of elements that stabilize the lattice chain $\lambda_{I^\sharp} \hat{\otimes} R$ agrees with $L^+P_I(R) = P_I(R[[t]])$.
(ii) Let $L_{I^\sharp} \in \mathcal{F}_I(R)$ be a lattice chain as above. Then locally for the étale topology on R, there exists an isomorphism $\lambda_{I^\sharp} \hat{\otimes} R \simeq L_{I^\sharp}$ realized by multiplication by an element $g \in LU_n(R)$.

Both of these statements essentially follow from the work in [43, App. to Ch. 3]. Indeed, to see (i) observe that we have $\lambda_{I^\sharp} \hat{\otimes}_k R = \lambda_{I^\sharp} \otimes_{k[[t]]} R[[t]]$ and we can consider $\lambda_{I^\sharp} \otimes_{k[[t]]} R[[t]]$ as a polarized periodic chain of $k[[u]] \otimes_{k[[t]]} R[[t]]$-modules (terminology of [43]). One can see that, even in the case when $n = 2m$ is even and when $\{m, m'\} \subset I$, the group of automorphisms of this chain that preserve the form $\phi \otimes_{k[[t]]} R[[t]]$ is equal to $L^+P_I(R) = P_I(R[[t]])$. Using $\lambda_i \otimes_{k[[t]]} R[[t]] \subset R((u))^n$ we see that these automorphisms are exactly the elements of $U_n(R((u))) = LU_n(R)$ that stabilize the lattice chain $\lambda_{I^\sharp} \hat{\otimes}_k R$.

Let us now discuss part (ii). We can see a priori that $\mathcal{F}_{I^\sharp}$ is represented by an ind-scheme of ind-finite type over k. By part (i), we obtain $LU_n/L^+P_I \to \mathcal{F}_{I^\sharp}$; it is enough to show (ii) for R strictly henselian. Now observe that [43, App. to Ch. 3], modified for this situation, shows that there is an isomorphism

$$\sigma_i : \lambda_i \hat{\otimes}_k R \xrightarrow{\sim} \mathcal{L}_i \tag{4.10}$$

of polarized chains of $R[[u]]$-modules (here we need to observe that the arguments of [43] extend to this case in which t is topologically nilpotent). Since both $\lambda_i \hat{\otimes}_k R$ and $\mathcal{L}_i$ are lattices in $R((u))^n$, the isomorphism $\sigma = \sigma_i[t^{-1}]$ is independent of i and is given by an element σ of $U_n(R((t))) = LU(R)$. $\quad\square$

Remark 4.2. A. By our discussion above, when P_I is not a parahoric subgroup of $U_n(K)$, then its special fiber has two connected components and $L^+P_I/L^+P_I^0 \simeq \{\pm 1\}$, where the isomorphism is given by the Kottwitz invariant. We will see in the following paragraphs that LU_n has two connected components also distinguished by the value ± 1 of the Kottwitz invariant. This will allow us to conclude that if P_I is not parahoric, then for the parahoric subgroup P_I^0 we have $LU_n/L^+P_I^0 \simeq \mathcal{F}_{I^\sharp} \sqcup \mathcal{F}_{I^\sharp}$.

B. Let us discuss here the variant of the special unitary group $SU_n = SU(W, \phi)$. In this case, let $\mathcal{F}'_{I^\sharp}$ be the functor which to a k-algebra R associates the set of $R[[u]]$-lattice chains as in (4.8) which satisfy, in addition to (a) and (b) above, also

(c) For all $q = 0, \ldots, k$, we have

$$\det(\mathcal{L}_{i_q}) = \det(\lambda_{i_q}) \subset R((u)) = \bigwedge^n R((u))^n. \tag{4.11}$$

The ind-group LSU_n acts on $\mathcal{F}'_{I^\sharp}$ and we obtain $\pi_I : LSU_n/L^+P_I' \to \mathcal{F}'_{I^\sharp}$. We will see that π_I is an isomorphism when n is odd, or when $n = 2m$ is even and I contains neither m nor m'. Indeed, we can always identify as before L^+P_I' with the stabilizer in LSU_n of the standard lattice chain $\lambda_{I^\sharp}$; the issue is again if each point $\mathcal{L}_{I^\sharp}$ in $\mathcal{F}'_{I^\sharp}(R)$ for R strictly henselian is in the orbit of the standard lattice chain $\lambda_{I^\sharp}$ by an element $g \in LSU_n(R) = SU_n(R((t)))$. By the proof of the theorem above, we can find $\sigma \in U_n(R((t)))$ such that $\sigma \cdot (\lambda_i \hat{\otimes}_k R) = \mathcal{L}_i$, for all i. Our additional condition (c) now guarantees that the determinant $a = \det(\sigma)$ is a unit in $R[[u]]^\times$. Since σ is unitary, we also have $\text{Norm}(a) = 1$ and the exact sequence (3.12) now implies that then there is $b \in R[[u]]^\times$ such that $a = \pm \bar{b} \cdot b^{-1}$. Now suppose first that $n = 2m + 1$ is odd. Then we can compose σ with the unitary automorphism $e_m \mapsto a^{-1}e_m$, $e_i \mapsto e_i$, for $i \neq m$, which stabilizes $\lambda_{I^\sharp}$. The resulting unitary automorphism has determinant 1 and takes $\lambda_i \hat{\otimes}_k R$ to $\mathcal{L}_i$. Suppose now that $n = 2m$ and that I contains neither m nor m'. Assume that $a = -\bar{b} \cdot b^{-1}$. Then the unitary transformation $e_m \mapsto be_{m+1}$, $e_{m+1} \mapsto \bar{b}^{-1}e_m$, $e_i \mapsto e_i$ if $i \neq m$, $m+1$, has

determinant a^{-1} and stabilizes $\lambda_{I^\sharp}$. Composing it with σ gives the desired result. If $a = \bar{b} \cdot b^{-1}$ we can use $e_m \mapsto be_m$, $e_{m+1} \mapsto \bar{b}^{-1}e_{m+1}$, $e_i \mapsto e_i$ if $i \neq m, m+1$ instead.

C. In the case when $n = 2m$ is even and $\{m, m'\} \subset I$, our definition of $I^\sharp$ is asymmetric (we favor λ_m over $\lambda_{m'}$). Note, however, that the element $g \in U_n(K)$ with $e_m \mapsto e_{m+1}$, $e_{m+1} \mapsto e_m$, and with $e_i \mapsto e_i$ for $i \neq m, m+1$ takes λ_m to $\lambda_{m'}$. The element g can be used to define an isomorphism between $\mathcal{F}_{I^\sharp}$ and $\mathcal{F}_{I^{\sharp'}}$, where $\mathcal{F}_{I^{\sharp'}}$ denotes the version where we favor $\lambda_{m'}$ over λ_m. A slightly different point of view is explained in [39]; there we consider the groups of unitary similitudes.

5. The Kottwitz map and connected components

5.a. We continue with the following assumptions: G is a connected reductive group over $K = k((t))$ with k algebraically closed. Let P be a parahoric group scheme of G in the sense of Section 2.a.2 and consider $\mathcal{F}_P := LG/L^+P$. By Theorem 1.4 the morphism $LG \to \mathcal{F}_P = LG/L^+P$ splits locally for the étale topology. Therefore, we have

$$\mathcal{F}_P(k) = LG(k)/L^+P(k) = G\big(k((t))\big)\big/P\big(k[[t]]\big). \tag{5.1}$$

Since by [23, Prop. 3], the Kottwitz homomorphism

$$\kappa = \kappa_G : G\big(k((t))\big) \to \pi_1(G)_I$$

is trivial on all parahoric subgroups of $G(k((t)))$ we also obtain a map

$$\bar{\kappa} : \mathcal{F}_P(k) \to \pi_1(G)_I.$$

Note that since $\mathcal{F}_P$ is of ind-finite type, the k-points of $\mathcal{F}_P$ are dense for the Zariski topology. In what follows, we will denote by π_0 the set of connected components.

Theorem 5.1. *The maps* $\kappa : LG(k) \to \pi_1(G)_I$, $\bar{\kappa} : \mathcal{F}_P(k) \to \pi_1(G)_I$ *are locally constant for the Zariski topology of the ind-schemes LG and $\mathcal{F}_P$. In fact, they induce bijections*

$$\pi_0(LG) \xrightarrow{\sim} \pi_1(G)_I, \qquad \pi_0(\mathcal{F}_P) \xrightarrow{\sim} \pi_1(G)_I.$$

Proof. Note that since P is a smooth affine group scheme over $k[[t]]$ with connected fibers, it follows from [20, p. 263], that L^+P is also a connected group scheme over k. Therefore, since by Theorem 1.4 the morphism $LG \to \mathcal{F}_P = LG/L^+P$ splits over points $\operatorname{Spec}(R) \to \mathcal{F}_P$ with values in strictly henselian rings R, we see that it induces a bijection $\pi_0(LG) \simeq \pi_0(\mathcal{F}_P)$ on the set of connected components.

We will divide the proof into three cases, that essentially correspond to the steps in the definition of the Kottwitz homomorphism:

A. Let us first discuss the case that $G = T$ is a torus. Then the connected component of the Neron model $P = \mathcal{T}^0$ is the unique parahoric subgroup scheme of T. By [42, Appendix, p. 380], $\mathcal{T}^0(k[[t]])$ coincides with the kernel of the Kottwitz homomorphism

$$\kappa_T : T(K) \to \pi_1(T)_I = X_*(T)_I$$

which is surjective. Therefore, our claims for $G = T$ will follow if we show that L^+P is the neutral component of the ind-scheme LT. Alternatively, it is enough to show that the connected component of $\mathcal{F}_P$ that contains the identity coset e mod L^+P is equal to $\{e$ mod $L^+P\}$. Note that $\mathcal{F}_P$ is of ind-finite type. Therefore, to show this statement it is enough to show the following:

Claim. *Let $\phi : Y = \operatorname{Spec}(R) \to \mathcal{F}_P$ be a morphism with Y a connected affine smooth curve over k and let y be a point in $Y(k)$ such that $\phi(y) = e$ mod L^+P. Then the morphism ϕ factors through the identity coset $\operatorname{Spec}(R) \to \operatorname{Spec}(k) \to \mathcal{F}_P$.*

To prove the claim observe that there is a (connected) étale cover $Z \to Y$ and a morphism $Z \to LT$ which lifts $Y \to \mathcal{F}_P$. By translating we can assume that there is a k-valued point $z \in Z(k)$ which maps to the unit element in LT. It will be enough to show that when $Z = \operatorname{Spec}(S)$ is a connected affine smooth curve and $z \in Z$ is a closed point, any morphism $a : Z \to LT$ with $a(z) = e$ factors through $Z \to L^+P$. Now a corresponds to an $S((t))$-valued point a of T. Let $\hat{S}_z \simeq k[[x]]$ be the completion of the local ring of S at z. Consider the scheme $W = \operatorname{Spec}(\hat{S}_z[[t]])$ with generic fiber $W_\eta = \operatorname{Spec}(\hat{S}_z((t)))$. Consider the composed morphism

$$f : W_\eta = \operatorname{Spec}(\hat{S}_z((t))) \to \operatorname{Spec}(S((t))) \to T$$

obtained from $Z \to LT$. By [6, 7.2/3] extended to locally finite type Neron models (this is done by replacing in the proof of 7.2/3 the use of 7.2/1(ii) by 10.1/3 [6]) we can now see that the morphism f extends to

$$\tilde{f} : W = \operatorname{Spec}(\hat{S}_z[[t]]) \to T^{lftN},$$

where T^{lftN} is the locally finite type Neron model of T (this is a strengthening of the Neron extension property to this situation). By our assumption, the specialization $\operatorname{Spec}(k[[t]]) \to T^{lftN}$ of this morphism factors through the neutral component, $\operatorname{Spec}(k[[t]]) \to P = T^0$; it follows that f actually extends to $\tilde{f} : W \to P = T^0$. Consider the corresponding ring homomorphism $\tilde{f}^* : \mathcal{O}(P) \to \hat{S}_z[[t]]$; by our construction, the composition of this homomorphism with $\hat{S}_z[[t]] \subset \hat{S}_z((t))$ is equal to

$$\mathcal{O}(P) \subset \mathcal{O}(P) \otimes_{k[[t]]} k((t)) = \mathcal{O}(T) \xrightarrow{a^*} S((t)).$$

We conclude that this last homomorphism takes values in $\hat{S}_z[[t]] \cap S((t)) = S[[t]]$; this shows that $Z \to LT$ factors through $Z \to L^+P$. This completes the proof in the case that $G = T$ is a torus. In particular, we see that in this case the Kottwitz homomorphism induces a bijection,

$$\bar{\kappa}_T : \mathcal{F}_{T^0}(k) \xrightarrow{\sim} X_*(T)_I. \tag{5.2}$$

Before we continue with the other cases let us remark that our basic task is to show that the Kottwitz homomorphism $\kappa_G : LG(k) \to \pi_1(G)_I$, or equivalently $\bar{\kappa}_G : \mathcal{F}_P(k) \to \pi_1(G)_I$ is locally constant. (Our argument in case A establishes this when G is a torus.) Assuming this we see that κ_G induces a group homomorphism

$$\pi_0(LG) \simeq \pi_0(\mathcal{F}_P) \to \pi_1(G)_I.$$

This is surjective, since κ_G is surjective. To show that $\pi_0(LG) \to \pi_1(G)_I$ is injective it is enough to show that the kernel $LG(k)_1 \subset LG(k)$ of κ_G lies in the neutral component of LG. By [23, Lemma 17], $LG(k)_1$ is generated by the union of all parahoric subgroups $L^+P(k) = P(k[\![t]\!])$ of G. As mentioned in the beginning of our proof, L^+P is connected for each parahoric P, hence the images of all multiplication morphisms $L^+P_{i_1} \times \cdots \times L^+P_{i_n} \to LG$ (with source an arbitrary finite product of parahorics) are contained in the neutral component of LG. We conclude that $LG(k)_1$ is contained in the neutral component of LG and so $\pi_0(LG) \to \pi_1(G)_I$ is injective.

In what follows, we will only deal with the claim that κ_G is locally constant since by the above this is enough to complete the proof.

B. The case that the derived group G_{der} is simply-connected. Then the Kottwitz homomorphism is the composition

$$LG(k) \to LD(k) \xrightarrow{\kappa_D} \pi_1(D)_I = \pi_1(G)_I$$

where D is the torus G/G_{der}. Our claim that κ_G is locally constant then follows from case A applied to $T = D$, since $LG(k) \to LD(k)$ is induced by a morphism of ind-schemes.

C. The general case. We can find a z-extension (see 2.a.2, Step 4)

$$1 \to S \to G' \to G \to 1.$$

As in case A it is enough to show that if Z is a connected affine smooth curve over k and a morphism $a : Z \to LG$ is given, the Kottwitz homomorphism is constant on the image $a(Z(k)) \subset LG(k)$. Let $z, z' \in Z(k)$ and consider $W = \mathrm{Spec}(\mathcal{O}_{Z,z;z'}((t)))$ where $\mathcal{O}_{Z,z;z'}$ is the semi-local ring of Z at z, z'. Since κ_G is surjective, we can assume without loss of generality that $\kappa_G(a(z)) = 0$. Since $\mathcal{O}_{Z,z;z'}$ is a UFD, the Picard group of W is trivial. Observe that for every finite separable extension L/K, we have $L \simeq k((t))$ and so $W \times_K L \simeq W$.

A standard argument using the Hochschild–Serre spectral sequence now gives $\mathrm{H}^1_{\mathrm{et}}(W, S_W) = (0)$ for the induced torus S: Indeed, it is enough to explain this for $S = \mathrm{Res}_{K'/K}\, \mathbf{G}_m$ with K'/K a finite separable extension. Let $\tilde{K} = k((\tilde{u}))$ be the Galois hull of K' over K; set $\Gamma = \mathrm{Gal}(\tilde{K}/K)$ and denote by H the subgroup that corresponds to K'. The Hochschild–Serre spectral sequence for $W \times_K \tilde{K} \to W$ now is

$$\mathrm{H}^p\big(\Gamma, \mathrm{H}^q_{\mathrm{et}}(W \times_K \tilde{K}, S_{W \times_K \tilde{K}})\big) \quad \Rightarrow \quad \mathrm{H}^{p+q}_{\mathrm{et}}(W, S_W).$$

Now $S_{W \times_K \tilde{K}}$ is a split torus. Hence, by the above

$$\mathrm{H}^1(W \times_K \tilde{K}, S_{W \times_K \tilde{K}}) = (0),$$

while also

$$\mathrm{H}^0(W \times_K \tilde{K}, S_{W \times_K \tilde{K}}) = \mathrm{Ind}_H^\Gamma(\mathbf{Z}) \otimes \big(\mathcal{O}_{Z,z;z'}((\tilde{u}))\big)^\times.$$

Shapiro's lemma now shows that the cohomology group $\mathrm{H}^1(\Gamma, -)$ of this last Γ-module is isomorphic to $\mathrm{H}^1(H, (\mathcal{O}_{Z,z;z'}((\tilde{u})))^\times)$. A similar argument as above applied to the H-cover $W \times_K \tilde{K} \to W \times_K K'$ now shows that since $\mathrm{H}^1(W \times_K K', S_{W \times_K K'}) = (0)$, we have $\mathrm{H}^1(H, (\mathcal{O}_{Z,z;z'}((\tilde{u})))^\times) = (0)$. We conclude that indeed $\mathrm{H}^1_{et}(W, S_W) = (0)$, as desired.

Since $S \to \mathrm{Spec}(k[[t]])$ is smooth, $\mathrm{H}^1_{et}(W, S_W) = (0)$ implies that the S-torsor given by $G' \to G$ splits after base-changing along the morphism $W \to G$ given by a. Therefore, the morphism $W \to G$ lifts to $W \to G'$; this gives $\mathrm{Spec}(\mathcal{O}_{Z,z;z'}) \to LG'$ that factors $a_{z;z'} : \mathrm{Spec}(\mathcal{O}_{Z,z;z'}) \to LG$. Now the image of $\mathrm{Spec}(\mathcal{O}_{Z,z;z'})$ lies in a single connected component of LG'. Since we know the conclusion of the theorem for G' (case B) we deduce that $\kappa_{G'}(z) = \kappa_{G'}(z')$. By the definition of κ_G this implies $\kappa_G(z) = \kappa_G(z')$. $\quad\square$

We obtain:

Corollary 5.2. *Assume that T is a torus over $K = k((t))$, where k is algebraically closed. Let $\mathcal{T}^0$ be the connected component of the Néron model of T over $k[[t]]$. Then the reduced locus $(LT/L^+\mathcal{T}^0)_{\mathrm{red}} = (\mathcal{F}_{\mathcal{T}^0})_{\mathrm{red}}$ is a discrete set which can be identified with $X_*(T)_I$.*

Corollary 5.3. *Assume that G is a reductive group over $K = k((t))$ with k algebraically closed.*

(a) *If G is split over K, then $\pi_0(LG) = \pi_1(G)$.*
(b) *If G is semi-simple and simply connected then LG is connected.*

We point out the analogy of the statement in (a) to the corresponding statement in topology. This statement (for $k = \mathbf{C}$) appears in [3, 1.2], and in [4, 4.5.4]. The heuristic that semi-simple simply connected groups over a non-archimedean field behave like connected groups in the classical theory appears quite often in the papers of Bruhat and Tits, comp., e.g., [8, 5.2.11].

6. Reducedness; reductions and examples

We continue with the following assumptions: k is a perfect field of characteristic p and G a connected reductive group over $K = k((t))$. Consider the generalized affine flag variety $\mathcal{F}_{\underline{a}}$ associated to G and to the facet $\underline{a}$ of the Bruhat–Tits building of $G(K)$. The following theorem is one of the main results of this paper.

Theorem 6.1. *Suppose that G is semi-simple, splits over a tamely ramified extension of K and that the order of the fundamental group $\pi_1(G)$ is prime to the characteristic of k. Then the ind-schemes LG and $\mathcal{F}_{\underline{a}}$, for any facet $\underline{a}$, are reduced.*

In this section, we will show that it will be enough to prove Theorem 6.1 under the additional assumptions that k is algebraically closed and G is simply connected. The proof of this result and therefore also of Theorem 6.1 will then be completed in Section 9. First of all, let us observe that the reduction to the case that k is algebraically closed is immediate since k is perfect. Observe also that, since $P_{\underline{a}}$ is a smooth affine scheme over $k[[t]]$, it follows from [20, p. 264], that $L^+ P_{\underline{a}}$ is a reduced scheme. Hence, since $LG \to \mathcal{F}_{\underline{a}} = LG/L^+ P_{\underline{a}}$ splits locally for the étale topology, $\mathcal{F}_{\underline{a}}$ is reduced if and only if LG is reduced.

Remark 6.2. (a) When k has characteristic 0 and G is split, then the result follows from Proposition 4.6 in [30]. This proof uses a theorem of Shafarevich which is particular to characteristic 0. For $G = SL_n$, the result is in [2].

(b) We do not know if the assumption that G splits over a tamely ramified extension of K is necessary. As we shall see in Remark 6.4, $LPGL_2$ is non-reduced in characteristic 2, so the

assumption $p \nmid \#\pi_1(G)$ appears necessary. We shall also see in Proposition 6.5 that LG is non-reduced for any reductive group G that is not semi-simple.

We will now show how to reduce the proof of Theorem 6.1 to the case that G is simply connected.

6.a.   Let k be algebraically closed and assume that G is semi-simple over K. Then by a result of Steinberg G is quasi-split [46]. Denote by $\tilde{G}$ its simply connected cover,

$$1 \to Z \to \tilde{G} \to G \to 1. \tag{6.1}$$

Here the center Z is a finite commutative group scheme over K. Let T be a maximal torus of G and consider its inverse image $\tilde{T}$ in $\tilde{G}$; this is a maximal torus of $\tilde{G}$. The group scheme Z is a subgroup of $\tilde{T}$ and hence it is a finite multiplicative group scheme over K (i.e., the Cartier dual Z^D of Z is a finite étale commutative group scheme over K). Actually, since $\tilde{G}$ is semi-simple and simply connected, $\tilde{T} = \prod_i \mathrm{Res}_{K_i/K}(\mathbf{G}_m)$ is an induced torus, e.g. [8, Prop. 4.4.16]. Hence, if Z is annihilated by n, we have

$$Z \subset \prod_i \mathrm{Res}_{K_i/K}(\mu_n). \tag{6.2}$$

Observe that

$$\pi_1(G) = Z(K^{\mathrm{sep}})(-1) \tag{6.3}$$

as I-modules, where (-1) denotes a (negative) Tate twist (and where $I = \mathrm{Gal}(K^{\mathrm{sep}}/K)$).

Now let R be a k-algebra and consider the exact sequence derived from (6.1) above,

$$1 \to Z\big(R((t))\big) \to \tilde{G}\big(R((t))\big) \to G\big(R((t))\big) \to \mathrm{H}^1_{\mathrm{fppf}}\big(R((t)), Z\big) \to \mathrm{H}^1_{\mathrm{fppf}}\big(R((t)), \tilde{G}\big). \tag{6.4}$$

By a result of Grothendieck [22, Thm. 11.7], $\mathrm{H}^1_{\mathrm{fppf}}(R((t)), \tilde{G}) = \mathrm{H}^1_{\mathrm{et}}(R((t)), \tilde{G})$. When R is Artinian with algebraically closed residue field this latter group is trivial by Section 1.a.3. In this case, we have an exact sequence

$$1 \to Z\big(R((t))\big) \to \tilde{G}\big(R((t))\big) \to G\big(R((t))\big) \to \mathrm{H}^1_{\mathrm{fppf}}\big(R((t)), Z\big) \to 1. \tag{6.5}$$

6.a.1.   In this subsection, we suppose in addition that the characteristic of k does not divide the order of $\pi_1(G)$. Then Z is finite étale over $k((t))$; for simplicity, we will identify Z with the I-module given by $Z(K^{\mathrm{sep}})$. Then we can write $Z = \pi_1(G)(1)$. Let us assume that R is a local k-algebra with residue field k', where k' is some algebraically closed field extension of k. Then, we see from (6.2) that we have

$$Z\big(R((t))\big) = Z\big(k((t))\big) = Z\big(k[\![t]\!]\big) = \big(\pi_1(G)(1)\big)^I. \tag{6.6}$$

These equalities imply that $Z(R((t))) \subset \tilde{T}^0(R[\![t]\!]) = (\prod_i \mathrm{Res}_{\mathcal{O}_{K_i}/\mathcal{O}_K} \mathbf{G}_m)(R[\![t]\!])$ for each maximal torus $\tilde{T}$ of $\tilde{G}$. Therefore, since each parahoric subgroup scheme $\tilde{P}_{\underline{a}}$ of $\tilde{G}$ contains the connected component of the Neron model $\tilde{T}^0$ for a maximal torus $\tilde{T}$ we conclude that $Z(R((t))) \subset$

$\tilde{P}_{\underline{a}}(R[[t]])$, for each $\underline{a}$. By [8, 1.7], the homomorphism $\tilde{G} \to G$ induces $\tilde{P}_{\underline{a}} \to P_{\underline{a}}$. Hence, the exact sequence (6.4) shows the existence of an immersion of ind-schemes

$$L\tilde{G}/L^+\tilde{P}_{\underline{a}} \hookrightarrow LG/L^+P_{\underline{a}}. \tag{6.7}$$

Remark 6.3. Observe that (6.6) is not true when $p = \mathrm{char}(k)$ divides the order of Z. For example, if R contains $a \neq 0$ with $a^p = 0$, then $at^{-1} + 1$ is in $\mu_p(R((t)))$ but not in $\mu_p(R[[t]])$. As a result, when p divides the order of Z, $LZ(R) = Z(R((t)))$ is not necessarily contained in the parahoric points $L^+\tilde{P}_{\underline{a}}(R) = \tilde{P}_{\underline{a}}(R[[t]])$. For example, if $p = 2$ and R contains $a \neq 0$ with $a^2 = 0$, the central element

$$z = \begin{pmatrix} 1 + at^{-1} & 0 \\ 0 & 1 + at^{-1} \end{pmatrix} \in SL_2\big(R((t))\big)$$

does not belong to $SL_2(R[[t]])$.

Now suppose also that R is Artinian with algebraically closed residue field k'. Then $R((t))$ is local henselian with residue field $k'((t))$, and we have

$$\mathrm{H}^1_{\mathrm{fppf}}\big(R((t)), Z\big) = \mathrm{H}^1_{\mathrm{et}}\big(R((t)), Z\big) = \mathrm{H}^1_{\mathrm{et}}\big(k'((t)), Z\big). \tag{6.8}$$

Recall that if B is a finite abelian group, and k' an algebraically closed field of characteristic prime to the order of B, then we have an isomorphism

$$\mathrm{H}^1_{\mathrm{et}}\big(k'((t)), B(1)\big) = k'((t))^\times \otimes_{\mathbf{Z}} B \xrightarrow{\sim} B \tag{6.9}$$

where the last isomorphism is given using the valuation $\mathrm{ord}\colon k'((t))^\times \to \mathbf{Z}$. Similarly, for the finite étale commutative group scheme $A(1)$ over $k((t))$ (which we may think of as an I-module with $\mathrm{Gal}(K^{\mathrm{sep}}/K) = I$), we have

$$\mathrm{H}^1_{\mathrm{et}}\big(k'((t)), A(1)\big) \xrightarrow{\sim} A_I. \tag{6.10}$$

Hence, under our assumptions, the exact sequence (6.5) becomes

$$1 \to \big(\pi_1(G)(1)\big)^I \to \tilde{G}\big(R((t))\big) \to G\big(R((t))\big) \xrightarrow{\kappa} \pi_1(G)_I \to 1. \tag{6.11}$$

(One can show that the homomorphism κ is given by the Kottwitz homomorphism κ_G.) Recall that by Theorem 5.1, $L\tilde{G}/L^+\tilde{P}_{\underline{a}}$ is connected. The exact sequence (6.11) now implies that the immersion (6.7) identifies $L\tilde{G}/L^+\tilde{P}_{\underline{a}}$ with the neutral component of $LG/L^+P_{\underline{a}}$. Furthermore, $LG/L^+P_{\underline{a}}$ is reduced if and only if $L\tilde{G}/L^+\tilde{P}_{\underline{a}}$ is reduced. This concludes the reduction of the proof of Theorem 6.1 to the simply connected case.

Remark 6.4. To see that the condition $p \nmid \#\pi_1(G)$ in the statement of Theorem 6.1 is necessary, suppose that k is an algebraically closed field of characteristic 2 and let $G = PGL_2$ so that $\tilde{G} = SL_2$ and $\pi_1(G) = \mu_2$. Consider the fpqc sheaf Q associated to the presheaf $R \mapsto (R((t)))^\times/(R((t))^\times)^2 R[[t]]^\times$, for a k-algebra R. We can see that Q is represented by an ind-group scheme which is of ind-finite type and has two connected components: $Q = \mathbf{Z}/2\mathbf{Z} \times Q^0$.

The ind-scheme Q^0 is also given by the fpqc quotient $(1+t^{-1}R[t^{-1}])^{\times}/((1+t^{-1}R[t^{-1}])^{\times})^2$; it is not hard to see that it is infinitesimal. Now suppose that R is an Artin local k-algebra with algebraically closed residue field. By comparing the exact sequences (6.5) for $G = \mathbf{G}_m$ and $G = PGL_2$, we see that we have an exact sequence

$$1 \to \mu_2\big(R((t))\big) \to SL_2\big(R((t))\big) \to PGL_2\big(R((t))\big) \to \big(R((t))\big)^{\times}/\big(R((t))^{\times}\big)^2 \to 1. \quad (6.12)$$

Similarly, we have

$$1 \to \mu_2\big(R[[t]]\big) \to SL_2\big(R[[t]]\big) \to PGL_2\big(R[[t]]\big) \to \big(R[[t]]\big)^{\times}/\big(R[[t]]^{\times}\big)^2 \to 1. \quad (6.13)$$

This gives an exact sequence of pointed sets

$$1 \to \mu_2\big(R((t))\big)/\mu_2\big(R[[t]]\big) \to \tilde{\mathcal{F}}(R) \xrightarrow{\phi(R)} \mathcal{F}(R) \to \big(R((t))\big)^{\times}/\big(R((t))^{\times}\big)^2 R[[t]]^{\times} \to 1 \quad (6.14)$$

where $\tilde{\mathcal{F}}$, $\mathcal{F}$ are the affine Grassmannians for SL_2, PGL_2 respectively and ϕ is the natural morphism (which is proper). When $R = k$, we have

$$1 \to \tilde{\mathcal{F}}(k) \to \mathcal{F}(k) \to \mathbf{Z}/2\mathbf{Z} \to 1. \quad (6.15)$$

Therefore, since $\tilde{\mathcal{F}}$, $\mathcal{F}$ are of ind-finite type and $\tilde{\mathcal{F}}$ is reduced by [13], the reduced locus of the neutral component $\mathcal{F}^0$ of $\mathcal{F}$ coincides with the scheme theoretic image $\phi(\tilde{\mathcal{F}})$. However, if there is an element in $(R((t)))^{\times}/(R((t))^{\times})^2 R[[t]]^{\times}$ whose class in $Q^0(R)$ is non-trivial, then there is also an R-valued point of $\mathcal{F}^0$ which does not belong to $\phi(\tilde{\mathcal{F}})(R)$. This shows that $\mathcal{F}$ is not reduced. In fact, the point of $\mathcal{F}^0(R)$ represented by

$$z = \begin{pmatrix} 1 + at^{-1} & 0 \\ 0 & 1 \end{pmatrix} \in PGL_2\big(R((t))\big)$$

does not lie in $\mathcal{F}_{\mathrm{red}}$. (Here again $a \in R$ satisfies $a \neq 0$ and $a^2 = 0$.) The morphism $\tilde{\mathcal{F}} \to \mathcal{F}$ induces a surjective radiciel morphism between $\tilde{\mathcal{F}} = \mathcal{F}_{SL_2}$ and the reduced locus of the neutral component of $\mathcal{F} = \mathcal{F}_{PGL_2}$. We can see that this is ind-finite and so it is a universal homeomorphism (in this context of ind-schemes).

The assumption in Theorem 6.1 that G be semi-simple, is also necessary, as is shown by the next proposition.

Proposition 6.5. *Let G be a reductive group over K which is not semi-simple. Then LG and $\mathcal{F}_{\underline{a}}$, for any facet $\underline{a}$, are not reduced. In particular, if $G = T$ is a non-trivial torus, then LT and $\mathcal{F}_{T^0}$ are not reduced.*

Proof. We first prove the last statement. Suppose that T is a non-trivial torus over K. Let K'/K be a finite Galois splitting field for T with Galois group Γ; denote by Γ_t the (cyclic) tame quotient of Γ by the wild inertia subgroup $\Gamma_w \subset \Gamma$. We write $K' = k((u))$ and $K_t = (K')^{\Gamma_w} = k((v))$ with $v = t^{1/m}$, where $m = \#\Gamma_t$. Then, for every k-algebra R, we have

$$LT(R) = T\big(R((t))\big) = \big(T\big(R((u))\big)\big)^{\Gamma} = \big(X_*(T) \otimes_{\mathbf{Z}} R((u))^{\times}\big)^{\Gamma}$$

where $X_*(T)$ is the Γ-module given by the cocharacter lattice. Consider the k-algebra $R = k[x]/(x^a)$, for $a > 1$ and set
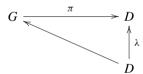
$$M = \left\{ 1 + \sum_{i=1}^{a-1} a_i x^i v^{-i} \;\middle|\; a_i \in k \right\} \subset R((v))^{\times}.$$

This is subgroup of $R((u))^{\times}$ which is fixed by Γ and is killed by p^b for some $b \gg 0$. Now find a non-zero element $w_0 \in X_*(T) \otimes_{\mathbf{Z}} \mathbf{Z}/p^b\mathbf{Z}$ which is fixed by the p-group Γ_w and consider the subgroup V generated by the "conjugates" $\gamma \cdot w_0$, $\gamma \in \Gamma$. The action of Γ on

$$V \otimes_{\mathbf{Z}} M \subset \left(X_*(T) \otimes_{\mathbf{Z}} R((u))^{\times} \right)^{\Gamma_w}$$

is inflated from an action of the cyclic group $\Gamma_t = \Gamma/\Gamma_w$ and there is a non-trivial element z_0 such that $\gamma_t \cdot z_0 = \chi(\gamma_t)z_0$ for a character $\chi : \Gamma_t \to k^{\times}$. Fix $n_\chi > 0$ such that $\gamma_t(v^{n_\chi}) = \chi(\gamma_t)v^{n_\chi}$. Then the R-valued point $z_0 \cdot v^{-n_\chi}$ is fixed by Γ; hence it gives an R-valued point of LT and a non-trivial point of $\mathcal{F}_{\mathcal{T}^0}$. We conclude that $\mathcal{F}_{\mathcal{T}^0}$ is not discrete and hence not reduced. This implies that LT is also not reduced.

Now we consider the general case. Let $G' = G_{\mathrm{der}}$ and $D = G/G'$. If G is not semi-simple, then D is a non-trivial torus and the previous considerations apply to D. There exists a commutative diagram

where π is the natural projection and where λ is an isogeny. This induces a commutative diagram of homomorphisms of ind-group schemes,

If LG were reduced, then $L\lambda$ would factor through $(LD)_{\mathrm{red}}$. But then there would exist $N > 0$ such that for any Artin local ring R and any $t \in LD(R)$ we have $t^N \in (LD)_{\mathrm{red}}(R)$. However, we can easily see that there exists a point of the above form for suitable $R = k[x]/(x^a)$ such that its Nth power is a non-trivial infinitesimal R-valued point of LD. $\quad\square$

6.b. Suppose that G is connected reductive but not necessarily semi-simple over $K = k((t))$ with k algebraically closed. Let us set $G' = G_{\mathrm{der}}$ and $D = G/G'$. Suppose that $\underline{a}$ is a facet in the Bruhat–Tits building of $G(K)$. Denote by $P'_{\underline{a}}$, $P_{\underline{a}}$ the corresponding parahoric group schemes, and $\mathcal{F}'_{\underline{a}}$, $\mathcal{F}_{\underline{a}}$ the corresponding affine flag varieties for G', G respectively. For simplicity, in what follows, we will omit the subscript $\underline{a}$. Consider the exact sequence

$$1 \to G' \xrightarrow{\phi} G \xrightarrow{\epsilon} D \to 1. \tag{6.16}$$

By [8, 1.7], there are homomorphisms of group schemes $P \to \mathcal{D}^0$, $P' \to P$ over $k[[t]]$; the composition $P' \to P \to \mathcal{D}^0$ is trivial. We obtain morphisms of ind-schemes

$$\mathcal{F}' \xrightarrow{\phi} \mathcal{F} \xrightarrow{\epsilon} \mathcal{F}_{\mathcal{D}^0}. \tag{6.17}$$

Using [8, 5.2.11], and Theorem 5.1 and its proof, we see that if $k' \supset k$ is algebraically closed, we have

$$(LG)^0(k') = \phi\big((LG')^0(k')\big) \mathcal{T}^0\big(k'[[t]]\big), \tag{6.18}$$

where $\mathcal{T}^0$ is the connected Neron model of any maximal torus T of G and $(LG)^0$, $(LG')^0$ denote the neutral components. We can deduce that $\phi : (\mathcal{F}')^0(k') \to \mathcal{F}^0(k')$ is surjective; by our description of $P'(k'[[t]])$, resp. $P(k'[[t]])$ as the stabilizers of $\underline{a}$ in $(LG')^0(k')$, resp. $(LG)^0(k')$, it is also injective, hence bijective. We conclude that

$$(\mathcal{F}')^0_{\mathrm{red}} \xrightarrow{\phi} (\mathcal{F})^0_{\mathrm{red}} \tag{6.19}$$

is a surjective radiciel morphism between the reduced loci of the neutral components of $\mathcal{F}'$ and $\mathcal{F}$.

We do not know whether this morphism is an isomorphism in general. In the tamely ramified case there is the following result.

Proposition 6.6. *Suppose that G splits over a tamely ramified extension of K. Then*

$$(\mathcal{F}_{G_{\mathrm{der}}})^0_{\mathrm{red}} \xrightarrow{\phi} (\mathcal{F}_G)^0_{\mathrm{red}} \tag{6.20}$$

is an isomorphism.

Proof. The essential ingredient is the following:

Lemma 6.7. *Let $1 \to T' \to T \to T'' \to 1$ be an exact sequence of tori over $K = k((t))$ such that T splits over a tamely ramified extension of K. Then there is an exact sequence of smooth group schemes over $\mathcal{O} = k[[t]]$*

$$1 \to \mathcal{S}' \to \mathcal{T}^0 \to \mathcal{T}''^0 \to 1 \tag{6.21}$$

which extends the above sequence of tori and is such that:

- *$\mathcal{T}^0$, $\mathcal{T}''^0$ are the connected Neron models of T, T'' respectively.*
- *The connected component of the group scheme $\mathcal{S}'$ is equal to $\mathcal{T}'^0$ and the group of connected components of the special fiber of $\mathcal{S}'$ is a subgroup of the torsion subgroup of $X_*(T')_I$.*

In fact, if T' is induced, then $X_*(T')_I$ is torsion-free and so $\mathcal{S}' = \mathcal{T}'^0$.

Proof. Assume that T splits over the tamely ramified (finite) Galois extension $\tilde{K}/K$ with Galois group Γ. Then the same is true for T' and T. Since we are then dealing with split tori, there is a short exact sequence

$$1 \to \mathcal{T}'_{\tilde{\mathcal{O}}} \to \mathcal{T}_{\tilde{\mathcal{O}}} \to \mathcal{T}''_{\tilde{\mathcal{O}}} \to 1 \tag{6.22}$$

between lft Néron models over $\tilde{\mathcal{O}}$. As in [12, proof of Theorem 4.2], we see that the fixed point schemes $(\mathcal{T}'_{\tilde{\mathcal{O}}})^{\Gamma}$, $(\mathcal{T}_{\tilde{\mathcal{O}}})^{\Gamma}$, $(\mathcal{T}''_{\tilde{\mathcal{O}}})^{\Gamma}$ are the lft Néron models $\mathcal{T}'$, $\mathcal{T}$, $\mathcal{T}''$ for T', T, T'' over $\mathcal{O}$. Hence, there is an exact sequence

$$1 \to \mathcal{T}' \to \mathcal{T} \to \mathcal{T}''. \tag{6.23}$$

In particular, $\mathcal{T}'$ is a closed subgroup scheme of $\mathcal{T}$. Hilbert's theorem 90 and the triviality of the Brauer group of K implies that $\mathrm{H}^1(K, T') = (0)$ (see e.g. [9, Lemma 4.3]) and so $1 \to T'(K) \to T(K) \to T''(K) \to 1$ is exact. Since by the Néron extension property $\mathcal{T}(\mathcal{O}) = T(K)$, $\mathcal{T}'(\mathcal{O}) = T'(K)$, $\mathcal{T}''(\mathcal{O}) = T''(K)$, we conclude that

$$1 \to \mathcal{T}'(\mathcal{O}) \to \mathcal{T}(\mathcal{O}) \to \mathcal{T}''(\mathcal{O}) \to 1 \tag{6.24}$$

is also exact. On the other hand, the targets of the Kottwitz invariant homomorphisms form the exact sequence

$$X_*(T')_I \xrightarrow{\phi_*} X_*(T)_I \to X_*(T'')_I \to 0, \tag{6.25}$$

where $\ker(\phi_*) \subset (X_*(T')_I)_{\mathrm{tor}}$. Now define $\mathcal{S}'$ to be the unique subgroup scheme of the lft Néron model $\mathcal{T}'$ which contains $\mathcal{T}'^0$ and with special fiber given by the connected components of the special fiber of $\mathcal{T}'$ which are parametrized by $\ker(\phi_*)$. We can conclude by the snake lemma and Theorem 5.1 that

$$1 \to \mathcal{S}'(\mathcal{O}) \to \mathcal{T}^0(\mathcal{O}) \to \mathcal{T}''^0(\mathcal{O}) \to 1 \tag{6.26}$$

is exact. (The exactness on the right follows also using [6, 9.6.2].) By the above, $\mathcal{S}'$ is a closed subgroup scheme of $\mathcal{T}^0$. The claim that (6.21) is exact everywhere except at $\mathcal{T}''^0$ immediately follows from (6.23) and (6.26). The claim that (6.21) is exact at $\mathcal{T}''^0$ also follows from the above and [8, Prop. 1.7.6]. Indeed, both $\mathcal{T}''^0$ and the quotient $\mathcal{T}^0/\mathcal{S}'$ are smooth (affine) commutative group schemes with T'' as generic fiber and with the same $\mathcal{O}$-valued points. $\square$

We will apply Lemma 6.7 to the exact sequence $1 \to T' \to T \to D \to 1$ where T', T are maximal tori in G', G respectively. Let T'_{sc} be the inverse image of T' in the simply connected cover of G'. Consider

$$0 \to X_*(T'_{\mathrm{sc}}) \to X_*(T') \to \pi_1(G') \to 0, \qquad 0 \to X_*(T'_{\mathrm{sc}})_I \to X_*(T')_I \to \pi_1(G')_I \to 0.$$

Here the first sequence is exact and $\pi_1(G')$ is a finite group. Also $X_*(T'_{\mathrm{sc}})$ is an induced I-lattice (the coroot lattice) so $X_*(T'_{\mathrm{sc}})_I = X_*(T'_{\mathrm{sc}})^I$ is torsion-free. Hence, the second sequence is also exact and we obtain

$$\left(X_*(T')_I\right)_{\mathrm{tor}} \hookrightarrow \pi_1(G')_I. \tag{6.27}$$

To complete the proof of Proposition 6.6, suppose that R is a strictly henselian local k-algebra; then so is $R[\![t]\!]$. Since $\mathcal{S}'$ is smooth over $k[\![t]\!]$ the homomorphism $\mathcal{T}^0(R[\![t]\!]) \to \mathcal{D}^0(R[\![t]\!])$ is surjective. Now suppose that we have $[g] \in LG^0(R)/L^+P(R) = \mathcal{F}^0(R)$ and that R is in addition reduced. Then by Corollary 5.2 we have $\epsilon(g) \in \mathcal{D}^0(R[\![t]\!])$. By modifying g by multiplying with the inverse of a lift of $\epsilon(g)$ in $\mathcal{T}^0(R[\![t]\!]) \subset P(R[\![t]\!])$ we can assume that $\epsilon(g) = 1$,

therefore $g \in LG'(R)$. In fact, $\kappa_{G'}(g)$ lies in the finite subgroup of $(X_*(T')_I)_{\mathrm{tor}} \subset \pi_1(G')_I$ given by Lemma 6.7 and (6.27). By modifying g further via multiplication by an element $s \in \mathcal{S}'(R[[t]]) \subset T'(R((t)))$ with $\kappa_{G'}(s) = -\kappa_{G'}(g)$ we can arrange so that the new g is also in $(LG')^0(R)$. This shows that, when G splits over a tame extension, then for every *reduced* strictly local k-algebra R, the map $\phi : \mathcal{F}'^0(R) \to \mathcal{F}^0(R)$ is surjective. Now $(\mathcal{F}')^0_{\mathrm{red}}$ and $(\mathcal{F})^0_{\mathrm{red}}$ are of ind-finite type. If $\phi : X \to Y$ is a morphism between reduced k-schemes of finite type, which is a universal homeomorphism and such that $X(R) \to Y(R)$ is surjective for every reduced strictly local k-algebra R, then ϕ is an isomorphism. This, combined with the above, allows us to conclude the proof. $\square$

7. Tame forms and liftings

7.a. In this section, we assume that k is algebraically closed. Suppose that G is an absolutely simple, simply connected, quasi-split group over $K = k((t))$ that splits over the (totally ramified) separable extension K'/K of degree e. This degree can take the values $e = 1$ (when G is split over K), 2 or 3 (see [48, §4], or [8, §4.1–4.2]). We have $e = 2$ when the local Dynkin diagram of G is of type B–C_n (for $n \geqslant 3$), C–B_n (for $n \geqslant 2$), C–BC_n (for $n \geqslant 1$), F_4^I. We have $e = 3$ only in one case, when the Dynkin diagram is of the form G_2^I. (The notations for the Dynkin diagrams are as in [48, Table 4.2, p. 60].)

We assume that $(e, \mathrm{char}(k)) = 1$ so that K'/K is tame and we set $K' = k((u))$ with $u^e = t$. Fix a generator τ of $\mathrm{Gal}(K'/K)$. Let us denote by H the *split* simple, simply connected, algebraic group over K such that $G_{K'} = H_{K'}$ (we fix a choice of this isomorphism). Consider the Weyl restriction of scalars $H' = \mathrm{Res}_{K'/K}(H_{K'})$; the automorphism $\mathrm{id} \otimes \tau$ of $\mathrm{Res}_{K'/K}(G \otimes_K K') = H'$ induces an automorphism σ of H' which can be written on $H_{K'} = H \otimes_K K'$ in the form $\sigma = \sigma_0 \otimes \tau$, where σ_0 is a Dynkin diagram automorphism of order e of the group H. In the cases B–C_n, C–BC_n (these correspond to G a special "ramified" unitary group as in Section 4), we have $H = SL_{2n}$, or $H = SL_{2n+1}$ respectively; the involution σ_0 is the usual transpose inverse involution. In the case C–B_n, we have $H = \mathrm{Spin}_{2n+2}$. In the case F_4^I, we have $H = E_6$. Finally, in the case G_2^I (then $e = 3$), we have $H = \mathrm{Spin}_8$ ("triality").

The automorphism σ gives an action on the Bruhat–Tits building $\mathcal{B}(H', K) = \mathcal{B}(H, K')$. By the main result of [40], the set of points $\mathcal{B}(H, K')^\sigma$ of $\mathcal{B}(H, K')$ fixed by σ can be identified ($(G(K)$-equivariantly) with the building $\mathcal{B}(G, K)$. Now let x be a point in $\mathcal{B}(G, K) \subset \mathcal{B}(H, K')$. Since H is simply connected, the stabilizer $H(K')^x$ of x in $H(K')$ gives the $k[[u]]$-valued points of the corresponding parahoric subgroup scheme P_x^H over $k[[u]]$. One can see that the automorphism σ extends to $\mathrm{Res}_{k[[u]]/k[[t]]}(P_x^H)$ (e.g. using [8, 1.7]). Now let us consider the scheme of fixed points $(\mathrm{Res}_{k[[u]]/k[[t]]}(P_x^H))^\sigma$. Since σ has order prime to the characteristic p, the scheme $(\mathrm{Res}_{k[[u]]/k[[t]]}(P_x^H))^\sigma$ is a group scheme which is smooth over $k[[t]]$. In fact, since its $k[[t]]$-valued points give the stabilizer $G(K)^x$ and G is simply connected, we have

$$P_x^G = \left(\mathrm{Res}_{k[[u]]/k[[t]]}\left(P_x^H\right)\right)^\sigma \tag{7.1}$$

with P_x^G the parahoric group scheme over $k[[t]]$ associated to $x \in \mathcal{B}(G, K)$.

Suppose now that in addition $\mathrm{char}(k) = p > 0$; we denote by $W = W(k)$ the ring of Witt vectors of k. Our goal is to extend this picture to a mixed characteristic situation (i.e., over W). We lift $k[[u]]/k[[t]]$ to the ring extension $W[[u]]/W[[t]]$ with $u^e = t$; we will continue to denote by τ the automorphism of $W[[u]]$ with lifts our choice of Galois generator with the same notation. First we observe that the group scheme P_x^H over $k[[u]]$ has a natural lift to a group scheme $\underline{P}_x^H$

over $W[\![u]\!]$. This can be obtained by applying [8, §3.2], to the base ring $A = W[\![u]\!]$ and to schematic root data for the split group H that correspond to x. More precisely, the needed ideals $f(a) \subset W[\![u]\!]$ in [8, Prop. 3.2.7], are given by $(u^{m(a)})$, where $m(a) \in \mathbf{Z}$ and where $(u^{m(a)}) \subset k[\![u]\!]$ are the corresponding ideals describing the schematic root data for P_x^H that correspond to $x \in \mathcal{B}(H, K')$; this choice guarantees that our lift is "horizontal in the W-direction." We can see again that we have an automorphism σ on $(\mathrm{Res}_{W[\![u]\!]/W[\![t]\!]}(\underline{P}_x^H))$ that covers $\tau : W[\![u]\!] \to W[\![u]\!]$. We set

$$\underline{P}_x^G = \left(\mathrm{Res}_{W[\![u]\!]/W[\![t]\!]}(\underline{P}_x^H)\right)^\sigma; \tag{7.2}$$

this is a smooth group scheme over $W[\![t]\!]$ that lifts the parahoric group scheme P_x^G over $k[\![t]\!]$. In fact, one could also construct $\underline{P}_x^G$ directly by applying [8, 3.9.4] to $A = W[\![t]\!]$ and to suitable "W-horizontal" schematic root data given by $x \in \mathcal{B}(G, K)$.

Denote by H_0 the Chevalley group scheme over W that corresponds to the split group H. Let us set $\underline{H} = H_0 \otimes_W W(\!(u)\!)$. This is a smooth affine group scheme over $W(\!(u)\!)$ that lifts H over $K' = k(\!(u)\!)$. The automorphism σ lifts to an automorphism of $\underline{H}' = \mathrm{Res}_{W(\!(u)\!)/W(\!(t)\!)}(\underline{H})$ over $W(\!(t)\!)$ and we can set

$$\underline{G} := \left(\mathrm{Res}_{W(\!(u)\!)/W(\!(t)\!)}(\underline{H})\right)^\sigma. \tag{7.3}$$

(The group scheme $\underline{G}$ over $W(\!(t)\!)$ can also be obtained as in [49, p. 217].)

7.b. We continue with the above assumptions and notations. In particular, we fix $x \in \mathcal{B}(G, K) \subset \mathcal{B}(H, K')$. Let us consider the following functors of W-algebras:

$$L\underline{H}(R) := \underline{H}(R(\!(u)\!)), \qquad L^+\underline{P}_x^H(R) := \underline{P}_x^H(R[\![u]\!]).$$

Also we have

$$L\underline{G}(R) = \left(\underline{H}(R(\!(u)\!))\right)^\sigma, \qquad L^+\underline{P}_x^G(R) = \left(\underline{P}_x^H(R[\![u]\!])\right)^\sigma \subset L\underline{G}(R).$$

Consider the fpqc sheaf $L\underline{H}/L^+\underline{P}_x^H$ over W which is associated to the presheaf $R \mapsto L\underline{H}(R)/L^+\underline{P}_x^H(R)$. Since $\underline{H}$ is split, we may refer to [13, Section 1], for the fact that this sheaf is represented by an ind-scheme over W (which is actually ind-proper); it is a (partial) affine flag variety for the (untwisted) loop group scheme $\underline{H}$ over W.

Proposition 7.1. *Consider the fpqc quotient sheaf $L\underline{G}/L^+\underline{P}_x^G$ over W.*

(a) *The fpqc sheaf $L\underline{G}/L^+\underline{P}_x^G$ can be identified with the neutral component of the fixed point ind-scheme $(L\underline{H}/L^+\underline{P}_x^H)^\sigma$.*

(b) *$L\underline{G}/L^+\underline{P}_x^G$ is representable by an ind-scheme which is ind-proper over W.*

(c) *The fiber of $L\underline{G}/L^+\underline{P}_x^G$ over k is isomorphic to the affine flag variety $LG/L^+P_x^G$ for G and the parahoric subgroup scheme P_x^G.*

Proof. Let us first discuss part (a). We first assume that P_x^H is an Iwahori group scheme for H; recall that x is fixed by σ. Then the maximal reductive quotient of the reduction of $\underline{P}_x^H$ over (u) is a split torus T_0 over W which is σ-stable. Let R be a local W-algebra. Recall that by [13], the

quotient morphism $L\underline{H} \to L\underline{H}/L^+\underline{P}_x^H$ splits locally for the Zariski topology of the ind-scheme $L\underline{H}/L^+\underline{P}_x^H$ and so we have

$$\left(L\underline{H}/\underline{P}_x^H\right)(R) = L\underline{H}(R)/L^+\underline{P}_x^H(R).$$

There is a natural "exact" sequence of pointed sets

$$1 \to \left(L\underline{H}(R)\right)^\sigma/\left(L^+\underline{P}_x^H(R)\right)^\sigma \to \left(L\underline{H}(R)/L^+\underline{P}_x^H(R)\right)^\sigma \to \mathrm{H}^1\left(\langle\sigma\rangle, \underline{P}_x^H\left(R[\![u]\!]\right)\right). \quad (7.4)$$

Now denote by $P^+(R)$ the kernel of the (surjective) reduction homomorphism

$$\underline{P}_x^H\left(R[\![u]\!]\right) \to T_0(R).$$

We can see that $\mathrm{H}^1(\langle\sigma\rangle, \underline{P}^+(R)) = (0)$ since $\langle\sigma\rangle$ has order prime to p and $P^+(R)$ is a projective limit of a system of groups which are iterated extensions of W-modules. Observe that the action of τ on $T_0(R)$ is trivial. Also recall that σ_0 is a diagram automorphism and hence acts on the cocharacter lattice $X_*(T_0)$ of the torus T_0 by a permutation representation. Therefore, we have

$$\mathrm{H}^1\left(\langle\sigma\rangle, \underline{P}_x^H\left(R[\![u]\!]\right)\right) = \mathrm{H}^1\left(\langle\sigma_0\rangle, T_0(R)\right) = \mathrm{H}^1\left(\langle\sigma_0\rangle, R^\times \otimes_{\mathbf{Z}} X_*(T_0)\right). \quad (7.5)$$

We can now see (the group $\langle\sigma_0\rangle$ is cyclic of order e), that $\mathrm{H}^1(\langle\sigma_0\rangle, R^\times \otimes_{\mathbf{Z}} X_*(T_0)) = \mathrm{H}^1(\langle\sigma_0\rangle, k^\times \otimes_{\mathbf{Z}} X_*(T_0))$, i.e., it is independent of R, and is a finite torsion group which we denote by $C(\langle\sigma_0\rangle, H)$. We obtain a map

$$\left(L\underline{H}/L^+\underline{P}_x^H\right)^\sigma \to C\left(\langle\sigma_0\rangle, H\right) \quad (7.6)$$

which is locally constant for the Zariski topology; this fact together with (7.4) completes the proof of (a) when P_x^H is Iwahori. To deal with the general case, suppose that x is a fixed point of σ in the building $\mathcal{B}(H, K')$. We claim that there is a σ-fixed point $z \in \mathcal{B}(H, K')$ such that P_z^H is Iwahori and $\underline{P}_z^H(R[\![u]\!]) \subset \underline{P}_x^H(R[\![u]\!])$: Indeed, consider the maximal reductive quotient of the reduction of $\underline{P}_x^H$ modulo (u). (This is a reductive group $\underline{Q}$ over W.) We can see that this supports an action of σ_0 which is given by a diagram automorphism. Hence we can find a σ_0-stable Borel subgroup $\underline{B}_Q \subset \underline{Q}$. This gives a σ-stable Iwahori subgroup P_z^H whose lifting $\underline{P}_z^H$ has the desired property. Now observe that

$$L\underline{H}/L^+\underline{P}_z^H \to L\underline{H}/L^+\underline{P}_x^H$$

is a σ-equivariant (Zariski locally trivial) smooth fibration with fiber over the origin the homogeneous space $\underline{P}_x^H/\underline{P}_z^H = \underline{Q}/\underline{B}_Q$. Note that all the fibers of the fixed point set $(\underline{Q}/\underline{B}_Q)^\sigma$ over W are non-empty. Indeed, it suffices by the Lefschetz fixed point formula to see that the trace of the automorphism σ_0 on the ℓ-adic cohomology of the flag variety Q/B_Q is non-zero. But this cohomology has a natural basis enumerated by the elements of the Weyl group of Q, and σ_0 acts by permuting the basis vectors. Consider the representation of the cyclic group generated by σ_0. On an irreducible constituent associated to a fixed vector, the trace of σ_0 is 1, on all other irreducible constituents it is 0. Since there are fixed vectors (e.g., the vector associated to the unit element in the Weyl group), the trace is non-zero. We conclude that

$$\left(L\underline{H}/L^+\underline{P}_z^H\right)^\sigma \to \left(L\underline{H}/L^+\underline{P}_x^H\right)^\sigma$$

is surjective. The proof of (a) in the general case now follows from these considerations and Theorem 5.1.

Part (b) now follows from (a) and the corresponding properties of $L\underline{H}/L^+\underline{P}_x^H$ since the action of σ respects the ind-scheme structure of $L\underline{H}/L^+\underline{P}_x^H$. In fact, we also see from (a) and its proof that for any local W-algebra R, we have

$$L\underline{G}(R)/L^+\underline{P}_x^G(R) = \left(L\underline{G}/L^+\underline{P}_x^G\right)(R). \tag{7.7}$$

Part (c) follows now from the above and our construction. $\quad\square$

Remark 7.2. Let us consider the example of the "ramified" special unitary group as in Section 4. Take $H = SL_n$, with σ_0 the involution

$$g \mapsto L_n \cdot \left(g^{\mathrm{tr}}\right)^{-1} \cdot L_n^{-1}$$

where L_n is the antidiagonal matrix of size n and $\tau(u) = \bar{u} = -u$. Then, $G = SU_n$ as in Section 4. In this case, we can obtain the group $\underline{G}$ over $W((t))$ as follows: Consider the $W\llbracket u \rrbracket$-lattice

$$\Lambda_0 = \mathrm{span}_{W\llbracket u \rrbracket}\{e_1, \ldots, e_n\}$$

with the (perfect) $W\llbracket u \rrbracket$-valued hermitian form Φ defined by $\Phi(e_i, e_j) = \delta_{i,n+1-j}$. First consider the group scheme $\underline{P}_0$ over $W\llbracket t \rrbracket$ whose S-valued points are the $W\llbracket u \rrbracket \otimes_{W\llbracket t \rrbracket} \mathcal{O}_S$-linear automorphisms A of $\Lambda_0 \otimes_{W\llbracket t \rrbracket} \mathcal{O}_S$ with determinant 1 which respect the form Φ, i.e., such that $\Phi(A \cdot v, A \cdot w) = \Phi(v, w)$ for all $v, w \in \Lambda_0 \otimes_{W\llbracket t \rrbracket} \mathcal{O}_S$. Then set $\underline{G} = \underline{P}_0 \otimes_{W\llbracket t \rrbracket} W((t))$. In fact, we can similarly also give $W\llbracket u \rrbracket$-lattices Λ_i, $\Lambda_{m'}$ by lifting the construction of the lattices λ_i, $\lambda_{m'}$ in Section 4 and consider group schemes $\underline{P}_I'$ over $W\llbracket t \rrbracket$ defined via Φ-unitary automorphisms that stabilize suitable chains of the Λ's. The group schemes $\underline{P}_I'$ give explicit realizations of the group schemes $\underline{P}_x^G$ constructed above. They lift the parahoric group schemes P_I' over $k\llbracket t \rrbracket$ which were constructed in Section 4.

Let us comment on the statement of Proposition 7.1(a) in this special case. For simplicity, we just restrict our consideration to the special fiber over k and to the case of an Iwahori subgroup.

(i) $n = 2m + 1 \geqslant 3$ odd. Take $I = \{0, \ldots, m\}$. Then P_I' is an Iwahori subgroup of $SU_n(K)$. In this case, by Remark 4.2(B), which identifies LSU_n/L^+P_I' with the functor of complete selfdual lattice chains of the correct determinant, we have

$$LSU_n/L^+P_I' = \left(LSL_n/L^+I_{SL_n}\right)^\sigma \tag{7.8}$$

where I_{SL_n} is a corresponding Iwahori of SL_n.

(ii) $n = 2m \geqslant 4$ even. Take $I = \{0, \ldots, m-2, m, m'\}$ so that P_I' is an Iwahori subgroup of $SU_n(K)$. By Remark 4.2(B) we have an "exact" sequence of pointed sets

$$1 \to LSU_n/L^+P_I' \to \left(LSL_n/L^+I_{SL_n}\right)^\sigma \xrightarrow{\iota} \mathbf{Z}/2\mathbf{Z} \to 0. \tag{7.9}$$

Again $(LSL_n/L^+I_{SL_n})^\sigma$ classifies complete self-dual lattice chains of the correct determinant and the map ι is given by the sign in the identity $a = \pm\bar{b} \cdot b^{-1}$ in Remark 4.2(B).

7.c. We continue with the above assumptions and notations, in particular G splits over the totally tamely ramified extension K'/K of degree $e = 1, 2$ or 3. Let Δ_G be the local Dynkin diagram of the quasi-split group G over the local field $K = k((t))$ (see for example [48, 1.8 and Table 4.2]). Each of these local Dynkin diagrams is (up to numbering) the Dynkin diagram of a unique generalized Cartan matrix of affine type in the sense of Kac (see [24, 4.8], in particular the tables Aff 1, 2, 3 on pp. 54–55).

Now suppose that F is a field that contains $W = W(k)$. We will denote by $\mathfrak{g}_{KM}(\Delta_G)_F$ or simply $\mathfrak{g}_{KM}(G)_F$ the *affine* Kac–Moody Lie algebra over F given by the corresponding generalized Cartan matrix $A(G) = \{a_{ij}\}_{1 \leqslant i,j \leqslant l+1}$ [24, Chapter 1]. Here l is the rank of the split group $G_{K'}$. In fact, we will only need to consider the factor algebra $\mathfrak{g}'_{KM}(G)_F$ of $\mathfrak{g}_{KM}(G)_F$ by the one-dimensional subspace generated by the derivation element. The Lie algebra $\mathfrak{g}'_{KM}(G)_F$ is generated over F by $3(l+1)$ generators e_i, f_i, h_i with the relations:

$$[h_i, h_j] = 0, \qquad [e_i, f_i] = h_i, \qquad [e_i, f_j] = 0, \quad \text{if } i \neq j,$$

$$[h_i, e_j] = a_{ij} e_j, \qquad [h_i, f_j] = -a_{ij} f_j,$$

$$(\operatorname{ad} e_i)^{1-a_{ij}} e_j = 0, \qquad (\operatorname{ad} f_i)^{1-a_{ij}} f_j = 0, \quad \text{if } i \neq j.$$

We also have the following description of $\mathfrak{g}'_{KM}(G)_F$:

If the group G is split, then $\underline{G} = H_0 \otimes_W W((t))$ and $\sigma = \mathrm{id}$. Then

$$\mathfrak{g}'_{KM}(G)_F \simeq F \cdot c \oplus \big(Lie(H_0) \otimes_W F[t, t^{-1}]\big)$$

with bracket described in [24, 7.1]. (The element c is central, $(Lie(H_0) \otimes_W F[t, t^{-1}])$ is the loop Lie algebra of H_0.) This is an *untwisted* affine Kac–Moody algebra (its generalized Cartan matrix is from table Aff 1 of [24, 4.8]). To explain the isomorphism, denote by Φ' the set of roots of the split group $G_{K'} = H_{K'}$ and fix a choice of a set of simple roots $a'_1, \ldots, a'_l$ and hence a set of positive roots Φ'^+. Denote by θ the highest root of Φ. Fix a Chevalley basis E_a, $a \in \Phi'$, $H_i = H_{a'_i}$, $1 \leqslant i \leqslant l$, for the Lie algebra of the group H_0 over W. The elements $e_i = E_{a'_i}$, $f_i = E_{-a'_i}$, $h_i = H_i$ for $1 \leqslant i \leqslant l$, $e_0 = t \otimes E_{-\theta}$, $f_0 = t^{-1} \otimes E_\theta$ and $h_0 = \frac{2}{(\theta|\theta)} \cdot c - \theta^\vee$ in $F \cdot c \oplus (Lie(H_0) \otimes_W F[t, t^{-1}])$ satisfy the defining relations for $\mathfrak{g}'_{KM}(H)_F$ above (the notation is as in [24, §7.4]).

If the group G is not split, then $\sigma = \sigma_0 \otimes \tau$ is not trivial. Define an automorphism $\tilde{\sigma}$ of

$$\mathfrak{g}'_{KM}(H)_F = F \cdot c' \oplus \big(Lie(H_0) \otimes_W F[u, u^{-1}]\big)$$

by $\tilde{\sigma}(c') = c'$, while for $x \in Lie(H_0)$, $f(u) \in F[u, u^{-1}]$, $\tilde{\sigma}(x \otimes f(u)) = \sigma_0(x) \otimes f(\tau(u))$. Then

$$\mathfrak{g}'_{KM}(G)_F \simeq \big(\mathfrak{g}'_{KM}(H)_F\big)^{\tilde{\sigma}},$$

the fixed point set of $\tilde{\sigma}$ on $\mathfrak{g}'_{KM}(H)_F$ (see [24, Theorem 8.3]). This is then a *twisted* affine Kac–Moody algebra (its generalized Cartan matrix is from table Aff 2 or Aff 3 of [24, 4.8]; in each case 2 or 3 is the order of σ). In this case, the elements e_i, f_i, h_i of $\mathfrak{g}'_{KM}(H)_F$ can also be given explicitly from the Chevalley generators E_a, $H_1, \ldots, H_l$, of the group H_0 over W using the formulas in [24, §8.3].

Note that there is a Lie algebra embedding

$$\left(Lie(H_0) \otimes_W F\left[u, u^{-1}\right]\right)^{\tilde{\sigma}} \hookrightarrow Lie\left(L(\underline{G} \otimes_W F)\right).$$

(The Lie algebra $Lie(L(\underline{G} \otimes_W F))$ also includes "power series" in t.)

8. Weyl groups and Schubert varieties

In this section, we assume that k is algebraically closed and that G is a connected reductive group over $K = k((t))$.

8.a. We start by recalling some facts on affine Weyl groups (cf. [48,7,8,42], and especially the appendix [23] to this paper). Let S be a maximal split torus in G and let T be its centralizer. Since k is algebraically closed, G is quasi-split and so T is a maximal torus in G. Let $N = N(T)$ be the normalizer of T; denote by $T(K)_1$ the kernel of the Kottwitz homomorphism $\kappa_T : T(K) \to X_*(T)_I$; then $T(K)_1 = \mathcal{T}^0(k[[t]])$ where $\mathcal{T}^0$ the connected Neron model of T over $k[[t]]$. By definition, the *Iwahori–Weyl group associated to S* is the quotient group

$$\tilde{W} = N(K)/T(K)_1,$$

cf. [23, Def. 7]. Since κ_T is surjective, the Iwahori–Weyl group $\tilde{W}$ is an extension of the relative Weyl group $W_0 = N(K)/T(K)$ by $X_*(T)_I$:

$$0 \to X_*(T)_I \to \tilde{W} \to W_0 \to 1. \tag{8.1}$$

We have ([23, Prop. 8], comp. also [42])

Proposition 8.1. *Let B_0 be the Iwahori subgroup of $G(K)$ associated to an alcove contained in the apartment associated to the maximal split torus S. Then $G(K) = B_0 \cdot N(K) \cdot B_0$ and the map $B_0 \cdot n \cdot B_0 \mapsto n \in \tilde{W}$ induces a bijection*

$$B_0 \backslash G(K)/B_0 \xrightarrow{\sim} \tilde{W}. \tag{8.2}$$

If P is the parahoric subgroup of $G(K)$ associated to a facet contained in the apartment corresponding to S, then

$$P \backslash G(K)/P \xrightarrow{\sim} \tilde{W}^P \backslash \tilde{W}/\tilde{W}^P, \quad \text{where } \tilde{W}^P := \left(N(K) \cap P\right)/T(K)_1. \tag{8.3}$$

In fact, if P is the (special) parahoric subgroup $P_{\underline{x}}$ that corresponds to a special vertex $\underline{x}$ in the apartment corresponding to S, then the subgroup $\tilde{W}^P \subset \tilde{W}$ maps isomorphically to W_0 under the quotient $\tilde{W} \to W_0$ and the exact sequence (8.1) represents the Iwahori–Weyl group as a semi-direct product

$$\tilde{W} = W_0 \ltimes X_*(T)_I, \tag{8.4}$$

see [23, Prop. 13].

Now let S_{sc}, T_{sc}, resp. N_{sc} be the inverse images of $S \cap G_{der}$, $T \cap G_{der}$, resp. $N \cap G_{der}$ in the simply connected covering G_{sc} of the derived group G_{der}. Then S_{sc} is a maximal split torus of G_{sc} and T_{sc}, resp. N_{sc} is its centralizer, resp. normalizer. Hence

$$W_a := N_{sc}(K)/T_{sc}(K)_1$$

is the Iwahori–Weyl group of G_{sc}. This group is also called the *affine Weyl group associated to S* and is a Coxeter group. Indeed, we can recover W_a in the following way: Let $N(K)_1$ be the intersection of $N(K)$ with the kernel $G(K)_1$ of the Kottwitz homomorphism $\kappa_G : G(K) \to \pi_1(G)_I$. Then the natural homomorphism

$$W_a = N_{sc}(K)/T_{sc}(K)_1 \xrightarrow{\sim} N(K)_1/T(K)_1 \tag{8.5}$$

is an isomorphism [23, Lemma 17] and there is an exact sequence

$$1 \to W_a \to \tilde{W} \xrightarrow{\kappa_G} \pi_1(G)_I = X_*(T)_I/X_*(T_{sc})_I \to 1, \tag{8.6}$$

cf. [23, Lemma 14]. Now let B_0 be the Iwahori subgroup of $G(K)$ associated to an alcove C in the apartment corresponding to S and let $\mathbf{S}$ be the set of reflections about the walls of C. Then by [8, 5.2.12], the quadruple $(G(K)_1, B_0, N(K)_1, \mathbf{S})$ is a double Tits system and $W_a = N(K)_1/T(K)_1$ is the affine Weyl group of the affine root system Φ_a of S. The affine Weyl group W_a acts simply transitively on the set of alcoves in the apartment of S. Since $\tilde{W}$ acts transitively on the set of these chambers, $\tilde{W}$ is the semi-direct product of W_a with the normalizer Ω of the base alcove C, i.e., the subgroup of $\tilde{W}$ which preserves the alcove,

$$\tilde{W} = W_a \rtimes \Omega. \tag{8.7}$$

We can identify Ω with $X_*(T)_I/X_*(T_{sc})_I = \pi_1(G)_I$, cf. [23, Lemma 14].

Let us write[2] $\mathbf{S} = \{s_i\}_{i \in I} \subset W_a$ for the finite set of reflections about the walls of C that generate the Coxeter group W_a. For each $w \in W_a$ its length $l(w)$ is the minimal number of factors in a product of s_i's representing w. Any such product realizing the minimum is called a *reduced decomposition* of w. We will denote by $\leqslant$ the corresponding Bruhat order. Recall its definition: Fix a reduced decomposition of $w \in W_a$. The elements $w' \leqslant w$ are obtained by replacing some factors in it by 1. (This set of such w''s is independent of the choice of the reduced decomposition of w.)

Let us denote by $\alpha_i \in \Phi_a$ the unique affine root with corresponding affine reflection equal to s_i. (Since the residue field k is algebraically closed, $\frac{1}{2}\alpha_i \notin \Phi_a$; see [48, 1.8].) We will denote by $\Delta = \Delta_G$ the (local) Dynkin diagram of the affine root system Φ_a. (This can be obtained from the subset $\{\alpha_i\}_{i \in I}$; see [7, 1.4] and [48, 1.8].) Suppose that the Coxeter system $(W_a, \mathbf{S})$ is a product of the irreducible systems $(W_a^k, \mathbf{S}^k)$ which correspond to the simple factors of the derived group G_{der}; we have $\mathbf{S} = \bigcup_k \mathbf{S}^k$ [7, 1.3]. For a subset $Y \subset \mathbf{S}$ such that $Y \cap \mathbf{S}^k \neq \mathbf{S}^k$ for all k, we denote by $W_Y \subset W_a$ the subgroup generated by s_i with $i \in Y$; we set $P_Y = B_0 \cdot W_Y \cdot B_0$. By general properties of Tits systems these are subgroups of $G(K)_1 \subset G(K)$; by [8, 5.2.12(i)], they are the parahoric subgroups of $G(K)$ that contain B_0. Using [7, 1.3.5] we see that we can

---

[2] Here we follow the tradition of parametrizing the set of simple affine roots by I. This should not be confused with the notation for the inertia group.

identify P_Y with the parahoric subgroup P_{C_Y} where C_Y is the facet consisting of a in the closure $\bar{C}$ of C for which Y is exactly the set of reflections $s \in \mathbf{S}$ which fix a.

Now let us fix a special vertex $\underline{x}$ in the apartment corresponding to S. We may assume that $\underline{x}$ is in $\bar{C}$. Then there exists a reduced root system ${}^x\Sigma$ such that the semi-direct product (8.4) (for G_{sc} instead of G) presents W_a as the affine Weyl group associated (in the sense of Bourbaki) to ${}^x\Sigma$,

$$W_a = W\left({}^x\Sigma\right) \ltimes Q^\vee\left({}^x\Sigma\right) \tag{8.8}$$

(cf. [7, 1.3.8], [48, 1.7, 1.9]). In other words, we have identifications $W_0 \simeq W({}^x\Sigma)$, $X_*(T_{sc})_I \simeq Q^\vee({}^x\Sigma)$ compatible with the semidirect product decompositions (8.4) and (8.8), cf. [23, Prop. 13].

Remark 8.2. The following variant of the Iwahori–Weyl group appears in [48, p. 34]. Set $W^\flat = N(K)/T(K)_b$ where $T(K)_b$ is the maximal bounded subgroup of $T(K)$ (this group is denoted by $\check{W}$ in [48]). We have an exact sequence

$$1 \to T(K)_b/T(K)_1 \to \tilde{W} \to W^\flat \to 1, \tag{8.9}$$

cf. [23, Rem. 10].

8.b. Let us now discuss Schubert cells and varieties.

Definition 8.3. Let P be a parahoric subgroup of G which corresponds to a facet in the apartment of the maximal split torus S and let $w \in \tilde{W}^P \backslash \tilde{W} / \tilde{W}^P$.

(a) The *Schubert cell* C_w is the reduced subscheme

$$L^+ P \cdot n_w \subset \mathcal{F}_P = LG/L^+P,$$

with $n_w \in N(K)$ a representative of $w \in \tilde{W}^P \backslash \tilde{W} / \tilde{W}^P$.[3]
(b) The *Schubert variety* S_w is the reduced scheme with underlying set the Zariski closure $\bar{C}_w$ of the Schubert cell C_w in $\mathcal{F}_P$. It is a projective variety over k.

Our main result is the following:

Theorem 8.4. *Suppose that G splits over a tamely ramified extension of K and that the order of the fundamental group of the derived group $\pi_1(G_{der})$ is prime to the characteristic of k. Then for each parahoric subgroup P of G and $w \in \tilde{W}^P \backslash \tilde{W} / \tilde{W}^P$ the Schubert variety S_w is normal, Frobenius split (when $\mathrm{char}(k) > 0$), and has rational singularities.*

Remark 8.5. The normality of Schubert varieties has been proved in the "Kac–Moody setting" by Kumar [28], by Mathieu [33], and by Littelmann [31]. That these Schubert varieties coincide in the case of SL_n with the ones defined in our context follows in characteristic 0 by an integrability result of Faltings. (See [2, App. to Sect. 7]. The general split case is similar, see also Section 9.f.) In characteristic $p > 0$, the comparison of the Schubert varieties in this article with

---

[3] Note that C_w is not a topological cell when P is not an Iwahori subgroup.

the Kac–Moody Schubert varieties is harder and follows from the normality theorem (see Section 9.h). The normality of Schubert varieties in our context is proved in the case of SL_n in [37]. Faltings [13] proved the above result in the case of an arbitrary split semi-simple simply connected group. More details about this proof are given in [19] and [14]. Our proof is an extension of Faltings' proof.

Remark 8.6. In fact, the conclusions of Theorem 8.4 also hold for the reduced Zariski closure of any (left) L^+B-orbit of the form $L^+B \cdot n_w$ in $\mathcal{F}_P$. This follows immediately from the proof, see 8.e.1.

8.c. Before dealing with the proof we define the "Demazure varieties" which are of central importance in our arguments. Let us fix an alcove C in the apartment corresponding to S and consider the corresponding Iwahori subgroup $B = B_0$. We would like to fix ideas and restrict our considerations to the Schubert cells (varieties) for $P = B$ and $w \in W_a \subset \tilde{W}$; by the above discussion and Theorem 5.1, the union of these is the (reduced) neutral component $\mathcal{F}_B^0 = (LG)^0/L^+B$ of $\mathcal{F}_B$. The advantage is that, at least set theoretically, these are the Schubert cells (varieties) for the double Tits systems $(G(K)_1, B, N(K)_1, S)$.

Proposition 8.7. (a) *The group scheme L^+B over k is a closed subgroup scheme of L^+P_Y for each $Y \subset S$ as in Section* 8.a.

(b) *The fpqc quotient L^+P_Y/L^+B is represented by a smooth projective homogeneous space Q_Y for the maximal reductive quotient $\bar{P}_Y^{red}$ of the special fiber $\bar{P}_Y$; the morphism $L^+P_Y \to L^+P_Y/L^+B \simeq Q_Y$ splits locally for the Zariski topology on Q_Y. When $Y = \{i\}$ then $Q_Y \simeq \mathbf{P}_k^1$.*

Proof. For every k-algebra R, we have $B(R[\![t]\!]) \subset P_Y(R[\![t]\!]) \subset G(R(\!(t)\!))$. Since both B, P_Y are affine over $k[\![t]\!]$, both L^+B, L^+P_Y are closed k-subschemes of the ind-scheme $LG = LP_Y = LB$. Part (a) follows.

To show part (b) we have to appeal to the construction of the parahoric group schemes $P = P_Y$ in [8, §4.6]. (Note that since k is algebraically closed, G is quasi-split and every unipotent group of finite type over k is split; in particular, "quasi-reductive = reductive.") We consider the reduction homomorphism, modulo (t) which defines an exact sequence

$$1 \to \tilde{U} \to L^+P \xrightarrow{q} \bar{P} \to 1. \tag{8.10}$$

Here $\tilde{U}$ is a pro-unipotent pro-algebraic group over k. Since $\bar{P}$ is affine and $\tilde{U}$ has a filtration by commutative pro-unipotent group schemes the $\tilde{U}$-torsor q is trivial. Let $\bar{P}^{red}$ be the maximal reductive quotient of $\bar{P}$. Then (8.10) defines an exact sequence

$$1 \to U \to L^+P \xrightarrow{q} \bar{P}^{red} \to 1 \tag{8.11}$$

where U is again a pro-unipotent pro-algebraic group over k. (We can apply this to either $P = P_Y$ or $P = P_\emptyset = B$.) By [8, Thm. 4.6.33], applied to $P = P_Y$ we see that $L^+B \subset L^+P_Y$ is the inverse image $q^{-1}(\bar{B})$, where $\bar{B}$ is a Borel subgroup of $\bar{P}_Y^{red}$. This implies that $L^+P_Y/L^+B = \bar{P}_Y^{red}/\bar{B} = Q_Y$ and that the quotient splits locally for the Zariski topology. In fact, when $Y = \{i\}$, then the derived group of $\bar{P}_Y^{red}$ is isomorphic to either SL_2 or PSL_2 and $Q_Y \simeq \mathbf{P}_k^1$. $\quad \square$

Proposition 8.8. *Consider $w \in W_a$ and fix a reduced decomposition $w = s_{i_1} \cdots s_{i_r}$. The Demazure variety $D(\tilde{w})$ is by definition the multiple "contracted product"*

$$D(\tilde{w}) := L^+ P_{i_1} \times^{L^+ B} \cdots \times^{L^+ B} L^+ P_{i_r} / L^+ B.$$

(This depends on the choice of reduced decomposition; we write $\tilde{w}$ instead of w to indicate this dependence.) It is a smooth projective variety over k of dimension equal to the length $l(w) = r$ of w and affords a surjective morphism

$$D(\tilde{w}) \xrightarrow{\pi_w} S_w \subset \mathcal{F}_B. \tag{8.12}$$

Proof. By definition, the contracted product in the statement is the quotient

$$\big(L^+ P_{i_1} \times \cdots \times L^+ P_{i_r}\big) \big/ \big(L^+ B\big)^r, \tag{8.13}$$

where the product group scheme $(L^+ B)^r$ acts on $L^+ P_{i_1} \times \cdots \times L^+ P_{i_r}$ from the right via

$$(p_1, \ldots, p_r) \cdot (b_1, \ldots, b_r) = \big(p_1 b_1, b_1^{-1} p_2 b_2, \ldots, b_{r-1}^{-1} p_r b_r\big). \tag{8.14}$$

Recall that, by Proposition 8.7, the quotient $L^+ P_i / L^+ B$ is isomorphic to the projective line $\mathbf{P}_k^1$. Let us write $w = w' \cdot s_{i_r}$. Note that forgetting the last coordinate gives a morphism $D(\tilde{w}) \to D(\tilde{w}')$ which is a locally trivial fibration with fiber $L^+ P_{i_r} / L^+ B \simeq \mathbf{P}_k^1$, comp. [18, 3.3.1]. The first statement of the proposition now follows by induction on the length of w. The morphism $\pi_w : D(\tilde{w}) \to \mathcal{F}_B = LG / L^+ B$ is given using the product in LG. $\square$

8.d.   Suppose here that G is absolutely simple, simply connected and splits over a ramified extension K'/K so that the setup and notations of Sections 7, 7.c applies. For simplicity, set $\mathfrak{g}$ for the affine Kac–Moody algebra $\mathfrak{g}'_{KM}(G)_F$ that corresponds to G, where $F = K_0$ is the fraction field of W. For $w \in W_a$ (which then can be identified with the affine Weyl group in the Kac–Moody theory [24, Ch. 3]) with reduced decomposition $\tilde{w}$, there is a Demazure variety $D_{\mathfrak{g}}(\tilde{w})$ over k in the Kac–Moody setting (see [33, p. 51]). The variety $D_{\mathfrak{g}}(\tilde{w})$ is given by a contracted product similar to the one above but with the groups $L^+ P_i$, $L^+ B$, replaced by the "parahoric" group schemes $\mathcal{P}_i$, $\mathcal{B}$ defined in [33, Ch. I]. The group schemes in [33] are defined using the Chevalley integral form $U(\mathfrak{g})^W$ of the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$; by definition, this is the subalgebra of $U(\mathfrak{g})$ which is generated over W by the divided powers $\frac{e_i^m}{m!}$, $\frac{f_i^m}{m!}$ and $\binom{h_i}{m}$ where e_i, f_i, h_i are as in Section 7.c. In particular, the group schemes $\mathcal{P}_i$, $\mathcal{B}$, depend on this presentation of $\mathfrak{g}'_{KM}(G)_F$. In what follows, we will see that the Demazure varieties $D(\tilde{w})$ above are naturally isomorphic to their analogues $D_{\mathfrak{g}}(\tilde{w})$ in the Kac–Moody theory.

It is enough to show this for the Iwahori subgroup scheme B which is given as the σ-fixed points of the inverse image under reduction modulo u of the Borel subgroup of H_0 that corresponds to Φ^+. Let also P be a parahoric subgroup scheme $P = P^G$ for G which we can write as the σ-fixed point scheme of $\mathrm{Res}_{k[[u]]/k[[t]]}(P^H)$ as in Section 7; we can assume $B \subset P$. We have a semi-direct product decomposition which is stable under the σ-action

$$L^+ P^H = M^H \ltimes U^H.$$

Here $M^H \simeq \overline{(P^H)}^{\mathrm{red}}$ is isomorphic to the maximal reductive quotient of the reduction of P^H modulo u and U^H is a pro-unipotent subgroup scheme over k. After taking σ-fixed points, this gives

$$L^+ P = M \ltimes U(P) \tag{8.15}$$

with $M \simeq \bar{P}^{\mathrm{red}}$ and U a pro-unipotent subgroup scheme over k. Denote by U_m^H the (σ-stable) normal subgroup scheme of U^H which is obtained as the kernel of reduction modulo u^m

$$P^H(m)(R) = \left\{ u \in P^H\left(R[[u]]\right) \,\middle|\, u \equiv 1 \in P^H\left(R[[u]]/(u^m)\right) \right\}$$

We also set $P(m) := (U_m^H)^\sigma$ a normal pro-unipotent subgroup scheme of U. Given a reduced decomposition $w = s_{i_1} \cdots s_{i_r}$, we can find integers $m_1, \ldots, m_r$ such that

$$P_{i_1}(m_1) \subset \cdots \subset P_{i_r}(m_r) \subset L^+ B. \tag{8.16}$$

Consider the quotients $Q_{i_1}^{m_1} = L^+ P_{i_1}/P_{i_1}(m_1), \ldots, Q_{i_r}^{m_r} = L^+ P_{i_r}/P_{i_r}(m_r)$, and $L^+ B^{m_1} = L^+ B/P_{i_1}(m_1), \ldots, L^+ B^{m_r} = L^+ B/P_{i_r}(m_r)$; these are all smooth algebraic groups over k. It is not hard to see that the Demazure variety $D(\tilde{w})$ can be identified with the quotient

$$D(\tilde{w}) = \left(Q_{i_1}^{m_1} \times \cdots \times Q_{i_r}^{m_r} \right) \big/ \left(L^+ B^{m_1} \times \cdots \times L^+ B^{m_r} \right) \tag{8.17}$$

where the action is induced by (8.14). A similar construction can be performed in the Kac–Moody set-up. We obtain:

$$D_{\mathfrak{g}}(\tilde{w}) = \left(\mathcal{Q}_{i_1}^{m_1} \times \cdots \times \mathcal{Q}_{i_r}^{m_r} \right) \big/ \left(\mathcal{B}^{m_1} \times \cdots \times \mathcal{B}^{m_r} \right) \tag{8.18}$$

with $\mathcal{Q}_i^{m_1} = \mathcal{P}_i/\mathcal{P}_i(m)$, similarly for $\mathcal{B}^{m_i}$. The algebraic groups $\mathcal{Q}_i^{m_1}$ and $\mathcal{B}^{m_i}$ both contain a central torus $\mathbf{G}_m$ that corresponds to the central element c. To show our claim $D(\tilde{w}) \simeq D_{\mathfrak{g}}(\tilde{w})$ is enough to show that there is an isomorphism $L^+ P_i/P_i(m) \simeq \mathcal{P}_i/(\mathbf{G}_m \times \mathcal{P}_i(m))$ which restricts to give $L^+ B/P_i(m) \simeq \mathcal{B}/(\mathbf{G}_m \times \mathcal{P}_i(m))$. In fact, these algebraic groups lift to smooth group schemes over W (using Section 7 and [33]). We can easily see from 7.c and [33] that such isomorphisms exist on the level of Lie algebras and hence between the generic fibers over the fraction field K_0 of W. Denote by $\mathfrak{q}_i^m$ the Lie algebra of $L^+ P_i/P_i(m)_{K_0} = \mathcal{P}_i/(\mathbf{G}_m \times \mathcal{P}_i(m))_{K_0}$ over K_0. Consider the W-module of distributions of these group schemes with support on the identity section (see [8, 1.3]). By the above, these can be identified with W-lattices in the K_0-algebra $U(\mathfrak{q}_i^m)$. Using [8, 3.5.1], we see that it is enough to show that the distribution lattices that correspond to the two group schemes coincide. Using the definition in [33], we can see that this amounts to checking that the image of $U(\mathfrak{g})^W$ under the natural map $U(\mathfrak{g}) \to U(\mathfrak{q}_i^m)$ coincides with the lattice of distributions of $L^+ P_i/P_i(m)_{K_0}$. This lattice can now be calculated in terms of the Chevalley basis E_a, H_i (cf. [26,32]). The expressions of e_i, f_i, h_i in terms of E_a, H_i in Section 7.c now allow us to show that the image of $U(\mathfrak{g})^W$ above agrees with this lattice.

We will return to a study of the Demazure varieties in the next section.

8.e. In the following few paragraphs we assume that the statement of Theorem 8.4 is true under the following conditions:

- G is absolutely simple, simply connected and splits over a tamely ramified extension of K,
- P is an Iwahori subgroup.

We will show how to deduce the full statement of the theorem from this assumption.

8.e.1. First of all observe that by Theorem 1.4 and Proposition 8.7 the natural morphism

$$\pi : \mathcal{F}_B = LG/L^+B \to \mathcal{F}_P = LG/L^+P_Y$$

is a proper and an étale locally trivial fibration with fibers $L^+P_Y/L^+B \simeq Q_Y$. The inverse image under π of a Schubert variety in $\mathcal{F}_P$ is a Schubert variety in $\mathcal{F}_B$. Hence, if all Schubert varieties in $\mathcal{F}_B$ are normal, Frobenius split and with rational singularities, the same is true for all Schubert varieties in $\mathcal{F}_P$ (for the transfer of the F-splitting of $\mathcal{F}_B$ to the F-splitting of $\mathcal{F}_P$, one uses that the direct image of $\mathcal{O}_{\mathcal{F}_B}$ is $\mathcal{O}_{\mathcal{F}_P}$, cf. [18, Prop. 2.4]). From here on we will assume that $P_Y = P = B$, i.e., is an Iwahori subgroup.

8.e.2. Suppose that G is semi-simple, simply connected, splits over a tame extension and that $P = B$. Then we can write

$$G = \prod_i \operatorname{Res}_{K_i/K} G_i, \qquad B = \prod_i \operatorname{Res}_{\mathcal{O}_{K_i}/\mathcal{O}_K} B_i, \qquad (8.19)$$

where G_i are absolutely simple and simply connected algebraic groups over K_i and B_i an Iwahori subgroup scheme of G_i. Each G_i splits over a tamely ramified extension of K_i (and K_i/K itself is tamely ramified). We obtain

$$\mathcal{F}_B \simeq \prod_i L \operatorname{Res}_{K_i/K}(G_i)/L^+ \operatorname{Res}_{\mathcal{O}_{K_i}/\mathcal{O}_K}(B_i) \qquad (8.20)$$

and the Schubert varieties in $\mathcal{F}_B$ are products of Schubert varieties in the affine flag varieties $L \operatorname{Res}_{K_i/K}(G_i)/L^+ \operatorname{Res}_{\mathcal{O}_{K_i}/\mathcal{O}_K}(B_i)$. Note however that $K_i = k((u)) \simeq K = k((t))$, $\mathcal{O}_{K_i} = k[[u]] \simeq \mathcal{O}_K = k[[t]]$ for each i. These induce isomorphisms $L \operatorname{Res}_{K_i/K}(G_i) \simeq LG_i$, $L^+ \operatorname{Res}_{\mathcal{O}_{K_i}/\mathcal{O}_K}(B_i) \simeq L^+B_i$ as in (1.3). Therefore the truth of the assertion of Theorem 8.4 for each G_i implies the assertion for G.

8.e.3. Suppose now that G is semi-simple, $p \nmid \#\pi_1(G)$, and that G splits over a tamely ramified extension. Let $\tilde{G}$ be the simply connected cover of G. Then $\tilde{G}$ satisfies the assumptions of the previous paragraph and by the discussion above, Theorem 8.4 is true for all Schubert varieties in $L\tilde{G}/L^+\tilde{B}$. By the work in Section 6.a.1, these Schubert varieties can be identified with the Schubert varieties S_w in the neutral component of LG/L^+B, i.e., the ones with $w \in W_a \subset \tilde{W}$. Since $\tilde{W} = W_a \rtimes \Omega$, translating by elements of Ω allows us to conclude that Theorem 8.4 is true for all Schubert varieties in LG/L^+B.

8.e.4. Finally suppose that G satisfies the assumptions of Theorem 8.4. Then it follows from Proposition 6.6 and Section 8.e.3 (using a similar argument to extend "outside the neutral component") that Theorem 8.4 is true for all Schubert varieties in LG/L^+B.

9. The proof of Theorem 8.4

By Section 8.e, it is enough to deal with the case that k is algebraically closed and G is absolutely simple, simply connected and splits over a tamely ramified extension of $K = k((t))$. We can also assume that $P = B$ is an Iwahori subgroup and for simplicity set $\mathcal{F} = \mathcal{F}_B = LG/L^+B$. Recall that then G is quasi-split and the set-up of Section 7 applies. We will continue with these assumptions for the rest of the section. Recall that we then have $W_a = \tilde{W}$, $G(K)_1 = G(K)$.

9.a. Choose a maximal torus T_0 of the Chevalley group $H = H_0 \otimes_k K$ over K which is defined over the field of constants k; we can assume that there is an action of the diagram automorphism σ_0 on T_0 also defined over k. Then $T := T_0 \otimes_K K'$ is a maximal torus of $G_{K'} = H_{K'}$ with an action of $\sigma = \sigma_0 \otimes \tau$; the fixed points $S = T^\sigma$ form a maximal split torus of G over K. We will denote by Φ the set of (relative) roots of G over K with respect to S, and by Φ_{aff} the set of affine roots. Also let Φ' be the (absolute) roots of $G_{K'} = H_{K'}$ with respect to T; the Galois group $\mathrm{Gal}(K'/K)$ acts on Φ'. The set of orbits of Φ' under $\mathrm{Gal}(K'/K)$ can be identified with Φ. Let us fix a choice of Chevalley generators for the Lie algebra of H_0; this provides us with Chevalley generators for the Lie algebra of $H = H_0 \otimes_k K$ and therefore with a Chevalley–Steinberg system $(x_a)_{a \in \Phi}$ [8, 3.2, 4.1]. We also fix a choice of a system of positive roots Φ^+ and hence a choice of an Iwahori subgroup B of $G(K)$. It is enough to show the Theorem for such a B.

For every affine root $\alpha \in \Phi_{\mathrm{aff}}$, there is a corresponding root subgroup scheme $U_\alpha \subset LG$ over k. In what follows, we will explain the construction of these subgroups in some detail. We will refer to [13, p. 46] for the construction of the affine root subgroups $U'_{\alpha'} \subset LH_{K'}$ for the split group $H_{K'}$; this construction works over a general base and in particular over the ring of Witt vectors W.

We first have to recall the shape of the root subgroup $U_a \subset G$ for $a \in \Phi$. This is a subgroup scheme over K. We distinguish the following cases (see [8, 4.1]):

(I) $\frac{1}{2}a, 2a \notin \Phi$. Then $x_a : U_a \simeq \mathrm{Res}_{L/K} \mathbf{A}^1_L$. In this case, $U_a \times_K K'$ is the direct product $\prod_{a'} U'_{a'}$ of root subgroups of $G_{K'} = H_{K'}$ for a' ranging over the $\mathrm{Gal}(K'/K)$-orbit in Φ' that corresponds to a and L is the fixed field of the stabilizer of a', in particular $L = K$ or K'; we have

$$U_a = \left(\prod_{a'} U'_{a'} \right)^\sigma. \tag{9.1}$$

There is a group scheme homomorphism $\pi_a : G^a := \mathrm{Res}_{L/K} SL_2 \to G$ such that π_a induces an isomorphism between the maximal unipotent subgroups U_+ and U_- of G^a normalized by $\pi_a^{-1}(S)$ and the subgroups U_a, U_{-a} of G respectively (see [8, 4.1.4]). The homomorphism π_a identifies the corrot lattice of G^a with a direct summand of the coroot lattice of G. Under our assumption that G is simply connected this implies that π_a is injective. The homomorphism π_a of course depends on our choice of the Chevalley–Steinberg system.

(II) $a, 2a \in \Phi$. Then K'/K is a separable (ramified) quadratic extension with Galois automorphism $\sigma(c) = \bar{c}$ and $x_a \colon U_a \simeq H(K', K)$ where $H(K', K)$ is the group scheme over K whose R-valued points (for R an K-algebra) are

$$H(K', K)(R) = \left\{ (c, d) \in (K' \otimes_K R) \times (K' \otimes_K R) \mid c\bar{c} = d + \bar{d} \right\}$$

with group law

$$(c, d) + (c', d') = (c + c', d + d' + \bar{c}c'). \tag{9.2}$$

One can see that U_{2a} is the subgroup scheme of U_a which corresponds, under x_a, to $\{(0, d) \mid d + \bar{d} = 0\} \subset H(K', K)(R)$. In this case, we can see that $U_a \times_K K'$ is the direct product

$$U'_{a'_1} U'_{a'_1 + a'_2} U'_{a'_2} \subset G_{K'} = H_{K'}$$

where a'_1, a'_2 are the two roots in Φ' in the Galois orbit corresponding to $a \in \Phi$. (Then $a'_1 + a'_2$ is also in Φ'.) We have

$$U_a = \left(U'_{a'_1} U'_{a'_1 + a'_2} U'_{a'_2} \right)^\sigma, \qquad U_{2a} = \left(U'_{a'_1 + a'_2} \right)^\sigma. \tag{9.3}$$

As above we can see that there is an injective group scheme homomorphism $\pi_a \colon G^a := SU_3 \to G$ such that π_a induces an isomorphism between the maximal unipotent subgroups U_+ and U_- of G^a normalized by $\pi_a^{-1}(S)$ and the subgroups U_a, U_{-a} of G respectively. Here again SU_3 is the quasi-split special unitary group for K'/K, and the form with matrix equal to the antidiagonal unit matrix. In case (II) we can also set $G^{2a} := G^a$, $\pi_{2a} := \pi_a$. (Then π_a and G^a make sense for any root a, but they really only depend on the "root ray" given by a.) Note that case (II) can only occur when $\operatorname{char}(k) \neq 2$.

9.b.   Recall that we denote by t, resp. u, uniformizers of K, resp. K', with $u^{[K':K]} = t$. Now consider the (hyperspecial) vertex x_0 in the Bruhat–Tits building $\mathcal{B}(H, K')$ that corresponds to the parahoric subgroup $H_0(k[\![u]\!])$; since σ acts on $H_0(k[\![u]\!])$, the vertex x_0 is fixed by σ and so it corresponds to a vertex x_0 in $\mathcal{B}(G, K)$. Affine roots α, α' for G, resp. $H_{K'}$, can then be written in the form $\alpha(x) = \nu(\alpha)(x - x_0) + m$, $\alpha'(x) = \nu(\alpha')(x - x_0) + m$ with $m \in \mathbf{Q}$ and with vector parts $a = \nu(\alpha) \in \Phi$, $a' = \nu(\alpha') \in \Phi'$. Here, we are using the valuation of K' with $\operatorname{val}(u) = 1/e$. For simplicity, we write $\alpha = a + m$, $\alpha' = a' + m$. The affine root subgroup $U'_{a'+m}$ in $H_{K'} = H_0 \otimes_k k(\!(u)\!)$ corresponds to $\{ r \cdot X_{a'} \cdot u^m \mid r \in k \}$ in the Lie algebra, where $X_{a'}$ is our choice (fixed above) of a Chevalley generator for H_0 and the root a'.

If $a = \nu(\alpha)$ falls in case (I), we set

$$U_\alpha = \left(\prod_{a'} U'_{a'+m} \right)^\sigma, \tag{9.4}$$

where $U'_{a'+m}$ is the affine root subgroup in $G_{K'} = H_{K'}$, and in the product a' ranges over the $\operatorname{Gal}(K'/K)$-orbit in Φ' that corresponds to a. In this case, we have $[L : K] \cdot m \in \mathbf{Z}$ and U_α is the k-subgroup scheme of LU_a such that

$$x_a \big(U_\alpha(R) \big) = R \cdot t^m = R \cdot u^{em}.$$

If either $\frac{1}{2}\nu(\alpha) \in \Phi$, or $2\nu(\alpha) \in \Phi$, (case (II)), then there are two subcases:

1(II) $\nu(\alpha) = a$ with $2a \in \Phi$. Then $m \in \mathbf{Z}$ and we set

$$U_\alpha = \left(U'_{a'_1+m} U'_{a'_1+a'_2+2m} U'_{a'_2+m} \right)^\sigma. \tag{9.5}$$

Then we have

$$x_a\big(U_\alpha(R)\big) = \left\{ \left(r \cdot t^{m/2}, \frac{(-1)^m}{2} r^2 t^m \right), r \in R \right\}.$$

2(II) $\nu(\alpha) = 2a$ with $a \in \Phi$. Then $2m$ is an odd integer and we set

$$U_\alpha = \left(U'_{a'_1+a'_2+2m} \right)^\sigma. \tag{9.6}$$

Then we have

$$x_{2a}\big(U_\alpha(R)\big) = \left\{ (0, r \cdot t^m), r \in R \right\}.$$

(See also [48, Example 1.15], which explains the odd ramified unitary groups. Observe of course, that in the notation of Bruhat–Tits the affine root subgroups are given as groups of field elements whose valuations are bounded from below, see Remark 9.1. Some more details about the calculation of the affine root system Φ_{aff} can be found in [39, §2.c].)

Note that for $(c, d) \in H(K', K)(R)$, we can write

$$(c, d) = \left(c, \frac{1}{2}c\bar{c} \right) + \left(0, d - \frac{1}{2}c\bar{c} \right) \tag{9.7}$$

(the sum in $H(K', K)(R)$).

For each affine root $\alpha = a + m$ there is a corresponding homomorphism

$$\phi_\alpha : SL_2 \to LG \tag{9.8}$$

which has the property that it induces an isomorphism between the unipotent upper triangular matrices resp. the unipotent lower triangular matrices and U_{α_i}, resp. $U_{-\alpha_i}$. This homomorphism can be constructed as follows:

First assume that $a/2$ is not a root. Recall the homomorphism $\pi_a : G^a \to G$ over K given in Section 9.a. It is enough to give a homomorphism $SL_2 \to LG^a$ corresponding to $\alpha = a + m$; then (9.8) is obtained by composing with $\pi_a : LG^a \to LG$. If $2a \notin \Phi$, G^a is either SL_2 or $\mathrm{Res}_{K'/K} SL_2$. Then we can reduce to the case $G = SL_2$. In this case, the homomorphism is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & bt^m \\ ct^{-m} & d \end{pmatrix}.$$

If $2a$ is a root, then $G^a = SU_3$ and we can reduce to the case $G = SU_3$. Then the homomorphism is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a^2 & -u^m ab & -(-1)^m t^m b^2/2 \\ -u^{-m} 2ac & ad + bc & (-1)^m u^m bd \\ -(-1)^m t^{-m} 2c^2 & (-1)^m u^{-m} 2cd & d^2 \end{pmatrix}.$$

Now assume that $a/2 = a'$ is a root. Then by considering $\pi_{a'} : G^{a'} \to G$ we can reduce to the case that $G = SU_3$. In this case, the homomorphism is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & 0 & -bt^m \\ 0 & 1 & 0 \\ -ct^{-m} & 0 & d \end{pmatrix}.$$

(Note that in this case m is a half integer and $t^m = u^{2m}$.)

For each simple affine root $\alpha_i = a_i + m_i$, the product $U_{-\alpha_i} U_{\alpha_i} U_{-\alpha_i}$ contains a representative of the reflection s_i. This can be seen as follows: By [48, 1.4] we see that s_i is the image of the unique element in the intersection $U_{-a_i} u U_{-a_i} \cap N$ for $u \in U_{\alpha_i} - \{1\}$ where N is the normalizer of the maximal torus T of G. Recall the homomorphism $\phi_{\alpha_i} : SL_2 \to LG$ constructed above. Since

$$U_- \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} U_- \cap N_{SL_2} = \left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}, \tag{9.9}$$

we can produce the element s_i by taking the image under $\phi_{\alpha_i} : SL_2 \to LG$ of the matrix on the right-hand side of Eq. (9.9) above.

For $w = s_{i_1} \cdots s_{i_r} \in W_a$, $\alpha \in \Phi_{\text{aff}}$, we see that we have

$$w^{-1} \cdot U_\alpha \cdot w = U_{\alpha \cdot w} \tag{9.10}$$

where we view w as acting via an affine transformation on the apartment of S.

Remark 9.1. The group schemes $U_\alpha \subset LG$ can be understood in the context of Bruhat–Tits theory as follows: Our choice of Chevalley basis gives a root datum valuation $(\phi_a)_{a \in \Phi}$. Given an affine root $\alpha = a + m$ as above, Bruhat and Tits [8] construct a smooth affine group scheme $\mathfrak{U}_\alpha$ over $k[[t]]$. This has generic fiber the root subgroup U_a and is such that $\mathfrak{U}_\alpha(k[[t]]) = \{u \in U_a(k((t))) \mid u = 1, \text{ or } \phi_a(u) \geqslant m\}$. Similarly, we have the subgroup scheme $\mathfrak{U}_{\alpha+}$ with $\mathfrak{U}_{\alpha+}(k[[t]]) = \{u \in U_a(k((t))) \mid u = 1, \text{ or } \phi_a(u) > m\}$. We can see that $U_\alpha \subset L^+\mathfrak{U}_\alpha$ becomes isomorphic to the quotient $\bar{\mathfrak{U}}_\alpha := L^+\mathfrak{U}_\alpha / L^+\mathfrak{U}_{\alpha+}$ with the obvious map.

Remark 9.2. Our construction also works to define affine root subgroups $\underline{U}_\alpha$, $\alpha \in \Phi_{\text{aff}}$ over the ring of Witt vectors. These are obtained via taking fixed points of products (as in (9.4), (9.5), (9.6)) of the affine root subgroups of [13, p. 46], for the (non-twisted) loop group LH over W. Using Section 7 we see that $\underline{U}_\alpha$ are subgroup schemes of $L\underline{G}$ that satisfy (9.10) with special fiber the group schemes U_α that we have constructed above. We also see that the group schemes G^a and the homomorphisms $\pi_a : G^a \to G$ lift to group schemes $\underline{G}^a$ and homomorphisms $\underline{\pi}_a : \underline{G}^a \to \underline{G}$ over W. This can be seen as follows. If G is split, $G^a = SL_2$ and the statement follows from the construction of the Chevalley group scheme $\underline{G}$ over $W[[t]]$. In general, there are two cases as in Section 9.a: Recall the group scheme $\underline{H}$ over $W[[u]]$ of Section 7 which is such that $\underline{G} = (\text{Res}_{W[[u]]/W[[t]]} \underline{H})^\sigma$. In case (I), we can construct $\underline{G}^a$ as the σ-fixed point scheme of $\prod_{\{a'\}} \underline{H}^{a'}$. In case (II), there is group scheme embedding $\underline{SL}_3 \to \underline{H}$ over $W[[u]]$ corresponding to the root subset $\{a'_1, a'_2, a'_1 + a'_2\}$. Then $\underline{G}^a = (\text{Res}_{W[[u]]/W[[t]]} \underline{SL}_3)^\sigma$.

9.c.   We now state a key ingredient of the proof:

Proposition 9.3 *(Key observation). The Lie algebra Lie(LG) of LG is spanned over k by the Lie algebras of L^+B and of U_α for all $\alpha \in \Phi_{\mathrm{aff}}$.*

Proof. The reductive group G over $K = k((t))$ contains the product

$$\prod_{a \in \Phi^+} U_{-a} \times T \times \prod_{a \in \Phi^+} U_a \tag{9.11}$$

as a Zariski open neighborhood of the origin. Hence, it is enough to show that the Lie algebras $Lie(U_a)$, $a \in \Phi$, and $Lie(T)$ are spanned over k by elements from the Lie algebras of L^+B and U_α for $\alpha \in \Phi_{\mathrm{aff}}$.

Let us first consider the unipotent subgroups. For each $a \in \Phi$, consider $u \in LU_a(R) = U_a(R((t)))$. Note that elements of $LU_a(R)$ of sufficiently high valuation lie in $L^+B(R) = B(R[[t]])$. Hence, we can see using (9.7) that we can write $u = u' + u_B$ with $u_B \in B(R[[t]])$ and u' a finite sum of elements in affine root subgroups $U_\alpha(R)$ with $\nu(\alpha) = a$. In particular, the result on Lie algebras follows.

Now let us consider the maximal torus T. We would like to show that each element of $T(R((t)))$ for R a local Artinian k-algebra with $R/M = k$, $M^2 = (0)$, which reduces to the identity modulo M, can be written in $G(R((t)))$ as a product of elements in $U_a(R((t)))$ for $a \in \Phi$. Then the result will follow by the above. Recall that G is quasi-split, absolutely simple, semi-simple and simply connected, and splits over the field K'/K, which is tame. The cocharacter group $X_*(T)$ coincides with the coroot lattice of the split group $G_{K'}$. Let Δ' be a basis for the roots Φ' of $G_{K'}$ over K' that corresponds to a Borel subgroup defined over K. The torus T splits into a product of induced tori T_b over the $\mathrm{Gal}(K'/K)$-orbits b on Δ'. This set of orbits can be identified with a basis of roots of Φ. For a root a in this basis, recall the homomorphism $\pi_a : G^a \to G$ given in Section 9.a. The torus T_b is equal to $\pi_a(T^a)$ with $T^a = \pi_a^{-1}(T)$ a maximal torus of G^a. Since π_a identifies the unipotent subgroups U_+ and U_- of G^a with U_a and U_{-a}, these considerations show that it is enough to consider the cases (corresponding to (I) and (II) of Section 9.a):

(I) $G = SL_2$ (split), and (II) $G = SU_3$ (for the ramified quadratic extension K'/K).

In case (I) our claim follows from the identity, cf. [13, p. 53]:

$$\begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix} = \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -c^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{9.12}$$

This indeed shows that these elements are in the subgroup generated by the root groups.

In case (II) let

$$t(d) = \mathrm{diag}\left(-d, \frac{\bar{d}}{d}, -\frac{1}{\bar{d}}\right),$$

with $d \in (R((u)))^\times$, be an element of the maximal torus $T \subset SU_3(R((t))) \subset SL_3(R((u)))$. If $d + \bar{d} = c\bar{c}$ with $c \in R((u))$ we can use the identity, cf. [47, p. 537]:

$$t(d) = \mathrm{diag}\left(-d, \frac{\bar{d}}{d}, -\frac{1}{\bar{d}}\right) = x(c, d) \cdot y\left(-\frac{\bar{c}}{\bar{d}}, \frac{1}{\bar{d}}\right) \cdot x\left(c\frac{\bar{d}}{d}, d\right) \cdot w \tag{9.13}$$

where

$$x(c,d) = \begin{pmatrix} 1 & -c & -d \\ 0 & 1 & \bar{c} \\ 0 & 0 & 1 \end{pmatrix}, \qquad y(c,d) = \begin{pmatrix} 1 & 0 & 0 \\ -c & 1 & 0 \\ -d & \bar{c} & 1 \end{pmatrix}, \qquad w = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \quad (9.14)$$

Here $x(c,d) \in U_+$, $y(c,d) \in U_-$ for the special unitary group SU_3. Note that $U_+ \simeq U_- \simeq H(K', K)$. In general, given an element $t(d) \in T(R((t))) \simeq R((u))^\times$, which reduces to the identity modulo M, we can find c in $R((t))$ with $c^2 = c\bar{c} = d + \bar{d}$ (since char $k \neq 2$). The identity (9.13) now allows us to write $t(d)$ as a product of unipotent matrices and the result follows. $\quad \square$

In fact, the following extension of the key Proposition 9.3 can be shown to hold by using Remark 9.2 and repeating the proof verbatim (we can redo the argument by replacing the local Artin k-algebra R with a local Artin W-algebra with residue field k).

Proposition 9.4. *The Lie algebra of LG is generated as a W-module by the Lie algebras of $L^+\underline{B}$ and $\underline{U}_\alpha$, $\alpha \in \Phi_{\mathrm{aff}}$.*

Corollary 9.5. *If R is an Artinian local k-algebra, then $LG(R)$ is generated by the subgroups $L^+B(R)$ and $U_\alpha(R)$ for $\alpha \in \Phi_{\mathrm{aff}}$.*

Proof. When R is a field, our claim follows from the Bruhat decomposition and the fact that $U_{-\alpha_i} U_{\alpha_i} U_{-\alpha_i}$ contains the reflection s_i for every simple affine root $\alpha_i = a_i + m_i$.

Now for a general Artinian R, choose a minimal ideal $I \subset R$ such that the assertion is true for R/I. Consider $g \in LG(R/I^2)$ and its reduction $\bar{g} \in LG(R/I)$. By assumption we can write $\bar{g}$ as a product of points in $L^+B(R/I)$, $U_\alpha(R/I)$. By smoothness, these points can be lifted to corresponding points with values in R/I^2 whose product is an element $g' \in LG(R/I^2)$ such that $h = g' \cdot g^{-1} \equiv 1 \bmod I/I^2$. Since h is given by an element of $Lie(LG) \otimes_k I/I^2$ our claim follows from the key Proposition 9.3 above. $\quad \square$

9.d. We now give some additional properties of the Demazure varieties $D(\tilde{w})$; we continue with the notations of Section 8.c. The proofs of the following two propositions are similar to proofs of corresponding statements in [13] (see also [19]). We will sketch the arguments and refer the reader to [13] and [18,19] for more details.

Proposition 9.6. (a) *For any reduced $\tilde{u} \leqslant \tilde{w}$ (i.e., a reduced decomposition obtained by omitting reflections from $w = s_{i_1} \cdots s_{i_r}$), there is a closed immersion $\sigma_{\tilde{u},\tilde{w}} : D(\tilde{u}) \to D(\tilde{w})$ and a commutative diagram*

$$\begin{array}{ccc} D(\tilde{u}) & \xrightarrow{\sigma_{\tilde{u},\tilde{w}}} & D(\tilde{w}) \\ \pi_u \downarrow & & \downarrow \pi_w \\ S_u & \longrightarrow & S_w \end{array} \qquad (9.15)$$

in which the bottom horizontal arrow is the natural closed immersion. For $\tilde{v} \leqslant \tilde{u} \leqslant \tilde{w}$, we have $\sigma_{\tilde{v},\tilde{w}} = \sigma_{\tilde{u},\tilde{w}} \cdot \sigma_{\tilde{v},\tilde{u}}$.

(b) *For every $\tilde{w}$, the morphism $\pi_w : D(\tilde{w}) \to S_w$ is proper and birational and has geometrically connected fibers.*

(c) *When $\mathrm{char}(k) > 0$, the varieties $D(\tilde{w})$ are Frobenius split compatibly with the closed immersions $\sigma_{\tilde{u},\tilde{w}} : D(\tilde{u}) \to D(\tilde{w})$, for all reduced $\tilde{u} \leqslant \tilde{w}$.*

Proof. Here the assertion (a) is clear. To see the birationality of the morphism in (b), note that the open subset of $D(\tilde{w})$ "where no factor in (8.13) lies in L^+B" maps isomorphically onto the Schubert cell C_w. This open subset is equal to $U_{\alpha_{i_1}} \cdot s_{i_1} \cdot U_{\alpha_{i_2}} \cdot s_{i_2} \cdots U_{\alpha_{i_r}} \cdot s_{i_r}$. The fact that the fibers are geometrically connected is shown by induction on $r = l(w)$. Indeed, if we write $\tilde{w} = s_i \tilde{u}$ with $l(u) = l(w) - 1$ then we can factor $\tilde{\pi}_w$ as

$$D(\tilde{w}) = L^+ P_i \times^{L^+B} D(\tilde{u}) \to L^+ P_i \times^{L^+B} S_u \to S_w.$$

The Frobenius splitting of $D(\tilde{w})$ is constructed by using the criterion of Mehta and Ramanathan, comp. [18, Prop. 2.5]. More precisely, one calculates the canonical bundle $\omega_{D(\tilde{w})}$ of $D(\tilde{w})$ as in [18, Prop. 3.19], and shows that there exists a global section s of $\omega_{D(\tilde{w})}^{-1}$ with divisor equal to the sum of the boundary divisor of $D(\tilde{w})$ and of an effective divisor which does not contain the origin of $D(\tilde{w})$. (The corresponding calculation in the general symmetrizable Kac–Moody case is also contained in [33, Ch. 8, 18].) As in [18, Cor. 3.23], it is checked that the resulting Frobenius splitting of $D(\tilde{w})$ is compatible with the closed immersions $\sigma_{\tilde{u},\tilde{w}}$, for all reduced $\tilde{u} \leqslant \tilde{w}$. $\square$

Let $\psi_w : \tilde{S}_w \to S_w$ be the normalization of S_w. We obtain a factorization of π_w

$$D(\tilde{w}) \xrightarrow{\tilde{\pi}_w} \tilde{S}_w \xrightarrow{\psi_w} S_w. \tag{9.16}$$

Proposition 9.7. (a) *The morphism ψ_w is a universal homeomorphism.*

(b) *For any $u \leqslant w$, there is a closed immersion $\tilde{S}_u \to \tilde{S}_w$ which lifts the natural closed immersion $S_u \to S_w$. If $\tilde{u} \leqslant \tilde{w}$ are reduced decompositions, then the diagram*

$$
\begin{array}{ccc}
D(\tilde{u}) & \xrightarrow{\sigma_{\tilde{u},\tilde{w}}} & D(\tilde{w}) \\
\tilde{\pi}_u \downarrow & & \downarrow \tilde{\pi}_w \\
\tilde{S}_u & \longrightarrow & \tilde{S}_w
\end{array}
\tag{9.17}
$$

commutes.

(c) *When $\mathrm{char}(k) > 0$, the varieties $\tilde{S}_w$ are Frobenius split, compatibly with the closed immersions $\tilde{S}_u \to \tilde{S}_w$, for all $u \leqslant w$.*

(d) *For $i > 0$, we have $R^i (\tilde{\pi}_w)_* (\mathcal{O}_{D(\tilde{w})}) = (0)$ and $(\tilde{\pi}_w)_* (\mathcal{O}_{D(\tilde{w})}) = \mathcal{O}_{\tilde{S}_w}$. The variety $\tilde{S}_w$ is Cohen–Macaulay and it has at most rational singularities.*

Proof. To prove (a), we note that by Proposition 9.6, the fibers of π_w are geometrically connected, and hence ψ_w is a universal homeomorphism. The diagram in (b) arises since (9.16) is the Stein factorization of π_w. Since ψ_w and ψ_u are universal homeomorphisms, the morphism $\tilde{S}_u \to \tilde{S}_w$ is a universal homeomorphism onto its image $\tilde{T}_u$ in $\tilde{S}_w$. The compatible F-splitting

of $D(\tilde{u})$ in $D(\tilde{w})$ induces an F-splitting of $\tilde{T}_u$, which is compatible with the F-splitting of $\tilde{S}_u$ induced from the F-splitting of $D(\tilde{u})$, i.e., the following diagram is commutative

$$
\begin{array}{ccc}
F_*(\mathcal{O}_{\tilde{T}_u}) & \longrightarrow & F_*(\mathcal{O}_{\tilde{S}_u}) \\
\downarrow & & \downarrow \\
\mathcal{O}_{\tilde{T}_u} & \longrightarrow & \mathcal{O}_{\tilde{S}_u}
\end{array}
$$

in which the horizontal arrows are injections. We can see that this implies that $\mathcal{O}_{\tilde{T}_u} = \mathcal{O}_{\tilde{S}_u}$, and hence the morphism $\tilde{S}_u \to \tilde{S}_w$ is a closed immersion, and the variety $\tilde{S}_w$ is F-split compatibly with the closed subscheme $\tilde{S}_u$.

To show (d), it remains to show that

$$
R^i (\tilde{\pi}_w)_*(\mathcal{O}_{D(\tilde{w})}) = (0) \quad \text{and} \quad R^i (\tilde{\pi}_w)_*(\omega_{D(\tilde{w})}) = (0) \quad \text{for } i > 0,
$$

and $(\tilde{\pi}_w)_*(\mathcal{O}_{D(\tilde{w})}) = \mathcal{O}_{\tilde{S}_w}$. Here the assertion concerning $\omega_{D(\tilde{w})}$ follows from the rest by the Grauert–Riemenschneider theorem (for Frobenius split varieties in positive characteristic, cf. [35]). The assertions concerning $\mathcal{O}_{D(\tilde{w})}$ are proved by induction on $l(w)$, cf. [19, Lemma 3.13]. Let $\tilde{w} = s_i \tilde{u}$ with $l(u) = l(w) - 1$. Let us factor $\tilde{\pi}_w$ as

$$
D(\tilde{w}) = L^+ P_i \times^{L^+ B} D(\tilde{u}) \to L^+ P_i \times^{L^+ B} \tilde{S}_u \to \tilde{S}_w.
$$

By induction we may assume that the higher direct images of $\mathcal{O}_{D(\tilde{w})}$ under the first morphism vanish. As in [13, p. 51], we can see that the second morphism has as geometric fibers either a point or a $\mathbf{P}^1$.

By the lemma of Mehta and Srinivas [19, Lemma 3.14], the vanishing of the higher cohomology groups of each fiber implies now, in the presence of a Frobenius splitting, the vanishing of the higher cohomology sheaves. This proves (d) when the characteristic of k is positive; the characteristic zero case follows using a semicontinuity argument, using the lifting over $W(k)$, cf. 9.e. This finishes the sketch of the proof of the proposition. $\quad\square$

Now consider the ind-scheme S over k defined as the inductive limit of the directed set of the Schubert varieties S_w, $w \in W_a$, taken with the Bruhat order: $S := \varinjlim S_w$; this is an ind-subscheme of the affine flag variety $\mathcal{F}$. In fact, since G is simple and simply connected, it coincides with the underlying reduced sub ind-scheme $(\mathcal{F})_{\text{red}}$ of $\mathcal{F}$. Using Proposition 9.7 we can also define an ind-scheme $\tilde{S}$ over k as the inductive limit of the directed set $\tilde{S}_w$, $w \in W_a$, taken with the Bruhat order,

$$
\tilde{S} := \varinjlim(\tilde{S}_w).
$$

We have

$$
\tilde{S} \xrightarrow{\psi} S = (\mathcal{F})_{\text{red}} \hookrightarrow \mathcal{F}
$$

and Theorem 8.4 will follow if we show that, under our conditions, both maps are isomorphisms.

Now for α_i, $i \in I$, a simple affine root, the subgroup scheme $L^+ P_i$ contains the corresponding reflection s_i and also the root subgroups U_{α_i}, $U_{-\alpha_i}$. The group scheme $L^+ P_i$ acts compatibly on $D(\tilde{w})$ and S_w, provided that $l(s_i w) < l(w)$. Hence $L^+ P_i$ acts also on the normalization $\tilde{S}_w$ when $l(s_i w) < l(w)$. The proof of the following is straightforward.

Lemma 9.8. *For each $w \in W_a$ and each $i \in I$, there is $w' \in W_a$ with $w \leqslant w'$ and $l(s_i w') < l(w')$.*

The lemma, together with the above discussion and the definition of $\tilde{S}_w$ using the Demazure varieties implies that for each $i \in I$, $L^+ P_i$ acts on $\tilde{S}$. Since W_a is generated by the reflections $s_i \in L^+ P_i$, $i \in I$, the above discussion then implies that the ind-scheme $\tilde{S}$ supports an action of $L^+ B$ and of U_α for all affine roots $\alpha \in \Phi_{\text{aff}}$. The same is of course also true for the ind-scheme S; the morphism $\psi : \tilde{S} \to S$ is equivariant with respect to these actions.

Proposition 9.9. *Under our assumptions (i.e. the group G is absolutely simple, simply connected and splits over a tamely ramified extension of K), we have*

$$S = \varinjlim S_w = \mathcal{F}.$$

Proof. Let us write the ind-scheme $\mathcal{F}$ as a union $\mathcal{F} = \bigcup_j \mathcal{F}_j$ of proper k-schemes (Proposition 7.1(b)). We have

$$\bigcup_{w \in W_a} (\mathcal{F}_j \cap S_w) \subset \mathcal{F}_j$$

and given j, there is $w \in W_a$ such that $(\mathcal{F}_j)_{\text{red}} \subset S_w$. Let x be a generic point of the support of the ideal sheaf of $\mathcal{O}_{\mathcal{F}_j}$ which defines the scheme theoretic intersection $\mathcal{F}_j \cap S$. Consider the ideal of definition I_w of $S_w \cap \mathcal{F}_j$ in the local ring $A = \mathcal{O}_{\mathcal{F}_j, x}$ with maximal ideal M. Observe that $I_{w'} \subset I_w$ if $w' \geqslant w$ and A is Noetherian. Then by the above, there is a $w \in W_a$ such that A/I_w is of finite length and is such that if $w' \geqslant w$ then $I_w = I_{w'}$. We conclude that there is $n > 0$ such that I_w, for all $w \in W_a$, contains M^n properly. This gives a point $x_n \in \mathcal{F}_j(A/M^n)$ such that $x_n \notin S(A/M^n)$. We can write $x_n = g_n \mod L^+ B(R)$, where $R = A/M^n$ and $g_n \in LG(R)$ for the Artin local ring $R = A/M^n$. Corollary 9.5 and the fact that $L^+ B(R)$ and $U_\alpha(R)$ act on S (see above) shows that x_n is in $S(R)$ which is a contradiction. $\square$

Corollary 9.10. *If G is semi-simple, simply connected and splits over a tamely ramified extension of K, then $\mathcal{F} = LG/L^+ B$ is a reduced ind-scheme.*

Proof. If G is in addition absolutely simple, this follows from Proposition 9.9. We can reduce to this case using an argument as in Section 8.e.2. $\square$

In view of our reductions in Section 6, this also completes the proof of Theorem 6.1.

9.e. Suppose now that $\text{char}(k) = p > 0$. Recall that under our assumptions, we have using Section 7, a lift of LG, $L^+ P$, $\mathcal{F} = LG/L^+ B$ over k to $L\underline{G}$, $L^+ \underline{P}$, $\underline{\mathcal{F}} = L\underline{G}/L^+ \underline{B}$ over the ring of Witt vectors $W = W(k)$. Also, by Section 9.a, we have the affine root subgroups $U_\alpha \subset LG$ and their lifts $\underline{U}_\alpha \subset L\underline{G}$ over W (see Remark 9.2).

Denote by K_0 the fraction field of W. For $w \in W_a$, we can define $\underline{S}_w$, the Schubert variety over W, as the reduced subscheme over W that is the Zariski closure of the "cell" $L^+\underline{B} \cdot n_w \subset L\underline{G}/L^+\underline{B}$. For a reduced expression $\tilde{w} = s_{i_1} \cdots s_{i_r}$, we can again define the Demazure variety over W as the quotient

$$\underline{D}(\tilde{w}) = \left(L^+\underline{P}_{i_1} \times \cdots \times L^+\underline{P}_{i_r}\right)/\left(L^+\underline{B}\right)^r. \tag{9.18}$$

This is smooth and projective over W, is an iterated $\mathbf{P}^1_W$-fibration, and supports a proper morphism induced by multiplication

$$\underline{D}(\tilde{w}) \xrightarrow{\underline{\pi}_w} \underline{S}_w. \tag{9.19}$$

Now set $\underline{\tilde{S}}_w = \operatorname{Spec}((\underline{\pi}_w)_*(\mathcal{O}_{\underline{D}(\tilde{w})}))$, i.e., $\underline{\tilde{S}}_w$ is the normalization of $\underline{S}_w$ in $\underline{D}(\tilde{w})$; then the structure morphism $\underline{\tilde{S}}_w \to \operatorname{Spec}(W)$ is flat. The following can be shown following the arguments of [13] (see [19, Prop. 3.15] et seq. for some more details):

Proposition 9.11. (a) *The formation of* $\underline{\tilde{S}}_w$ *commutes with base change on* W. *In particular, there is a natural isomorphism*

$$\underline{\tilde{S}}_w \otimes_W k \simeq \tilde{S}_w. \tag{9.20}$$

(b) *If* $u \leqslant w$, *then there is a closed immersion* $\underline{\tilde{S}}_u \to \underline{\tilde{S}}_w$ *which lifts the natural closed immersion* $\underline{S}_u \hookrightarrow \underline{S}_w$ *such that a diagram as* (9.17) *commutes.*

Using Proposition 9.11(b), we can define an ind-scheme $\underline{\tilde{S}}$ over W as the inductive limit of the directed set $\underline{\tilde{S}}_w$, $w \in W_a$, taken with the Bruhat order:

$$\underline{\tilde{S}} := \varinjlim(\underline{\tilde{S}}_w).$$

In fact, because of Proposition 9.11(a), we have

$$\underline{\tilde{S}} \otimes_W k \simeq \tilde{S}. \tag{9.21}$$

We can see that the group schemes $L^+\underline{B}$ and $\underline{U}_\alpha$, $\alpha \in \Phi_{\mathrm{aff}}$, act on $\underline{S} = \varinjlim \underline{S}_w$ and $\underline{\tilde{S}}$.

9.f. Recall that K_0 is the fraction field of $W(k)$. Here we show that, for any w, $\tilde{S}_w \otimes_W K_0 = S_w \otimes_W K_0$, i.e., $S_{w,K_0} := S_w \otimes_W K_0$ is normal. It is enough to show this statement with K_0 replaced by its algebraic closure $\bar{K}_0$. Observe that the normality of Schubert varieties in affine flag varieties associated to (affine) Kac–Moody Lie algebras is known by the work of Kumar, Mathieu, Littelmann [28,33,31]. Our strategy is essentially to show that the (characteristic zero) Schubert varieties $S_{w,\bar{K}_0}$ coincide with corresponding Schubert varieties in the theory of (affine) Kac–Moody Lie algebras.

Recall that by Section 7 we are dealing with an absolutely simple, simply connected (quasi-split) group over $\bar{K}_0((t))$ which is one of the types described in the beginning of that paragraph. All these types are realized over $\mathbf{C}((t))$ and by the Lefschetz principle it is enough to show that the Schubert varieties $S_{w,\mathbf{C}}$ are normal for groups G over $\mathbf{C}((t))$ as in Section 7. For simplicity, in the rest of the paragraph, we will omit the subscript $\mathbf{C}$. Recall our notation $\mathfrak{g}'_{KM}(G)$ (and

$\mathfrak{g}_{KM}(G)$) for the affine Kac–Moody Lie algebra over $\mathbf{C}$ associated to the local Dynkin diagram of G (see 7.c).

Now consider the embedding $\mathcal{F}_G = LG/L^+B \hookrightarrow \mathcal{F}_H = LH/L^+B_H$ described in Section 7. The affine flag variety $\mathcal{F}_H$ for the split group $H = H_0 \otimes_{\mathbf{C}} \mathbf{C}((t))$ is the affine flag variety for H_0 considered e.g. in [13]. We can give a projective embedding $\mathcal{F}_H \hookrightarrow \mathbf{P}(V)$ by suitably embedding LH into LSL_n and composing the resulting closed immersion $\mathcal{F}_H \hookrightarrow \mathcal{F}_{SL_n}$ with the standard projective emdedding of $\mathcal{F}_{SL_n} \hookrightarrow \mathbf{P}(V)$ given by its lattice description. (Here V is a countably infinite-dimensional complex vector space and $\mathbf{P}(V)$ stands for the corresponding projective space which is naturally an ind-scheme over $\mathbf{C}$.) We obtain a projective embedding $\mathcal{F}_G \hookrightarrow \mathbf{P}(V)$; this induces a line bundle $\mathcal{L}$ on $\mathcal{F}_G$; which we can pull back to S_w, $\tilde{S}_w$ and $D(\tilde{w})$ for each $w \in W_a$. By Proposition 9.7 we see that, when $w' \leqslant w$, there are pull back (restriction) homomorphisms

$$\Gamma(\tilde{S}_w, \mathcal{L}) \to \Gamma(\tilde{S}_{w'}, \mathcal{L}). \tag{9.22}$$

By the Frobenius splitting of the normalizations in characteristic p (i.e., over k) we can see (cf. [33, Lemma 137]) that the analogous restriction homomorphisms (9.22) in characteristic p are surjective. By semicontinuity the homomorphisms (9.22) themselves are then also surjective. Now after passing to the $\mathbf{C}$-duals, we can consider the inductive limit of injective maps

$$\tilde{E}(\mathcal{L}) := \varinjlim\left(\Gamma(\tilde{S}_w, \mathcal{L})^*\right). \tag{9.23}$$

The rest of the argument is as in [13, p. 52] (which also follows [28], see also [29]). This complex vector space $\tilde{E}(\mathcal{L})$ admits actions by L^+P_i. For each index i there is an sl_2-triple $\{e_i, f_i, h_i\}$ in the Lie algebra of LG; this is given via the homomorphism $\phi_i : SL_2 \to LG$ of Section 9.b. The Lie algebra of L^+P_i contains all the elements e_j and h_j and the element f_i and we can see that these satisfy the additional relations $[e_j, f_i] = 0$ for $i \neq j$, $[h_r, e_s] = a_{rs}e_s$, and $[h_r, f_i] = -a_{ri}f_i$, where $\{a_{rs}\}$ is the generalized Cartan matrix. The action of the Lie algebra of L^+P_i on $\tilde{E}(\mathcal{L})$ is locally finite since it respects the direct limit structure of (9.23). Using [29, Cor. 1.3.10], we can now see that the action of e_i, f_i, h_i on $\tilde{E}(\mathcal{L})$ combine to a $\mathfrak{g}'_{KM}(G)$-module structure (cf. [28, Proposition 2.11]). In fact, $\tilde{E}(\mathcal{L})$ is an integrable $\mathfrak{g}'_{KM}(G)$-module with a highest weight vector of multiplicity one and hence is irreducible. (This highest weight vector is given by the evaluation of global sections of $\mathcal{L}$ at the "origin" $w \in L^+Bw \subset \tilde{S}_w$. Note that [28, Proposition 2.11] shows a similar statement for $\tilde{S}_w$ replaced by the Demazure varieties $D(\tilde{w})$; however, we can see, using Proposition 9.7, semicontinuity and the projection formula, that $\Gamma(D(\tilde{w}), \mathcal{L}) \simeq \Gamma(\tilde{S}_w, \mathcal{L})$. By 8.d our Demazure varieties for G coincide with the Kac–Moody Demazure varieties in [28].) Similarly, we can consider the inductive limit $E(\mathcal{L}) := \varinjlim(\Gamma(S_w, \mathcal{L})^*)$; a similar argument shows that this is also an integrable $\mathfrak{g}'_{KM}(G)$-module with a highest weight vector of multiplicity one. For each $w \in W_a$ we have a surjective homomorphism $\Gamma(\tilde{S}_w, \mathcal{L})^* \to \Gamma(S_w, \mathcal{L})^*$ which respects the action of L^+P_i, obtained as the dual of the pull-back homomorphism. Hence, we obtain a surjective $\mathfrak{g}'_{KM}(G)$-module homomorphism $\tilde{E}(\mathcal{L}) \to E(\mathcal{L})$; by irreducibility this then has to be an isomorphism. Since $\Gamma(\tilde{S}_{w'}, \mathcal{L})^* \to \Gamma(\tilde{S}_w, \mathcal{L})^*$ is injective for $w' \leqslant w$, we can conclude that $\Gamma(\tilde{S}_w, \mathcal{L})^* \to \Gamma(S_w, \mathcal{L})^*$ is an isomorphism for each $w \in W_a$. The same argument applies to $\mathcal{L}^{\otimes n}$, $n > 0$: We obtain $\Gamma(S_w, \mathcal{L}^{\otimes n}) \simeq \Gamma(\tilde{S}_w, \mathcal{L}^{\otimes n})$ from which we can easily deduce $\tilde{S}_w = S_w$, i.e., that S_w is normal (cf. [13] or [28]).

9.g.   Here we complete the proof of Theorem 8.4. (The argument is due to Faltings [14].) We continue to assume that $\mathrm{char}(k) = p > 0$. Consider the morphisms

$$\underline{D}(\tilde{w}) \to \underline{\tilde{S}}_w \xrightarrow{\psi_w} \underline{S}_w$$

over the ring of Witt vectors W. Let $e_0 : \mathrm{Spec}(W) \to \underline{S}_w$ be the point that corresponds to the origin of $\underline{\mathcal{F}} = L^+ \underline{G}/L^+ \underline{B}$. This lifts to the obvious neutral point of $D(\tilde{w})$ and therefore also to a point of $\underline{\tilde{S}}_w$. Denote by A_w, resp. $\tilde{A}_w$, B_w the local ring of $\underline{S}_w$, resp. $\underline{\tilde{S}}_w$, $\underline{D}(\tilde{w})$ at the corresponding neutral points (lifting $\bar{e}_0 := e_0 \otimes_W k$) of the special fibers. We have

$$A_w \hookrightarrow \tilde{A}_w \hookrightarrow B_w$$

where $B_w = W[t_1, \ldots, t_r]_{(p, t_1, \ldots, t_r)}$. This ring supports a filtration by the ideals $F^n(B_w) = (t_1, \ldots, t_r)^n$; the associated graded pieces are free W-modules. We set $F^n(A_w) = F^n(B_w) \cap A_w$ which is a filtration of A_w by ideals. Let us denote by $x \in \underline{S}_w(A_w)$ the tautological point $\mathrm{Spec}(A_w) \to \underline{S}_w$; it gives points $x_n \in \underline{S}_w(A_w/F^n(A_w))$. For simplicity of notation, we set $A_w^n := A_w/F^n(A_w)$.

Claim. *For each $n \geqslant 1$, there is an element $g_n \in L\underline{G}(A_w^n)$ which belongs to the subgroup generated by $L^+ \underline{B}(A_w^n)$ and $\underline{U}_\alpha(A_w^n)$ for $\alpha \in \Phi_{\mathrm{aff}}$ such that $x_n = g_n \cdot e_0$.*

Proof. We will argue using induction on n. For $n = 1$ the claim is true since $A_w^1 = W$ and $x_1 = e_0$. Suppose now the claim is true for n. Let us write $x_n = g_n \cdot e_0$; since $L^+ \underline{B}$ and $\underline{U}_\alpha$ are smooth over W, we can lift g_n to an element g'_{n+1} which belongs to the subgroup generated by $L^+ \underline{B}(A_w^{n+1})$ and $\underline{U}_\alpha(A_w^{n+1})$. Consider $x'_{n+1} = g'_{n+1} \cdot e_0$; by our construction we have

$$x'_{n+1} \bmod F^n(A_w) \equiv x_n \bmod F^n(A_w). \tag{9.24}$$

Since $\underline{S}_w \subset \underline{\mathcal{F}} = L\underline{G}/L^+ \underline{B}$, we can write $x_{n+1} = h_{n+1} \cdot e_0$. Now we can see that the element $g'_{n+1} \cdot h_{n+1}^{-1}$ of $L\underline{G}(A_w/F^{n+1}(A_w))$ satisfies $g'_{n+1} \cdot h_{n+1}^{-1} \equiv 1 \bmod F^n(A_w)/F^{n+1}(A_w)$. Therefore (cf. (1.5)), it is given by an element of

$$Lie(L\underline{G}) \otimes_W F^n(A_w)/F^{n+1}(A_w).$$

Now our claim follows from the fact (Proposition 9.4) that the Lie algebra $Lie(L\underline{G})$ is spanned over W by the Lie algebras of $L^+ \underline{B}$ and $\underline{U}_\alpha$ for all $\alpha \in \Phi_{\mathrm{aff}}$.   □

Since the element $g_n \in L\underline{G}(A_w^n)$ obtained in the claim above acts on $\underline{\tilde{S}}$, we obtain the following: There is $w' > w$ in W_a such that $x_n \in \underline{S}_w(A_w^n)$ lifts to an element $\tilde{x}_n \in \underline{\tilde{S}}_{w'}(A_w^n)$. Now recall that by Section 9.f, $\underline{\tilde{S}}_w \otimes_W K_0 = \underline{S}_w \otimes_W K_0$. Hence, we have

$$\tilde{x}_n \otimes_W K_0 \in \underline{\tilde{S}}_{w'}\big((A_w^n) \otimes_W K_0\big) \cap \underline{S}_w\big((A_w^n) \otimes_W K_0\big) = \underline{\tilde{S}}_w\big((A_w^n) \otimes_W K_0\big)$$

which implies $x_n \in \underline{\tilde{S}}_w(A_w^n)$. Furthermore, our arguments show that $\tilde{x}_n$ are compatible when we vary the n. Hence, we obtain a splitting β of the (injective) ring homomorphism

$$\alpha : \hat{\mathcal{O}}_{\underline{S}_w, \bar{e}_0} \hookrightarrow \hat{\mathcal{O}}_{\underline{\tilde{S}}_w, \bar{e}_0} \tag{9.25}$$

induced by $\psi_w : \tilde{\underline{S}}_w \to \underline{S}_w$ on the formal completions (i.e., $\alpha := \hat{\psi}_w^*$). We have $\beta \cdot \alpha = \mathrm{id}$. Now consider the coherent sheaf

$$(\psi_w)_* (\mathcal{O}_{\tilde{\underline{S}}_w}) / \mathcal{O}_{\underline{S}_w}$$

on $\underline{S}_w \subset \underline{\mathcal{F}}$. This sheaf is $L^+\underline{B}$-equivariant. Using Section 9.f we see that it is annihilated by a power of p. Assume that its support is non-empty. Then it has to contain the unique fixed point $\bar{e}_0$ of the action of L^+B (in characteristic p) and therefore,

$$\hat{\mathcal{O}}_{\tilde{\underline{S}}_w, \bar{e}_0} / \alpha(\hat{\mathcal{O}}_{\underline{S}_w, \bar{e}_0})$$

is non-trivial and p^m-torsion for some $m > 0$. For $f \in \hat{\mathcal{O}}_{\tilde{\underline{S}}_w, \bar{e}_0}$ now write

$$\alpha \cdot \beta(p^m \cdot f) = \alpha \cdot \beta \cdot \alpha(f') = \alpha(f') = p^m \cdot f.$$

Since $\hat{\mathcal{O}}_{\tilde{\underline{S}}_w, \bar{e}_0}$ is flat over W, we obtain $\alpha \cdot \beta(f) = f$, so $\alpha \cdot \beta = \mathrm{id}$. This shows α is an isomorphism which is a contradiction. Therefore, $(\psi_w)_*(\mathcal{O}_{\tilde{\underline{S}}_w}) = \mathcal{O}_{\underline{S}_w}$ and so ψ_w is an isomorphism. This concludes the proof when $\mathrm{char}(k) = p > 0$. The case that $\mathrm{char}(k) = 0$ follows by semicontinuity, or more directly by the arguments in Section 9.f.

9.h. Here we discuss the relation with the Schubert varieties in the theory of Kac–Moody flag varieties. Suppose that k is algebraically closed and G is absolutely simple, simply connected and splits over a tamely ramified extension. As in Section 9.f we can associate to G (via its extended Dynkin diagram) a Kac–Moody algebra $\mathfrak{g} := \mathfrak{g}_{KM}(G)$ over k. For $w \in W_a$ with reduced decomposition $\tilde{w}$, there is a Demazure variety $D_{\mathfrak{g}}(\tilde{w})$ in the Kac–Moody setting (see [33, p. 51]) given by a contracted product similar to the one in 8.c. By Section 8.d, $D_{\mathfrak{g}}(\tilde{w})$ is equivariantly isomorphic to the Demazure variety $D(\tilde{w})$ defined in Section 8.c. Now let λ be a regular dominant integral weight of $\mathfrak{g}$ and denote by $L(\lambda)$ the corresponding highest weight module over k with highest weight vector v_λ. Set $S_{w,\lambda}^{\mathfrak{g}}$ for the Zariski closure of $B \cdot w\lambda$ in the (infinite-dimensional) projective space $\mathbf{P}(L(\lambda))$; this is a projective scheme over k. We have a birational proper morphism of k-schemes

$$p_w : D(\tilde{w}) = D_{\mathfrak{g}}(\tilde{w}) \to S_{w,\lambda}^{\mathfrak{g}}.$$

In this case, Mathieu [33] and Littelmann [31] prove that $S_{w,\lambda}^{\mathfrak{g}}$ is normal with rational singularities and as an abstract scheme is independent of λ. We can see by induction that the k-schemes S_w and $S_{w,\lambda}^{\mathfrak{g}}$ are stratawise isomorphic and so homeomorphic as topological spaces. In fact (cf. [33, proof of Lemme 33]), we can find a homeomorphism

$$\tau_w : S_w \xrightarrow{\sim} S_{w,\lambda}^{\mathfrak{g}}$$

such that $\tau_w \cdot \pi_w = p_w$. By [33] and [31], resp. Theorem 8.4, the structure sheaf of $S_{w,\lambda}^{\mathfrak{g}}$, resp. S_w, is given as $p_{w*}(\mathcal{O}_{D(\tilde{w})})$, resp. $\pi_{w*}(\mathcal{O}_{D(\tilde{w})})$. Since a scheme is determined by the corresponding locally ringed topological space, we see that τ_w actually gives an isomorphism of schemes. These isomorphisms combine to give an isomorphism of ind-schemes

$$LG/L^+B = \varinjlim S_w \xrightarrow{\sim} \varinjlim S_{w,\lambda}^{\mathfrak{g}}. \tag{9.26}$$

Recall that the first equality is Proposition 9.9. On the other hand, $\varinjlim S_{w,\lambda}^{\mathfrak{g}} = \varinjlim \tilde{S}_{w,\lambda}^{\mathfrak{g}}$ is by definition the Kac–Moody flag variety of [33,31].

10. A coherence conjecture

In this section, we formulate a conjecture which relates amongst each other the dimensions of the spaces of global sections of natural "ample" line bundles on the various partial affine flag varieties attached to a fixed loop group. In the next section we demonstrate an example of the kind of application of this conjecture we have in mind.

We continue to assume that k is algebraically closed and that G is a simply connected semi-simple and absolutely simple group over $K = k((t))$ which splits over the tamely ramified extension K' of K. Since this is the case we are mostly interested in, we also assume $\operatorname{char}(k) = p > 0$. We continue to use the notation of the previous section; in particular B is an Iwahori subgroup. Recall that we denote by **S** the corresponding set of simple affine roots.

10.a. We first explain the construction of certain ample line bundles.

Proposition 10.1. *Denote by $\mathcal{F} = LG/L^+B$ the full affine flag variety. There is an isomorphism*

$$\operatorname{Pic}(\mathcal{F}) \xrightarrow{\sim} \mathbf{Z}^{\mathbf{S}} = \bigoplus_{i \in \mathbf{S}} \mathbf{Z}\epsilon_i$$

from the Picard group of line bundles on $\mathcal{F}$ to $\mathbf{Z}^{\mathbf{S}}$, defined by the degrees of the restrictions to the projective lines corresponding to the simple affine roots.

Proof. We can assume that B is an Iwahori subgroup given as in Section 9.a. Let us recall that by the projective line $\mathbf{P}_{\alpha_i}^1 \hookrightarrow \mathcal{F}$ that corresponds to the simple affine root α_i we mean the image of L^+P_i/L^+B in $\mathcal{F}$; this coincides with the image of $\mathbf{P}^1 \to \mathcal{F} = LG/L^+B$ given by the homomorphism $\phi_{\alpha_i}: SL_2 \to LG$ defined in Section 9.b. The main point now is that our previous results allow us to refer to the Kac–Moody context. Use Section 7.c with the choice of positive roots and Chevalley basis the same to the choice that gives B as in Section 9.a. Then, in view of (9.26), our statement is given by [33, XVIII, Prop. 28] (see also [33, XII, Lemme 76] and its proof). These references construct a line bundle $\mathcal{L}(\lambda)$ for each weight $\lambda \in (\bigoplus \mathbf{Z} \cdot h_i)^*$. Roughly speaking, the line bundle $\mathcal{L}(\lambda)$ can be identified with the homogeneous line bundle on the affine flag variety attached to the character λ of the Iwahori subgroup in Kac–Moody theory. The degree of the restriction of $\mathcal{L}(\lambda)$ to the projective line corresponding to the simple affine root α_i is then $\lambda(h_i)$ and so in our notation $\epsilon_i = h_i^*$. $\quad\square$

Remarks 10.2. (i) By restricting a character λ of the Iwahori subgroup as above to the center of the Kac–Moody central extension, we obtain a homomorphism

$$c: \operatorname{Pic}(\mathcal{F}) = \mathbf{Z}^{\mathbf{S}} \to \mathbf{Z}. \tag{10.1}$$

The image $c(\mathcal{L})$ of a line bundle $\mathcal{L}$ on $\mathcal{F}$ is called its *central charge*. It is given explicitly as follows.

Let us identify **S** with $\{0, \ldots, l\}$, and let $A = (a_{ij})_{i,j=0,\ldots,l}$ be the corresponding generalized Cartan matrix. By [24, (6.1.1)], there exists an element $(a_0^\vee, \ldots, a_l^\vee) \in (\mathbf{Z}_{\geqslant 0})^{l+1}$ such that

$$\left(a_0^\vee, \ldots, a_l^\vee\right) \cdot A = (0, \ldots, 0),$$

and such that $a_i^\vee = 1$ for at least one i with $0 \leqslant i \leqslant l$. This element is obviously uniquely determined. Then

$$c(\epsilon_i) = a_i^\vee, \quad i = 0, \ldots, l. \tag{10.2}$$

Indeed, this follows from the formula $K = \sum_i a_i^\vee \alpha_i^\vee$ [24, p. 80] for the canonical generator of the center of $\mathfrak{g}'_{KM}(G)$, once ϵ_i is identified with h_i^* as in the proof of Proposition 10.1 above.

(ii) Let G be split and write $G = G_0 \otimes_k K$. Let $P = G_0 \otimes_k k[[t]]$ be the corresponding maximal parahoric subgroup. Let $T \subset G_0$ be a maximal torus and B the Iwahori subgroup obtained as the inverse image of a Borel subgroup containing T under the reduction map

$$L^+ P \to G_0.$$

In [13, p. 54], Faltings constructs a central extension

$$1 \to \mathbf{G}_m \to \tilde{L}G \to LG \to 1 \tag{10.3}$$

which acts on all line bundles on $\mathcal{F}$. The Lie algebra of $\tilde{L}G$ is the formal version of $\mathfrak{g}'_{KM}(G)$. We may consider T as a subgroup of $L^+ P$ (*constant loops*). Let $\tilde{T}$ be the inverse image of T in $\tilde{L}G$. Then $X^*(\tilde{T})$ may be identified with the character group of the Iwahori subgroup in Kac–Moody theory and we obtain a canonical isomorphism

$$X^*(\tilde{T}) \xrightarrow{\sim} \mathrm{Pic}(\mathcal{F}). \tag{10.4}$$

Let $\{\alpha_1, \ldots, \alpha_l\}$ be the set of simple roots of T with respect to the fixed Borel subgroup, and let $\alpha_0 = 1 - \theta$ be the remaining simple affine root. Then we may identify $X^*(T)$ with $\bigoplus_{i=1}^l \mathbf{Z}\epsilon_i$, whereas $X^*(\tilde{T})$ is identified with $\bigoplus_{i=0}^l \mathbf{Z}\epsilon_i$ via Proposition 10.1 and (10.4). In terms of these identifications, the canonical inclusion $\iota : X^*(T) \hookrightarrow X^*(\tilde{T})$ is given as

$$\iota(\epsilon_i) = \epsilon_i - a_i^\vee \epsilon_0, \quad i = 1, \ldots, l. \tag{10.5}$$

Indeed, this follows from the equalities

$$\deg\left(\mathcal{L}\left(\iota(\epsilon_i)\right) \mid \mathbf{P}^1_{\alpha_i}\right) = \delta_{ij}, \quad i, j = 1, \ldots, l,$$
$$c\left(\mathcal{L}\left(\iota(\epsilon_i)\right)\right) = 0, \quad i = 1, \ldots, l.$$

The central extension (10.3) has a unique splitting over $L^+ P$. This also yields a splitting of the exact sequence

$$0 \to X^*(T) \xrightarrow{\iota} X^*(\tilde{T}) \xrightarrow{c} \mathbf{Z} \to 0. \tag{10.6}$$

In terms of the identification $X^*(\tilde{T}) = \bigoplus_{i=0}^l \mathbf{Z}\epsilon_i$, this splitting is given by sending $1 \in \mathbf{Z}$ to ϵ_0. This follows from [13, Thm. 7] (the last two bullets) and the above.

10.a.1.   The result of Proposition 10.1 in the case of a split group $G = G_0 \otimes_k K$ is shown in [13, Cor. 12], by using the "big cell" in $\mathcal{F}$. Proposition 10.1 can be shown in all cases, except when G is of type $A_{2m}^{(2)}$, by appealing to the results of [13] without using the identification (9.26) of the two types of affine flag varieties and the results of [33]. In this paragraph, we take some time to explain this claim; our arguments are also useful for the rest of the paper.

To see the injectivity of the homomorphism in the statement of the proposition, consider a line bundle on $\mathcal{F}$ with zero restriction degrees. This line bundle then has trivial restriction to any Demazure variety $D(\tilde{w})$ and hence to any Schubert variety S_w, cf. [13, proof of Cor. 12]. The injectivity assertion therefore follows from Proposition 9.9. To show the rest of our claim, we present G as the invariants under σ in $H' = \mathrm{Res}_{K'/K}(H_{K'})$, where H is a split group and where $\sigma = \sigma_0 \otimes \tau$, as in Section 7.

In fact, we will show that under the restriction homomorphism

$$\mathrm{Pic}(\mathcal{F}_H) \to \mathrm{Pic}(\mathcal{F}) \hookrightarrow \mathbf{Z}^{\mathbf{S}} \tag{10.7}$$

the image of $\mathrm{Pic}(\mathcal{F}_H)$ is the subgroup $\bigoplus_{i \in \mathbf{S}} \mathbf{Z}\kappa(i)\epsilon_i$. Here we set $\kappa(i) = 2$ or 1, according as the vector part of the affine root α_i is a multipliable root or not. Recall that $\kappa(i) = 1$ for all i unless G is of type $A_{2m}^{(2)}$. It is enough to show that the image of the restriction homomorphism contains the elements $\kappa(i)\epsilon_i$. Recall from Section 9 that we fixed a Chevalley–Steinberg system relative to the splitting of G over K'. We may write any affine root α in the form

$$\alpha = \nu(\alpha) + m, \quad m \in \frac{1}{[K':K]} \cdot \mathbf{Z},$$

cf. Section 9. Here $a = \nu(\alpha) \in \Phi$ is a root of (G, S). Now a is the restriction of an absolute root a' of $(G \otimes_K K', T')$, and the absolute roots a' with this property form a single orbit $\{a'\}$ under σ. Recall also from Section 9 the group homomorphism $\pi_a : G^a \to G$. Set $H_{K'}^a := G^a \times_K K'$. We have $H^a = \prod_{\{a'\}} SL_{2/K'}$ in case (I) and $H_{K'}^a \simeq SL_{3/K'}$ in case (II). The homomorphism π_a induces injective homomorphisms

$$\pi'_a : H_{K'}^a \to H_{K'} \tag{10.8}$$

and we can see that these give corresponding injective maps between affine flag varieties

$$\pi'_a : \mathcal{F}_{H^a} = LH^a/L^+B^{H^a} \to \mathcal{F}_H = LH/L^+B^H. \tag{10.9}$$

Note that by definition, $\mathbf{P}_\alpha^1 \hookrightarrow \mathcal{F}_G \subset \mathcal{F}_H$ factors through

$$\mathbf{P}_\alpha^1 \hookrightarrow \mathcal{F}_{G^a} \subset \mathcal{F}_{H^a}.$$

Consider now the simple affine root $\alpha_i = a_i + m_i$ that corresponds to the index i. Note that $\alpha'_i = a'_i + m_i$, for all a'_i in the σ-orbit that correspond to a_i, are simple affine roots for $H_{K'}$. Appealing to the case of a split group, we find a line bundle on the affine flag variety $\mathcal{F}_H$ with restriction of degree 1 to the projective line corresponding to one $\alpha'_i = a'_i + m_i$ in the σ-orbit, and degree 0 to the projective lines corresponding to all other simple affine roots of $H_{K'}$. By restriction this induces a line bundle on $\mathcal{F}_G \subset \mathcal{F}_H$. We claim that this line bundle has image $\kappa(i)\epsilon_i$ in $\mathbf{Z}^{\mathbf{S}}$ under our homomorphism.

Suppose that $\beta = b + m$ is a simple affine root of G. First assume that b is not in the ray of the root a. Then by appealing to the injectivity of the homomorphism of the proposition applied to H^b we can see that the restriction of this line bundle to $\mathcal{F}_{H^b}$ is trivial. Hence, the restriction of this line bundle to $\mathbf{P}^1_\beta$ is also trivial.

It remains to deal with the case that b is in the ray of a_i. Note that the simple affine roots of H^b can be identified with a subset of the set of simple affine roots of H.

First assume that neither $2a_i$ nor $a_i/2$ are roots. Then $G^b = G^{a_i} = \mathrm{Res}_{L/K} SL_2$ with $L = K$ or K'. The embedding $\mathbf{P}^1_{\alpha_i} \hookrightarrow \mathcal{F}_{G^{a_i}} \subset \mathcal{F}_G \subset \mathcal{F}_H$ factors through a commutative diagram

$$
\begin{array}{ccc}
\mathbf{P}^1_{\alpha_i} & \longrightarrow & \mathcal{F}_{G^{a_i}} \\
\downarrow & & \downarrow \\
\prod_{\{\alpha_i'\}} \mathbf{P}^1_{\alpha_i'} & \longrightarrow & \prod_{\{a_i'\}} \mathcal{F}_{SL_2}.
\end{array}
\tag{10.10}
$$

Here again $\{a_i'\}$ is the σ-orbit corresponding to a_i, $\{\alpha_i'\} = \{a_i' + m_i\}$, and $\mathcal{F}_{H^{a_i}} = \prod_{\{a_i'\}} \mathcal{F}_{SL_2} \subset \mathcal{F}_H$. The left vertical arrow is the diagonal embedding. We can see that the pull-back of our line bundle on $\mathcal{F}_H$ to $\prod_{\{\alpha_i'\}} \mathbf{P}^1_{\alpha_i'}$ has degree 1 on one of the factors and is trivial on all the others. Hence our claim follows.

Now assume that either $2a_i$ or $a_i/2$ is a root. Then $G^a = SU_3$, $H^a = SL_{3/K'}$ and we can reduce to the case $G = SU_3$. In this case, there are two simple affine roots β_1, β_0 which correspond to the σ-orbits $\{\alpha_1 + 0, \alpha_2 + 0\}$ and $\{-(\alpha_1 + \alpha_2) + \frac{1}{2}\}$ of affine roots of SL_3 (with the standard notation). The homomorphism $\phi_{\beta_1} : SL_2 \to LSU_3$ factors through the "constants" $(SL_3)^{\sigma_0} = SO_3$. (This follows from the definition of ϕ_{β_1} in Section 9.b since in this case $m = 0$.) Therefore the embedding $\mathbf{P}^1_{\beta_1} \hookrightarrow \mathcal{F}_{SU_3}$ factors through a commutative diagram

$$
\begin{array}{ccc}
\mathbf{P}^1_{\beta_1} & \longrightarrow & \mathcal{F}_{SU_3} \\
\downarrow & & \downarrow \\
SL_3/B & \longrightarrow & \mathcal{F}_{SL_3}
\end{array}
\tag{10.11}
$$

and the left vertical morphism identifies $\mathbf{P}^1_{\beta_1}$ with the space of isotropic flags.

The homomorphism $\phi_{\beta_0} : SL_2 \to LSU_3 \to LSL_3$ agrees (up to an automorphism of the target) with the homomorphism $\phi_{\alpha_0} : SL_2 \to LSL_3$ associated to the simple affine root $\alpha_0 = -(\alpha_1 + \alpha_2) + \frac{1}{2}$ of $SL_{3/K'}$. This gives a commutative diagram

$$
\begin{array}{ccc}
\mathbf{P}^1_{\beta_0} & \longrightarrow & \mathcal{F}_{SU_3} \\
\downarrow & & \downarrow \\
\mathbf{P}^1_{\alpha_0} & \longrightarrow & \mathcal{F}_{SL_3}
\end{array}
\tag{10.12}
$$

where the left vertical map is an isomorphism.

If $\alpha_i = \beta_1$ then our line bundle restricts under the lower horizontal map in (10.11) to one of the two ample generators of the Picard group of SL_3/B. Either of them has restriction of degree 2 on the $\mathbf{P}^1$ of isotropic flags and so the degree of its restriction to $\mathbf{P}^1_{\beta_1}$ is 2. Using the commutative diagram (10.12) we can see that its restriction to $\mathbf{P}^1_{\beta_0}$ is 0. This shows our claim in this case. The case of $\alpha_i = \beta_0$ is similar (the restriction of the line bundle to $\mathbf{P}^1_{\beta_0}$, resp. $\mathbf{P}^1_{\beta_1}$, has degree 1, resp. degree 0).

10.a.2. It will be convenient for the rest of the paper to introduce the following notation for the partial affine flag varieties associated to G. If Y is a non-empty subset of the set of simple affine roots $\mathbf{S}$, we set $W^Y = W_{\mathbf{S}-Y}$, $P^Y = P_{\mathbf{S}-Y}$ and $\mathcal{F}^Y = LG/L^+P^Y = \mathcal{F}_{\mathbf{S}-Y}$. Therefore, $\mathcal{F} = \mathcal{F}^{\mathbf{S}}$.

Assume that the weight $\lambda = \sum n_i \epsilon_i$ is dominant, i.e. $n_i \geqslant 0$ for all i. It follows from its construction (see the references in the proof of Proposition 10.1) that the corresponding line bundle $\mathcal{L}(\lambda)$ comes by pull-back from an ample line bundle $\mathcal{L}(\lambda)^Y$ on $\mathcal{F}^Y$ for $Y = Y(\lambda) := \{\alpha_i \in \mathbf{S} \mid n_i > 0\}$. Here ample is meant in the sense that the restriction of $\mathcal{L}(\lambda)^Y$ to any Schubert variety is ample.

For $Y \subset \mathbf{S}$, let us now consider the line bundles

$$\mathcal{L}(Y) = \mathcal{L}\left(\sum_{i \in Y} \epsilon_i\right)^Y, \qquad \mathcal{L}'(Y) = \mathcal{L}\left(\sum_{i \in Y} \kappa(i)\epsilon_i\right)^Y$$

on $\mathcal{F}^Y$. Let $Y = \{i_1, \ldots, i_s\}$ and consider the closed embedding of ind-schemes,

$$\mathcal{F}^Y \hookrightarrow \mathcal{F}^{i_1} \times \mathcal{F}^{i_2} \times \cdots \times \mathcal{F}^{i_s}. \tag{10.13}$$

Then $\mathcal{L}(Y)$ is the restriction of the line bundle

$$\mathcal{L}(i_1) \boxtimes \mathcal{L}(i_2) \boxtimes \cdots \boxtimes \mathcal{L}(i_s) \tag{10.14}$$

on $\mathcal{F}^{i_1} \times \mathcal{F}^{i_2} \times \cdots \times \mathcal{F}^{i_s}$ and similarly for $\mathcal{L}'(Y)$. Notice here that the restriction of $\mathcal{L}(Y)$ to a Schubert variety S_w in $\mathcal{F}^Y$ is ample, and hence by the Frobenius splitting of S_w we have $H^i(S_w, \mathcal{L}(Y)) = 0$ for $i > 0$, and similarly for any positive tensor power of $\mathcal{L}(Y)$ and for any closed subset of $\mathcal{F}^Y$ which is a finite union of Schubert varieties (or more generally B-orbits). These statements are also true for $\mathcal{L}'(Y)$.

Remark 10.3. For $G = SL_n$ resp. Sp_{2r}, the sheaf $\mathcal{L}(Y) = \mathcal{L}'(Y)$ is explicitly constructed in [19, 3.2.2]. As we shall see below we can also obtain a relatively explicit description for $\mathcal{L}'(Y)$ in the case of the ramified special unitary group of dimension n as in Section 4. If $n = 2m + 1$ is odd, resp. $n = 2m$ is even, then we may identify the set of simple affine roots $\mathbf{S}$ with $\{0, \ldots, m\}$, resp. $\{0, \ldots, m-2, m, m'\}$, by associating to $i \in \{0, \ldots, m\}$, resp. $i \in \{0, \ldots, m-2, m, m'\}$, the unique simple reflection that *does not* stabilize the lattice λ_i. For $Y \subset \mathbf{S}$ let $I = I(Y)$ be the corresponding subset of $\{0, \ldots, m\}$, resp. $i \in \{0, \ldots, m-2, m, m'\}$. Recall the definition of $I^\sharp = \{i_0 < \cdots < i_k\}$ as in 4.b, and let $\mathcal{F}'_{I^\sharp}$ be the corresponding variety of special hermitian lattice chains (see (4.8) and (4.11)). By Remark 4.2 the partial affine flag variety $\mathcal{F}^Y$ for SU_n can be identified with a connected component of $\mathcal{F}'_{I^\sharp}$.

Given a Schubert variety S_w in $\mathcal{F}'_{I^\sharp}$, we can find $N > 0$ such that all corresponding lattices lie between $t^{-N}\lambda_0$ and $t^N\lambda_0$. We get a closed embedding

$$S_w \hookrightarrow \prod_{i_q \in I^\sharp} \mathrm{Grass}(t^{-N}\lambda_0/t^N\lambda_0, n_{i_q}), \tag{10.15}$$

where $n_{i_q} = \dim_k \lambda_{i_q}/t^N\lambda_0$. The restriction of $\mathcal{L}'(Y)$ to S_w is then the pullback of the exterior tensor product of the very ample generators of the Picard groups of the various Grassmann varieties occurring in this product. Indeed, this follows from the corresponding expression for these line bundles in the case of SL_n and the fact that we can obtain the line bundle $\mathcal{L}'(Y)$ as a restriction via the embedding $\mathcal{F}_{SU_n} \hookrightarrow \mathcal{F}_{SL_n}$. (Cf. 10.a.1. Here we view SU_n as an outer form of SL_n.)

10.b.   We now let T be the centralizer torus of S in G and denote by T_{ad} the image of T in the adjoint group. To $\mu \in X_*(T_{\mathrm{ad}})$ we attach its image λ in the coinvariants $X_*(T_{\mathrm{ad}})_I$. By (8.1) we can consider λ as an element in the Iwahori–Weyl group $\tilde{W}_{\mathrm{ad}}$ of G_{ad}. After the choice of a special vertex x in the chosen alcove we can write a commutative diagram

$$
\begin{array}{ccc}
W_a & \simeq & W(^x\Sigma) \ltimes Q^\vee(^x\Sigma) \\
\downarrow & & \downarrow \\
\tilde{W}_{\mathrm{ad}} & \hookrightarrow & W(^x\Sigma) \ltimes P^\vee(^x\Sigma).
\end{array}
$$

Here $Q^\vee(^x\Sigma)$, resp. $P^\vee(^x\Sigma)$, is the group of coroots, resp. of coweights, of the finite root system $^x\Sigma$, cf. [42, §3] or (8.8), and $\lambda \in P^\vee(^x\Sigma)$. In the sequel we write W_0 for the finite Weyl group $W(^x\Sigma)$. We extend in the obvious way the Bruhat order from W_a to $\tilde{W}_{\mathrm{ad}} = W_a \rtimes \Omega_{\mathrm{ad}}$.

We consider the *admissible subset of $\tilde{W}_{\mathrm{ad}}$ associated to the coweight μ,*

$$\mathrm{Adm}(\mu) = \{w \in \tilde{W}_{\mathrm{ad}} \mid w \leqslant w_0(\lambda) := w_0\lambda w_0^{-1} \text{ for some } w_0 \in W_0\}. \tag{10.16}$$

The set $\mathrm{Adm}(\mu)$ only depends on the geometric conjugacy class of the one-parameter subgroup μ, cf. [42, §3]. Note that all elements of $\mathrm{Adm}(\mu)$ have the same image in $\tilde{W}_{\mathrm{ad}}/W_a = \Omega_{\mathrm{ad}}$. Denoting by τ this common image of all elements of $\mathrm{Adm}(\mu)$ in Ω_{ad}, we can define the subset $\mathrm{Adm}(\mu)^\circ$ of W_a by

$$\mathrm{Adm}(\mu)^\circ = \mathrm{Adm}(\mu) \cdot \tau^{-1}$$

so that $\mathrm{Adm}(\mu) = \{w \in \tilde{W}_{\mathrm{ad}} \mid w = w' \cdot \tau \text{ with } w' \in \mathrm{Adm}(\mu)^\circ\}$.

For a non-empty subset Y of the set of simple affine roots, let $Y^\circ \subset \mathbf{S}$ be the subset that corresponds to the set of simple reflections of the form $\{\tau \cdot s_i \cdot \tau^{-1} \mid i \in Y\}$ where s_i is the reflection corresponding to the simple affine root parametrized by $i \in Y$.

We may define the subset $\mathrm{Adm}^Y(\mu)$ of $\tilde{W}_{\mathrm{ad}}$ by $W^Y \cdot \mathrm{Adm}(\mu) \cdot W^Y$ and the subset $\mathrm{Adm}^Y(\mu)^\circ$ of W_a by

$$\mathrm{Adm}^Y(\mu)^\circ := W^Y \cdot \mathrm{Adm}(\mu)^\circ \cdot W^{Y^\circ}.$$

Since $\tau \cdot W^Y \cdot \tau^{-1} = W^{Y^\circ}$ this is also equal to $(W^Y \cdot \mathrm{Adm}(\mu) \cdot W^Y) \cdot \tau^{-1}$.

Let

$$\mathcal{A}(\mu)^\circ = \bigcup_{w \in \mathrm{Adm}(\mu)^\circ} S_w \subset \mathcal{F}, \qquad \mathcal{A}^Y(\mu)^\circ = \bigcup_{w \in \mathrm{Adm}^Y(\mu)^\circ} L^+ B \cdot n_w \subset \mathcal{F}^{Y^0} \qquad (10.17)$$

be the closed reduced subset, union of all Schubert varieties (resp. $L^+ B$-orbits) for w in the μ-admissible set (translated into the neutral component).

We are interested in the dimension

$$h_Y^{(\mu)}(a) = \dim H^0\big(\mathcal{A}^Y(\mu)^\circ, \mathcal{L}'(Y^\circ)^{\otimes a}\big). \qquad (10.18)$$

This is given by a polynomial in a. By our previous remarks on the comparison between our set-up and the Kac–Moody theory, and using Littelmann's path model [31], we can write

$$h_Y^{(\mu)}(a) = \left| \left\{ \text{LS-paths } \pi \text{ of shape } a \cdot \left(\sum_{\alpha_i \in Y^\circ} \kappa(i)\epsilon_i \right) \,\Big|\, i(\pi) \in \mathrm{Adm}^Y(\mu)^\circ / W^{Y^0} \right\} \right|.$$

Here $i(\pi)$ denotes the *initial direction* of π in W_a. We see therefore that the polynomial $h_Y^{(\mu)}$ has a purely combinatorial description. We want to give a (conjectural) explicit formula for the polynomials $h_Y^{(\mu)}$.

Assume that $\mu \in X_*(T_{\mathrm{ad}})$ is given by a cocharacter $\mu : \mathbf{G}_m \to G_{\mathrm{ad}} \otimes_K K^{\mathrm{sep}}$ over the separable closure K^{sep} of K (here we abuse notation by using the same symbol for the cocharacter of G_{ad}). In fact, since G is split over K', μ is given by $\mu : \mathbf{G}_m \to H_{\mathrm{ad}} \otimes_K K'$ (the notation is as in Section 7).

Let us first suppose that μ is miniscule. Let $e = [K' : K]$ and let $\{\mu = \mu^{(1)}, \ldots, \mu^{(e)}\}$ be the orbit of μ under the automorphism σ_0 of H. Denote by $P(\mu^{(j)})$ a corresponding maximal parabolic subgroup of $H_{K'} = G_{K'}$. Let $X(\mu^{(j)}) = H/P(\mu^{(j)})$ be the corresponding homogeneous space over K', and let $\mathcal{L}(\mu^{(j)})$ be the ample generator of the Picard group of $X(\mu^{(j)})$. Note that

$$X(\mu) = G_{K'}/P(\mu) = \left(\prod_{j=1}^e X(\mu^{(j)}) \right)^{\sigma_0} \subset \prod_{j=1}^e X(\mu^{(j)}) \qquad (10.19)$$

and that the restriction of $\boxtimes_{j=1}^e \mathcal{L}(\mu^{(j)})$ to this fixed point scheme is $\mathcal{L}(\mu)^{\otimes e}$. We introduce the polynomial in a,

$$h^{(\mu)}(a) = \dim H^0\big(X(\mu), \mathcal{L}(\mu)^{\otimes e \cdot a}\big). \qquad (10.20)$$

Now assume that $\mu = \mu_1 + \cdots + \mu_r$ is a sum of miniscules. Then we set

$$h^{(\mu)} = h^{(\mu_1)} \cdot h^{(\mu_2)} \cdots \cdots h^{(\mu_r)}. \qquad (10.21)$$

Note that this is independent of the way μ is written as a sum of miniscules. Also note that

$$h^{(\mu)}(a) = \dim H^0\big(X(\mu_1) \times X(\mu_2) \times \cdots \times X(\mu_r), \big(\mathcal{L}(\mu_1) \boxtimes \mathcal{L}(\mu_2) \boxtimes \cdots \boxtimes \mathcal{L}(\mu_r)\big)^{\otimes e \cdot a}\big).$$

Remark 10.4. It is possible to give an explicit expression for the polynomial $h^{(\mu)}$. For instance, if $G = SL_n$ and $\mu = \mu_r = (1^r, 0^{n-r})$ is a minuscule coweight, then $X(\mu)$ is the Grassmannian $Gr(r, n)$ of r-planes in a vector space of dimension n, and $h^{(\mu)}(a)$ equals the degree of the irreducible representation of PGL_n of highest weight $a \cdot (\epsilon_1 + \cdots + \epsilon_r)$.

Conjecture 10.5. *Assume that μ is minuscule, or a sum of minuscule coweights. Let $Y \subset \mathbf{S}$. Then*

$$h_Y^{(\mu)}(a) = h^{(\mu)}\big(|Y| \cdot a\big), \quad \forall a = 1, 2, \ldots.$$

In particular, taking $Y = \{\alpha_i\}$ for $\alpha_i \in \mathbf{S}$, the polynomial $h_{\{\alpha_i\}}^{(\mu)}(a)$ is independent of α_i.

Remark 10.6. A somewhat weaker version of this conjecture would assert the existence of a polynomial $h^{(\mu)}$ for which the above identity holds. It is this coherence statement that is really important for the applications of this conjecture we have in mind. We do not know whether the hypothesis on μ is necessary for this version of the conjecture to hold true.

Theorem 10.7. *Let $G = SL_n$ or Sp_{2n}. Then Conjecture* 10.5 *holds true.*

Proof. The essential additional ingredient here comes from the results of Görtz on flatness of local models for Shimura varieties [18,19]. Let us first consider the case where $G = SL_n$ and where $\mu = (1^r, 0^{n-r})$ is minuscule. Let us number the set of simple affine roots in the usual way by $s_0, \ldots, s_{n-1}$. Denoting by K_0 the fraction field of W, let Λ_i be the W-lattice in $V = K_0^n$ generated in terms of the standard basis $e_1, \ldots, e_n$ by $p^{-1}e_1, \ldots, p^{-1}e_i, e_{i+1}, \ldots, e_n$. Then s_i is the unique simple affine reflection that does *not* fix the lattice Λ_i. In what follows we will identify a subset Y of $\mathbf{S}$ with a subset of $\{0, 1, \ldots, n-1\}$.

Consider the local model associated to (GL_n, μ) over $\mathrm{Spec}(W)$, and to $I \subset \{0, \ldots, n-1\}$, cf. [18]. This is a flat scheme M_I over $\mathrm{Spec}(W)$ with generic fiber the Grassmannian $GL_n/P(\mu)$ and with reduced special fiber. In fact, this special fiber is isomorphic to $\mathcal{A}^I(\mu)^\circ$, cf. [18]. For $I = \{i_1, \ldots, i_s\}$, we have a closed embedding

$$M_I \hookrightarrow M_{i_1} \times_W \cdots \times_W M_{i_s}. \tag{10.22}$$

(Here we simply write M_i instead of $M_{\{i\}}$.) Recall that M_i is a Grassmannian over W which represents the functor

$$M_i(R) = \big\{\mathcal{E}_i \subset \Lambda_i \otimes_W R \mid \mathcal{E}_i \text{ locally on } \mathrm{Spec}(R) \text{ a direct summand of rank } r\big\}.$$

In the generic fiber, this closed embedding is the diagonal embedding of $GL_n/P(\mu)$ into the s-fold product of $GL_n/P(\mu)$ with itself.

In the special fiber this closed embedding is part of a commutative diagram, in which we denote by $\mathcal{F}^{I^\circ}$ the partial affine flag variety $LSL_n/L^+P^{I^\circ}$,

$$\begin{array}{ccc}
\bar{M}_I & \hookrightarrow & \bar{M}_{i_1} \times \cdots \times \bar{M}_{i_s} \\
\cup\uparrow & & \cup\uparrow \\
\downarrow & & \downarrow \\
\mathcal{F}^{I^\circ} & \hookrightarrow & \mathcal{F}^{\{i_1\}^\circ} \times \cdots \times \mathcal{F}^{\{i_s\}^\circ}.
\end{array} \tag{10.23}$$

Claim. *The restriction of $\mathcal{L}'(I^\circ) = \mathcal{L}(I^\circ)$ to $\bar{M}_I$ lifts to an invertible sheaf $\mathfrak{L}(I^\circ)$ on M_I with generic fiber isomorphic to the $|I|$th power of the ample generator $\mathcal{L}_0$ of the Picard group of the Grassmannian.*

Since by (10.14) we have $\mathcal{L}(I^\circ) = \mathcal{L}(\{i_1\}^\circ) \boxtimes \cdots \boxtimes \mathcal{L}(\{i_s\}^\circ) \mid \mathcal{F}^{I^\circ}$, it suffices to consider the case where $I = \{i\}$ consists of a single element. Then by (10.22) the claim will follow in general. Recall that r is defined by $\mu = (1^r, 0^{n-r})$. The embedding $\bar{M}_i \hookrightarrow \mathcal{F}^{\{i\}^\circ}$ is given by associating to the point of $\bar{M}_i$ represented by the submodule $\mathcal{E}_i$ of $\Lambda_i \otimes_W R = (\lambda_i \otimes_{k[\![t]\!]} k) \otimes_k R$ the lattice $L_{\mathcal{E}_i}$ inside $\lambda_{i\,R} := \lambda_i \otimes_{k[\![t]\!]} R[\![t]\!]$ obtained by taking the inverse image of $\mathcal{E}_i$ under the surjective map

$$\lambda_i \otimes_{k[\![t]\!]} R[\![t]\!] \rightarrow \lambda_i \otimes_{k[\![t]\!]} R. \tag{10.24}$$

Here $\lambda_i \subset k[\![t]\!]^n$ is the $k[\![t]\!]$-lattice generated by $t^{-1}e_1, \ldots, t^{-1}e_i, e_{i+1}, \ldots, e_n$. We obtain

$$\bar{M}_i(R) = \left\{ L \mid L \text{ lattice with } t \cdot \lambda_{i\,R} \subset L \subset \lambda_{i\,R} \text{ with } \operatorname{rk}(L/t \cdot \lambda_{i\,R}) = r \right\}.$$

Notice that $P^{\{i\}^\circ}$ is the stabilizer of the $k[\![t]\!]$-lattice $\lambda_i^r := (1, \ldots, 1, t, \ldots, t) \cdot \lambda_i$ where there are $n - r$ copies of t. The lattice λ_i^r has the same nth exterior power with L and so, locally on R, we can write $L = g \cdot \lambda_i^r$ with $g \in SL_n(R(\!(t)\!))$. The restriction of $\mathcal{L}(\{i\}^\circ)$ to this space of lattices has fiber $\bigwedge^r(L/t\lambda_{i\,R})$ at the point represented by L, comp. Remark 10.3 above. It follows that the line bundle $\mathfrak{L}(\{i\}^\circ)$ on M_i with fiber $\bigwedge^r \mathcal{E}_i$ at the point represented by the submodule $\mathcal{E}_i$ of $\Lambda_i \otimes R$ is the desired lifting.

From the claim we obtain, by Frobenius splitness and semi-continuity (for flat schemes),

$$\begin{aligned}
h^{(\mu)}(|I| \cdot a) &= \dim H^0\big(SL_n/P(\mu), \mathcal{L}_0^{\otimes |I| a}\big) \\
&= \dim H^0\big(M_I \otimes_W K_0, \mathfrak{L}(I^\circ)^{\otimes a}\big) \\
&= \dim H^0\big(M_I \otimes_W k, \mathfrak{L}(I^\circ)^{\otimes a}\big) \\
&= \dim H^0\big(\mathcal{A}_I(\mu)^\circ, \mathcal{L}(I^\circ)^{\otimes a}\big) \\
&= h_I^{(\mu)}(a),
\end{aligned}$$

which proves the conjecture in this case.

More generally, let $G = SL_n$ and let $\mu = \mu_1 + \cdots + \mu_m$ be a sum of minuscule coweights. (Note that, in fact, any coweight of PGL_n is a sum of minuscule coweights.) In this case let K' be a totally ramified extension of degree m of $K_0 = \operatorname{Frac}(W)$, and consider the local model M_I attached to the group $G = \operatorname{Res}_{K'/K_0}(GL_n)$, the coweight $(\mu_1, \ldots, \mu_m)$, and to $I \subset \{0, \ldots, n-1\}$, comp. [38]. Then again M_I is a flat W-scheme whose special fiber can be embedded in the partial flag variety $LGL_n/L^+ P^I$. In fact, the special fiber is reduced and can be identified with $\mathcal{A}_I(\mu)^\circ$. In this case the *geometric* generic fiber of M_I is the product of the Grassmannians $SL_n/P(\mu_1), \ldots, SL_n/P(\mu_m)$. We claim that the same reasoning as before yields the formula

$$h_I^{(\mu)}(a) = \prod_{j=1}^m \dim H^0\big(SL_n/P(\mu_j), \mathcal{L}_j^{\otimes |I| a}\big). \tag{10.25}$$

Here $\mathcal{L}_j$ denotes the ample generator of the Picard group of $SL_n/P(\mu_j)$.

To see this, we may again assume that $I = \{i\}$ consists of a single element. Then M_i is contained in the naive local model M_i^{naive}. Let E be the reflex field for $(G, (\mu_1, \ldots, \mu_m))$, cf. [38]. Then a point of M_i^{naive} with values in an $\mathcal{O}_E$-algebra R is given by

$$M_i^{\mathrm{naive}}(R) = \left\{ \mathcal{E} \subset \Lambda_i \otimes_W R \, \middle| \, \text{a } \mathcal{O}_{K'} \otimes_W R\text{-submodule which is locally on } \mathrm{Spec}(R) \right.$$
$$\left. \text{a direct summand as } R\text{-module with } \det(a|\mathcal{E}) = \prod_j \varphi_j(a)^{r_j}, \, \forall a \in \mathcal{O}_{K'} \right\}.$$

Here $\Lambda_i \subset K'^n$ denotes the $\mathcal{O}_{K'}$-lattice generated by $\pi'^{-1}e_1, \ldots, \pi'^{-1}e_i, e_{i+1}, \ldots, e_n$, where π' is uniformizer of $\mathcal{O}_{K'}$, and $\varphi_1, \ldots, \varphi_m$ denote the various K_0-embeddings of K' into a fixed algebraic closure $\bar{K}_0$ of K_0 and r_j is defined by $\mu_j = (1^{r_j}, 0^{n-r_j})$. The special fiber of M_i^{naive} is embedded into $\mathcal{F}^{\{i\}^\circ}$ by associating to a point represented by the submodule $\mathcal{E}$ of

$$\Lambda_i \otimes_W R = \Lambda_i \otimes_{\mathcal{O}_{K'}} (\mathcal{O}_{K'}/p\mathcal{O}_{K'}) \otimes_k R = (\lambda_i/t^m\lambda_i) \otimes_k R$$

the lattice $L_{\mathcal{E}}$ equal to the inverse image of $\mathcal{E}$ under the natural map $\lambda_{iR} \to \lambda_{iR}/t^m\lambda_{iR}$. Here again λ_{iR} denotes $\lambda_i \otimes_{k[[t]]} R[[t]]$. Hence $L_{\mathcal{E}}$ is squeezed as

$$t^m \lambda_{iR} \subset L_{\mathcal{E}} \subset \lambda_{iR}. \tag{10.26}$$

The value of $\mathcal{L}^{\{i\}^\circ}$ on $L_{\mathcal{E}}$ is $\bigwedge^{\max}(L_{\mathcal{E}}/t^m\lambda_{iR})$. Consider the line bundle $\mathfrak{L}^{\mathrm{naive}}$ on M^{naive} with fiber $\bigwedge^{\max}(\mathcal{E})$ at the point represented by the submodule $\mathcal{E}$. If $\mathcal{E}$ represents a point of the geometric generic fiber, then

$$\mathcal{E} = \bigoplus_{j=1}^m \mathcal{E}_{\varphi_j}, \tag{10.27}$$

where $K'^n \otimes_{K_0} \bar{K}_0 = \bigoplus(\bar{K}_0^n)_{\varphi_j}$ and $\mathcal{E}_{\varphi_j} := \mathcal{E} \cap (\bar{K}_0^n)_{\varphi_j}$ is a subspace of dimension r_j. Now

$$\bigwedge{}^{\max} \mathcal{E} = \bigwedge{}^{\max}(\mathcal{E}_{\varphi_1}) \otimes \cdots \otimes \bigwedge{}^{\max}(\mathcal{E}_{\varphi_m}), \tag{10.28}$$

and this is the fiber at $\mathcal{E} = (\mathcal{E}_{\varphi_1}, \ldots, \mathcal{E}_{\varphi_m})$ of the exterior tensor product of the line bundles $\mathcal{L}_1, \ldots, \mathcal{L}_m$ on the factor Grassmannians. It follows that the restriction $\mathfrak{L}$ of $\mathfrak{L}^{\mathrm{naive}}$ to the local model is a lifting of the restriction of $\mathcal{L}^{\{i\}^\circ}$ to the special fiber of M_i, and has geometric generic fiber equal to the exterior tensor product of $\mathcal{L}_1, \ldots, \mathcal{L}_m$. The formula (10.25) above now follows as in the minuscule case.

The case $G = Sp_{2n}$, and when μ is a multiple of the unique minuscule coweight is analogous. Instead of the local models for $\mathrm{Res}_{K'/K_0}(GL_n)$ one uses the local models for $\mathrm{Res}_{K'/K_0}(GSp_{2n})$, cf. [19] and [38]. □

Remark 10.8. In the previous proof we used the local models for the group GL_n over K_0, resp. for $\mathrm{Res}_{K'/K_0}(GL_n)$. In fact, the same proof works, if K_0 is replaced by the fraction field of any complete discrete valuation ring with residue field k. In particular, one can take this dvr of equal characteristic, e.g. $k[[t]]$.

11. Local models for the ramified unitary groups

In this section we will show how to embed the special fiber of the local model attached to a ramified unitary group into an affine flag variety. The structure results of the preceding sections have then interesting consequences for these local models (see also [39]).

We use a notation that is modeled on that of Section 4. Let F be a complete discretely valued field with ring of integers $\mathcal{O}_F$ and perfect residue field k of characteristic $\neq 2$. Let F'/F be a ramified quadratic extension and let $\pi' \in F'$ be a uniformizer with $\bar{\pi}' = -\pi'$. Let V be a F'-vector space of dimension $n \geqslant 2$ and let

$$\phi : V \times V \to F'$$

be a F'/F-hermitian form. We assume that ϕ is split. This means that there exists a basis $e_1, \ldots, e_n$ of V such that

$$\phi(e_i, e_{n-j+1}) = \delta_{ij}, \quad \forall i, j = 1, \ldots, n.$$

As in Section 4 we have two associated F-bilinear forms,

$$(x, y) = \mathrm{Tr}_{F'/F}\big(\phi(x, y)\big), \qquad \langle x, y \rangle = \mathrm{Tr}_{F'/F}\big(\pi'^{-1} \cdot \phi(x, y)\big).$$

For any $\mathcal{O}_{F'}$-lattice Λ in V we set, as in Section 4,

$$\hat{\Lambda} = \big\{v \in V \mid \phi(v, \Lambda) \subset \mathcal{O}_{F'}\big\} = \big\{v \in V \mid \langle v, \Lambda \rangle \subset \mathcal{O}_{F'}\big\}.$$

Similarly, as in Section 4, we set

$$\hat{\Lambda}^s = \big\{v \in V \mid (v, \Lambda) \subset \mathcal{O}_F\big\},$$

so that $\hat{\Lambda}^s = \pi'^{-1} \cdot \hat{\Lambda}$. For $i = 0, \ldots, n-1$, set

$$\Lambda_i = \mathrm{span}_{\mathcal{O}_{F'}}\big\{\pi'^{-1}e_1, \ldots, \pi'^{-1}e_i, e_{i+1}, \ldots, e_n\big\}.$$

Then we may complete these lattices to a periodic lattice chain by putting for $j \in \mathbf{Z}$ of the form $j = kn + i$ with $0 \leqslant i < n$,

$$\Lambda_j = \pi'^{-k} \cdot \Lambda_i.$$

This lattice chain is selfdual and periodic. More precisely, we have $\hat{\Lambda}_j = \Lambda_{-j}$, for all j, and $\pi' \cdot \Lambda_j = \Lambda_{j-n}$.

Now we fix non-negative integers r, s with $n = r + s$. We set $E = F'$ if $r \neq s$ and $E = F$ if $r = s$ (this is the reflex field of the local model we are about to define). We now define as follows a functor M^{naive} on the category of $\mathcal{O}_E$-schemes. A point of M^{naive} with values in an $\mathcal{O}_E$-scheme S is given by a $\mathcal{O}_{F'} \otimes_{\mathcal{O}_F} \mathcal{O}_S$ submodule

$$\mathcal{E}_j \subset \Lambda_j \otimes_{\mathcal{O}_F} \mathcal{O}_S \tag{11.1}$$

for each $j \in \mathbf{Z}$. The following conditions are imposed.

(a) As an $\mathcal{O}_S$-module, $\mathcal{E}_j$ is locally on S a direct summand of rank n.
(b) For each $j < j'$, there is a commutative diagram

$$
\begin{array}{ccc}
\Lambda_j \otimes_{\mathcal{O}_F} \mathcal{O}_S & \longrightarrow & \Lambda_{j'} \otimes_{\mathcal{O}_F} \mathcal{O}_S \\
\cup & & \cup \\
\mathcal{E}_j & \longrightarrow & \mathcal{E}_{j'}
\end{array}
$$

and for each j, the isomorphism $\pi' : \Lambda_j \to \Lambda_{j-n}$ induces an isomorphism of $\mathcal{E}_j$ with $\mathcal{E}_{j-n}$.
(c) We have $\mathcal{E}_{-j} = \mathcal{E}_j^\perp$ where $\mathcal{E}_j^\perp$ is the orthogonal complement of $\mathcal{E}_j$ under the perfect pairing

$$
(\Lambda_{-j} \otimes_{\mathcal{O}_F} \mathcal{O}_S) \times (\Lambda_j \otimes_{\mathcal{O}_F} \mathcal{O}_S) \to \mathcal{O}_S
$$

given by $\langle , \rangle \otimes_{\mathcal{O}_F} \mathcal{O}_S$.

Next note that $\mathcal{E}_j$ is an $\mathcal{O}_{F'} \otimes_{\mathcal{O}_F} \mathcal{O}_S$-module, hence $\mathcal{O}_{F'}$ and $\mathcal{O}_E$ act on it. We require further that

(d) For each j, the characteristic polynomial of the action of π' equals

$$
\det\big((T\,\mathrm{id} - \pi') \mid \mathcal{E}_j\big) = (T - \pi')^r \cdot (T + \pi')^s \subset \mathcal{O}_E[T].
$$

This concludes the definition of the functor M^{naive}, which is obviously representable by a projective scheme over $\mathrm{Spec}(\mathcal{O}_E)$. We call M^{naive} the *naive* local model associated to the group $GU(V, \phi)$, the signature type (r, s) and for the complete lattice chain Λ_j, $j \in \mathbf{Z}$. Note that the geometric generic fiber of M^{naive} is isomorphic to the Grassmannian of r-planes in n-space.

We can generalize this definition to incomplete selfdual periodic lattice chains as follows. Let I be a non-empty subset of the set of simple roots, which we identify as in Remark 10.3 with a subset of $\{0, \ldots, m\}$ in case $n = 2m + 1$ is odd, resp. with a subset of $\{0, \ldots, m - 2, m, m'\}$ in case $n = 2m \geqslant 4$ is even. We also recall that we associated to I a totally ordered subset $I^\sharp$ of $\{0, \ldots, m - 1, m\}$ resp. $\{0, \ldots, m - 1, m, m'\}$. Here $I = I^\sharp$ unless n is even and $\{m, m'\} \subset I$. In the last case, $I^\sharp$ is obtained from I by replacing m' by $m - 1$ and keeping all other elements. It is obvious that, conversely, $I^\sharp$ determines uniquely I. In order to obtain transition morphisms between naive local models in all cases, we will only consider subsets I which in case $n = 2m$ have the property that if $m' \in I$, then $\{m, m'\} \subset I$. (We refer to Remark 4.2C for an explanation of why we do not loose much information by only considering subsets I with this property.) Now I defines an incomplete lattice chain by including all Λ_j in the chain with j of the form $j = kn \pm i$ with $i \in I^\sharp$. Correspondingly, we obtain a functor $M_{I^\sharp}^{\mathrm{naive}}$ by only giving the submodules $\mathcal{E}_j$ for j in this subset of $\mathbf{Z}$. Therefore the case of complete lattice chains is obtained as $M^{\mathrm{naive}} = M_{\{0,\ldots,m\}}^{\mathrm{naive}}$. We have forgetful morphisms of projective schemes

$$
M^{\mathrm{naive}} \to M_{I^\sharp}^{\mathrm{naive}} \quad \text{and} \quad M_{I'^\sharp}^{\mathrm{naive}} \to M_{I^\sharp}^{\mathrm{naive}} \quad \text{for } I' \supset I. \tag{11.2}
$$

We now construct natural embeddings of the special fibers $\bar{M}_{I^\sharp}^{\mathrm{naive}} = M_{I^\sharp}^{\mathrm{naive}} \otimes_{\mathcal{O}_E} k$ into a variety of hermitian lattice chains of the type considered in Section 4. For this we fix identifications compatible with the actions of π', resp. u,

$$\Lambda_j \otimes_{\mathcal{O}_F} k = \lambda_j \otimes_{k[\![t]\!]} k$$

which sends the natural bases of each side to one another. We therefore also obtain a $k[\![u]\!]/(u^2)$-module chain isomorphism

$$\Lambda_\bullet \otimes_{\mathcal{O}_F} k \simeq \lambda_\bullet \otimes_{k[\![t]\!]} k, \tag{11.3}$$

which is in fact an isomorphism of polarized periodic module chains [43]. Let R be a k-algebra. For an R-valued point of $M_{I^\sharp}^{\mathrm{naive}}$ we have

$$\mathcal{E}_j \subset \Lambda_j \otimes_{\mathcal{O}_F} R = (\lambda_j \otimes_{k[\![t]\!]} k) \otimes_k R, \quad j = kn \pm i,\ i \in I^\sharp.$$

Let $L_j := u^{-1} L_{\mathcal{E}_j}$ where $L_{\mathcal{E}_j} \subset \lambda_j \otimes_{k[\![t]\!]} R[\![t]\!]$ is the inverse image of $\mathcal{E}_j$ under the canonical projection

$$\lambda_j \otimes_{k[\![t]\!]} R[\![t]\!] \to \lambda_j \otimes_{k[\![t]\!]} R. \tag{11.4}$$

If $I^\sharp = \{i_0 < i_1 < \cdots < i_k\}$, then

$$\cdots \subset L_{i_0} \subset L_{i_1} \subset \cdots \subset u^{-1} L_{i_0} = L_{i_0+n} \subset \cdots$$

is a periodic lattice chain in $R(\!(u)\!)^n$ which satisfies conditions (a) and (b) in Section 4.b, and which therefore defines an R-valued point of $\mathcal{F}_{I^\sharp}$. We obtain in this way a morphism

$$\iota_{I^\sharp} : M_{I^\sharp}^{\mathrm{naive}} \otimes_{\mathcal{O}_E} k \hookrightarrow \mathcal{F}_{I^\sharp}, \tag{11.5}$$
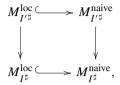
which is a closed immersion (of ind-schemes).

For $I' \supset I$, we have a commutative diagram

$$
\begin{array}{ccc}
M_{I'^\sharp}^{\mathrm{naive}} \otimes_{\mathcal{O}_E} k & \hookrightarrow & \mathcal{F}_{I'^\sharp} \\
\downarrow & & \downarrow \\
M_{I^\sharp}^{\mathrm{naive}} \otimes_{\mathcal{O}_E} k & \hookrightarrow & \mathcal{F}_{I^\sharp}.
\end{array}
$$

The horizontal morphisms are equivariant for the actions of the stabilizer groups of Section 4, $L^+ P_{I'}$ resp. $L^+ P_I$, in the sense of [38, Section 6].

Examples show [39] that $M_{I^\sharp}^{\mathrm{naive}}$ is not flat over $\mathcal{O}_E$ in general. We define $M_{I^\sharp}^{\mathrm{loc}}$ to be the flat closure of the generic fiber $M_{I^\sharp}^{\mathrm{naive}} \otimes_{\mathcal{O}_E} E$ in $M_{I^\sharp}^{\mathrm{naive}}$. Since the generic fibers $M_{I^\sharp}^{\mathrm{naive}} \otimes_{\mathcal{O}_E} E$ are

all identical, we obtain for $I' \supset I$ commutative diagrams of projective morphisms resp. closed embeddings,

$$
\begin{array}{ccc}
M^{\mathrm{loc}}_{I'^\sharp} & \hookrightarrow & M^{\mathrm{naive}}_{I'^\sharp} \\
\downarrow & & \downarrow \\
M^{\mathrm{loc}}_{I^\sharp} & \hookrightarrow & M^{\mathrm{naive}}_{I^\sharp},
\end{array}
$$

whose generic fibers are all the identity morphism. Note that, since the generic fiber of $M^{\mathrm{loc}}_{I^\sharp}$ is connected, so is its special fiber.

Remark 11.1. As we have seen in Section 4 the stabilizer group P_I is not always a parahoric subgroup of U_n, but that its intersection P'_I with SU_n always is a parahoric subgroup. Furthermore, the use of the lower index is "complementary" to its use elsewhere in the paper (e.g., if $I = \{0, \ldots, m\}$ resp. $I = \{0, \ldots, m-2, m, m'\}$, then P_I is an Iwahori subgroup of U_n). To avoid a notational conflict, we will denote for the rest of the paper by P^I the parahoric subgroup of SU_n corresponding to the subset I of the set of simple affine roots, as in Section 10 above. The corresponding parahoric subgroup of U_n will be denoted by $P^I(U_n)$.

Lemma 11.2. *Let μ be the geometric cocharacter of $U_n \otimes_K \bar{K} \simeq GL_n$ given by $\mu = (1^r, 0^s)$, Let $\mathcal{A}^I(\mu) = \bigcup_{w \in W^I \backslash \mathrm{Adm}^I(\mu)/W^I} S_w$, a closed reduced subset of $LU_n/L^+ P^I(U_n)$.[4] Then $\mathcal{A}^I(\mu)$ is contained in the image of $M^{\mathrm{loc}}_{I^\sharp} \otimes_{\mathcal{O}_E} k$ under $\iota_{I^\sharp}$.*

Proof. It suffices to prove this for $I = \{0, \ldots, m\}$, resp. $I = \{0, \ldots, m-2, m, m'\}$. In this case, it suffices to lift the points in an open subset of each maximal stratum S_w to the generic fiber of M^{loc}, where $w = w_0(\lambda)$ for some $w_0 \in W_0$. The proof of this statement can be found in [39, §3.d]. □

Theorem 11.3. *Denote by μ_r the image in the adjoint group of the cocharacter $(1^r, 0^s)$ of U_n. Assume the validity of Conjecture 10.5 for the pair $(G, \mu) = (SU_n, \mu_r)$ and for I°. Then the special fiber of $M^{\mathrm{loc}}_{I^\sharp}$ is reduced and in fact isomorphic to $\mathcal{A}^I(\mu)^\circ$. Its irreducible components are normal and with rational singularities.*

Proof. As already mentioned above, $\iota_{I^\sharp}$ is equivariant for the action of $L^+ P_I$. Hence the image of $\iota_{I^\sharp}$ is set-theoretically a union of Schubert varieties. After identifying the special fiber $\bar{M}^{\mathrm{loc}}_{I^\sharp}$ of $M^{\mathrm{loc}}_{I^\sharp}$ with its image under $\iota_{I^\sharp}$, we have by Lemma 11.2 a closed embedding $\mathcal{A}^I(\mu) \subset \bar{M}^{\mathrm{loc}}_{I^\sharp}$. We can identify $\mathcal{A}^I(\mu)$ with the reduced closed subset $\mathcal{A}^I(\mu)^\circ$ of the partial affine flag variety LSU_n/P^{I°, and similarly for $\bar{M}^{\mathrm{loc}}_{I^\sharp}$ (note that $\bar{M}^{\mathrm{loc}}_{I^\sharp}$ is connected and so by our results in Section 6, $(\bar{M}^{\mathrm{loc}}_{I^\sharp})_{\mathrm{red}}$ lies in a translate of LSU_n/P^{I° inside $\mathcal{F}_{I^\sharp}$, cf. Proposition 6.6). We consider the natural ample line bundle $\mathcal{L}'(I^\circ)$ on LSU_n/P^{I° and denote by the same symbol its restrictions to $\mathcal{A}^I(\mu)^\circ$ and $(\bar{M}^{\mathrm{loc}}_{I^\sharp})_{\mathrm{red}}$. We will show that the restriction of $\mathcal{L}'(I^\circ)$ to $(\bar{M}^{\mathrm{loc}}_{I^\sharp})_{\mathrm{red}}$ lifts to a line bundle $\mathfrak{L}'(I^\circ)$ on $M^{\mathrm{loc}}_{I^\sharp}$ with geometric generic fiber isomorphic to the $2|I|$-fold tensor power of the ample

---

[4] In Section 10, $\mathrm{Adm}^I(\mu)$ is only defined in the case of an adjoint group, but the general definition is the same.

generator $\mathcal{L}(\mu_r)$ of the Picard group of the Grassmannian of r-planes in n-space. Then, to prove that the inclusions $\mathcal{A}^I(\mu)^\circ \subset (\bar{M}_{I^\sharp}^{\mathrm{loc}})_{\mathrm{red}} \subset \bar{M}_{I^\sharp}^{\mathrm{loc}}$ are equalities, it suffices to prove that

$$\dim H^0\big(\mathcal{A}^I(\mu)^\circ, \mathcal{L}'(I^\circ)^{\otimes a}\big) = \dim H^0\big(\bar{M}_{I^\sharp}^{\mathrm{loc}}, \bar{\mathfrak{L}}'(I^\circ)^{\otimes a}\big) \tag{11.6}$$

for all $a \gg 0$, or also $\chi(\mathcal{A}^I(\mu)^\circ, \mathcal{L}'(I^\circ)^{\otimes a}) = \chi(\bar{M}_{I^\sharp}^{\mathrm{loc}}, \bar{\mathfrak{L}}'(I^\circ)^{\otimes a})$ (equality of the Hilbert polynomials). This follows by Conjecture 10.5 for I°, since by the flatness of $M_{I^\sharp}^{\mathrm{loc}}$ over $\mathcal{O}_E$ we have

$$\chi\big(\bar{M}_{I^\sharp}^{\mathrm{loc}}, \mathfrak{L}(I^\circ)^{\otimes a}\big) = \chi\big(M_{I^\sharp}^{\mathrm{loc}} \otimes_{\mathcal{O}_E} \bar{E}, \mathcal{L}(\mu_r)^{\otimes 2a|I|}\big) = h^{(\mu)}\big(|I| \cdot a\big).$$

It remains to construct the line bundle $\mathfrak{L}' = \mathfrak{L}'(I^\circ)$ and for this we may replace the local model by the naive local model. Furthermore, by the Kunneth type argument used in the proof of (10.7) we may assume that $I^\sharp$ consists of a single element $\{i\}$.[5] We claim in this case that the line bundle $\mathfrak{L}'$ with value $\bigwedge^{\mathrm{max}} \mathcal{E}$ in the point of $M_{\{i\}}^{\mathrm{naive}}$ represented by the subspace $\mathcal{E}$ of $\Lambda_i \otimes_{\mathcal{O}_F} \mathcal{O}_S$ has the required property. Let us calculate this line bundle on the special fiber and the geometric generic fiber.

If $\mathcal{E}$ represents a point of the special fiber which is mapped to the lattice $u^{-1} L_\mathcal{E}$ under $\iota_{\{i\}}$, then we have an inclusion of lattices as in (10.3),

$$u^N \lambda_i \subset u^{-1} L_\mathcal{E} \subset u^{-N} \lambda_i.$$

By Remark 10.3, the value of $\mathcal{L}'(\{i\}^\circ)$ at $u^{-1} L_\mathcal{E}$ is equal to $\bigwedge^{\mathrm{max}}(u^{-1} L_\mathcal{E}/u^N \lambda_i)$. Hence the restriction $\iota_{\{i\}}^* \mathcal{L}'(\{i\}^\circ)$ is indeed isomorphic to the restriction of $\mathfrak{L}'$ to the special fiber $\bar{M}_{\{i\}}^{\mathrm{naive}}$.

Let us now consider the pullback of $\mathfrak{L}'$ to the geometric generic fiber of $M_{\{i\}}^{\mathrm{naive}}$. Now

$$V \otimes_F F' = V_1 \oplus V_2,$$

according to the two F-automorphisms of F'. A point $\mathcal{E}$ of the geometric generic fiber corresponds to a pair of subspaces, $\mathcal{E}_1$ of V_1 and $\mathcal{E}_2$ of V_2, of dimension r and s respectively, which are the mutual annihilators of each other under the pairing $V_1 \times V_2 \to F'$ induced by the hermitian form ϕ on V. We therefore obtain a closed embedding

$$M_{\{i\}}^{\mathrm{naive}} \otimes_{\mathcal{O}_E} F' \hookrightarrow Gr(r, n)_{F'} \times Gr(s, n)_{F'}.$$

(Note that this is an example of (10.19).) Now

$$\bigwedge^{\mathrm{max}} \mathcal{E} = \bigwedge^{\mathrm{max}} \mathcal{E}_1 \otimes \bigwedge^{\mathrm{max}} \mathcal{E}_2.$$

Hence the pullback of $\mathfrak{L}'$ to the geometric generic fiber is isomorphic to the exterior tensor product of the two ample generators of the Picard groups of $Gr(r, n)$, resp. $Gr(s, n)$. By identifying $M_{\{i\}}^{\mathrm{naive}} \otimes_{\mathcal{O}_E} F'$ with the Grassmannian $Gr(r, n)_{F'}$, this pull back becomes the second tensor power of the ample generator of the Picard group of $Gr(r, n)$. This proves our claim.

---

[5] Or, that $n = 2m$ and that $I^\sharp = \{m - 1, m\}$. This case is handled in the same way.

This establishes the first part of the theorem. The second part follows because the inverse image of $\mathcal{A}^I(\mu)^\circ$ under the projection $LSU_n/L^+B \to LSU_n/L^+P^{I^\circ}$ is a union of Schubert varieties, which, by Theorem 8.4, are normal, simultaneously Frobenius split and with rational singularities. Hence the same holds also for the irreducible components of $\mathcal{A}^I(\mu)^\circ$. $\square$

Remark 11.4. For some index sets I, one can prove directly the statements in Theorem 11.3, and then deduce Conjecture 10.5 for I°. We mention the following fact.

Suppose that $I = \{m\}(= I^\sharp)$ if $n = 2m$ is even and that $I = \{0\}$ if n is odd. Then the local model M_I^{loc} has reduced irreducible special fiber which is normal, Frobenius split, and has only rational singularities. In fact, $\bar{M}_I^{\mathrm{loc}} = \mathcal{A}^I(\mu)$.

Indeed, in [39, §5], we show that, in these cases, the special fiber is irreducible and contains an open subset which is reduced and equal to an open subset of the Schubert cell corresponding to the image in $W^I \backslash \tilde{W}/W^I$ of the element λ in $\tilde{W}$ arising from μ. The underlying reduced scheme $(\bar{M}_I^{\mathrm{loc}})_{\mathrm{red}}$ is therefore the Schubert variety $\mathcal{A}^I(\mu)$ in $LU_n/L^+P^I(U_n)$. We deduce from Theorem 8.4 that $(\bar{M}_I^{\mathrm{loc}})_{\mathrm{red}}$ is normal, Frobenius split and has only rational singularities. Now an application of Hironaka's lemma [21, IV.5.12.8] to the flat $\mathcal{O}_E$-scheme M_I^{loc} shows that $(\bar{M}_I^{\mathrm{loc}})_{\mathrm{red}} = \bar{M}_I^{\mathrm{loc}} = \mathcal{A}^I(\mu)$.

We note that, even though for I as above the parahoric subgroup P_I' is a special maximal parahoric subgroup of SU_n, the local model M_I^{loc} is rarely smooth. This is in contrast to what happens for the groups $G = GSp_{2n}$ and $G = GL_n$, cf. [18,19]. More precisely, for such I, M_I^{loc} is smooth only in the following cases: $r = 0$, or $r = n$, or $n = 2m$ is even and $r = 1$ or $r = n - 1$.

Acknowledgments

References

[1] S. Anantharaman, Schémas en groupes, espaces homogènes et espaces algébriques sur une base de dimension 1, Bull. Soc. Math. France Mem. 33 (1973) 5–79.

[2] A. Beauville, Y. Laszlo, Conformal blocks and generalized theta functions, Comm. Math. Phys. 164 (1994) 385–419.

[3] A. Beauville, Y. Laszlo, C. Sorger, The Picard group of the moduli of G-bundles on a curve, Compos. Math. 112 (1998) 183–216.

[4] A. Beilinson, V. Drinfeld, Quantization of Hitchin's integrable system and Hecke eigensheaves, preprint, available under http://www.ma.utexas.edu/benzvi/.

[5] M. Borovoi, Abelian Galois cohomology of reductive groups (English summary), Mem. Amer. Math. Soc. 132 (626) (1998).

[6] S. Bosch, W. Lütkebohmert, M. Raynaud, Néron Models, Ergeb. Math. Grenzgeb. (3), vol. 21, Springer-Verlag, Berlin, 1990.

[7] F. Bruhat, J. Tits, Groupes réductifs sur un corps local, Inst. Hautes Études Sci. Publ. Math. 41 (1972) 5–251.

[8] F. Bruhat, J. Tits, Groupes réductifs sur un corps local. II. Schémas en groupes. Existence d'une donné radicielle valuée, Inst. Hautes Études Sci. Publ. Math. 60 (1984) 197–376.

 [9] C.-L. Chai, Néron models for semiabelian varieties: Congruence and change of base field, Asian J. Math. 4 (4) (2000) 715–736.
[10] M. Demazure, P. Gabriel, Groupes algébriques. Tome I: Géométrie algébrique, généralités, groupes commutatifs. Avec un appendice Corps de classes local par Michiel Hazewinkel, Masson & Cie, Éditeur, Paris, 1970, North-Holland Publishing Co., Amsterdam.
[11] M. Demazure, A. Grothendieck, Structure des groupes réductifs, Lecture Notes in Math., vol. 153, Springer-Verlag, Berlin, 1970.
[12] B. Edixhoven, Néron models and tame ramification, Compos. Math. 81 (1992) 291–306.
[13] G. Faltings, Algebraic loop groups and moduli spaces of bundles, J. Eur. Math. Soc. (JEMS) 5 (2003) 41–68.
[14] G. Faltings, Course at Bonn University, winter semester 2004.
[15] D. Gaitsgory, Construction of central elements in the affine Hecke algebra via nearby cycles, Invent. Math. 144 (2001) 253–280.
[16] W.-T. Gan, J.-K. Yu, Schémas en groupes et immeubles des groupes exceptionnels sur un corps local. I. Le groupe G_2, Bull. Soc. Math. France 131 (3) (2003) 307–358.
[17] H. Garland, The arithmetic theory of loop algebras, J. Algebra 53 (2) (1978) 480–551.
[18] U. Görtz, On the flatness of models of certain Shimura varieties of PEL-type, Math. Ann. 321 (2001) 689–727.
[19] U. Görtz, On the flatness of local models for the symplectic case, Adv. Math. 176 (2003) 89–115.
[20] M. Greenberg, Schemata over local rings. II, Ann. of Math. (2) 78 (1963) 256–266.
[21] A. Grothendieck, Elements de geometrie algebrique. IV. Etude locale des schemas et des morphismes de schémas. II, Inst. Hautes Études Sci. Publ. Math. 24 (1965), 231 pp. (in French).
[22] A. Grothendieck, Le groupe de Brauer. III. Exemples et compléments. Dix Exposés sur la Cohomologie des Schémas, North-Holland, Amsterdam, 1968, Masson, Paris, pp. 88–188.
[23] T. Haines, M. Rapoport, On parahoric subgroups (appendix to this paper).
[24] V. Kac, Infinite-dimensional Lie Algebras, third ed., Cambridge University Press, Cambridge, 1990.
[25] C. Kaiser, Ein getwistetes fundamentales Lemma für die GSp_4, Dissertation, Rheinische Friedrich-Wilhelms-Universität Bonn, Bonn, 1997, Bonner Math. Schriften, vol. 303, Universität Bonn, Mathematisches Institut, Bonn, 1997.
[26] B. Kostant, Groups over **Z**, in: Algebraic Groups and Discontinuous Subgroups, Boulder, CO, 1965, in: Proc. Sympos. Pure Math., vol. 9, Amer. Math. Soc., Providence, RI, 1966, pp. 90–98.
[27] R. Kottwitz, Isocrystals with additional structure. II, Compos. Math. 109 (1997) 255–339.
[28] S. Kumar, Demazure character formula in arbitrary Kac–Moody setting, Invent. Math. 89 (1987) 395–423.
[29] S. Kumar, Kac–Moody Groups, Their Flag Varieties and Representation Theory, Progr. Math., vol. 204, Birkhäuser Boston Inc., Boston, MA, 2002.
[30] Y. Laszlo, C. Sorger, The line bundles on the moduli of parabolic G-bundles over curves and their sections, Ann. Sci. École Norm. Sup. (4) 30 (1997) 499–525.
[31] P. Littelmann, Contracting modules and standard monomial theory for symmetrizable Kac–Moody algebras, J. Amer. Math. Soc. 11 (1998) 551–567.
[32] G. Lusztig, Study of a **Z**-form of the coordinate ring of a reductive group, arXiv: 0709.1286.
[33] O. Mathieu, Formules de caractères pour les algèbres de Kac–Moody générales, Astérisque 159–160 (1988).
[34] O. Mathieu, Construction d'un groupe de Kac–Moody et applications, Compos. Math. 69 (1) (1989) 37–60.
[35] V. Mehta, W. van der Kallen, On a Grauert–Riemenschneider vanishing theorem for Frobenius split varieties in characteristic p, Invent. Math. 108 (1992) 11–13.
[36] J. Milne, K.-y. Shih, Conjugates of Shimura Varieties. Hodge Cycles, Motives, and Shimura Varieties, Lecture Notes in Math., vol. 900, Springer-Verlag, Berlin, 1982, pp. 280–356.
[37] G. Pappas, M. Rapoport, Local models in the ramified case I. The EL-case, J. Algebraic Geom. 12 (2003) 107–145.
[38] G. Pappas, M. Rapoport, Local models in the ramified case II. Splitting models, Duke Math. J. 127 (2005) 193–250.
[39] G. Pappas, M. Rapoport, Local models in the ramified case III. Unitary groups, preprint, math.AG/0702286.
[40] G. Prasad, J.-K. Yu, On finite group actions on reductive groups and buildings, Invent. Math. 147 (2002) 545–560.
[41] A. Pressley, G. Segal, Loop Groups, Oxford Math. Monogr., Oxford Science Publications, The Clarendon Press, Oxford University Press, New York, 1986.
[42] M. Rapoport, A guide to the reduction modulo p of Shimura varieties, Astérisque 298 (2005) 271–318.
[43] M. Rapoport, Th. Zink, Period Spaces for p-Divisible Groups, Ann. of Math. Stud., vol. 141, Princeton University Press, 1996.
[44] J.-P. Serre, Groupes de Grothendieck des schémas en groupes réductifs déployés, Inst. Hautes Études Sci. Publ. Math. 34 (1968) 37–52.
[45] J.-P. Serre, Galois Cohomology, Corrected reprint of the 1997 English edition, Springer Monogr. Math., Springer-Verlag, Berlin, 2002.

[46] R. Steinberg, Regular elements of semi-simple algebraic groups, Inst. Hautes Études Sci. Publ. Math. 25 (1965) 49–80.

[47] R. Steinberg, Generators, relations and coverings of algebraic groups, J. Algebra 71 (1981) 527–543.

[48] J. Tits, Reductive groups over local fields. Automorphic forms, representations and *L*-functions, in: Proc. Sympos. Pure Math., Part 1, Oregon State Univ., Corvallis, OR, 1977, in: Proc. Sympos. Pure Math., vol. XXXIII, Amer. Math. Soc., Providence, RI, 1979, pp. 29–69.

[49] J. Tits, Groups and group functors attached to Kac–Moody data, in: Workshop Bonn 1984, Bonn, 1984, in: Lecture Notes in Math., vol. 1111, Springer-Verlag, Berlin, 1985, pp. 193–223.

[50] J. Tits, Groupes associès aux algébres de Kac–Moody, in: Séminaire Bourbaki, vol. 1988/89, Astérisque 177–178 (1989) 7–31, Exp. No. 700.

[51] J.-K. Yu, Smooth models associated to concave functions in Bruhat–Tits theory, preprint, 2002.

Appendix: On parahoric subgroups

T. Haines [a,1], M. Rapoport [b]

[a] *Department of Mathematics, University of Maryland, College Park, MD 20742, USA*
[b] *Mathematisches Institut der Universität Bonn, Beringstrasse 1, 53115 Bonn, Germany*

Abstract

We give the proofs of some simple facts on parahoric subgroups and on Iwahori Weyl groups used in [T. Haines, The base change fundamental lemma for central elements in parahoric Hecke algebras, preprint, 2008; G. Pappas, M. Rapoport, Twisted loop groups and their affine flag varieties, Adv. Math. 219 (2008) 118–198; M. Rapoport, A guide to the reduction modulo *p* of Shimura varieties, Astérisque 298 (2005) 271–318].

© 2008 Published by Elsevier Inc.

Let G be a connected reductive group over a strictly henselian discretely valued field L. Kottwitz defines in [6] a functorial surjective homomorphism

$$\kappa_G : G(L) \longrightarrow X^*\big(\hat{Z}(G)^I\big). \qquad (1)$$

Here $I = Gal(\bar{L}/L)$ denotes the absolute Galois group of L.

Let $\mathcal{B}$ be the Bruhat–Tits building of the adjoint group of G. Then $G(L)$ operates on $\mathcal{B}$.

Definition 1. A *parahoric subgroup* of $G(L)$ is a subgroup of the form

$$K_F = \text{Fix}(F) \cap \text{Ker}\,\kappa_G,$$

for a facet F of $\mathcal{B}$. An *Iwahori subgroup* of $G(L)$ is the parahoric subgroup associated to an alcove of $\mathcal{B}$.

E-mail addresses: tjh@math.umd.edu (T. Haines), rapoport@math.uni-bonn.de (M. Rapoport).

Remark 2. If $F' = gF$, then

$$K_{F'} = g K_F g^{-1}.$$

In particular, since $G(L)$ acts transitively on the set of all alcoves, all Iwahori subgroups are conjugate.

We shall see presently that this definition coincides with the one of Bruhat and Tits [5, 5.2.6]. They associate to a facet F in $\mathcal{B}$ a smooth group scheme $\mathcal{G}_F$ over Spec O_L with generic fiber G and the open subgroup $\mathcal{G}_F^\circ$ of it with the same generic fiber and with connected special fiber, and define the parahoric subgroup attached to F as $P_F^\circ = \mathcal{G}_F^\circ(O_L)$; from their definition it follows that $P_F^\circ \subseteq \mathrm{Fix}(F)$. We denote by $G(L)_1$ the kernel of κ_G.

For a facet F, let as above P_F° be the associated parahoric subgroup in the BT-sense. Our first goal is to prove the following proposition.

Proposition 3. *For any facet F in $\mathcal{B}$, we have $P_F^\circ = K_F$.*

Proof. (a) If $G = T$ is a torus, then $T(L)_1$ is the unique Iwahori subgroup, cf. [8, Notes at the end, no. 1]. Hence the result follows in this case.

(b) If G is semisimple and simply connected, then $G(L)_1 = G(L)$. The assertion therefore follows from [5, 4.6.32], which also proves that $\mathcal{G}_F = \mathcal{G}_F^\circ$, where $\mathcal{G}_F$ is the group scheme defined in [5, 4.6.26].

(c) Let G be such that G_{der} is simply connected. Let $D = G/G_{\mathrm{der}}$. We claim that there is a commutative diagram with exact rows

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & \mathcal{G}_{\mathrm{der},F}(O_L) & \longrightarrow & \mathcal{G}_F^\circ(O_L) & \longrightarrow & \mathcal{D}^\circ(O_L) & \longrightarrow & 1 \\
 & & \| & & \downarrow & & \| & & \\
1 & \longrightarrow & K_{\mathrm{der}} & \longrightarrow & K_F & \longrightarrow & D(L)_1. & &
\end{array}
$$

The bottom row involves parahoric subgroups associated in our sense to the facet F, and the top row involves those defined in [5] and comes by restricting the exact sequence

$$1 \longrightarrow G_{\mathrm{der}}(L) \longrightarrow G(L) \longrightarrow D(L) \longrightarrow 1$$

to O_L-points of the appropriate Bruhat–Tits group schemes (and more precisely in the case of $\mathcal{D}^\circ(O_L)$, the group $\mathcal{D}$ is the lft Neron model of D_L; cf. [8, Notes at the end, no. 1]). The vertical equalities result from (a) and (b) above. The vertical arrow is an inclusion (but we need to justify its existence, see below) making the entire diagram commutative.

Let us first *construct* the top row. The key point is to show that the map $G(L) \longrightarrow D(L)$ restricts canonically to a surjective map $\mathcal{G}_F^\circ(O_L) \longrightarrow \mathcal{D}^\circ(O_L)$. We shall derive this from the corresponding statement involving the lft Neron models of D and of a maximal torus $T \subset G$. Let S denote a fixed maximal L-split torus in G, and define $T = \mathrm{Cent}_G(S)$, a maximal torus in G since by Steinberg's theorem G is quasisplit. Also define $T_{\mathrm{der}} := G_{\mathrm{der}} \cap T = \mathrm{Cent}_{G_{\mathrm{der}}}(S_{\mathrm{der}})$, where $S_{\mathrm{der}} = (S \cap G_{\mathrm{der}})^\circ$, a maximal L-split torus in G_{der}.

Consider the lft Neron models $\mathcal{T}$, $\mathcal{T}_{\mathrm{der}}$, and $\mathcal{D}$ associated to T, T_{der}, and D, cf. [3]. The map $\mathcal{T}(O_L) \longrightarrow \mathcal{D}(O_L)$ is surjective, and this implies that $\mathcal{T}^\circ(O_L) \longrightarrow \mathcal{D}^\circ(O_L)$ is also surjective, cf. [3, 9.6, Lemma 2]. By [5, 4.6.3 and 4.6.7] we have decompositions $\mathcal{G}_F^\circ(O_L) = \mathcal{T}^\circ(O_L)\mathfrak{U}_F(O_L)$ and $\mathcal{G}_{\mathrm{der},F}(O_L) = \mathcal{T}_{\mathrm{der}}(O_L)\mathfrak{U}_F(O_L)$, where $\mathfrak{U}_F$ denotes the group generated by certain root-group O_L-schemes $\mathfrak{U}_{F,a}$, which depend on F; these all fix F. These remarks show that $G(L) \longrightarrow D(L)$ restricts to a map $\mathcal{G}_F^\circ(O_L) \longrightarrow \mathcal{D}^\circ(O_L)$, and also that the latter map is *surjective*.

The kernel of the latter map is contained in the subgroup of $G_{\mathrm{der}}(L)$ which fixes F, hence is precisely $K_{\mathrm{der}} = \mathcal{G}_{\mathrm{der},F}(O_L)$. This completes our discussion of the first row of the diagram above.

Since G_{der} is simply connected, $G(L)_1$ is the inverse image of $D(L)_1$ under the natural projection $G \longrightarrow D$; hence $\mathcal{G}_F^\circ(O_L)$ belongs to $G(L)_1$ hence is contained in K_F; this yields the inclusion which fits in the above diagram and makes it commutative. A diagram chase then shows that the inclusion $\mathcal{G}_F^\circ(O_L) \longrightarrow K_F$ is a bijection.

(d) To treat the general case choose a z-extension $G' \longrightarrow G$, with kernel Z where the derived group of G' is simply connected. Since $X_*(Z)_I = X^*(\hat{Z}^I)$ is torsion-free, the induced sequence

$$0 \longrightarrow X^*(\hat{Z}^I) \longrightarrow X^*(\hat{Z}(G')^I) \longrightarrow X^*(\hat{Z}(G)^I) \longrightarrow 0 \tag{2}$$

is exact. We therefore obtain an exact sequence

$$1 \longrightarrow Z(L)_1 \longrightarrow G'(L)_1 \longrightarrow G(L)_1 \longrightarrow 1.$$

It follows that we also have an exact sequence

$$1 \longrightarrow Z(L)_1 \longrightarrow K_F' \longrightarrow K_F \longrightarrow 1.$$

As in (c) one shows that $G'(L) \to G(L)$ maps $\mathcal{G}_F'^\circ(O_L) = P_F'^\circ$ onto $\mathcal{G}_F^\circ(O_L) = P_F^\circ$; in particular, since $P_F'^\circ = K_F'$ by (c), we deduce $P_F^\circ \subset G(L)_1$ and thus $P_F^\circ \subseteq K_F$. The equality $P_F^\circ = K_F$ then follows from $P_F'^\circ = K_F'$. $\square$

Remark 4. Proposition 3 makes sense and still holds true if F is replaced with any bounded non-empty subset $\Omega \subset \mathcal{B}$ which is contained in an apartment. Indeed, one can follow the same proof, making only the following adjustment in the proof of (b) where $G = G_{\mathrm{sc}}$: although [5, 4.6.32] is restricted to Ω contained in a facet, the equality $\mathcal{G}_\Omega^\circ(O_L) = \mathcal{G}_\Omega(O_L)$ (= fixer of Ω) which we need requires only the connectedness of the group scheme $\mathcal{T}$ occurring in [5, 4.6.32], which holds here by [5, 4.6.1].

Let F be a facet contained in the apartment associated to the maximal split torus S. Let T be the centralizer of S, a torus since by Steinberg's theorem G is quasisplit. Let N be the normalizer of S. Let K_F be the parahoric subgroup associated to F. Let

$$\kappa_T : T(L) \longrightarrow X^*(\hat{T}^I) = X_*(T)_I$$

be the Kottwitz homomorphism associated to T, and $T(L)_1$ its kernel.

Lemma 5.

$$T(L) \cap K_F = T(L)_1.$$

Proof. By functoriality of the Kottwitz homomorphisms, we see $T(L)_1 \subset G(L)_1$. The elements of $\operatorname{Ker} \kappa_T$ act trivially on the apartment associated to S, hence the inclusion "$\supset$" is obvious. The converse follows from the fact that $T(L)_1$ equals $\mathcal{T}^\circ(O_L)$ where $\mathcal{T}^\circ$ is the identity component of the lft Neron model of T (cf. [8, Notes at the end, no. 1]) and the fact that $\mathcal{T}^\circ(O_L)$ is the centralizer of S in K_F, comp. [5, 4.6.4] or [7, 6.3]. $\quad\square$

Let K_0 be the Iwahori subgroup associated to an alcove contained in the apartment associated to S.

Lemma 6.

$$N(L) \cap K_0 = T(L)_1.$$

Proof. An element of the LHS acts trivially on the apartment associated to S, hence is contained in $T(L) \cap K_0 = T(L)_1$ by the previous lemma. $\quad\square$

Definition 7. Let $S \subset T \subset N$ be as before (maximal split torus, contained in a maximal torus, contained in its normalizer). The *Iwahori–Weyl group* associated to S is

$$\tilde{W} = N(L)/T(L)_1.$$

Let $W_0 = N(L)/T(L)$ (relative vector Weyl group). We obtain an obvious exact sequence

$$0 \longrightarrow X_*(T)_I \longrightarrow \tilde{W} \longrightarrow W_0 \longrightarrow 0. \tag{3}$$

Proposition 8. *Let K_0 be the Iwahori subgroup associated to an alcove contained in the apartment associated to the maximal split torus S. Then*

$$G(L) = K_0.N(L).K_0$$

and the map $K_0 n K_0 \mapsto n \in \tilde{W}$ induces a bijection

$$K_0 \backslash G(L)/K_0 \simeq \tilde{W}.$$

More generally, let K resp. K' be parahoric subgroups associated to facets F resp. F' contained in the apartment associated to S. Let

$$\tilde{W}^K = \big(N(L) \cap K\big)/T(L)_1,$$
$$\tilde{W}^{K'} = \big(N(L) \cap K'\big)/T(L)_1.$$

Then

$$K \backslash G(L)/K' \simeq \tilde{W}^K \backslash \tilde{W}/\tilde{W}^{K'}.$$

Proof. To F there are associated the subgroups

$$U_F \subset P_F \subset G(L),$$

cf. [4, 7.1] (where P_F is denoted $\hat{P}_F$) or [7, 8.8]. By [4, 7.1.8] or [7, 8.10] we have

$$P_F \subset U_F.N(L).$$

Since U_F is contained in the parahoric subgroup $P_F^\circ \subset P_F$, the equality $G(L) = P_F.N(L).P_{F'}$ ([4, 7.4.15] resp. [7, 8.17]) implies the first assertion

$$G(L) = K_F.N(L).K_{F'}.$$

To see the second assertion, we follow closely the proof of [4, 7.3.4]. Assume that $n, n' \in N(L)$ with $n' \in P_F^\circ n P_{F'}^\circ$, i.e.

$$n'.n^{-1} \in P_F^\circ \cdot P_{nF'}^\circ.$$

We choose points $f \in F$ and $f' \in n \cdot F'$ with

$$P_F^\circ = P_f^\circ, \qquad P_{n.F'}^\circ = P_{f'}^\circ.$$

We choose an order on the root system of S, with associated vector chamber D such that

$$f \in f' + D.$$

Then $U_f^- \cdot U_{f'}^- \subset U_{f'}^-$, in the notation of [7, 8.8]. Hence, by [5, 4.6.7] (comp. [7, 8.10]),

$$
\begin{aligned}
P_f^\circ.P_{f'}^\circ &= \left[\left(N(L) \cap P_f^\circ \right) \cdot U_f^+.U_f^- \right] \cdot \left[U_{f'}^- U_{f'}^+ \cdot \left(N(L) \cap P_{f'}^\circ \right) \right] \\
&= \left(N(L) \cap P_f^\circ \right) \cdot U_f^+.P_{f'}^\circ \\
&= \left(N(L) \cap P_f^\circ \right) \cdot U_f^+.U_{f'}^+.U_{f'}^- \left(N(L) \cap P_{f'}^\circ \right).
\end{aligned}
$$

It follows that there exist $m_1 \in N(L) \cap P_f^\circ$ and $m_2 \in N(L) \cap P_{f'}^\circ$ such that

$$m_1 \cdot n'n^{-1}m_2 \in U^+(L).U^-(L).$$

From the usual Bruhat decomposition it follows that

$$
\begin{aligned}
m_1 \cdot n'n^{-1}m_2 &= 1, \quad \text{i.e.,} \\
m_1 n' \left(n^{-1} m_2 n \right) &= n.
\end{aligned}
$$

Since $m_1 \in N(L) \cap K_F$ and $n^{-1}m_2n \in N(L) \cap K_{F'}$, the last equality means that

$$n \equiv n' \quad \text{in } \tilde{W}^K \backslash \tilde{W} / \tilde{W}^{K'}. \qquad \square$$

Remark 9 *(Descent).* Let σ denote an automorphism of L having fixed field $L^\natural$ such that L is the strict henselization of $L^\natural$. Let us assume G is defined over $L^\natural$; we may assume S, and hence T and N, are likewise defined over $L^\natural$ [5, 5.1.12]. Assume F and F' are σ-invariant facets in $\mathcal{B}$. Write $K(L^\natural) = K^\sigma$ and $K'(L^\natural) = K'^\sigma$. We have a canonical bijection

$$K(L^\natural)\backslash G(L^\natural)/K'(L^\natural) \xrightarrow{\sim} \left[K\backslash G(L)/K'\right]^\sigma. \tag{4}$$

To prove that the map is surjective, we use the vanishing of $H^1(\langle\sigma\rangle, K)$ and $H^1(\langle\sigma\rangle, K')$. To prove it is injective, we use the vanishing of $H^1(\langle\sigma\rangle, K \cap gK'g^{-1})$ for all $g \in G(L^\natural)$. The vanishing statements hold because K, K', and $K \cap gK'g^{-1}$ are the O_L-points of group schemes over $O_{L^\natural}$ with connected fibers, by virtue of Proposition 3 and Remark 4.

Next, note that, using a similar cohomology vanishing argument,

$$\tilde{W}^\sigma = N(L^\natural)/T(L^\natural) \cap T(L)_1 =: \tilde{W}(L^\natural),$$
$$\left(\tilde{W}^K\right)^\sigma = N(L^\natural) \cap K/T(L^\natural) \cap T(L)_1 =: \tilde{W}^K(L^\natural).$$

Now suppose that F and F' are σ-invariant facets contained in the closure of a σ-invariant alcove in the apartment of $\mathcal{B}$ associated to S. Then the canonical map

$$\tilde{W}^K(L^\natural)\backslash\tilde{W}(L^\natural)/\tilde{W}^{K'}(L^\natural) \to \left[\tilde{W}^K\backslash\tilde{W}/\tilde{W}^{K'}\right]^\sigma \tag{5}$$

is bijective. Indeed, note first that $\tilde{W}^K$ and $\tilde{W}^{K'}$ are parabolic subgroups of the quasi-Coxeter group $\tilde{W}$ (see Lemma 14 below), and that any element $x \in \tilde{W}$ has a *unique* expression in the form wx_0w' where $w \in \tilde{W}^K$, $w' \in \tilde{W}^{K'}$, such that x_0 is the unique minimal-length element in $\tilde{W}^K x_0 \tilde{W}^{K'}$, and wx_0 is the unique minimal-length element in $wx_0\tilde{W}^{K'}$. Secondly, note that σ preserves these parabolic subgroups as well as the quasi-Coxeter structure and therefore the Bruhat-order on $\tilde{W}$. These remarks imply the bijectivity just claimed. Putting (4) and (5) together, we obtain a bijection

$$K(L^\natural)\backslash G(L^\natural)/K'(L^\natural) \xrightarrow{\sim} \tilde{W}^K(L^\natural)\backslash\tilde{W}(L^\natural)/\tilde{W}^{K'}(L^\natural).$$

Remark 10. We now compare the Iwahori Weyl group with a variant of it in [9].

In [9, p. 32], the following group is introduced. Let $T(L)_b$ be the maximal bounded subgroup of $T(L)$. The *affine Weyl group* in the sense of [9, p. 32], associated to S is the quotient $\tilde{W}' = N(L)/T(L)_b$. We obtain a morphism of exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X_*(T)_I & \longrightarrow & \tilde{W} & \longrightarrow & W_0 & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \Lambda & \longrightarrow & \tilde{W}' & \longrightarrow & W_0 & \longrightarrow & 0.
\end{array}$$

Here $\Lambda = X_*(T)_I/torsion$. Of course,

$$\Lambda = \mathrm{Hom}\left(X^*(T)^I, \mathbf{Z}\right) = T(L)/T(L)_b$$

(see [6, 7.2]). It follows that the natural homomorphism from $\tilde{W}$ to $\tilde{W}'$ is surjective, with finite kernel isomorphic to $T(L)_b/T(L)_1$.

The affine Weyl group in the sense of [9] also appears in a kind of Bruhat decomposition, as follows. Let

$$v_G : G(L) \longrightarrow X^*\big(\hat{Z}(G)^I\big)/torsion$$

be derived from κ_G in the obvious way. Let C be an alcove in the apartment corresponding to S. We consider the subgroup

$$\tilde{K}_0 = \mathrm{Fix}(C) \cap \mathrm{Ker}\, v_G. \tag{6}$$

Then K_0, the Iwahori subgroup corresponding to C, is a normal subgroup of finite index in $\tilde{K}_0$. In fact $\mathrm{Ker}\, v_G$ is the group denoted G^1 in [5, 4.2.16], and so by [5, 4.6.28], $\tilde{K}_0$ is the group denoted there $\hat{P}_C^1$, in other words the fixer of C in G^1. Using [5, 4.6.3, 4.6.7], we have $K_0 = T(L)_1 \mathfrak{U}_C(O_L)$ and $\tilde{K}_0 = T(L)_b \mathfrak{U}_C(O_L)$, where $\mathfrak{U}_C$ is the group generated by certain root-group O_L-schemes $\mathfrak{U}_{C,a}$ which fix C. It follows that $\tilde{K}_0 = T(L)_b K_0$, and

$$T(L)_b/T(L)_1 \simeq \tilde{K}_0/K_0.$$

In [9, 3.3.1], the affine Weyl group $\tilde{W}'$ appears in the Bruhat decomposition with respect to $\tilde{K}_0$

$$\tilde{K}_0 \backslash G(L)/\tilde{K}_0 \simeq \tilde{W}'. \tag{7}$$

The group $T(L)_b/T(L_1)$ acts freely on both sides of the Bruhat decomposition

$$K_0 \backslash G(L)/K_0 = \tilde{W}$$

of Proposition 8, and we obtain (7) by taking the quotients.

Remark 11. In [5] the building of $G(L)$ (sometimes called the enlarged building $\mathcal{B}^1$) is also considered; it carries an action of $G(L)$. There is an isomorphism $\mathcal{B}^1 = \mathcal{B} \times V_G$, where $V_G :=$ $X_*(Z(G))_I \otimes \mathbb{R}$. Given a bounded subset Ω in the apartment of $\mathcal{B}$ associated to S, there is a smooth group scheme $\hat{\mathcal{G}}_\Omega$ whose generic fiber is G and whose O_L-points $\hat{\mathcal{G}}(O_L)$ is the subgroup of $G(L)$ fixing $\Omega \times V_G$, in other words the subgroup $\hat{P}_\Omega^1$ in G^1 fixing Ω. We have $(\hat{\mathcal{G}}_\Omega)^\circ = \mathcal{G}_\Omega^\circ$. Thus, the above discussion and Proposition 3 show that

$$(\hat{\mathcal{G}}_\Omega)^\circ(O_L) = \hat{\mathcal{G}}_\Omega(O_L) \cap \mathrm{ker}\,\kappa_G.$$

Proposition 12. *Let K be associated to F as above. Let $\mathcal{G}_F^\circ$ be the O_L-form with connected fibres of G associated to F, and let $\bar{\mathcal{G}}_F^\circ$ be its special fiber. Then $\tilde{W}^K$ is isomorphic to the Weyl group of $\bar{\mathcal{G}}_F^\circ$.*

Proof. Let $\bar{\mathcal{S}}^\circ \subset \bar{T}^\circ \subset \bar{\mathcal{G}}_F^\circ$ be the tori associated to S resp. T, cf. [5, 4.6.4] or [7, 6.3]. The natural projection $\bar{\mathcal{G}}_F^\circ \longrightarrow \bar{\mathcal{G}}_{F,\mathrm{red}}^\circ = \bar{\mathcal{G}}_F^\circ/R_u(\bar{\mathcal{G}}_F^\circ)$ induces an isomorphism

$$\bar{\mathcal{S}}^{\circ} \xrightarrow{\sim} \bar{\mathcal{S}}^{\circ}_{\mathrm{red}},$$

where the index "red" indicates the image group in $\bar{\mathcal{G}}^{\circ}_{F,\mathrm{red}}$. Let $\bar{W}$ denote the Weyl group of $\bar{\mathcal{S}}^{\circ}_{\mathrm{red}}$ and consider the natural homomorphism

$$N(L) \cap K_F \longrightarrow \bar{W}.$$

The surjectivity of this homomorphism follows from [5, 4.6.13] or [7, 6.10]. An element n of the kernel centralizes $\bar{\mathcal{S}}^{\circ}_{\mathrm{red}}$ and hence also $\bar{\mathcal{S}}^{\circ}$. But then n centralizes $\mathcal{S}$ because this can be checked via the action of n on $X_*(\mathcal{S})$. But the centralizer of S in K_F is $T(L) \cap K_F$, cf. [5, 4.6.4], or [7, 6.3]. Hence $n \in T(L) \cap K_F$, which proves the claim. $\quad\square$

We will give the Iwahori–Weyl group $\tilde{W}$ the structure of a quasi-Coxeter group, that is, a semi-direct product of an abelian group with a Coxeter group. Consider the real vector spaces $V = X_*(T)_I \otimes \mathbb{R} = X_*(S) \otimes \mathbb{R}$ and $V' := X_*(T_{\mathrm{ad}})_I \otimes \mathbb{R}$, where T_{ad} denotes the image of T in the adjoint group G_{ad}. The relative roots $\Phi(G, S)$ for S determine hyperplanes in V (or V'), and the relative Weyl group W_0 can be identified with the group generated by the reflections through these hyperplanes. The homomorphism

$$T(L) \to X_*(T)_I \to V$$

derived from κ_T can be extended canonically to a group homomorphism

$$\nu : N(L) \to V \rtimes W_0$$

where $V \rtimes W_0$ is viewed as a group of affine-linear transformations of V; see [9, §1]. Using ν, Tits defines in [9, 1.4] the set of affine roots Φ_{af}, which can be viewed as a set of affine-linear functions on V or on V'. There is a unique reduced root system Σ such that the affine roots Φ_{af} consist of the functions on V (or V') of the form $y \mapsto \alpha(y) + k$ for $\alpha \in \Sigma$, $k \in \mathbb{Z}$. The group generated by the reflections through the walls in V (or in V') through the hyperplanes coming from Φ_{af} is the affine Weyl group $W_{\mathrm{af}}(\Sigma)$, which we also sometimes denote W_a. The group $W_{\mathrm{af}}(\Sigma)$ can be given the structure of a Coxeter group, as follows.

The apartment $\mathcal{A}$ in $\mathcal{B}$ associated to S is a torsor for V'; we identify it with V' by fixing a special vertex x in $\mathcal{A}$. Assume x belongs to the closure of the alcove $C \subset \mathcal{A}$. Let $\mathbf{S}$ denote the reflections in V' through the walls of C. Then $(W_{\mathrm{af}}(\Sigma), \mathbf{S})$ is a Coxeter group. Writing a superscript x to designate subgroups of the affine linear transformations $\mathrm{Aff}(V')$ of $\mathcal{A}$ with x as "origin," we have a semi-direct product

$$^x W_{\mathrm{af}}(\Sigma) = Q^\vee \rtimes {}^x W(\Sigma),$$

where $Q^\vee = Q^\vee(\Sigma)$ is the coroot lattice for Σ viewed as translations on V', and where $^x W(\Sigma)$ is the finite Weyl group, isomorphic to W_0 and consisting of the elements in $^x W_{\mathrm{af}}(\Sigma)$ which fix x.

The group $N(L)$ acts on the apartment $\mathcal{A}$ via the homomorphism ν when the latter is viewed as taking values in the group $^x W_{\mathrm{af}}(\Sigma)$ of affine-linear transformations on $\mathcal{A}$ endowed with x as the "origin."

Let T_{sc} resp. N_{sc} be the inverse images of $T \cap G_{\mathrm{der}}$ resp. $N \cap G_{\mathrm{der}}$ in G_{sc}. Let S_{sc} denote the split component of T_{sc}. Then S_{sc} is a maximal split torus in G_{sc}, and T_{sc} resp. N_{sc} is its centralizer resp. normalizer. Let $\tilde{W}_{\mathrm{sc}} = N_{\mathrm{sc}}(L)/T_{\mathrm{sc}}(L)_1$ be the Iwahori Weyl group of G_{sc}. By [5, 5.2.10], this can be identified with $W_{\mathrm{af}}(\Sigma)$. More precisely, note that the Kottwitz homomorphism for G_{sc} is trivial; also parahoric subgroups in $G_{\mathrm{sc}}(L)$ are simply the stabilizers or also the pointwise fixers of facets in $\mathcal{B}$. Let $K_{\mathrm{sc},0}$ be the Iwahori subgroup of $G_{\mathrm{sc}}(L)$ associated to a chamber $C \subset \mathcal{A}$ and let $\mathbf{S}$ be the set of reflections about the walls of C. Then [5, 5.2.10] $(G_{\mathrm{sc}}(L), K_{\mathrm{sc},0}, N_{\mathrm{sc}}(L), \mathbf{S})$ is a double Tits system and $\nu: N_{\mathrm{sc}}(L) \to V' \rtimes {}^x W(\Sigma)$ induces an isomorphism $\tilde{W}_{\mathrm{sc}} = {}^x W_{\mathrm{af}}(\Sigma)$. In particular, $\tilde{W}_{\mathrm{sc}}$ is a Coxeter group.

Proposition 13. *Let $x \in \mathcal{B}$ be a special vertex in the apartment corresponding to the maximal split torus S, and let K be the associated maximal parahoric subgroup of $G(L)$. The subgroup $\tilde{W}^K$ projects isomorphically to the factor group W_0, and the exact sequence (3) presents $\tilde{W}$ as a semi-direct product*

$$\tilde{W} = X_*(T)_I \rtimes W_0.$$

Proof. In view of Lemma 5 we need only to see that $\tilde{W}^K$ maps surjectively to W_0. In the case of G_{sc}, we have an isomorphism $\tilde{W}_{\mathrm{sc}} = {}^x W_{\mathrm{af}}(\Sigma)$ which sends $\tilde{W}^{K_{\mathrm{sc}}}$ onto ${}^x W(\Sigma)$, which is identified with $N_{\mathrm{sc}}(L)/T_{\mathrm{sc}}(L) = N(L)/T(L) = W_0$. The desired surjectivity follows. $\square$

By a result of Borovoi [1], there is an exact sequence

$$0 \to X_*(T_{\mathrm{sc}}) \to X_*(T) \to X^*(\hat{Z}(G)) \to 0.$$

Since $X_*(T_{\mathrm{sc}})$ is an induced Galois module [5, 4.4.16], $X_*(T_{\mathrm{sc}})_I$ is torsion-free and so the map $X_*(T_{\mathrm{sc}})_I \to X_*(T)_I$ is injective, and just as with (2),

$$X_*(T)_I / X_*(T_{\mathrm{sc}})_I = X^*(\hat{Z}(G)^I).$$

Using Proposition 13 for both $\tilde{W}$ and $\tilde{W}_{\mathrm{sc}} \cong W_{\mathrm{af}}(\Sigma) =: W_a$, we deduce the following lemma.

Lemma 14. *There is an exact sequence*

$$1 \longrightarrow W_a \longrightarrow \tilde{W} \longrightarrow X^*(\hat{Z}(G)^I) \longrightarrow 1.$$

The subgroup $\Omega \subset \tilde{W}$ consisting of the elements which preserve the alcove C maps isomorphically onto $X^*(\hat{Z}(G)^I)$, defining a quasi-Coxeter structure on $\tilde{W}$,

$$\tilde{W} = W_a \rtimes \Omega.$$

Let ν' denote the composition of ν with the map $V \to V'$. By [9, 1.7], Φ_{af} is stable under $\nu'(N(L)) \subset \mathrm{Aff}(V')$, hence in particular under the group of translations $\nu'(T(L))$, which may be identified with $X_*(T_{\mathrm{ad}})_I$. (Note that $X_*(T_{\mathrm{ad}})_I$ is torsion-free, by [5, 4.4.16].)

Lemma 15. *There are natural inclusions*

$$Q^\vee \subset X_*(T_{\mathrm{ad}})_I \subset P^\vee,$$

where $P^\vee \subset V'$ is the set of coweights for Σ. Moreover, ν' gives an identification

$$X_*(T_{\mathrm{sc}})_I = Q^\vee.$$

Proof. The translations $\nu'(T(L))$ permute the special vertices in $\mathcal{A}$, and $P^\vee$ acts simply transitively on them [2, chap. VI, §2, Prop. 2]; hence $\nu'(T(L)) \subset P^\vee$. The equality $X_*(T_{\mathrm{sc}})_I = Q^\vee$ follows from the isomorphism $\tilde{W}_{\mathrm{sc}} = {}^x W_{\mathrm{af}}(\Sigma)$ deduced above from [5, 5.2.10], and this implies $Q^\vee \subset X_*(T_{\mathrm{ad}})_I$. $\square$

Remark 16. The affine Weyl group associated to S also appears in [5] in a slightly different way. In [5, 5.2.11], the following subgroup of $G(L)$ is introduced,

$$G(L)' = \text{ subgroup generated by all parahoric subgroups of } G(L). \tag{8}$$

In [5, 5.2.11] it is shown that this coincides with

$$G(L)' = T(L)_1.\varphi\big(G_{\mathrm{sc}}(L)\big), \tag{9}$$

where $\varphi : G_{\mathrm{sc}} \longrightarrow G$ is the natural homomorphism from the simply connected cover of G_{der}.

Lemma 17. *We have $G(L)' = G(L)_1$.*

Proof. The inclusion "$\subset$" is obvious. If $G = T$ is a torus the equality is obvious, and so is the case when G is semisimple and simply connected. Next assume that G_{der} is simply connected and let $D = G/G_{\mathrm{der}}$. We obtain a commutative diagram

$$
\begin{array}{ccc}
G(L) & \longrightarrow & X^*(\hat{Z}(G)^I) \\
\downarrow & & \| \\
D(L) & \longrightarrow & X^*(\hat{D}^I).
\end{array}
$$

The exact sequence $1 \longrightarrow T_{\mathrm{der}} \longrightarrow T \longrightarrow D \longrightarrow 1$ induces a morphism of exact sequences

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & T_{\mathrm{der}}(L) & \longrightarrow & T(L) & \longrightarrow & D(L) & \longrightarrow & 1 \\
& & \downarrow & & \downarrow{\scriptstyle \kappa_T} & & \downarrow{\scriptstyle \kappa_D} & & \\
& & X_*(T_{\mathrm{der}})_I & \longrightarrow & X_*(T)_I & \longrightarrow & X_*(D)_I & \longrightarrow & 0.
\end{array}
$$

All vertical maps are surjective. Now let $g \in G(L)_1$. Then by a simple diagram chase there exists $t \in T(L)_1$ with the same image in D as g, and hence $g = tg' \in T(L)_1 \cdot \varphi(G_{\mathrm{der}}(L)) = G(L)'$.

To treat the general case, choose a z-extension

$$1 \longrightarrow Z \longrightarrow \tilde{G} \xrightarrow{\pi} G \longrightarrow 1,$$

with $\tilde{G}_{\mathrm{der}}$ simply connected. We obtain a morphism of exact sequences with surjective vertical maps

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & Z(L) & \longrightarrow & \tilde{G}(L) & \longrightarrow & G(L) & \longrightarrow & 1 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
& & X^*(\hat{Z}^I) & \longrightarrow & X^*(\hat{Z}(\tilde{G})^I) & \longrightarrow & X^*(\hat{Z}(G)^I) & \longrightarrow & 0.
\end{array}
$$

The diagram shows that $G(L)_1 = \pi(\tilde{G}(L)_1)$. On the other hand, the equality $G(L)' = \pi(\tilde{G}(L)')$ is easy to see. We conclude by the equality $\tilde{G}(L)' = \tilde{G}(L)_1$ which we already know. $\square$

The other way of introducing the affine Weyl group is now as follows. Let $N(L)' = N(L) \cap G(L)'$ and $W' = N(L)'/T(L)_1$. Let K_0 be the Iwahori subgroup associated to an alcove C in the apartment corresponding to S and let $\mathbf{S}$ be the set of reflections about the walls of C. Then [5, 5.2.12] $(G(L)', K_0, N(L)', \mathbf{S})$ is a double Tits system and W' is the affine Weyl group of the affine root system of S. In particular, the natural homomorphism $W' \longrightarrow W_a$ is an isomorphism.

Acknowledgment

References

[1] M.V. Borovoi, Abelian Galois cohomology of reductive algebraic groups, Mem. Amer. Math. Soc. 626 (1998).
[2] N. Bourbaki, Eléments de mathématique. Fasc. XXXIV. Groupes et algèbres de Lie. Chapitre IV: Groupes de Coxeter et systèmes de Tits. Chapitre V: Groupes engendrés par des réflexions. Chapitre VI: Systèmes de racines, in: Actualités Scientifiques et Industrielles, vol. 1337, Hermann, Paris, 1968, 288 pp. (in French).
[3] S. Bosch, W. Lütkebohmert, M. Raynaud, Néron Models, Springer-Verlag, 1990.
[4] F. Bruhat, J. Tits, Groupes réductifs sur un corps local, Inst. Hautes Études Sci. Publ. Math. 41 (1972) 5–251.
[5] F. Bruhat, J. Tits, Groupes réductifs sur un corps local. II. Schémas en groupes. Existence d'une donnée radicielle valuée, Inst. Hautes Études Sci. Publ. Math. 60 (1984) 197–376.
[6] R. Kottwitz, Isocrystals with additional structure. II, Compos. Math. 109 (1997) 255–339.
[7] E. Landvogt, A Compactification of the Bruhat–Tits Building, Lecture Notes in Math., vol. 1619, Springer-Verlag, Berlin, 1996, viii+152 pp.
[8] M. Rapoport, A guide to the reduction modulo p of Shimura varieties, Astérisque 298 (2005) 271–318.
[9] J. Tits, Reductive groups over local fields, in: Automorphic Forms, Representations and L-Functions, Proc. Sympos. Pure Math., Part 1, Oregon State Univ., Corvallis, OR, 1977, in: Proc. Sympos. Pure Math., vol. XXXIII, Amer. Math. Soc., Providence, RI, 1979, pp. 29–69.