Uniqueness and Stability for Boundary Value Problems with Weakly Coupled Systems of Nonlinear Integro-Differential Equations and Application to Chemical Reactions

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Submitted by W. F. Ames

This paper is dedicated to Professor H. Gätler on the occasion of his sixty-fifth birthday

1. Introduction

With \( y \) and \( f \) as elements of suitable spaces, \( I[y] = f \) with a linear homogeneous operator \( L \) possesses a unique solution if the Fredholm alternative theorem applies and \( L[z] = 0 \) possesses only the trivial solution \( z = 0 \). In case of systems of equations, subsequently, \( L \) will denote the pertinent vectorial linear homogeneous operators and \( y \) function vectors; \( P \) will denote vectorial operators which are not necessarily linear and homogeneous and \( Y \) will denote a set of functions \( y \) which are admissible for \( P \). For classes of systems \( P[y] = 0 \), the existence of at most one solution \( y \in Y \) can be reduced to the exclusion of nontrivial solutions \( z \in Y \) of the linear problem \( L[z; P, y_1, y_2] = 0 \) where \( z = y_1 - y_2 \) and \( L[z; P, y_1, y_2] = P[y_1] - P[y_2] \). The definition of the operators \( P \) and \( L \) is assumed to include boundary conditions which appear in the problem. The operator \( L[z; P, y_1, y_2] \) is always defined, though not necessarily uniquely, provided a Lipschitz condition is assumed in \( Y \) for \( P \) and \( y_1, y_2 \in Y \).

In [4, p. 448], \( L[z; P, y_1, y_2] \) is given for the operator \( P \) of a system of quasilinear hyperbolic equations. According to well-known theorems, linear homogeneous systems \( L[z; P, y_1, y_2] = 0 \) possess only the trivial solution in the following cases: hyperbolic systems (e.g., [4, p. 445]), parabolic systems (e.g., [14, p. 263]), systems of ordinary differential equations with initial
conditions (e.g., [7]), etc. The situation is more complicated if eigensolutions \( z \neq 0 \) of \( L[z; P, y_1, y_2] = 0 \) exist. Therefore, this paper applies predominantly to boundary value problems with ordinary or elliptic integro-differential equations. However, as for instance [12] shows, a boundary value problem with a system of parabolic differential equations may have elliptic properties provided the boundary conditions establish a relation between the end points of the interval of the parabolic variable.

According to Theorem 1, \( P[y] = 0 \) possesses at most one solution \( y_1 = Y \) if \( L[z; P, y_1, y_2] = 0 \) with \( y_1, y_2 \in Y \) possesses only the trivial solution as is true in particular if \( L \) is the operator of a problem of monotonic type; i.e., \( L[\eta; P, y_1, y_2] \geq 0 \) yields \( \eta \geq 0 \). Because of Theorem 2, an admissible operator \( P \) is the operator of a problem of monotonic type if this is true for \( L \); for one differential equation \( L[u] + F(x, u) = 0 \) in \( G \subset \mathbb{R}^n \) with linear boundary conditions, this already has been shown in [2, p. 47]. Here and subsequently, vectorial inequalities are to be taken by components. In Theorem 4, it is assumed that a class \( C^* \) of operators \( P^* \) can be generated from \( P \) by variation of data in \( P \); the theorem expresses that the solution \( y \) of \( P[y] = 0 \) is stable, i.e., depends continuously on the data which are varied in the family \( P^* \).

According to [13] and [14], \( L[z; P, y_1, y_2] \) is a parabolic or elliptic or ordinary operator of a problem of monotonic type provided

(a) the system \( L \) is weakly coupled, i.e., in the line \( i \) of this system, there are no derivatives of \( z_j \) with \( j \neq i \);

(b) the integro-differential operator \( L_i \) is ordinary of first (or second) order or parabolic (or elliptic) of second order; in \( L_i[z; P_i, y_1, y_2] \), boundary conditions are prescribed as follows: one "initial condition" for \( z_i(x) \) if \( L_i \) is of first order in \( G \subset \mathbb{R} \), one boundary condition for \( z_i(x) \) at each end point of the interval \( G \subset \mathbb{R} \) if \( L_i \) is a second order operator, boundary conditions for \( z_i(x) \) to render \( L_i \) well-posed if \( L_i \) is parabolic or elliptic;

(c) the coefficients in \( L \) are defined and satisfy certain inequalities and conditions on the quasimonotonic coupling of the lines of the operator \( L \), which are defined subsequent to Theorem 2, and

(d) in the domain \( G \times Y \) of the operator \( P \), there exists a test function \( v \) for every pair \( y_1, y_2 \in Y \) with the properties \( v > 0 \) and \( L[v; P, y_1, y_2] > 0 \).

Condition (a) may require the rearrangement of the lines in the system \( L \) by use of linear combinations. The boundary conditions required by (b) in general preclude the replacement of a given system of integro-differential equations by a system of first (second) order unless \( G \subset \mathbb{R} \) and the boundary conditions are suitable according to (b). The inequalities in (c) mainly ensure that the operator \( L \) is parabolic or elliptic. As shown subsequently, an
operator $\hat{L}$ with quasimonotonic coupling of its lines is obtained from $L$ if certain coefficients in $L$ are replaced by their modulus. According to Theorem 3, $L[z; P, y_1, y_2] = 0$ possesses only the trivial solution if $\hat{L}$ is the operator of a problem of monotonic type.

A construction method for $v$ in condition (d) is not given in [13] or in related papers employing such test functions; e.g., in [11, p. 23], the existence of solutions of one quasilinear elliptic differential equation with boundary conditions is shown by use of sequences $v_n$ where $v_0$ is a test function in the above sense whose construction is not discussed in [11]. The same applies to the first terms of such pairs of sequences in [3, p. 278]. In the present paper, an iterative construction of $v$ is presented that applies to systems of ordinary differential equations of second order with boundary conditions of the first kind; $\hat{L}$ for $\lambda = 1$ is embedded in the operator $\hat{L}_\alpha$ of an eigenvalue problem $\hat{L}_\alpha[v] = 0$. According to Theorems 5 and 6, the solution $v$ of $\hat{L}_\alpha[v] = 1$ is a test function if $\lambda \in [0, |\lambda_1|)$, where $\lambda_1$ is the eigenvalue with smallest modulus of $\hat{L}_\alpha$. An iterative approximation of $v$ yields the sequences $v^{(n)}$ and $\lambda^{(n)}$, where $v^{(n)}$ is a test function for $\lambda \in [0, \lambda^{(n)})$. It is shown in Theorem 7 that a point of accumulation $\hat{\lambda}$ with $\hat{\lambda} < |\lambda_1|$ does not exist. If $\lambda^{(n)} > 1$ for any $n \in \mathbb{N}$, $v^{(n)}$ is a test function for $\hat{L}$.

This exclusion of eigenvalues $\lambda \in [0, |\lambda_1|)$ does not require the operator $\hat{L}_\alpha$ to be self-adjoint or full-definite. These conditions usually are required in the theory of eigenvalue problems with one differential equation of even order. In [7, p. 286], a lower bound of the $n$th eigenvalue is given in case of one special system of two second order self-adjoint differential equations.

The proofs of Theorems 2–4, 6, and 7 make use of the theory of differential and integral inequalities, e.g., [14]. Closely related with this theory is the one of the maximum principle, [10] and [14]. In [10], this principle is employed to yield uniqueness theorems for the case of only one linear parabolic or elliptic second order differential equation with boundary conditions. In the context of this principle, it is also possible to derive uniqueness theorems for weakly and quasi monotonically coupled parabolic systems of second order differential equations, see [10, p. 188–193].

Essentially different from the uniqueness considerations employed here are: contraction theorems and the use of energy integrals for linear problems, [4] and [6].

2. The Operators $P$ and $L$

Suppose $G \subset \mathbb{R}^n$ is a bounded domain, i.e., $\bar{G}$ is compact. Unbounded domains $G$ are admissible if points at infinity can be introduced so as to render $\bar{G}$ compact and to define an operator preserving the solution(s) in
every point $x \in G$. Required smoothness conditions on the boundary $\partial G$ of $G$ will be mentioned subsequently.

The most general case to be treated here is the following system of second order integro-differential equations which includes boundary conditions:

$$P_i[y] = f_i\left(x, y(x), \frac{\partial y_i(x)}{\partial x_m}, \frac{\partial^2 y_i(x)}{\partial x_m \partial x_k}, \int_G K_i(x, \xi, y(\xi)) \, d\xi\right) = 0,$$

$$x \in G, \quad i = 1(1) N, \quad m, k = 1(1) n, \quad y = (y_1, ..., y_N).$$

The integrals in (1) are of the Stieltjes-type and thus may include $y(x)$ at individual points $x \in G$. The $f_i(x, U, V, W, Z)$ are supposed to be weakly monotone decreasing in the matrix $W$ for all admissible $x, U, V, Z$ as defined in [14, pp. 181, 256]. Therefore, elliptic or parabolic equations are special cases of (1). This weakly monotone decrease is also assumed for $x \in \partial G$ where, however, $\partial^2 y_i/\partial x_m \partial x_k$ denotes only derivatives tangential to $\partial G$; i.e., second order derivatives are admissible for those $x \in \partial G$ where $\partial G$ possesses suitable tangential spaces. Because of the unusual admission of second order derivatives in $f_i$ for $x \in \partial G$, parabolic problems are tractable without special consideration of those parts of $\partial G$ where the domain operator is valid. In the elliptic case, usually second order derivatives do not appear in the boundary operators. For $x \in \partial G$, $\partial y_i/\partial x_m$ is assumed to represent the outer normal derivative $\partial / \partial n_a$ (e.g., [14, p. 245]) and $f_i$ in these points of $\partial G$ is assumed to be weakly monotone increasing in $V$. The $f_i$ are assumed to satisfy a Lipschitz-condition in $G$ with respect to $y \in Y$ and the derivatives of $y$. This completes the definition of the class of admissible operators which includes systems of integral equations and/or ordinary equations. Since the existence problem is not treated here, the assumptions on the smoothness of $\partial G$ are sufficient.

Functions in the admissible class for $P$ are required to be continuous in $G$ and to possess continuous derivatives wherever derivatives appear in the problem. Weaker assumptions on the class of admissible functions are given in [14].

If $u$ and $w$ are admissible for $P$, the operator $L[z; P, u, w]$ is defined as follows with $z_i = u_i - w_i$, $i = 1(1) N$:

$$P_i[u] - P_i[w] =$$

$$\left[ f_i\left(x, u, \frac{\partial u_i}{\partial x_m}, \frac{\partial^2 u_i}{\partial x_m \partial x_k}, \int_G K_i(x, \xi, u) \, d\xi\right) - f_i\left(x, (w_1, w_2, ..., w_N), \frac{\partial u_i}{\partial x_m}, \frac{\partial^2 u_i}{\partial x_m \partial x_k}, \int_G K(x, \xi, u) \, d\xi\right)\right] \cdot [u_1 - w_1]$$

$$= \left[ \frac{\partial u_i}{\partial x_m}, \frac{\partial^2 u_i}{\partial x_m \partial x_k}, \int_G K_i(x, \xi, u) \, d\xi\right]$$
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\[ f_i \left( x, (w_1, w_2, u_3, \ldots, u_N) \right) \frac{\partial u_i}{\partial x_m} + \frac{\partial^2 u_i}{\partial x_m \partial x_k} \int_{\mathcal{G}} K_i(x, \xi, u) \, d\xi \]

\[ - f_i \left( x, w, \frac{\partial u_i}{\partial x_m} + \frac{\partial^2 u_i}{\partial x_m \partial x_k} \int_{\mathcal{G}} K_i(x, \xi, u) \, d\xi \right) \]

\[ \left[ u_2 - w_2 \right] + \ldots \]

\[ f_i \left( x, w, \frac{\partial^2 w_i}{\partial x_m \partial x_k} \int_{\mathcal{G}} K_i(x, \xi, u) \, d\xi \right) \]

\[ - f_i \left( x, w, \frac{\partial^2 w_i}{\partial x_m \partial x_k} \int_{\mathcal{G}} K_i(x, \xi, u) \, d\xi \right) \]

\[ \int_{\mathcal{G}} K_i(x, \xi, u) \, d\xi - \int_{\mathcal{G}} K_i(x, \xi, (w_1, u_2, \ldots, u_N)) \, d\xi \]

\[ \frac{K_i(x, \xi, u) - K_i(x, \xi, (w_1, u_2, \ldots, u_N))}{u_i(\xi) - w_i(\xi)} \cdot \left[ u_i(\xi) - w_i(\xi) \right] \, d\xi + \ldots \]

\[ f_i \left( x, w, \frac{\partial^2 w_i}{\partial x_m \partial x_k} \int_{\mathcal{G}} K_i(x, \xi, (w_1, \ldots, w_{N-1}, u_N)) \, d\xi \right) \]

\[ - f_i \left( x, w, \frac{\partial^2 w_i}{\partial x_m \partial x_k} \int_{\mathcal{G}} K_i(x, \xi, w) \, d\xi \right) \]

\[ \int_{\mathcal{G}} K_i(x, \xi, (w_1, \ldots, w_{N-1}, u_N)) \, d\xi - \int_{\mathcal{G}} K_i(x, \xi, w) \, d\xi \]

\[ \frac{K_i(x, \xi, (w_1, \ldots, w_{N-1}, u_N)) - K_i(x, \xi, w)}{u_N(\xi) - w_N(\xi)} \cdot \left[ u_N(\xi) - w_N(\xi) \right] \, d\xi \]

\[ = L_i[u - w; P_i, u, w], \quad x \in \mathcal{G}, \quad i = 1(1) N, \quad m, k = 1(1) n. \] (2)

This linear homogeneous operator \( L \) has the structure

\[ L_i[z; P_i, u, w] \]

\[ = - \sum_{j=1}^{N} a_{ij}(x) z_j + \sum_{m=1}^{n} b_{im}(x) \frac{\partial z_i}{\partial x_m} - \sum_{m,k=1}^{n} c_{imk}(x) \frac{\partial^2 z_i}{\partial x_m \partial x_k} \]

\[ - \sum_{j=1}^{N} \int_{\mathcal{G}} K_{ij}(x, \xi) z_j(\xi) \, d\xi, \quad \text{for} \ x \in \mathcal{G}, \]

\[ = \sum_{j=1}^{N} d_{ij}(x) z_j + e_{ij}(x) \frac{\partial z_i}{\partial u_a} - \sum_{m,k=1}^{n} g_{imk}(x) \frac{\partial^2 z_i}{\partial x_m \partial x_k} \]

\[ - \sum_{j=1}^{N} \int_{\mathcal{G}} M_{ij}(x, \xi) z_j(\xi) \, d\xi, \quad \text{for} \ x \in \partial \mathcal{G}, \quad i = 1(1) N, \] (3)
where condition (c) in the Introduction requires for \( i = 1(1)N, m \) and \( k = 1(1)n \) that (a) \( e_i > 0 \), (b) the matrix with elements \( g_{imk} \) for \( x \in \partial G \) to admit second order derivatives only in tangential spaces of \( \partial G \), and (c) the matrices with elements \( c_{imG} \) and \( g_{imk} \) to be positive semidefinite. The coefficients \( a_{ij} \), ... in (3) depend on \( x, u, w \) and on derivatives of \( u \) and \( w \):

\[
\begin{align*}
 a_{ij} &= a_{ij} \left( x, u(x), w(x), \frac{\partial u_i}{\partial x_m}, \frac{\partial w_i}{\partial x_m}, \frac{\partial^2 u_i}{\partial x_m \partial x_k}, \frac{\partial^2 w_i}{\partial x_m \partial x_k}, \int_{\overline{G}} K_i(x, \xi, u) \, d\xi, \right. \\
 &\left. \int_{\overline{G}} K_i(x, \xi, w) \, d\xi, \right) , \quad i, j = 1(1) N, \quad m, k = 1(1) n. 
\end{align*}
\]

(4)

The coefficients in (3) are not necessarily continuous; they exist provided the respective denominators in (2) are nonzero. Because of the assumed Lipschitz-condition, the coefficients in (3) are bounded at every \( x \in \overline{G} \) as \( u \to w \).

3. The Linear Operator \( L \) as a Problem of Monotonic Type

With \( P \) and \( y \) as defined in (1), the following problem is considered:

\[
L[z; P, y_1, y_2] = 0, \quad x \in \overline{G}.
\]

Theorem 1. If for \( P \) given and \( y_1 \neq y_2 \) with \( y_1, y_2 \in Y \) the problem (5) possesses only the trivial solution \( z = 0, x \in \overline{G} \), then \( y_1 \) and \( y_2 \) cannot be solutions simultaneously of (1).

Proof. Contrary to the assumption \( y_1 \neq y_2 \), there follows \( y_1 - y_2 = 0 \) from \( 0 = P[y_1] - P[y_2] = L[y_1 - y_2; P, y_1, y_2] \), and this completes the proof.

In order to deduce from this theorem the uniqueness of the solution \( y \in Y \) of (1), it must be known for every pair of functions \( y_1, y_a \in Y \) that (5) possesses only the trivial solution \( z = 0 \) for \( x \in \overline{G} \). For this, it is sufficient that \( L \) is the operator of a problem of monotonic type with admission of the equality sign in every inequality. Collatz has introduced this property as follows: from the operator inequality \( Q[u] \leq Q[w], x \in \overline{G} \), there follows the inequality \( u \leq w, x \in \overline{G} \) for every pair of admissible functions \( u, w \). If \( Q \) is the operator of a problem of monotonic type, \( Q[u] = 0 \) possesses at most one solution \( u \) since \( 0 = Q[u] \leq Q[w] = 0 \) yields \( u \leq w \) and \( Q[u] \geq Q[w] \) yields \( u \geq w \), i.e., \( u = w \). If \( L[z; P, y_1, y_2] \) is a linear homogeneous operator of monotonic type, the necessarily existing solution \( z = 0 \) is the only one.
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THEOREM 2. If \( L[z, P, y_1, y_2] \) is the operator of a problem of monotonic type for every pair \( y_1, y_2 \in Y \), then \( P \) has this property for \( y \in Y \).

Proof. Suppose \( 0 \leq P[y_1] - P[y_2] = L[y_1 - y_2, P, y_1, y_2] \). Since \( L \) by assumption is of monotonic type, the inequality \( 0 \leq L[y_1 - y_2, P, y_1, y_2] \) yields \( 0 \leq y_1 - y_2 \); i.e., \( P[y_1] \leq P[y_2] \) yields \( y_1 \leq y_2 \) and thus \( P \) is of monotonic type; this completes the proof.

The conditions (a)-(d) in the Introduction are sufficient for \( L \) to possess the property of monotonic type. Condition (c) requires the validity of the inequalities \( a_{ij}, d_{ij} \geq 0 \) for \( i \neq j \) and \( K_{ij}, M_{ij} \geq 0 \) for \( i, j = 1(1)N \) which establish the quasimonotonic coupling of \( L \).

Even if these inequalities for \( a_{ij}, d_{ij} \) and \( K_{ij}, M_{ij} \) are not satisfied by \( L \), the uniqueness of the solution \( y \) of \( P[y] = 0 \) also can be shown:

THEOREM 3. If a test function \( v > 0 \) in \( G \) exists with

\[
\tilde{L}_i[v; P, y_1, y_2] = \begin{cases} 
- a_{ij}v_j - \sum_{j=1}^{N} \left| a_{ij} \right| v_j + \sum_{m=1}^{n} b_{im} \frac{\partial v_j}{\partial x_m} - \sum_{m, k=1}^{n} c_{imk} \frac{\partial^2 v_j}{\partial x_m \partial x_k} \\
- \sum_{j=1}^{N} \int_{\partial G} |K_{ij}| v_j \, d\xi > 0, \quad x \in G, \\
- d_{ij}v_i - \sum_{j=1}^{N} \left| d_{ij} \right| v_j + \sum_{m, k=1}^{n} g_{imk} \frac{\partial^2 v_j}{\partial x_m \partial x_k} \\
- \sum_{j=1}^{N} \int_{\partial G} |M_{ij}| v_j \, d\xi > 0, \quad x \in \partial G, \quad i = 1(1)N,
\end{cases}
\]

then (1) possesses at most one solution and \( \tilde{L} \) is an operator of monotonic type.

Proof. Because of the structure of \( \tilde{L}_i \) and the inequalities in (6), the conditions (a)-(d) in the Introduction are satisfied and \( \tilde{L} \) thus is of monotonic type, [13]. The assumption that (5) possesses a solution \( z_0 \neq 0 \) in \( G \) will be shown to yield a contradiction. Assume

\[
a_{ij}^+(x) = a_{ij}(x) \quad \text{if} \quad a_{ij} > 0 \quad \text{and} \quad 0 \quad \text{otherwise},
\]

\[
a_{ii}^- = a_{ii} - a_{ii}^+ \leq 0
\]

\[
K_{ij}^+(x, \xi) = K_{ij}(x, \xi) \quad \text{if} \quad K_{ij} > 0 \quad \text{and} \quad 0 \quad \text{otherwise},
\]

\[
K_{ii}^- = K_{ii} - K_{ii}^+ \leq 0
\]
and, correspondingly, \( d_{ij}^+, d_{ij}, M_{ij}, M_{ij}^+ \). The system

\[
\hat{L}_i[(\hat{z}, \hat{z}); P, y_1, y_2] =
\begin{align*}
- a_{ii}^{+} \hat{z}_i - \sum_{j=1}^{N} \left( a_{ij}^{+} \hat{z}_j + a_{ij}^{-} \hat{z}_j^+ \right) + \sum_{m=1}^{n} b_{im} \frac{\partial \hat{z}_i}{\partial x_m} - \sum_{m,k=1}^{n} c_{imk} \frac{\partial^2 \hat{z}_i}{\partial x_m \partial x_k} \\
- \sum_{j=1}^{N} \int_{G} \left( K_{ij}^{+} \hat{z}_j + K_{ij}^{-} \hat{z}_j^+ \right) d\xi = 0, \quad x \in G, \ i = 1(1) N, \\
- d_{ij}^{+} \hat{z}_i - \sum_{j=1}^{N} \left( d_{ij}^{+} \hat{z}_j + d_{ij}^{-} \hat{z}_j^+ \right) + e_i \frac{\partial \hat{z}_i}{\partial n} - \sum_{m,k=1}^{n} g_{imk} \frac{\partial^2 \hat{z}_i}{\partial x_m \partial x_k} \\
- \sum_{j=1}^{N} \int_{G} \left( M_{ij}^{+} \hat{z}_j + M_{ij}^{-} \hat{z}_j^+ \right) d\xi = 0, \quad x \in \partial G, \ i = 1(1) N,
\end{align*}
\]

possesses the nontrivial solution \( \hat{z} = z = z_0 \) because it is identical with (5). The system (7) has the property of quasimonotonic coupling with respect to the vector \((\hat{z}, -\hat{z})\) with \(2N\) components and it possesses the positive test function \((\hat{z}, z) = (v, v)\) from (6). Therefore, the linear homogeneous operator \((\hat{L}, \hat{I})\) is a problem of monotonic type, according to [13], and (7) possesses only the trivial solution. This is a contradiction to the assumption \(z_0 \neq 0\) and completes the proof of Theorem 3 because of Theorem 1.
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4. STABILITY OF THE SOLUTION

In order to show the stability of the solution \( y \in Y \) of \( P[y] = 0 \), this system will now be embedded in a class \( C^* \) of admissible operators \( P^* \) with solutions \( y^* \) of

\[
P^*_i[y^*] = f^*_i(x, y^*, \frac{\partial y^*}{\partial x_m}, \frac{\partial^2 y^*}{\partial x_m \partial x_k}, \int_G K_i(x, \xi, y^*(\xi)) \, d\xi) = 0,
\]

\( x \in G, \quad y^* \in Y, \quad i = 1(1) N. \) \( (8) \)

The class \( C^* \) is obtained from \( P \) by small variations of those data (coefficients) in \( f \) for which \( |P^*[y]| < \epsilon \). Following [4, p. 227], stability is defined as the continuous dependency of \( y^* \) on the data in \( P^* \) for \( P^* \to P \) and thus \( y^* \to y \) as \( \epsilon \to 0 \). For \( P^* \) given, \( L^*, \hat{L}^*, L^*, \hat{L}^* \), and \( \hat{L}^* \) are defined corresponding to (3), (6), and (7). It will be assumed:

(a*) Corresponding to (6), it can be shown by use of one test function \( v \) that the operators \( \hat{L}^* \) belonging to \( P^* \in C^* \) are of monotonic type, where

\[
\inf_{x \in G} \hat{L}^*[v; P^*, y, y^*] > 0 \quad \text{with} \quad y^*, y \in Y.
\]

The assumptions (a*) and (b*) may require the restriction of the variation of data in \( P \) to suitable coefficient functions. It follows from (a*) and (b*) that every problem \( P^*[y^*] = 0 \) with \( P^* \in C^* \) has at most one solution \( y^* \in Y \).

The following simple examples will serve to illustrate the stability problem at hand. The solution \( y(x; \lambda) \) of

\[
-y'' - \lambda y = 1, \quad x \in (0, 1), \quad y(0) = y(1) = 0,
\]

\( (9) \)

does not depend continuously on \( \lambda \). In the parabolic differential equation

\[
\frac{\partial y}{\partial t} - a(x) \frac{\partial^2 y}{\partial x^2} = 1 \quad \text{in} \quad G = \{(x, t): 0 < x < 1, 0 < t < \infty\},
\]

\( a(x) = 0 \quad \text{for} \quad x \in [0, 0.5], \quad a(x) = x - 0.5 \quad \text{for} \quad x \in (0.5, 1], \) \( (10) \)

variations of \( a(x) \) in \( x \in [0, 0.5] \) are restricted to nonnegative values of \( a \) to preserve the parabolic properties of the differential equation.

**Theorem 4.** Under the assumptions (a*) and (b*), the solution \( y \in Y \) of \( P[y] = 0, x \in G \) is stable with respect to variations of data of \( P \) in the class \( C^* \).

**Proof.** Because of (1) and (2),

\[
L^*[y - y^*; P^*, y, y^*] = P^*[y] - P^*[y^*] = P^*[y], \quad x \in G, \quad (11)
\]

\( L^* \) being the operator corresponding to (6).
where \( y \) and \( \mathbf{P^*} \) are given and \(|\mathbf{P^*[y]}| < \varepsilon\) with \( \varepsilon \in \mathbb{R}^+\). Corresponding to (6) and (7) and because of assumptions (a*) and (b*),

\[
L^*[i^*(v, -v); \mathbf{P_i^*}, y, y^*] = \hat{L}^{i^*}[i^*(v, -v); \mathbf{P_i^*}, y, y^*] > 0, \quad x \in \mathcal{G}, \quad i = 1(1) N. \tag{12}
\]

Because of assumption (b*), an \( \alpha \in \mathbb{R}^+ \) exists in

\[
\alpha\hat{L}_{i^*}^*[i^*(v, -v); \mathbf{P_i^*}, y, y^*] \geq \hat{L}_{i^*}^*[i^*(y - y^*, -y + y^*); \mathbf{P_i^*}, y, y^*] = \mathbf{P_i^*[y]} = L_{i^*}^*[y - y^*, -y + y^*; \mathbf{P_i^*}, y, y^*] \geq -\alpha L_{i^*}^*[i^*(v, -v); \mathbf{P_i^*}, y, y^*], \quad x \in \mathcal{G}, \quad i = 1(1) N. \tag{13}
\]

Since \((\hat{L}^*, \hat{L}_{\varepsilon}^*)\) is the operator of a problem of monotonic type,

\[
\alpha(\mathbf{v}, \mathbf{v}) \geq (\mathbf{y} - \mathbf{y}^*, -\mathbf{y} + \mathbf{y}^*) \geq -\alpha(\mathbf{v}, \mathbf{v}), \quad x \in \mathcal{G}. \tag{14}
\]

If \( \varepsilon \to 0 \) in \(|\mathbf{P^*[y]}| < \varepsilon\) for \( \mathbf{P^*} \to \mathbf{P} \), it follows from (11)–(13) that \( \alpha(\varepsilon) \to 0 \) is admissible, which completes the proof of Theorem 4.

**Example.** The boundary value problem

\[
P[y] = \begin{cases} 
-y'' - a(x)y + Ay^3 + g(x) \text{ in } x \in (-1, 1) \\
y(-1) = y(1)
\end{cases} = 0, \tag{15}
\]

\[
a(x) = \frac{1 - x}{4}, \quad A = 1,
\]

and \( \mathbf{P^*} \in C^* \) with

\[
\mathbf{P^*[y^*]} = \begin{cases} 
-y^{**} - [a(x) + a_1(x)]y^* + Ay^3 - g(x) \text{ in } x \in (-1, 1) \\
y^*(-1) = y^*(1)
\end{cases} = 0, \tag{16}
\]

\[
A = 1
\]

will be considered; \( \mathbf{P^*} \) is of monotonic type since \( \mathbf{v} = \cos \pi x/3 \) is a test function because of \( \mathbf{v} > 0 \) in \( x \in (-1, 1) \) and

\[
\mathbf{L^*[v; P^*, y, y^*]} = \begin{cases} 
-v'' - [(1 - x)|4 + a_1(x)|v + A[y^{**} + y^2 + yy^*] \text{ in } x \in (-1, 1) \\
v(-1) = v(1)
\end{cases} \geq \varepsilon_1 > 0, \tag{17}
\]
provided $a_i(x) < 0.5$ with $\epsilon_1 \leq 0.01$. There is no need to employ the operators $\hat{P}^*, \hat{L}^*$, and $\hat{L}_x^*$ since $P$ does not represent a system. Because of $|P^*[y]| = |a_i(x) y| < \epsilon$ a sufficiently large $\alpha \in \mathbb{R}^+$ can be determined in

$$\alpha L^*[v; P^*, y, y^*] \geq L^*[y - y^*; P^*, y, y^*] = P^*[y] \geq \alpha L^*[v; P^*, y, y^*],$$

$x \in (-1, 1)$. (18)

Since $\alpha(\epsilon) \to 0$ is admissible together with $\epsilon \to 0$, $y^* \to y$ for $P^* \to P$ in $x \in [-1, 1]$; i.e., the solution $y$ of (15) is stable with respect to $a(x)$. For $A = -1$, however, $L^*[v; P^*, y_1, y_2] > 0$ in (17) holds true only for functions $y$ and $y^*$ from a suitable bounded set $Y$. Stability theorems for problems of monotonic type are mentioned in [14].

5. ON THE ITERATIVE CONSTRUCTION OF A TEST FUNCTION

In this section, the operator $P$ is restricted as follows:

(a) $P$ represents a system of ordinary differential operators on the interval $x \in (0, 1)$ with boundary conditions of the first kind; these boundary conditions can be solved for the boundary values of $y$;

(b) the coefficients $a_{ij}$ in the operator $\mathbf{L}$ are continuous for $x \in [0, 1]$ and $y_1, y_2 \in Y$.

An iterative construction of a sequence of functions $v^{(n)}$ will be presented which after $n$ steps yields a test function for $\hat{L}$ provided condition (35) given subsequently is satisfied. Here (5), with $\mathbf{L}$ belonging to $\mathbf{P}$, can be represented by

$$L_i(z; \mathbf{P}, y_1, y_2) = \begin{cases} -z''_i - \sum_{j=1}^{N} a_{ij}(x) z_j \text{ in } x \in (0, 1), \\ z_i(0) = z_i(1), \end{cases} = 0, \quad i = 1(1) N. \quad (19)$$

The following eigenvalue problem is adjoined to (19):

$$L_{i\lambda}(\zeta; \mathbf{P}, y_1, y_2) = \begin{cases} -\zeta''_i - \lambda \sum_{j=1}^{N} |a_{ij}(x)| \zeta_j \text{ in } x \in (0, 1) \\ \zeta_i(0) = \zeta_i(1) \end{cases} = 0, \quad i = 1(1) N. \quad (20)$$

If there exist eigenvalues of (20), $\lambda_1$ denotes the one with the smallest modulus.
The problem (20) is of monotonic type for $\lambda \in [0, \lambda_{\text{max}}]$ if a suitable test function $v$ is known for $\lambda = \lambda_{\text{max}}$. If $\lambda_{\text{max}} > 1$, the problem $P[y] = 0$ possesses at most one solution because of Theorem 3. If the numbers

$$\bar{a}_{ij} = \max_{x \in [0,1]} |a_{ij}(x)|, \quad i, j = 1(1) N,$$

are sufficiently small, it may be possible to verify that the function $V_{\ast}^* = \sin(\pi(x + \epsilon))/(1 + 2\epsilon)$, $x \in [0, 1]$, $\epsilon \in \mathbb{R}^+$, $i = 1(1) N$ satisfies

$$\mathcal{L}_n[V^*; y_1, y_2] > 0, \quad \text{in} \quad \{(x, \lambda): 0 \leq x \leq 1, 0 \leq \lambda \leq \lambda_{\text{max}}\}, \quad (21)$$

with $\lambda_{\text{max}} > 1$. Then $V^*$ is a test function for (20).

If (21) is not true, the iterative method given subsequently for the construction of a test function may be employed. Prior to the presentation of this method, the auxiliary Theorems 5 and 6 will be given.

**Theorem 5.** The solution $y(x, \lambda)$ of the boundary value problem

$$L_{\lambda}[y; P, y_1, y_2] = \left\{ \begin{array}{l} -y_i'' - \lambda \sum_{j=1}^{N} |a_{ij}(x)| y_j \quad \text{in} \quad x \in (0, 1) \\ y_i(0) = 1, \quad y'_i(0) = 0, \quad i = 1(1) N \end{array} \right\} = 1, \quad i = 1(1) N \quad (22)$$

is continuous in the set $H_{\lambda} = \{(x, \lambda): 0 \leq x \leq 1, \lambda \in \mathbb{R}, \lambda \neq \lambda_{ev}\}$ where $\lambda_{ev}$ denotes the real values in the set of eigenvalues of (20).

**Proof.** Since the $|a_{ij}(x)|$ are continuous by assumption, the initial value problem

$$-y_i'' - \lambda \sum_{j=1}^{N} |a_{ij}(x)| y_j = 1, \quad \lambda \in \mathbb{R}, \quad y_i(0) = 1, \quad y'_i(0) = 0, \quad i = 1(1) N, \quad (23)$$

possesses one and only one solution $y_{0i}(x, \lambda)$ with $N$ components $y_{0i}(x, \lambda)$ for every fixed $\lambda \in \mathbb{R}$ in the set $H = \{(x, \lambda): 0 \leq x \leq 1, \lambda \in \mathbb{R}\}$, e.g., [7, p. 48]. The $N$ initial value problems

$$-y''_i - \lambda \sum_{j=1}^{N} |a_{ij}(x)| y_{ji} = 0, \quad x \in (0, 1), \quad \lambda \in \mathbb{R}, \quad y_{ji}(0) = 0, \quad (24)$$

$$y_{ji}'(0) = 1, \quad y_{ji}(0) = 0 \quad \text{for} \quad j \neq p, \quad i, p = 1(1) N$$
also possess one and only one solution \( y_p(x, \lambda), \lambda = 1(1)N \) in \( H \) for every given \( \lambda \). According to \([8, 116-117], y_0(x, \lambda) \) and \( y_p(x, \lambda) \) are continuous in \( H \). The expression

\[
y(x, \lambda) = y_0(x, \lambda) + \sum_{p=1}^{N} c_p y_p(x, \lambda) \quad \text{in} \quad H, \quad c_p \in \mathbb{R},
\]

is a continuous function of \( x, \lambda \), and the \( c_p \) in the set \( \{(x, \lambda, c_p): 0 \leq x \leq 1, \lambda \in \mathbb{R}, c_p \in \mathbb{R}, p = 1(1)N\} \). The vector \( c \) with components \( c_p \) is determined as solution of the system of linear equations

\[
\sum_{p=1}^{N} c_p y_{p0}(1, \lambda) = 1 - y_{00}(1, \lambda), \quad j = 1(1) N.
\]

According to \([15, p. 181], \) there exists one and only one solution \( c \) of \((26)\) provided \( \lambda \) is not an eigenvalue of \((20)\). Under this condition, the matrix with elements \( y_{p0}(1, \lambda) \) is regular. Then, each \( c_x \) can be expressed as the ratio of the determinants \( i^{-1} \) and \( B \) where \( B \neq 0 \) and the elements of \( A_p \) and \( B \) depend continuously on \( \lambda \). Therefore, \( c \) also depends continuously on \( \lambda \) unless \( \lambda \) is an eigenvalue of \((20)\). Finally, then \( y(x, \lambda) \) in \((25)\) is a continuous function of \( x \) and \( \lambda \) in \( H_\lambda \). This completes the proof of Theorem 5.

**Theorem 6.** The solution \( v(x, \lambda) \) of

\[
\tilde{L}_{0i}[v; \mathbf{P}, \mathbf{y}_1, \mathbf{y}_2] = 1, \quad x \in [0, 1], \quad i = 1(1)N
\]

is a test function \( v \) for \((20)\) with \( v > 0 \) in the set \( H_\lambda = \{(x, \lambda): 0 \leq x \leq 1, \lambda \in \mathbb{R}, 0 \leq \lambda < |\lambda_1|\} \) where \( \lambda_1 \) is the eigenvalue of \((20)\) with the smallest modulus.

**Proof.** For \( \lambda = 0 \), the solution of \((27)\) is \( v_i(x, 0) = -x^2/2 + x/2 + 1 \) with \( v(x, 0) > 0 \) in \( x \in [0, 1] \). According to Theorem 5, the solution \( v(x, \lambda) \) of \((27)\) is a continuous function of \( x \) and \( \lambda \) in the set \( H_\lambda \) since there is no eigenvalue with \( |\lambda| < |\lambda_1| \). If there exist values \( v_i(x, \hat{\lambda}) = 0 \) with \( \hat{\lambda} \in [0, |\lambda_1|] \), it follows from the continuity of \( v(x, \lambda) \) in the compact set \( \{(x, \lambda): 0 \leq x \leq 1, 0 \leq \lambda \leq \hat{\lambda}\} \) that there exists a smallest value \( \hat{\lambda} \) for which

\[
v(x, \lambda) > 0 \quad \text{for} \quad \{(x, \lambda): 0 \leq x \leq 1, 0 \leq \lambda < \hat{\lambda}\}, \quad v_i(\hat{x}, \hat{\lambda}) = 0,
\]

for a pair of numbers \((i, \hat{x})\) and \( \hat{x} \in [0, 1] \). Because of \((22)\) and \((28)\), \(-v_i'(\hat{x}, \hat{\lambda}) \geq 1 \); therefore, values \( v_i(x, \lambda) < 0 \) must exist in a vicinity of \( \hat{x} \) which is a contradiction to the choice of \( \hat{\lambda} \). This completes the proof of
Theorem 6 which is also true for \( \lambda \in [0, \lambda_{10}) \) where \( \lambda_{10} \) is the smallest positive eigenvalue of (20) if there exists such a value.

The solution \( v(x, \lambda) \) of (27) will now be approximated iteratively. For this purpose, (27) is replaced by the equivalent system of integral equations

\[
J_{\lambda}[v] = v(x) - \lambda \sum_{j=1}^{N} \int_{0}^{1} G(x, \xi) |a_{ij}(\xi)| v_j(\xi) \, d\xi
\]

\[
= -\frac{x^2}{2} + \frac{x}{2} + 1, \quad x \in [0, 1], \quad i = 1(1)N, \quad \lambda \in \mathbb{R}_0^+
\]

\( G(x, \xi) = \xi(1-x) \) for \( \xi < x \), \( G(x, \xi) = x(1-\xi) \) for \( x < \xi \).

According to [9, p. 301], this system can be transformed into a single integral equation whose kernel is bounded because of the assumed continuity of the \( a_{ij}(x) \). This single integral equation can be solved iteratively by use of the method of successive approximations which converges uniformly in \( x \in [0, 1] \) for \( |\lambda| < |\lambda_1| \), e.g., [9, p. 58]. Since the single integral equation is equivalent to the system (29), the iteration defined by

\[
v^{(n)}_i(x, \lambda) = \lambda \sum_{j=1}^{N} \int_{0}^{1} G(x, \xi) |a_{ij}(\xi)| v^{(n-1)}_j(\xi, \lambda) \, d\xi - \frac{x^2}{2} + \frac{x}{2} + 1,
\]

\( x \in [0, 1], \quad v^{(0)}_i = 0, \quad i = 1(1)N, \quad n \in \mathbb{N}, \quad \lambda \in \mathbb{R}_0^+ \)

also converges uniformly for \( x \in [0, 1] \) and \( |\lambda| < |\lambda_1| \). Subsequently, a real sequence \( \lambda^{(n)} \) is defined by

\[
\lambda^{(n)} = \sup_{x \in [0, 1], i = 1(1)N} \inf_{n \in \mathbb{N}} \frac{v^{(n)}_i(x, \lambda)}{\sum_{j=1}^{N} \int_{0}^{1} G(x, \xi) |a_{ij}(\xi)| v^{(n)}_j(\xi, \lambda) \, d\xi}, \quad n \in \mathbb{N}.
\]

such that \( J_{\lambda}[v^{(n)}] \geq 0 \) in \( \{v(x, \lambda) : 0 \leq x \leq 1, \ 0 \leq \lambda < \lambda^{(n)} \} \). Then \( v^{(n)}_i(x, \lambda) > 0 \) is a test function, for \( \lambda \in [0, \lambda^{(n)}) \) where \( i = 1(1)N \).

**Theorem 7.** The limsup of the real sequence \( \lambda^{(n)} \) is not smaller than \( |\lambda_1| \).

**Proof.** It will be assumed that the sequence \( \lambda^{(n)} \) possesses a point of accumulation \( \hat{\lambda} < |\lambda_1| \). The approximation of \( v(x, \lambda^*) \) by the sequence \( v^{(n)} \) converges uniformly for any fixed \( \lambda^* \in [0, |\lambda_1|] \), e.g., \( \lambda^* = 0.5[\hat{\lambda} + |\lambda_1|] \). Therefore, there exists an \( \epsilon \in \mathbb{R}^+ \) such that for every sufficiently large \( \bar{n} \in \mathbb{N} \)

\[
v^{(\bar{n})}_i(x, \lambda^*) \geq v_i(x, \lambda^*) - \epsilon > 0, \quad J_{\lambda^*}[v^{(\bar{n})}] \geq 1 - \epsilon > 0, \quad x \in [0, 1],
\]

\( i = 1(1)N \)
or, because of (29),
\[ v_i^{(n)}(x, \lambda^*) > \lambda^* \sum_{j=1}^{N} \int_{0}^{1} G(x, \xi) |a_{ij}(\xi)| v_j^{(n)}(\xi, \lambda^*) \, d\xi, \]
\[ x \in [0, 1], \quad i = 1(1) N. \]  
(33)

The integrals in (33) are nonnegative because of \( G(x, \xi) \geq 0 \) and since \( v_i^{(n)}(x, \lambda^*) > 0, \ x \in [0, 1] \) due to (32). Because of (31) and (33),
\[ \lambda^{(n)} > \lambda^* > \bar{\lambda} \]
and this contradiction to the assumed existence of the point of accumulation \( \bar{\lambda} \leq |\lambda_1| \) completes the proof of Theorem 7.

Therefore, the solution of (27) can be approximated iteratively by use of (30) in the set \( \{(x, \lambda): 0 < \lambda \leq 1, 0 < A < |\lambda_1|\} \); in this set, \( v^{(n)}(x, \lambda) \) is a test function for (20), and the system under consideration \( P[y] = 0 \) possesses at most one solution \( y \) because of Theorem 3 if
\[ \lambda^{(n)} > 1, \]  
(35)

Since \( \lambda_1 \) is not known a priori, \( \lambda^{(n)} \) for every \( n \in \mathbb{N} \) thus may have a value smaller than one. In practical applications, a sequence \( \lambda^{(n)} \) will be constructed by use of a sequence \( v^{(n)}(x, \lambda^{(n-1)}) \) of test functions in \( \{(x, \lambda): 0 \leq x \leq 1, 0 < \lambda < |\lambda^{(n-1)}|\} \) instead of \( v^{(n)}(x, \lambda) \). This has been carried out in the subsequent example.

By use of the iteration method (30) and Theorem 7, a sequence of lower bounds \( \lambda^{(n)} \), \( n \in \mathbb{N} \), of the modulus of the eigenvalue \( \lambda_1 \) of (20) may be constructed.

The sequences \( v^{(n)}(x, \lambda) \) and \( \lambda^{(n)}, \ n \in \mathbb{N}, \) can also be constructed for an eigenvalue problem with ordinary first order differential equations provided one boundary condition is given for each equation; in this case only the kernel \( G(x, \xi) \) has to be replaced. The construction of the sequences \( v^{(n)}(x, \lambda) \) and \( \lambda^{(n)} \) for a system of elliptic differential equations is possible; however, the kernels in the equivalent system of integral equations in general are not available explicitly.

\textbf{Example.}  The eigenvalue problem
\[ \hat{L}[\xi; P, y_1, y_2] = \begin{cases} -\xi' - \lambda[(2 + 2x) \xi_1 + (2 + 4x) \xi_2] & \\
-\xi_2' - \lambda[(2 + 6x) \xi_1 + (2 + 8x) \xi_2] \quad \text{in } x \in (0, 1) & \\
\xi_i(0) = \xi_i(1) & 
\end{cases} = 0, \]
\[ i = 1, 2, \ldots \]  
(36)
is considered. By use of \( v^{(1)}_1 = \sin[n(x + 10^{-4})/(1 + 2 \cdot 10^{-4})], \ i = 1, 2, \) the iteration yields \( \lambda^{(1)} = 0.548, \ \lambda^{(2)} = 0.945, \ \lambda^{(3)} = 1.03, \) i.e., \( v^{(3)}(x, \lambda^{(2)}) \) is
a test function; from the existence of $v^q > 0$ it follows that every nonlinear system $P[y] = 0$ from which $L[\xi; P, y_1, y_2]$ is derived, possesses at most one solution $y$. In this example, the right-hand side term 1 in (27) has been replaced by an arbitrarily small positive constant $\epsilon$.

6. Application to Chemically Reacting Flowing Systems

The fields of velocity $A(x, t)$, pressure $p(x, t)$, concentrations $c_i(x, t)$ with $i = 1(1)N - 1$, and temperature $T(x, t)$ are considered in the domain $G = \{(x, t): 0 < x_1 < \infty, 0 < x_2 < H, 0 < x_3 < L, 0 < t < \infty\}, H, L \in \mathbb{R}^+$. It is assumed that (a) $A$ and $p$ do not depend on the state vector $y = c_1, \ldots, c_{N-1}, T$ and (b) $c_j \leq c_{N-1}$ for $j = 1(1)N - 2$. According to [1, p. 571], the transport equations for this system are

$$P_i[y] = \frac{\partial y_i}{\partial t} + (A \cdot \text{grad}) y_i - \text{div}(D_i \text{grad} y_i) - r_i = 0 \text{ in } G, \quad i = 1(1)N,$$

(37)

and boundary conditions where the $r_i$ represent the rates of chemical or heat production, respectively, per unit volume. The $r_i$ are assumed to satisfy Lipschitz-conditions; i.e., there exist $q_{ij}, \tilde{q}_{ij} \in \mathbb{R}$ such that

$$q_{ij} \leq \frac{\Gamma_i(w_1, \ldots, w_{j-1} u, \ldots, u_N) - \Gamma_i(w_1, \ldots, w_j u, \ldots, u_N)}{u_j - w_j} \leq \tilde{q}_{ij},$$

$$i, j = 1(1)N.$$  

(38)

It is assumed that the transport coefficients $D_i$ are real positive numbers in case of laminar flow or given nonnegative functions of $x, t$ in case of turbulent flow. For simplicity, only boundary conditions of the first kind are considered. An a priori set of functions $Y$ with $u, w \in Y$ may be determined by use of [5]. According to (2),

$$L_i[z, P, u, w] =$$

$$= \begin{cases} 
\frac{\partial x_i}{\partial t} + \sum_{m=1}^{3} \left[ \left( A_m - \frac{\partial D_i}{\partial x_m} \right) \frac{\partial x_i}{\partial x_m} - D_i \frac{\partial^2 x_i}{\partial x_m^2} \right] \\
- \frac{\Gamma_i(u) - \Gamma_i(w_1, \ldots, u_N)}{u_1 - w_1} x_1 - \ldots \\
- \frac{\Gamma_i(w_1, \ldots, w_{N-1}, u_N) - \Gamma_i(w)}{u_N - w_N} x_N = 0 \text{ in } G, \quad i = 1(1)N, \\
Z_i = 0 \text{ on the parabolic boundary.} 
\end{cases}$$

(39)
This system possesses the trivial solution $\mathbf{z} = \mathbf{u} - \mathbf{w} = \mathbf{0}$ only, e.g., [14, p. 263], i.e., transient solutions of (37) are unique. In the steady-state case, eigensolutions of the elliptic system $L_{\phi}\mathbf{z}; \mathbf{P}, \mathbf{u}, \mathbf{w} = \mathbf{0}$ are possible; at most one steady-state solution of (37) exists if $L_{\phi}\mathbf{z}; \mathbf{P}, \mathbf{u}, \mathbf{w}$ is of monotonic type. This is true if a test function $V_i^* = \sin(\pi x_1 + \epsilon)/(\beta + 2\epsilon), \ i = 1(1)N, V_i^* > 0$ in $x_2 \in [0, H]$ satisfies the inequalities

$$\frac{dV_i^*}{dx_2} \left[ A_2 - \frac{\partial D_i}{\partial x_2} \right] - D_i \frac{d^2 V_i^*}{dx_2^2} - \sum_{i=1}^{N} V_j^* \max\{-q_{ij}, \tilde{q}_{ij}\} > 0,$$

in $x_2 \in [0, H], \ i = 1(1)N.$

Then (37) possesses at most one solution. The stability of this solution can be treated according to Section 4. This method of proving the uniqueness and stability of a solution of (37) is simpler and more general in many respects than the pertinent theory in [5]. For $A = 0, (37)$ applies to catalyst particles.

References
