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Letter to the editor

On the conditions for the coincidence of two cubic Bézier curves

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ABSTRACT

In a recent article, Wang et al. [2] derive a necessary and sufficient condition for the coincidence of two cubic Bézier curves with non-collinear control points. The condition reads that their control points must be either coincident or in reverse order. We point out that this uniqueness of the control points for polynomial cubics is a straightforward consequence of a previous and more general result of Barry and Patterson, namely the uniqueness of the control points for rational Bézier curves. Moreover, this uniqueness applies to properly parameterized polynomial curves of arbitrary degree.

1. Introduction

Given two integral (i.e., polynomial) cubic Bézier curves, what are the conditions for these cubics to be coincident? By *coincident* we mean that they define the same locus of points, i.e., they are *geometrically equivalent* according to the terminology introduced in [1]. This problem was recently tackled in [2], where a necessary and sufficient condition was obtained for two cubics with non-collinear control points to be coincident, namely that their control point lists be either coincident or the reverse of each other.

In essence, this result is tantamount to the uniqueness of the cubic Bézier representation. Therefore, it can be derived in a straightforward manner in a broader context: the uniqueness of the Bézier points of rational curves, as shown in this short note. In Section 2, we briefly review the fundamental results regarding this uniqueness of the Bézier points, and apply them in Section 3 to the particular case of polynomial curves.

2. The uniqueness of the control points for rational Bézier curves

Given a degree-n rational Bézier curve $\mathbf{b}(t),\ t \in [0, 1]$, suppose that we want to obtain other rational Bézier representations that are geometrically equivalent to $\mathbf{b}(t)$. Berry and Patterson [3] proved that there exist only three procedures:

(1) Projective (i.e., Moebius or rational bilinear) transformation of the parameter space [4], which does not modify the degree.
(2) Nonlinear rational parameter substitution $t(s)$ of degree $m \geq 2$. The new representation, called *improperly parameterized* [5], has degree mn.
(3) Introduction of *base points* [6], also called *generalized degree elevation* [1]. We multiply $\mathbf{b}(t)$ by a rational representation $r(t)$ of the unit function. If $r(t)$ has degree m, then the degree-elevated curve has degree $m + n$.

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The customary standard Bézier degree elevation [7] fits into this scheme as the limit case of introducing a base point at infinity. In addition, Denker and Herron [1] proved that generalized degree elevation is tantamount to a combination of Moebius transformations and standard degree elevation, already studied in [8].

Degenerate curves [9] are those that can be obtained from a lower-degree representation by either procedure (2) or procedure (3), or a combination of them. Since a Moebius reparameterization (1) is the only one of these three procedures that does not change the control points, a fundamental property of non-degenerate representations is the uniqueness of their Bézier points [3]:

Theorem 1. *All non-degenerate Bézier representations* $\mathbf{b}(t),\ t \in [0, 1]$, *of a rational curve segment relate to one another exclusively by Moebius transformations of the parameter space that retain the endpoints of the domain. In consequence, the Bézier points of the segment are uniquely defined (aside from reverse order).*

Remark 1. Rather than dealing with arbitrary rational parameterizations, we are dealing with rational Bézier representations $\mathbf{b}(t)$, by definition over a unit domain. The original result in [3] considered thus Moebius transformations that keep 0 and 1 fixed. With the addendum "aside from reverse order" we have slightly expanded this to encompass transformations reversing the direction of the segment (and hence the order of the Bézier points).

3. The particular case of polynomial curves

Let us apply the results of the previous section to the case of rational curves with unit weights, that is, polynomial curves. Observe that the only ways to derive geometrically equivalent representations of a given curve are restricted now to those particular instances of procedure (1)–(3) that preserve the polynomial character of a parameterization:

(1) Linear transformation of the parameter.
(2) Nonlinear polynomial parameter substitution $t(s)$ of degree $m \geq 2$.
(3) Introduction of a base point at infinity (i.e., standard degree elevation).

Let us see how these observations help us rewrite Theorem 1 in a more specific manner for polynomial curves. The only allowable Moebius transformations $t(s)$ involved in Theorem 1 is restricted now to linear transformations (1) that preserve the endpoints, namely either the identity ($s = t$) or a reversal ($t = 1-s$). Consider also what happens to the control points of a properly parameterized Bézier curve. It must be either non-degenerate, a case in which Theorem 1 applies, or degenerate, i.e., obtained via standard degree elevation (3) from a non-degenerate representation of lower degree. Since once again by Theorem 1 this non-degenerate curve has a unique set of control points, the unique procedure of standard degree elevation generates a unique set of control points (for a given final degree). Therefore, in Theorem 1, non-degeneracy can be replaced by the milder condition of proper parameterization:

Corollary 1. *The Bézier points (for a given degree) of a properly parameterized polynomial segment are uniquely defined (aside from the reverse order).*

Moreover, as Sederberg [5] notes, any improperly parameterized cubic (or quadratic) curve is a multiply defined straight line. In consequence, Corollary 1 becomes even more specific for cubics:

Corollary 2. *The cubic Bézier points of a polynomial segment (different from a straight line) are uniquely defined (aside from reverse order).*

By the linear precision property of Bézier curves [7], Corollary 2 is equivalent to Theorem 2.4 in [2], which gives the necessary and sufficient condition for the coincidence of two cubic Bézier curves with non-collinear control points. This condition is derived through detailed analysis of the relationship between two different parameterizations $\mathbf{b}(s), \mathbf{b}(t),\ s, t \in [0, 1]$, of the same segment, reaching the conclusion that they either coincide ($s = t$) or one is the reversal of the other ($t = 1 - s$).

Finally, this uniqueness of the Bézier representation does not extend to arbitrary polynomial curves $\mathbf{b}(t)$ of degree ≥ 4, as already observed in [2]. Indeed, Corollary 1 states that it applies only to curves with proper parameterization. Sederberg [5] presents an algorithm for detecting and correcting improper parameterizations. More recent results on this topic are due to Pérez-Díaz [10].

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References

[1] W.A. Denker, G.J. Herron, Generalizing rational degree elevation, Comput. Aided Geom. Design 14 (5) (1997) 399–406.
[2] W.-K. Wang, H. Zhang, X.-M. Liu, J.-C. Paul, Conditions for coincidence of two cubic Bézier curves, J. Comput. Appl. Math. 235 (17) (2011) 5198–5202.
[3] T.G. Berry, R.R. Patterson, The uniqueness of Bézier control points, Comput. Aided Geom. Design 14 (9) (1997) 877–880.
[4] R.R. Patterson, Projective transformations of the parameter of a Bernstein–Bézier curve, ACM Trans. Graph. 4 (4) (1985) 276–290.
[5] T.W. Sederberg, Improperly parametrized rational curves, Comput. Aided Geom. Design 3 (1) (1986) 67–75.
[6] G. Farin, NURB Curves and Surfaces: From Projective Geometry to Practical Use, second ed., AK Peters, 1999.
[7] G. Farin, Curves and Surfaces for Computer Aided Geometric Design, fifth ed., Morgan Kaufmann, 2001.
[8] G. Farin, R. Worsey, Reparameterization and degree elevation for rational Bézier curves, in: G. Farin (Ed.), NURBS for Curve and Surface Design, SIAM, 1991, pp. 47–58.
[9] T.W. Sederberg, Degenerate parametric curves, Comput. Aided Geom. Design 1 (4) (1984) 301–307.
[10] S. Pérez-Díaz, On the problem of proper reparametrization for rational curves and surfaces, Comput. Aided Geom. Design 23 (4) (2006) 307–323.