JOURNAL OF ALGEBRA 36, 98–127 (1975)

2-Transitive Groups Whose 2-Point Stabilizer has 2-Rank 1*

MICHAEL ASCHBACHER

California Institute of Technology, Pasadena, California 91109

Communicated by M. Hall, Jr.

Received April 3, 1974

THEOREM 1. *Let G^{Ω} be a doubly transitive permutation group in which the stabilizer of 2 points has 2-rank 1. Then either*

(1) *G has a regular normal subgroup, or*

(2) $G \leqslant \mathrm{Aut}(L)$ *and* L^{Ω} *is* $L_2(q)$, $Sz(q)$, $U_3(q)$, *or* $R(q)$, *in its natural doubly transitive representation, or* $L_2(11)$ *or* M_{11} *on* 11 *letters.*

$R(q)$ denotes a group of Ree Type on $q^3 + 1$ letters.

For odd degree, Theorem 1 is a corollary to the classification of finite groups with a proper 2-generated core [2]. For even degree, Theorem 1 is a corollary to the following theorem:

THEOREM 2. *Let G^{Ω} be a doubly transitive group of even degree in which a Sylow 2-subgroup of the stabilizer of 2 points is cyclic, quaternion, or dihedral. Then either*

(1) G^{Ω} *has a regular normal subgroup, or*

(2) $G \leqslant \mathrm{Aut}(L)$, *and* L^{Ω} *is* $L_2(q)$, $U_3(q)$, $R(q)$, A_6, *or* A_8, *in its natural doubly transitive representation, or* M_{11} *on* 12 *letters.*

The proof of Theorem 2 involves work of M. O'Nan [17] and of the author [3] on doubly transitive groups in which the stabilizer of a point is local.

The author would like to thank Professor Michael O'Nan for a careful reading of this manuscript, leading to a number of improvements.

1. NOTATION

Let G be a permutation group on a set Ω, $X \subseteq G$, and $\Delta \subseteq \Omega$. Then $F(X)$ is the set of fixed points of X on Ω. $G(\Delta)$ and G_{Δ} are the global and pointwise stabilizer of Δ in G, respectively. Set $G^{\Delta} = G(\Delta)/G_{\Delta}$ with induced permutation representation.

* Research partially supported by NSF GP-35678.

Usually G^{Ω} is 2-transitive, $\alpha, \beta \in \Omega$, $H = G_{\alpha\beta}$, t is an involution with cycle (α, β), $D^* = D\langle t\rangle$, $U \in \mathrm{Syl}_2(D)$, and $U^* = U\langle t\rangle \in \mathrm{Syl}_2(D^*)$.

"Regular normal subgroup" is abbreviated by RNS and "fixed point free" is abbreviated by FPF.

Most of the group theoretic notation is standard and taken from [8].

Given groups A and B, AYB denotes the central product of A and B with identified centers.

$\mathrm{Fit}(G)$ is the Fitting subgroup of G. $E(G)$ is the product of all quasisimple subnormal subgroups of G. $F^*(G) = \mathrm{Fit}(G)\, E(G)$.

$S(q)$ is the group of transformations $x \to ax^{\theta} + b$ on $GF(q)$, where $0 \neq a$ and b are in $GF(q)$ and $\theta \in \mathrm{Aut}(GF(q))$.

2. Preliminary Results

LEMMA 2.1. (Manning, [16]) *Let G^{Ω} be a transitive permutation group, $\alpha \in \Omega$, $H = G_{\alpha}$, and $X \subseteq H$. Let k be the number of orbits of H on $X^G \cap H$, $r = |X^H|$, $s = |X^G \cap H|$, and $m = |F(X)|$. Then*

(1) *$N(X)^{F(X)}$ has exactly k orbits, and*

(2) $|\alpha^{N(X)}| = mr/s$.

2.1 will be applied to situations where X is an ordered or an unordered set.

LEMMA 2.2. *Let Q be a subgroup of prime order in G, R a 2-subgroup of G, and $Z = C_K(Q)$. Assume $RQ \trianglelefteq G$, $R = [R, Q]$, $Z \trianglelefteq G$, $m(Z) \leqslant 2$, and G is transitive on $(R/Z)^{\#}$. Then one of the following holds:*

(1) $\Phi(R) = 1 = Z$,

(2) $RQ \cong SL_2(3)$,

(3) *G is transitive on $Z^{\#}$ and R is a Suzuki 2-group.*

Proof. Assume (1) does not hold. Then by 2.3 in [3], $\Omega_1(R) \leqslant Z$. Further if $Z \leqslant Z(G)$ the proof shows $RQ \cong SL_2(3)$. We may take $G = O^2(G)$, so as $m(Z) \leqslant 2$, we may assume G is transitive on $Z^{\#}$ and hence R is a Suzuki 2-group.

LEMMA 2.3. *Let U be a dihedral 2-group of order $2r$ and assume U^* is an extension of U by an involution t. Then U^* is isomorphic to one of the following:*

(1) $B_r = \langle v, u, s \colon v^r = u^2 = s^2 = 1, v^u = v^{-1}, u^s = u, v^s = v^{r/2+1}\rangle$,

(2) D_{4r},

(3) Z_4YU,

(4) $Z_2 \times U$.

Proof. If $|U| = 4$ the result is trivial, so assume $|U| > 4$. Let $V = \langle v\rangle$ be the cyclic subgroup of index 2 in U and $u \in U - V$. Then $V \trianglelefteq U^*$.

Suppose V is self-centralizing. Then by 5.4.8 in [8], U^* is either dihedral or $W = C_{U^*}(\mho^1(V))$ is modular. In the latter case we may pick $t \in W$. Then $\langle t, a\rangle = \Omega_1(W) \trianglelefteq U^*$ with ta conjugate to a under V, so we may pick u to centralize t. That is $U^* \cong B_r$.

Next assume $x \in U^* - U$ centralizes V. If $V \leqslant \langle x\rangle$ then $\langle x\rangle$ is a cyclic subgroup of index 2 in U^*, so U^* is dihedral. Thus we may take $x = t$ to be an involution. Let w be an element of order 4 in V. Then either $[u, t] = 1$ and $U^* \cong Z_2 \times U$ or $[u, tw] = 1$ and $U^* \cong Z_4YU$.

LEMMA 2.4. *Let G^Ω be a transitive permutation group whose degree is a power of 2. Assume for each pair of distinct points α and β in Ω that there is a unique* FPF *involution with cycle (α, β). Then if G^Ω is primitive or $O_2(G_\alpha) = 1$ then G has a* RNS.

Proof. Let $H = G_\alpha$ and Δ the set of FPF involutions. If $s, t \in \Delta$ and st is a p-element acting FPF on Ω, then as the degree of G is a power of 2, $p = 2$. On the other hand if $st \in H$ then s and t are both FPF involutions with cycle (α, α^t), so $s = t$. It follows that st is always a 2-element, so by a result of Baer [4], $T = \langle \Delta\rangle$ is a 2-group. So if G^Ω is primitive then T is regular. Also $T_\alpha \leqslant O_2(G_\alpha)$ so if $O_2(G_\alpha) = 1$ then again T is regular.

LEMMA 2.5. *Let X be a group acting on the group Y of odd order and assume*

(1) *X has a normal 2-group T of order at least 4 and X acts transitively on $T^\#$.*

(2) *If $t \in T^\#$ then $[Y, t]$ is cyclic.*

Then $[T, Y] = 1$.

Proof. See 2.9 in [5].

LEMMA 2.6. *Let X, Y, and Z be groups with X acting on Y and Y acting on Z, such that*

(1) *Y has odd order.*

(2) *X has a normal 2-group T or order at least 4 and X acts transitively on $T^\#$.*

(3) *If* $t \in T$ *and* $y \in Y$ *is inverted by* t, *then* y *acts semiregularly on* Z. *Then* $[T, Y] = 1$.

Proof. This follows from 2.5, and 2.4 in [5].

LEMMA 2.7. *Let* G^{Ω} *be a transitive permutation group,* $\alpha \in \Omega$, $H = G_{\alpha}$, a *an involution in* $Z^*(H)$, $m = |F(a)|$, $n = |\Omega|$ *and* Δ *the set of* FPF *involutions in* G. *Assume*

(i) T *is an elementary 2-subgroup normal in* $C(a)$ *with* $T/\langle a \rangle$ *regular on* $F(a)$ *and* $T^{\#} = (C(a) \cap \Delta) \cup (C(a) \cap a^G)$.

(ii) *Every 2 points of* Ω *is fixed by some conjugate of* a.

(iii) $C(a)^{F(a)}$ *is 3/2-transitive of rank* $r \leqslant 4$. *If* $r = 4$ *then* $\langle a \rangle \in \mathrm{Syl}_2(H)$.

Then one of the following holds:

(1) G *has a* RNS *and* $n = m^2$.

(2) G *is an extension of* $L_2(8)$ *or* $L_2(32)$ *and* $n = 28$ *or* 496, *respectively.*

(3) $G \cong Z_2 \times S_4$ *and* $n = 8$, $G \cong Z_2 \times A_5$ *and* $n = 12$, *or* $G \cong A_5$ *and* $n = 6$.

Proof. Let (γ, γ^a) be a cycle in a. $a \in Z^*(H)$ and by (i) and 2.1, $a^G \cap H = a^H$, so a centralizes some conjugate b of a fixing γ and γ^a. Suppose a fixes a second such conjugate c. Then as $a \in Z^*(H)$ and $a^G \cap H = a^H$, bc has odd order. But $b, c \in T$, so bc is a 2-element. Thus b is the unique conjugate of a fixing γ and γ^a, and centralizing a. Let $K = O(G_{\gamma\gamma^a})$. It follows that $C_K(a) \leqslant C_K(b)$. Also $a \in O_2(C(b))$, so $C_K(b) \leqslant C_K(a)$. Thus ab centralizes K.

Next, let $n = |\Omega|$, $m = |F(a)|$, $\Gamma = a^G \cap T$ and $|\Gamma| = k$. Then $|T| = 2m$ and $(n - m)/m = |\Gamma| - 1 = k - 1$. So $n = mk$.

Suppose $T^{\#}$ is fused in G. Then Shult's fusion theorem [19] implies $\langle a^G \rangle \cong L_2(2m)$. As $C(a)^{F(a)}$ is 3/2-transitive of rank at most 4 we conclude G is an extension of $L_2(4)$, $L_2(8)$, or $L_2(32)$ on 6, 28, or 496 letters, respectively. Thus we may assume $T^{\#}$ is not fused.

Suppose $C(a)^{F(a)}$ is 2-transitive. Then $C_H(a)$ has 2 orbits on $T - \langle a \rangle$, so as $T^{\#}$ is not fused, $k = m$. Then the first paragraph implies there exists a unique element of Δ with cycle (α, β) for each $\alpha, \beta \in \Omega$, so by 2.4, G^{Ω} has a RNS.

So we may assume $C(a)^{F(a)}$ is of rank 4 and $\langle a \rangle$ is Sylow in H. $k = r(m - 1)/3 + 1$, $1 \leqslant r \leqslant 6$. If $r = 3$ then $k = m$ and as above G^{Ω} has a RNS. If $k = 1$ or 5 then $k \equiv \pm(2/3) \bmod m$, so as $|G: H| = mk$, $|N(\Gamma)^{\Gamma}| \equiv 2 \bmod 4$ and in particular $N(\Gamma)^{\Gamma}$ is solvable. If r is even then k is odd, $T \in \mathrm{Syl}_2(G)$, and clearly $N(\Gamma)^{\Gamma}$ is solvable.

So $N(\Gamma)^{\Gamma}$ is solvable 3/2-transitive of rank at most 7. Thus N^{Γ} is regular, primitive, or a Frobenius group, and in any event has a RNS.

$a^G \cap H = a^H$. Also $a \in Z^*(H)$ and any two points of Ω are fixed by some conjugate of a, so a fixes a point in each orbit of H. Thus $a \notin Z(H)$.

Suppose k is odd. Then T is an abelian Sylow 2-group of G and $\langle a^G \rangle$ is the direct product of a 2-group with simple groups isomorphic to $L_2(2^i)$, with a projecting on each factor. As $a \notin Z(H)$, $\langle a^G \rangle$ is not a 2-group. So if $|T| = 8$, then $\langle a^G \rangle \cong L_2(8)$ and $T^\#$ is fused or $\langle a^G \rangle \cong Z_2 \times A_5$ and $n = 12$. Thus we may take $|T| > 8$, so that $N(T/\langle a \rangle)$ acts irreducibly on $T/\langle a \rangle$, and again we conclude $\langle a^G \rangle$ is simple and $T^\#$ is fused.

So k is even. Then there exists a 2-element u in $N(T) - T$ with $u^2 \in T$. Suppose $m = 4$. Then $k = 2, 4$ or 6. Also as $T \in \mathrm{Syl}_2\, C(b)$ for each $b \in \Gamma$, $C_\Gamma(u)$ is empty. Thus $k \neq 6$, and if $k = 4$ then as above G^Ω has a RNS. So take $k = 2$, Then $n = 8$. If G possesses elements of order 5 or 7 then G^Ω and then $C(a)^{F(a)}$ is 2-transitive, so no such elements exists, and G is a $\{2, 3\}$-group. As H contains a Sylow 3-group of G, $O_3(G) = 1$. Then $X = O_2(G)$ is transitive on Ω and as $a \notin Z(H)$, $a \notin X$, so X is regular. H contains an element y or order 3 acting nontrivially on X, so as $G \not\cong SL_2(3)$, X is elementary. Thus G is as in (3).

So assume $m > 4$, and let Q^Γ be the RNS for N^Γ. If k is not a power of 2, then N^Γ is not primitive and therefore is Frobenius. k is even so Q is not a p-group. But then N^Γ has rank greater than 7, a contradiction.

Thus k is a power of 2. As $m > 4$, $N(T/\langle a \rangle)$ acts irreducibly on $T/\langle a \rangle$ and thus if $T \leqslant P \in \mathrm{Syl}_2(Q)$ we find $T = (Z(P) \cap T) \times \langle a \rangle$. So as $C_\Gamma(u)$ is empty, $Z(P) \cap T^\# = \Delta \cap T$. So as above, $n = m^2$ and G^Ω has a RNS X.

LEMMA 2.8. *Let $p = 3$ or 5, $H \leqslant GL_3(p)$ and assume $O_p(H) = 1$, H has dihedral Sylow 2-groups, and H has no normal 2-compliment. Then either $A_4 \leqslant H \leqslant S_4$, or $p = 5$ and $A_5 \leqslant H \leqslant S_5$.*

Proof. $p^2 + p + 1$ is a prime and if $p^2 + p + 1$ divides the order of a subgroup H of $GL_3(p)$ with dihedral Sylow 2-groups, then H has a normal 2-complement. Thus if $p = 3$ then H is a $\{2, 3\}$-group, so as $O_3(H) = 1$, $A_4 \leqslant H \leqslant S_4$.

So we may take $p = 5$ and H a $\{2, 3, 5\}$-group. $GL_3(5)$ has a Sylow 3-group of order 3, so as H has no normal 2-complement, $O(H)$ is a $3'$-group. Then as $O_5(H) = 1$, $O(H) = 1$. So either $A_4 \leqslant H \leqslant S_4$ or $A_5 \leqslant H \leqslant S_5$.

LEMMA 2.9. *Let G be a group, a an involution in G, $S \in \mathrm{Syl}_2(C(a))$, and $T \trianglelefteq N(S)$. Then*

(1) *If $a \in T$ then $a^G \cap Z(S) \subseteq T$.*

(2) *If $a \notin T$, each of $aT^\#$ and $T^\#$ is fused, and a is fused to an element of $\langle a \rangle T$, then $aT = a^G \cap T\langle a \rangle$ and $S \notin \mathrm{Syl}_2(G)$.*

Proof. In (1) if $a^g \in Z(S)$ then we may choose $g \in N(S)$. (1) implies (2).

3. 2-Transitive Groups

In this section G^Ω is a 2-transitive group, $\alpha, \beta \in \Omega$, $H = G_\alpha$, $D = G_{\alpha\beta}$, t is an involution with cycle (α, β), $D^* = D\langle t \rangle$, $U \in \mathrm{Syl}_2(D)$, and $U^* = U\langle t \rangle \in \mathrm{Syl}_2(D^*)$. Set $n = |\Omega|$.

Lemma 3.1. *Assume n is even and G is solvable. Then $G \leqslant S(n)$.*

Proof. See [15].

Lemma 3.2. *Assume G has a* RNS *T of even order and a cyclic subgroup X which acts transitively on Ω. Then $G^\Omega = S_4$.*

Proof. Let $2^n = |T|$ and $X = \langle x \rangle$. As T^Ω is transitive, $x = td$, where $t \in T$, and d is a 2-element fixing 2 or more points of Ω. Then $x^2 = [t, d^{-1}]\, d^2$ and by induction on i, $x^{2^i} = [t, d^{-1}, d^{-2}, \ldots, d^{-2^{i-1}}]\, d^{2^i}$.

Let $u = d^{2^{n-2}}$. As d fixes 2 or more points, $|d| < |\Omega| = 2^n$ and hence u is an involution. X^Ω is regular, so $x^{2^{n-1}} \neq 1$ and thus $[t, d^{-1}, \ldots, d^{-2^{n-3}}, u] \neq 1$.

Let $T_{n-2} = C_T(u)$ and $T_{n-i}/T_{n-i+1} = C_{T/T_{n-i+1}}(d^{2^{n-i}})$. Then as $u^2 = 1$, $|T : T_{n-2}| \leqslant |T_{n-2}|$, so $|T/T_{n-2}| \leqslant 2^{[n/2]}$. Similarly by induction on i, $|T/T_{n-i}| \leqslant 2^{n/2^{i-1}}$. Now if $n \geqslant 4$ then $n \leqslant 2^{n-2}$, so $|T/T_1| \leqslant 2^{n/2^{n-2}} \leqslant 2$, and if $n = 3$ then $|T/T_1| \leqslant 2^{[3/2]} = 2$. We may assume $n \geqslant 3$, so $[T, d] \leqslant T_1$.

Now by induction on $k = n - i$ we find

$$[T, d^{-1}, \ldots, d^{-2^k}] \leqslant T_{k+1} = C_{T/T_{k+2}}(d^{2^{k+1}}).$$

In particular

$$[t, d^{-1}, \ldots, d^{-2^{n-3}}] \in T_{n-2} = C_T(u).$$

Therefore $[t, d^{-1}, \ldots, d^{-2^{n-3}}, u] = 1$, a contradiction.

Lemma 3.3. *Assume n is odd and G has dihedral Sylow 2-subgroups. Then either*

(1) *G has a* RNS, *or*

(2) *$G \leqslant \mathrm{Aut}(L)$ and L^Ω is A_5, A_7, or $L_3(2)$ in its natural 2-transitive representation, $L_2(11)$ on 11 letters, or A_7 on 15 letters.*

Proof. We may assume G has no RNS, so $O(G) = 1$. Then by [11], $G \leqslant \mathrm{Aut}(L)$, $L \cong L_2(q)$, q odd, or A_7. If $L \cong L_2(q)$, then [7] yields the result. One can inspect the maximal subgroups of A_7 to determine its representations.

LEMMA 3.4. *Assume G has wreathed, semidihedral, dihedral or abelian Sylow 2-subgroups and n is even. Then either*

(1) *G has a* RNS

(2) $G \leqslant \mathrm{Aut}(L)$ *and L^Ω is $L_2(q)$, $U_3(q)$, $R(q)$, or A_6 in its natural doubly transitive representation, or M_{11} on 12 letters.*

Proof. Either G has a RNS or G is contained in the automorphism group of a simple group L, so we may assume the latter. L is a group of known type. Now apply [7], unless $G = M_{11}$. By inspection of the character table of M_{11}, if G is M_{11} then $n = 12$.

LEMMA 3.5. *Let X be weakly closed in D with respect to G and assume $n = |F(X)|^2$. Then G has a* RNS.

Proof. This follows from 2.1 and a result of Wagner [20].

LEMMA 3.6. *Let a be an involution in D with $C(a)^{F(a)}$ transitive. Set $e = |a^G \cap D^* - D|$, $r = |a^D|$, $s = |a^H \cap D|$, and $m = |F(a)|$. Then $n = m(m-1)e/s + m$.*

Proof. Let Γ be the set of pairs (a^g, c) with c a cycle in a^g. Then $|a^G|(n-m)/2 = |\Gamma| = n(n-1)e/2$. Also as $C(a)^{F(a)}$ is transitive, $|a^G| = n|a^H|/m$. Finally by 2.1,

$$|a^H| = |H: C_H(a)| = (n-1)|D: C_D(a)|/|C_H(a): C_D(a)|$$
$$= (n-1)r/(m-1)r/s.$$

4. PRELIMINARIES TO THEOREM 2

In this section we continue the hypothesis and notation of Section 3. In addition assume n is even and U is cyclic, quaternion or dihedral.

LEMMA 4.1. *Assume G has a* RNS *T, U is cyclic or dihedral, and t is a* FPF *involution. Then either $t \in T$ or $n = 8$ and $H \cong L_3(2)$.*

Proof. Assume $t \notin T$. As T^Ω is transitive, $T\langle t\rangle = T\langle u\rangle$ where $\langle u\rangle = T\langle t\rangle \cap H$ has order 2. So $t = us$, $s \in T$. Now $|F(u)| = |C_T(u)| = m$ and $n \leqslant m^2$. If $n = m^2$ then $C_T(u) = [T, u]$ so that $t = us \in u^T$, impossible as t is FPF.

So $n < m^2$. Then by 3.1, H is not solvable. Let $L/O(H) = E(H/O(H))$. Then $\bar{L} = L/O(H)$ has dihedral Sylow 2-groups. So either $\bar{U} \leqslant \bar{L} \cong L_2(q)$ or A_7, or $\overline{UL} \cong PGL_2(q)$.

Suppose u inverts an element $x \in H$ acting FPF on $T^{\#}$. Then $C_T(u) \cap C_T(ux) \leqslant C_T(x) = 1$, so as $|T| \leqslant |C_T(a)|^2$ for each involution $a \in H$, we get $n = m^2$. So no such x exists.

Now if $u \in \bar{L} \cong L_2(q)$ then u inverts cyclic groups $\bar{X}_\epsilon$ of order $(q - \epsilon)/2$, $\epsilon = \pm 1$, so there are conjugates Y_ϵ of X_ϵ in D. Further if $q \equiv 1 \bmod 4$, u inverts a group $\bar{Q}$ of order q, so some conjugate Q_1 of Q is in D. Then $Y = \langle U \cap L, Y_1, Y_2, Q_1 \rangle \leqslant D$. It follows that either all involutions in $U \cap L$ are fused in $Y \leqslant D$ or $q = 7$ and $\overline{L \cap D} \cong S_4$. Similarly if $\bar{L} \cong A_7$ we conclude $\overline{L \cap D} \cong A_6$ and all involutions of U are fused in D. Finally if $u \in U - L$ and $\overline{UL} \cong PGL_2(q)$, then $u^H \cap U = u^U$, so $u^G \cap D = u^D$.

Thus either $u^H \cap D = u^D$ and by 2.1, $C(u)^{F(u)}$ is 2-transitive, or $\bar{L} \cong L_2(7)$ and $\overline{L \cap D} \cong S_4$.

In the former case $C_H(u)$ is transitive on $C_T(u)^{\#}$ and then on $uC_T(u)^{\#}$. But for $r \in [T, u] \leqslant C_T(u)$, $ur \in u^T$, so $t = us \in (ur)^H \leqslant u^G$, a contradiction.

In the latter case let $\bar{H}_1$ be a subgroup of order 7 in $\bar{L}$. Then H_1T is solvable and 2-transitive, so by 3.1, $Fit(H_1)$ and then also $Fit(H)$ is cyclic. So $L^\infty \cong L_2(7)$. Now let Δ be the set of pairs (u^h, γ), where $\alpha \neq \gamma \in F(u^h)$ and $h \in H$. Then $(m - 1)|u^H| = |\Delta| = |u^G \cap D|(n - 1)$. $|u^H| = 21$ and $|u^G \cap D| = 9$, so $n = 7(m - 1)/3 + 1$. But $n = 2^i$ and $m = 2^j$ with $i > j$. So $0 \equiv 2^i = n = 7(m - 1)/3 + 1 \equiv -4/3 \bmod 2^j$, and then $m = 4$ and $n = 8$.

LEMMA 4.2. *Assume $n \equiv 2 \bmod 4$. Then G is contained in the automorphism group of $L_2(q)$, $U_3(q)$ or A_6, acting in its natural 2-transitive representation.*

Proof. By [1], G contains a simple normal subgroup M with M^Ω 2-transitive and $G \leqslant \mathrm{Aut}(M)$. Now $M \cap U$ is cyclic, quaternion or dihedral. In the first two cases [1] implies the desired result. So we may take $M = G$ and assume U is dihedral. Then $U^* \in \mathrm{Syl}_2(G)$ and as G contains no subgroup of index 2, $|F(u)| \equiv n \equiv 2 \bmod 4$, for each involution $u \in U^*$.

By 2.3, U^* has one of 4 forms. In the last two cases U^* is not Sylow in a simple group unless $U^* \cong E_8$. In that case we appeal to 3.4.

Suppose $U^* \cong B_r$. Then $\langle v^2, u \rangle = C_U(s)$ is dihedral and as $|F(s)| \equiv 2 \bmod 4$, $C_U(s)$ contains a subgroup W of index 2 with $\langle W, s \rangle$ conjugate to a subgroup of U. But $\langle W, s \rangle$ is neither cyclic or dihedral.

It follows that U^* is dihedral. Now appeal to 3.4.

LEMMA 4.3. *Let a and b be commuting, conjugate involutions. Assume $C(a)^{F(a)}$ is 2-transitive with* RNS $T_0^{F(a)}$ *and b acts* FPF *on $F(a)$. Then $b \in T_0$.*

Proof. Assume $b \notin T_0$. By 4.1, $C_H(a)^{F(a)} \cong L_3(2)$ and $|F(a)| = 8$. Let $T \in \mathrm{Syl}_2(T_0)$ and $S = TU \in \mathrm{Syl}_2(C(a))$. Set $\overline{C(a)} = C(a)/O(C(a))$.

If U is cyclic or dihedral then $C_H(a)$ has a normal 2-compliment. So U is quaternion and even $U_{F(a)} = \langle a\rangle$. As $| T/\langle a\rangle| > 4$ and $C_H(a)$ is transitive on $(\bar{T}/\langle\bar{a}\rangle^{\#})$, T is elmentary. As $b \notin T$, $\langle a\rangle = Z(S)$.

The initial arguments in Janko's characterization of M_{23} [22] now show G has one class of involutions. Therefore as $C(a)$ is 2-constrained, signalizer functor arguments show $O(C(a)) = 1$. [21] Hence [22] implies $G = M_{23}$. But a subgroup of M_{23} isomorphic to $SL_2(7)$ does not act nontrivially on a subgroup of odd order, so M_{23} does not have a representation of the required sort.

5. Semiregular Groups

In this section assume the following hypothesis:

Hypothesis 5.1. *$Q \neq 1$ is a subgroup of odd order of the group G, $\Omega = Q^G$, and $H = N_G(Q)$. Represent G by conjugation on Ω and assume $H \neq G$ and Q acts semiregularly on $\Omega - Q$.*

Theorem 5.2. *Let $K \leqslant G$, p a prime, and $P \in \mathrm{Syl}_p(Q)$. Then*

(1) *P is strongly closed in S with respect to G for any $P \leqslant S \in \mathrm{Syl}_p(G)$.*

(2) *K acts transitively on the set*

$$\{Q^g \colon | K \cap Q^g |_p \neq 1\}$$

(3) *If $K \cap Q \neq 1$ and $K \nleqslant H$ then the pair $(K, K \cap Q)$ has hypothesis* 5.1.

(4) *If $K \trianglelefteq G$ either $G = HK$ or $K \cap Q = 1$ and the pair $(G/K, QK/K)$ has* 5.1.

(5) *Assume $G = \langle \Omega\rangle$ and P is not cyclic. Then $G = G'Q$, G' is quasi-simple, and $Q \cap G' \neq 1$.*

(6) *If $K \trianglelefteq G$ and $K \leqslant H$ then $K \leqslant Z(G)$.*

Proof. See Section 3 of [3].

Lemma 5.3. *Let $h \in H$ be centralized by a Sylow 2-subgroup of H and assume $h^2 \neq 1$ but h is inverted in G. Then $C_Q(h) = 1$.*

Proof. Assume $C_Q(h) \neq 1$ and choose p to be a prime divisor of the order of $C_Q(a)$ and $P \in \mathrm{Syl}_p(C_Q(h))$. Choose t with $h^t = h^{-1}$ and let $L = \langle P^{C(h)}\rangle$. By 5.2.2, t normalizes L, and then by 5.2.1, $L\langle t\rangle \leqslant LN(P) \leqslant LH \leqslant C(h)\, H$. So we may choose t to be a 2-element in H. But this is impossible as a Sylow 2-subgroup of H centralizes h.

HYPOTHESIS 5.4. *(G, Q) has hypothesis* 5.1. *a is an involution with $\langle a \rangle$ Sylow in H. The stabilizer of any two points of Ω is of even order. G acts faithfully on Ω. $C(a)^{F(a)}$ has a* RNS *$T_0^{F(a)}$, $T \in \mathrm{Syl}_2(T_0)$ is elementary of order at least* 8, *and $C(a)$ is normal of index at most* 3 *in a subgroup X (possibly not contained in G) doubly transitive on $F(a)$ and acting on $Y = O(C(a)_{F(a)})$.*

LEMMA 5.5. *Assume Hypothesis* 5.4. *Then G satisfies* (1) *or* (2) *of Lemma* 2.7.

Proof. Suppose $y \in Y^{\#}$ is inverted by $t \in T$. Then by 5.3, y acts semiregularly on Q. We conclude from 2.6 that T centralizes Y. Then $T = O_2(C(a)) \trianglelefteq C(a)$. Now 2.7, yields the result.

HYPOTHESIS 5.6. *Hypothesis* 5.1 *is satisfied. H contains no nontrivial cyclic normal subgroups. If $1 \neq A$ is a normal abelian subgroup of H then $C(A)$ is semiregular on $\Omega - \{Q\}$, and is of odd order.*

LEMMA 5.7. *Assume hypothesis* 5.1. *Let X be a* 4*-group in H with $|F(X)| = 2^m > 2$ and let B be an elementary abelian subgroup of Q which is normal in H. Assume*

(1) *$C(X)^{F(X)}$ has an elementary* RNS *Y.*

(2) *P is a subgroup of $C_H(X)$ of odd order such that $P^{F(X)}$ is of prime order p and* FPF *on $F(X) - \{Q\}$.*

(3) *If $x \in X^{\#}$ with $F(x) \neq F(X)$, then $C(x)^{F(x)}$ has a* RNS *of order 2^{2m}. Then $[P, B] = 1$ and Hypothesis* 5.6 *is not satisfied.*

Proof. Let $x \in X^{\#}$. By hypothesis $C(x)^{F(x)}$ has a RNS W. If $F(x) \neq F(X)$ then $|W| = |Y|^2$, $Y = C_W(X)$, and the representation of P on Y is equivalent to its representation on W/Y under the map $Yw \to [w, X]$. In particular $P^{F(x)}$ is semiregular on $W^{\#}$. Now $C_B(x)$ is also semiregular on $W^{\#}$ and normalized by P with $\Phi(B) = 1$, so $[P, C_B(x)] \leqslant Q_{F(x)} = 1$.

Therefore $B = \prod_{X^{\#}} C_B(x) \leqslant C(P)$. Assume Hypothesis 5.6. Then we may take $Q = C(B)$, so that $P \leqslant Q$. Now P is the unique subgroup of order p in $C_Q(x)$, so $P \trianglelefteq \prod_{X^{\#}} C_Q(x) = Q$. Hence we may take $P \leqslant B$, and then $P = C_B(x)$, each $x \in X^{\#}$. So $P = B \trianglelefteq H$, contrary to Hypothesis 5.6.

6. 2-TRANSITIVE SEMIREGULAR GROUPS

In this section we operate under the following hypothesis:

HYPOTHESIS 6.1. *Hypothesis* 5.1 *holds with G^{Ω} doubly transitive. $Q = C_G(Q)$ and a is an involution inverting Q with $|F(a)| > 2$.*

LEMMA 6.2. *$H = QC_H(a)$ and $a^H \cap D = \{a\}$ for all $Q^t \in F(a)$ and all $D = H \cap H^t$.*

Proof. As $Q = C_G(Q)$ and a inverts Q, $Q\langle a\rangle \trianglelefteq H$. As Q has odd order, $H = QC_H(a)$. If $Q^t \in F(a)$, then $Q\langle a\rangle \cap D = \langle a\rangle$ as Q is semiregular on $\Omega - Q$.

LEMMA 6.3. *$C(a)^{F(a)}$ is 2-transitive and a fixes a unique point in each Q orbit.*

Proof. As $a^H \cap D = \{a\}$, $C_H(a)$ is transitive on $F(a) - Q$ by 2.1. Let $Q \neq Q^g \in F(a)$. If $C_{Q^g}(a) \neq 1$ then $C_{Q^g}(a)$ moves Q to a point $Q^x \in F(a)$ inverted by a. So we may choose Q^g inverted by a. So $C_{H^g}(a)$ is transitive on $F(a) - Q^g$. Thus as $|F(a)| > 2$, $C(a)^{F(a)}$ is 2-transitive.

$a^H = a^Q$ and H is transitive on the nontrivial Q-orbits, so a fixes a point in each such orbit. As $C_Q(a) = 1$, a fixes a unique point in each orbit.

LEMMA 6.4. *Let $Y \leqslant G_{F(a)}$ with $C_Q(Y) \neq 1$ and let $L = \langle a^{C(Y)}\rangle$. Then*

(1) *$L^{F(Y)}$ is transitive*

(2) *$C_L(a)^{F(a)}$ is transitive.*

Proof. By 6.2 and 6.3, $a^G \cap H = a^Q$. So $a^G \cap C_H(Y) = a^{C_Q(Y)}$. Given points $\gamma, \delta \in F(Y)$, Y centralizes the conjugate b of a fixing γ and δ. Now there exists a conjugate c of a fixing a unique point of $F(a)$ and $F(b)$. Then $a, b \in c^{C(Y)}$ so $C(Y)$ is transitive on the conjugates Δ of a fixing 2 or more points of $F(Y)$. Then $\Delta = a^{C(Y)}$ and $L = \langle \Delta\rangle$.

Let $k + 1 = |F(Y)|$ and $m + 1 = |F(a)|$. By 6.3, $k = m \mid C_Q(Y)|$. As $C(Y)^\Delta$ is transitive and $C(a)^{F(a)}$ is 2-transitive, $k = m \mid C_{Q^g}(Y)|$ for each $Q^g \in F(Y)$. Thus $| C_Q(Y)| = | C_Qg(Y)|$ and by 5.2.2, $L^{F(Y)}$ is transitive. As $a^{C(Y)} \cap H = a^{C_H(Y)}$, 2.1 implies $C_L(a)^{F(a)}$ is transitive.

LEMMA 6.5. *Let p be an odd prime and $K = O_p(G_{F(a)})$. Assume either:*

(1) *$C(a)^{F(a)}$ contains no transitive subgroups with cyclic Sylow 2-groups,*

or

(2) *$C(a)^{F(a)}$ is an extension of $L_2(q)$, $q \equiv -1 \bmod 4$, on $q + 1$ letters, and if $U^{F(a)} \neq 1$ then $U_{F(a)} \leqslant C(K)$.*

Then $C(K\langle a\rangle)^{F(a)}$ is transitive.

Proof. Let X be an abelian subgroup of K. Then there exists $Y \leqslant X$ with $C_Q(Y) \neq 1$ and X/Y cyclic. By 6.4, $C(\langle a\rangle\ Y)^{F(a)}$ is transitive. Assume $X \trianglelefteq N_G(Y) \cap C(a)$ and if $U^{F(a)} \neq 1$ then $u \in N(X)$, for some $u \in U - U_{F(a)}$.

Suppose there exists no 2-element $t \in C(\langle a \rangle X)$ acting nontrivially on $F(a)$. We may take $X = \Omega_1(X)$, so $|X/Y| = p$. Let $S \in \mathrm{Syl}_2(C(\langle a \rangle Y))$. Then S acts on X/Y, so $S/C_S(X)$ is cyclic. By assumption $C_S(X) \leqslant G_F(a)$, so $S^{F(a)}$ is cyclic. Therefore Hypothesis (1) cannot hold and then $C(a)^{F(a)}$ is an extension of $L_2(q)$. As $S^{F(a)}$ is cyclic we get $U^{F(a)} \neq 1$. We may assume $T = \langle u, S \rangle$ is a 2-group. Then $T^{F(a)}$ is Sylow in $C(a)^{F(a)}$ and is dihedral with $|T: S| = 2$. But then T normalizes $[S, X]$ which is of order p, so $T/C_T(X)$ is cyclic and then as $T^{F(a)}$ is dihedral, $C_T(X)^{F(X)} \neq 1$, contrary to assumption.

So there exists a 2-element $t \in C(\langle a \rangle X)$ acting nontrivally on $F(a)$.

Let X_1 be a critical subgroup of K. (That is X_1 is characteristic in K of exponent p and class at most 2, such that all nontrivial p'-automorphisms of K act nontrivally on X_1.) Let $X_2 = Z(X_1)$, and let Y_2 be a subgroup of index at most p in X_2 with $C_Q(Y_2) \neq 1$.

If $X_2 = Y_2$ we may choose $Y_2 \leqslant Y_1$ of index at most p in X_1 with $C_Q(Y_1) \neq 1$. Now arguing as above there exists a 2-element $t \in C(\langle a \rangle X_1)$ acting nontrivial on $F(a)$. If $X_2 \neq Y_2$ let $X_3 \in SCN(X_1)$. Then $X_3 = Y_3 X_2$ for some $Y_2 \leqslant Y_3$ of index p in X_3 with $C_Q(Y_3) \neq 1$, so $X_3 \trianglelefteq N_G(Y_3) \cap C(a)$. As u induces an automorphism of order at most 2 on K we may choose $u \in N(X_3)$. We conclude there exists a 2-element $t \in C(X_3\langle a \rangle)$ acting nontrivially on $F(a)$. As $X_3 \in SCN(X_1)$, the Thompson $A \times B$ lemma implies $[t, X_1] = 1$.

So in any event we may choose $[t, X_1] = 1$. Then as X_1 is critical, $[t, K] = 1$. So $C(\langle a \rangle K)^{F(a)} \neq 1$. But $K \trianglelefteq C(a)$, so $C(\langle a \rangle K)^{F(a)} \trianglelefteq C(a)^{F(a)}$. Then as $C(a)^{F(a)}$ is 2-transitive, it follows that $C(\langle a \rangle K\rangle)^{F(a)}$ is transitive.

7. Proof of Theorem 2

For the remainder of this paper G is counterexample of minimal orler, to Theorem 2, $\alpha, \beta \in \Omega$, $H = G_\alpha$, $D = G_{\alpha\beta}$, t is an involution with cycle (α, β), $D^* = D\langle t \rangle$, $U \in \mathrm{Syl}_2(D)$, and $U^* = U\langle t \rangle \in \mathrm{Syl}_2(D^*)$, and $n = |\Omega|$. Let $V = \langle v \rangle$ be a cyclic subgroup of index 2 in U, and let a be the involution in V.

Lemma 7.1. $O_\infty(G) = 1$.

Proof. G has no RNS.

Lemma 7.2. *G possesses no proper normal 2-transitive subgroup.*

Proof. If $G_0 \lhd G$ with G_0^Ω 2-transitive, then G_0 satisfies the hypothesis of Theorem 2, and then, by minimality of G, satisfies the conclusion of Theorem 2. This forces G to also satisfy the conclusion of Theorem 2.

LEMMA 7.3. $n \equiv 0 \bmod 4$.

Proof. See 4.2.

LEMMA 7.4. *Let u be an involution in G. Then $|F(u)| \equiv 0 \bmod 4$.*

Proof. We may assume $u \in U$. Then by 7.3, u induces an even permutation on Ω. So $|F(u)| \equiv n \equiv 0 \bmod 4$.

LEMMA 7.5. *Assume U is dihedral and let $x \in U$ with $x^2 \neq 1$. Then either*

(1) *$\{x, x^{-1}\} = x^G \cap U$ and $C(x)^{F(x)}$ is 2-transitive, or*

(2) *$\{x, x^{-1}\} \subset x^G \cap U$ and $|F(x)| = 2$.*

Proof. $\{x, x^{-1}\} = x^D \cap U$ and by 2.1, $C(x)^{F(x)}$ is 2-transitive if and only if $x^D \cap U = x^G \cap U$. But as U is dihedral and $x^2 \neq 1$, $X = \langle x \rangle$ is weakly closed in U with respect to G, so by 2.1, $N(X)^{F(X)}$ is 2-transitive. As $|F(X)|$ is even, $O^2(N(X))^{F(X)}$ is also 2-transitive unless $|F(x)| = 2$. But as X is cyclic, $O^2(N(X)) \leqslant C(X)$.

LEMMA 7.6. *If $1 \neq A$ is an abelian normal subgroup of H then $C_H(A)$ is of odd order and acts semiregularly on $\Omega - \alpha$. Further $G = \langle A, A^g \rangle = G'A$ with G' simple and $A \cap G' \neq 1$. A is not cyclic.*

Proof. Assume A is not semiregular on $\Omega - \alpha$. Then by [17], G is an extension of $L_m(q)$ acting on $m - 1$ dimensional projective space. As n is even, $m \geqslant 4$, so U is not cyclic, quaternion or dihedral.

So A acts semiregularly on $\Omega - \alpha$. Then by 3.3, Theorem 3 in [2] and Theorem 4 in [3], $G = \langle A, A^g \rangle$ and $C_H(A)$ acts semiregularly on $\Omega - \alpha$. Next, by [12], $C_H(A)$ has odd order. Finally, by Theorem 3 in [3], A is not cyclic.

Now the pair (G, A) satisfies hypothesis 5.1, so everything else follows from 5.2.

LEMMA 7.7. $\mathrm{Fit}(H) \neq 1$ *if and only if* $E(H) = 1$. *In any event* $\mathrm{Fit}(H)$ *has odd order.*

Proof. By 7.6, $\mathrm{Fit}(H)$ is of odd order and if $\mathrm{Fit}(H) \neq 1$, then $E(H) \leqslant C_H(\mathrm{Fit}(H))$ is of odd order.

LEMMA 7.8. *If U is dihedral then U does not act semiregularly on $\Omega - F(U)$.*

Proof. Assume U is dihedral and acts semiregularly on $\Omega - F(U)$. Then $H(F(U)) = X$ is strongly embedded in H, so by [6], $H/O(H) \cong L_2(4)$ and $X = O(H)\, N_H(U)$. As $O(H) \leqslant H(F(U))$ and $N(U)^{F(U)}$ is 2-transitive, 7.6 implies $O(H) = 1$. So $H \cong L_2(4)$. Then $D = U$ or $N_H(U)$ and $n - 1 = 15$ or 5. As $n \equiv 0 \bmod 4$, $D = U$ and $n = 16$. But U is weakly closed in D and $|F(U)| = 4$, so 3.5 yields a contradiction.

LEMMA 7.9. *Assume $C(a)^{F(a)}$ is 2-transitive and let $W = U_{F(a)}$ and $S \in \mathrm{Syl}_2(C(a))$. Then*

(1) *If U is dihedral $|U: W| \leqslant 2$.*

(2) *Either $C(a)^{F(a)}$ has a* RNS *$T_0^{F(a)}$ or a characteristic subgroup $L_0^{F(a)}$ isomorphic to $L_2(q)$, $q \equiv -1 \bmod 4$, on $q + 1$ letters.*

(3) *$L_0 = G_{F(a)} C_{L_0}(W)$ and $C_{L_0}(W)/O(C_{L_0}(W))$ is isomorphic to $Z(W) \times L_2(q)$ or $Z(W)\, Y SL_2(q)$ with $S \in \mathrm{Syl}_2(G)$ in the latter case.*

(4) *$T_0 = C(a)_{F(a)}\, C_{T_0}(W)$ and letting $W \leqslant T \in \mathrm{Syl}_2(T_0)$ either*

(i) *$T = WYE$, where $E = [T, N_H(T) \cap C(W)]$ is elementary or quaternion of order 8, or*

(ii) *$|F(a)| = 4$, $W \cong Q_8$, and $C_T(W)$ is elementary or quaternion, or*

(iii) *U is quaternion, $\Phi(T) = 1$, and $W = \langle a \rangle$.*

Proof. Minimality of G and 7.4 imply either $C(a)^{F(a)}$ has a RNS $T_0^{F(a)}$ or a characteristic subgroup $L_0^{F(a)}$ isomorphic to $L_2(q)$, $U_3(q)$, $R(q)$, $q \equiv -1 \bmod 4$, M_{11} or A_8. By a Frattini argument, $C(a) = G_{F(a)}X$, where $X = N(W) \cap C(a)$. As W is cyclic, dihedral or quaternion and X centralizes a, either $O^2(X) \leqslant C(W)$, or $W \cong Q_8$ and $O^2(X)/C(W) \cap O^2(X) \cong Z_3$. So $C(W)$ covers L_0 or T_0 as the case may be.

Assume L_0 exists and let $A = C_{L_0}(W)$ and $\bar{A} = A/O(A)$. Then $A = Z(W)\, B$, where $B = O^2(A)$. If $L_0^{F(a)} \cong R(q)$ M_{11} or $U_3(q)$, then the multiplier of $L_0^{F(a)}$ is of odd order, so $\bar{A} = Z(\bar{W}) \times \bar{B}$ and $\bar{B} \cong L_0^{F(a)}$. If $\bar{B} = R(q)$ then the outer automorphism group of B is of odd order and $|B \cap U| = 2$. So $U = W \times (B \cap U)$ and U is dihedral of order 4. Now by the Z^*-theorem, a is conjugate to an element $b \neq a$ of $C(a)$, and as B has one class of involutions we may pick $b \in U$. As $C(a)^{F(a)}$ is 2-transitive, 2.1 implies b is fused to a in D. So $U^{\#}$ is fused in D. Then all involutions in $C(a)$ are conjugate to a, so $S \in \mathrm{Syl}_2(G)$. Now 3.4 implies a contradiction.

Suppose $\bar{B} \cong U_3(q)$. Then $B \cap U$ is cyclic of order greater than 2, so U is not cyclic, quaternion or dihedral. Similarly $\bar{B} \not\cong M_{11}$.

Suppose $L_0^{F(a)} \cong A_8$. Then $U^{F(a)}$ is dihedral and $H^{F(a)}$ is not solvable, so U is quaternion and $\bar{B} \cong \hat{A}_8$. Then a is the unique involution in the center of S, so $S \in \mathrm{Syl}_2(G)$. Now by Theorem A in [9], G is McLaughlin's group. But then G does not have a 2-transitive representation.

This yields (2). To complete (3) we remark that if $\bar{A} \cong Z(W)\, YSL_2(q)$ then a is the unique involution in the center of S, so $S \in \mathrm{Syl}_2(G)$.

Assume U is dihedral. Then $C_H(a)$ has a normal 2-compliment, so $H^{F(a)}$ is solvable, and then with 3.1, $U^{F(a)}$ is cyclic. This yields (1).

Assume T_0 exists and let $W \leqslant T \in \mathrm{Syl}_2(T_0)$. If $W = \langle a \rangle$ then as $C(a) \cap N(T)$ is transitive on $(T/\langle a \rangle)^{\#}$, $\Phi(T) = 1$. Hence if U is quaternion we may assume $W \neq \langle a \rangle$, so that $C_H(a)$ and then $C(a)^{F(a)}$ is solvable.

Assume $|F(a)| > 4$. Then with 3.1, there exists $Q^{F(a)} \trianglelefteq (C_H(W) \cap N(T))^{F(a)}$ of prime order and $T = WE$ where $E = [C_T(W), Q]$. $O^2(C(a))$ is transitive on $(E/(W \cap E))^{\#}$ and as $W \cap E \leqslant Z(W)$ is not quaternion, $O^2(C(a))$ centralizes $W \cap E$. Hence $\Phi(E) = 1$ by 2.2.

So take $|F(a)| = 4$. If $C(W)^{F(a)}$ is 2-transitive we argue as above. Hence W is quaternion of order 8 and there is a 3-element $x \in C_H(a)$ inducing an automorphism of order 3 on W. Let $E/\langle a \rangle$ be an x-invariant compliment for $W/\langle a \rangle$ in $C_T(W)$. Then E is elementary or quaternion of order 8.

LEMMA 7.10. *Assume U is dihedral. Then one of the following holds*:

(1) *$C(U)^{F(U)}$ is 2-transitive.*

(2) *$|F(U)| = 2$ and $|U| > 4$.*

(3) *$|U| = 4$, $C(U)^{F(U)}$ has a* RNS, *$U^* \cong E_8$, and $U^{\#}$ is fused in H but not in D.*

Proof. By 2.1, $N(U)^{F(U)}$ is 2-transitive of even degree. Further $O^2(N(U)) \leqslant C(U)$ unless $|U| = 4$ and $O^2(N(U)/(O^2(N(U)) \cap C(U) \cong Z_3$. Finally $O^2(N(U))^{F(U)}$ is 2-transitive unless $|F(U)| = 2$. Thus we may assume $|U| = 4$. If $|F(U)| > 2$ [5] implies either $N(U)^{F(U)}$ has a RNS or a characteristic subgroup isomorphic to $L_2(q)$. In the latter case $C(U)^{F(U)}$ is 2-transitive and in the former $C(U)$ covers the RNS. Thus we may take $|F(U)| = 2$, and U^* dihedral. Then $C(a)^{F(a)}$ is 2-transitive by Lemma 4 in [1], so as U^* is dihedral, 7.9 implies a Sylow 2-group of $C(a)$ is semidihedral and Sylow in G. Now appeal to 3.4.

LEMMA 7.11. *If U is dihedral then U^* is not dihedral.*

Proof. Assume U and U^* are dihedral. Then by 7.10, $|U| > 4$. Now there exists $x \in U^*$ with $x^2 = v$, so $|F(v)| \equiv n \equiv 0 \bmod 4$. Then by 7.5, $C(v)^{F(v)}$ is 2-transitive and as $|F(v)| \equiv 0 \bmod 4$, there exists an involution b, distinct from a, centralizing v. But we may choose $b \in U^*$.

LEMMA 7.12. *Let U be dihedral and $X \leqslant U$. Then $C(X)^{F(X)}$ is transitive except possibly if $V \leqslant X$ and $U^* \cong B_{|V|}$.*

Proof. If $V \nleqslant X$ or $U^* \ncong B_{|V|}$, then by 2.3 and 7.11, $C_{U^*}(Y) \nleqslant U$ for each subgroup Y of U isomorphic to X.

LEMMA 7.13. *Let X be a 4-group in U, and $W = U_{F(X)}$. Then*

(1) *$N(X)^{F(X)}$ is 2-transitive.*

(2) *$C(X)^{F(X)}$ has either a* RNS *$T_0^{F(X)}$ or a characteristic subgroup $L_0^{F(X)} \cong L_2(q)$, $q \equiv -1 \bmod 4$, on $q+1$ letters.*

(3) *Assume $C(X)^{F(X)}$ is not 2-transitive and let $T \in \mathrm{Syl}_2(T_0)$. Then $W = X$, $T = W \times E$, E is elementary, and $E = [T, N_H(T) \cap C(W)]$ unless $|E| = 4$. In any event $|E| = 2^{2i}$ and if $U \neq X$ then i is odd.*

Proof. By 7.12, $C(X)^{F(X)}$ is transitive. So as $X^H \cap U = X^U$, $N(X)^{F(X)}$ is 2-transitive by 2.1. $|N_U(X): X| \leqslant 2$, so minimality of G implies either $N(X)^{F(X)}$ (and then even $C(X)^{F(X)}$) has a RNS $T_0^{F(X)}$ or a characteristic subgroup $L_0^{F(X)} \cong L_2(q)$ or $R(q)$. As X is self-centralizing in U, in the latter case we have $L_0^{F(X)} \cong L_2(q)$ and $q \equiv -1 \bmod 4$.

Assume $C(X)^{F(X)}$ is not 2-transitive. By 2.1, X is fused in H but not in D. By (2), $C(X)^{F(X)}$ has a RNS $T_0^{F(X)}$. If $|F(X)| > 4$, then by 2.2, either T has the factorization claimed or $|F(X)| = 16$ and T is a Suzuki 2-group. Assume the latter. If $X \neq U$ then $|F(Y)| = 4$, where $Y = N_U(X)$. $Y^H \cap U = Y^U$ so $N(Y)^{F(Y)}$ is 2-transitive and then $C(Y)^{F(Y)} = A_4$. But now $C(X)^{F(X)}$ is 2-transitive. So $U = X$ and $T \in \mathrm{Syl}_2(G)$, so by [23], $G \cong U_3(4)$, a contradiction.

Assume $|F(X)| = 4$ and let $h \in H$ induces an automorphism of order 3 on X. We may assume T is not abelian so $X = Z(T) = \Omega_1(T)$. Now by [14] T is homocyclic. By 7.10, $X \neq U$, so $N_U(X)\, T = S \in \mathrm{Syl}_2(N(X))$ and S is wreathed of order 32. Further X is characteristic in S, so $S \in \mathrm{Syl}_2(G)$. Now 3.4 yields a contradiction.

Finally as X is fused in H but not in D, $|N_H(X): N_D(X)| = 3$, so $|F(X)| \equiv 1 \bmod 3$ and then $|E| = |F(X)| = 2^{2i}$. If $U \neq X$ then $|U^{F(X)}| = 2$. $C(N_U(X)^{F(N_U(X))}$ is 2-transitive of degree 2^i, so as $C(X)^{F(X)}$ is not 2-transitive, $2^i + 1 \equiv 0 \bmod 3$ and then i is odd.

LEMMA 7.14. *Assume $|U| > 4$, U is dihedral, and let B be the cyclic subgroup of order 4 in U. Then $N(B)^{F(B)}$ has* RNS *or is an extension of $L_2(q)$, $q \equiv -1 \bmod 4$.*

Proof. By 2.1, $N(B)^{F(B)}$ is 2-transitive. Notice $U^{F(B)}$ is dihedral, or cyclic of order at most 2, and if $U^{F(B)} \neq 1$ then $C(B)^{F(B)}$ is a normal subgroup index 2.

Suppose $|F(B)| \equiv 2 \bmod 4$. If $|U| > 8$ then a generator of B is rooted in U, so $2 \equiv |F(B)| \equiv n \equiv 0 \bmod 4$, a contradiction. So $|U| = 8$ and

$|U^{F(B)}| \leqslant 2$. So minimality of G and remarks in the last paragraph imply $|F(B)| = 2$.

So we may take $|F(B)| \equiv 0 \bmod 4$. Then again minimality of G and the first paragraph give the desired result.

8. The Case $a \in Z^*(H)$

In this section we assume $a \in Z^*(H)$ and produce a contradiction.

Lemma 8.1. *$C(u)^{F(U)}$ is 2-transitive for each involution $u \in U$.*

Proof. If $m(U) = 1$ then $\langle u \rangle$ is weakly closed in U and 2.1 applies. If U is dihedral then as $a \in Z^*(H)$, U has a normal 2-complement in H. Then $u^H \cap U = u^U$, so $C_H(u)$ is transitive on $F(u) - \alpha$ by 2.1. But by 7.12, $C(u)^{F(u)}$ is transitive.

As $a \in Z^*(H)$, $O(H) \neq 1$, so there exists an abelian normal subgroup $A \neq 1$ of H. By 7.6, $C_H(A)$ is semiregular on $\Omega - \alpha$. Let Q be maximal with respect to containing $C_H(A)$, being normal in H, and acting semiregularly on $\Omega - \alpha$. By 7.6, Q is of odd order.

Lemma 8.2. *Assume U is dihedral and let u be in involution in U. Then $|U^{F(u)}| \leqslant 2$ and if $u \notin Z(U)$ then*

(1) $L = \langle C_Q(u)^{C(u)} \rangle \neq 1 \neq C_A(u)$.

(2) $U_{F(u)} = \langle u \rangle$.

(3) *Either L has a* RNS *or $L \simeq L_2(q)$, $q \equiv -1 \bmod 4$*

(4) *If Y is* RNS *for L then $uY = u^G \cap Y\langle u \rangle$.*

Proof. By 7.9, $|U^{F(a)}| \leqslant 2$. Thus we may take $u \notin Z(U)$. Then u is conjugate to ua in U. As $A = C_A(u)\, C_A(a)\, C_A(ua)$ and $[A, a] \neq 1$ by 7.6, we get $1 \neq C_A(u) \leqslant L$. If $U_{F(u)} \neq \langle u \rangle$ then $F(u) = F(ua) \subseteq F(a)$ and $A \leqslant C(a)$. This yields (2). Now as $|C_U(u)| = 4$, $|U^{F(u)}| \leqslant 2$.

By 7.4 and minimality of G, either $L/Z(L) = L^{F(u)}$ has a RNS or is isomorphic to $L_2(q)$, $U_3(q)$, $R(q) \equiv -1 \bmod 4$, or $L_2(8)$. As $|U^{F(u)}| \leqslant 2$, $L^{F(u)} \neq U_3(q)$. If $L^{F(u)} \simeq L_2(q)$ then $L \simeq SL_2(q)$ or $L_2(q)$. But in the former case a Sylow 2-subgroup of $C(u)$ is semidihedral, while by 7.2, $C(\langle u, a \rangle)^{F(\langle u, a \rangle)}$ is transitive.

If $L^{F(u)}$ has a RNS $Y^{F(u)}$, then by 2.2 either Y is regular on $F(u)$ or $L \simeq SL_2(3)$. The latter is impossible as above.

Let $S \in \mathrm{Syl}_2(C(u))$, and $x \in U$ with $u^x = ua$. Then $uC_{Y^x}(u) \subseteq u^{Y^x}$ and $\langle u, a \rangle\, C_{Y^x}(u) = \langle u, a \rangle\, C_Y(u)$. Also $C_H(u)$ is transitive on $Y^\#$. As $S' = Z(S) \cap Y$, 2.9 implies (4).

It remains to show $L^{F(u)} \not\cong R(q)$ or $L_2(8)$, so assume otherwise. Then $S = \langle u \rangle \times (L \cap S)$ with $L \cap S^{\#}$ fused in L. So all involutions in S are conjugate to u or a and then we may take $\langle U, S \rangle \leqslant R \in \mathrm{Syl}_2(C(a)) \subseteq \mathrm{Syl}_2(G)$. Further letting $X = \langle u, a \rangle$, $C(X)^{F(X)} \cong L_2(q)$, so by 7.9, $q = 3$ and $C(a)^{F(a)}$ has a RNS $T_0^{F(a)}$. Then $L \cong L_2(8)$. Let $T \in \mathrm{Syl}_2(T_0)$. By 7.9, $T = W \times E$, where $W = U_{F(a)}$ and E is elementary.

Suppose $a \neq w$ is an involution in W. If $F(a) = F(w)$ then

$$Q = C_Q(a)\, C_Q(w)\, C_Q(aw) \leqslant G(F(a)),$$

contradicting 7.6. So $F(a) \subset F(w)$. Then $\langle C_Q(w)^{C(w)} \rangle$ has a RNS Y_1 of rank 8. So $m(C(w)) \geqslant 9$, while $m(R) = 6$, a contradiction. So W is cyclic. Similarly $u^G \cap T$ is empty, so as $S^{\#} \subseteq a^G \cup u^G$, and $C_H(a)$ is transitive on $E^{\#}$, all involutions in T are in a^G. But now considering the transfer of G to S/T, G has a 2-transitive subgroup of index 2 contradicting 7.2.

LEMMA 8.3. *Assume $C(a)^{F(a)}$ has a* RNS $T_0^{F(a)}$ *and let $W = U_{F(a)} \leqslant T \in \mathrm{Syl}_2(T_0)$. Then*

(1) *$W = \langle a \rangle$ and $\Phi(T) = 1$.*

(2) *$a^G \cap C(a) \subseteq T_0$ and each involution in T is either* FPF *or conjugate to a.*

LEMMA 8.4. *Assume $C(a)F^{(a)}$ has a characteristic subgroup $L_0^{F(a)}$ isomorphic to $L_2(q)$, $q > 3$. Then*

(1) *$U_{F(a)} = \langle a \rangle$ and $a^G \cap C(a) \subseteq L_0$.*

(2) *$L_0/O(L_0) \cong Z_2 \times L_2(q)$.*

(3) *G has a class of* FPF *involutions.*

We prove 8.3 and 8.4 together. Set $W = U_{F(a)}$, and let $W \leqslant S \in \mathrm{Syl}_2(C(a))$. As $a \in Z^*(H)$, each $a \neq b = a^g \in S$ acts FPF on $F(a)$ by 8.1. Thus in 8.3, $b \in T$ by 4.3, while in 8.4, $b \in L_0$.

Assume $C(a)^{F(a)}$ has a characteristic subgroup $L_0^{F(a)}$ isomorphic to $L_2(q)$, $q \equiv -1 \bmod 4$. By 7.9, $\bar{L}_0 = C_{L_0}(W)/O(C_{L_0}(W)) \cong Z(W) \times L_2(q)$ or $Z(W)\, Y SL_2(q)$ with $S \in \mathrm{Syl}_2(G)$ in the latter case.

Suppose W is dihedral. As usual $F(a) \subset F(u)$ for some involution $u \in W$. Now by 8.2.3, $q = 3$ and $\bar{L}_0 \cong Z(W) \times L_2(3)$. So in 8.4, $m(W) = 1$.

Suppose U is quaternion and $W \neq U$. Then there exist elements $u \in U - W$ and $w \in W$ of order 4. u and w induce odd and even permutations on $F(a)$, and then even and odd permutations on $\Omega - F(a)$, respectively. So $n - q - 1 \equiv 0 \bmod 2 \mid w \mid$, a contradiction.

Suppose $\bar{L}_0 \cong Z(W)\, Y SL_2(q)$. Then $S \cap L_0 = WYX$ where $X = \langle x, y \rangle$

is quaternion. Choose $|y| \geqslant |x|$. Recall in this case $S \in \mathrm{Syl}_2(G)$. Suppose $U = W$ and let $e + 1$ be the exponent of S. We may choose $t \in a^G$ with $C_S(t) \in \mathrm{Syl}_2(C(a) \cap C(t))$. But a is a root of degree 2^e in $C_S(t)$ while t is not, a contradiction. So $W < U$, $W = \langle v \rangle$ is cyclic, and U is dihedral or cyclic. If $U = \langle u \rangle$ then $C_S(t) = \langle ux, y \rangle$ is abelian of index 2 in S. Then as $C_S(t)$ is Sylow in $C(a) \cap C(t)$, $C_S(t)$ must be homocyclic. As xt is an involution in $S - C_S(t)$, S is wreathed, contradicting 3.4. So $U = \langle u, v \rangle$ is dihedral. Then $C_S(t) = \langle v, ux \rangle$ and $C_S(t)' = \langle v^2 \rangle$, impossible as $a \in \langle v^2 \rangle$ and $C_S(t)$ is Sylow in $C(t) \cap C(a)$.

With 7.9, the above yields (2) of 8.4 and in 8.3 implies $T = W \times E$, where $\Phi(E) = 1$.

Suppose W is dihedral. Then we have shown we are in 8.3. We may choose $u \in W^{\#}$ with $F(a) \subset F(W)$. Then apply 5.7 to $X = \langle a, w \rangle$, using 7.9 and 8.2, to obtain a contradiction.

So we may assume $m(W) = 1$. Then $\langle a \rangle E = \langle a^G \cap S \rangle \trianglelefteq N_G(S)$, so $WZ(E) = C_S(\langle a \rangle E) \trianglelefteq N(S)$. Then by 2.9, $W = \langle a \rangle$.

Assume we are in 8.4. If S is abelian, 3.4 implies $S \notin \mathrm{Syl}_2(G)$, so S contains FPF involutions. Thus $1 \neq S' \cap Z(S) \leqslant L_0'$, so by 2.9 $S \notin \mathrm{Syl}_2(G)$ and the involution t in $S' \cap Z(S)$ is not fused to a. t is 2-central and we may assume t is not FPF, so t is fused to $u \in U$. Then replacing a by u we get a contradiction by symmetry, since u is 2-central.

So we are in 8.3. $a^G \cap S \subseteq T$, hence if 8.3 is false, U is a 4-group and some $u \in U - \langle a \rangle$ is fused into T. Further by 7.9 we may take $E = [T, C_H(a)]$ and $S' \cap Z(S) \leqslant E$. Now we argue as in the last paragraph.

LEMMA 8.5. *$C(a)^{F(a)}$ has a characteristic subgroup $L_0^{F(a)}$ isomorphic to $L_2(q)$, $3 < q \equiv -1 \bmod 4$.*

Proof. Assume not. Then by 7.9, $C(a)^{F(a)}$ has a RNS $T_0^{F(a)}$. Let $T \in \mathrm{Syl}_2(T_0)$. By 8.3, 4.3, and 2.7 it suffices to show T centralizes $C(a)_{F(a)}$.

Assume first $L = \langle C_Q(a)^{C(a)} \rangle \neq 1$. Then $[L, C(a)_{F(a)}] = 1$ and $T = \langle a \rangle E \leqslant \langle a \rangle L$.

So $L = 1$ and a inverts Q. Then G satisfies Hypothesis 6.1. Now by 3.2 and 6.5, $C(O_p(G_{F(a)}\langle a \rangle)^{F(a)}$ is transitive for each odd prime p and thus covers T. It follows that T centralizes $\mathrm{Fit}(O(G_{F(a)}))$ and then $O(G_{F(a)})$. By 8.3, $G_{F(a)} = \langle a \rangle O(G_{F(a)})$, and the proof is complete.

LEMMA 8.6. *Let $t \in a^G$ and $W = \langle a, t \rangle$. Then*

$$n - 1 = (q + 1)\, q \,|\, C_D(a) \colon C_D(W)| / | C_D(t) \colon C_D(W)| + q.$$

Proof. Let $s = |a^D|$, $e = |t^D|$. By 3.6 and 8.4, $n - 1 = q(q + 1)\, e/s + q$.

LEMMA 8.7. *a inverts Q and $a \in Z(D)$.*

Proof. Assume $L = \langle C_Q(a)^{C(a)} \rangle \neq 1$. By 8.4, $L \simeq L_2(q)$ and $L_0 = L \times C(a)_{F(a)}$. Let $X = L \cap D$. Then $X = [C_D(a), t]$ is of order $(q-1)/2$ and is centralized by U. So by 5.3, X acts semiregularly on $Q^{\#}$, and then Q is nilpotent. Let $Y = [C_D(t), a]$, where $t = a^g$. a and t are in the center of some $S \in \mathrm{Syl}_2(C(a))$, so $S \in \mathrm{Syl}_2(C(t))$. Also $Y \leqslant [C(t), a] \leqslant L^g$ and is normalized by S. It follows that $Y = 1$. Then by 8.6, $n - 1 = q(q^2+1)/2$. So $C_Q(a)$ is Sylow in Q. Then as Q is nilpotent, 7.6 yields a contradiction.

LEMMA 8.8. $[t, G_{F(a)}] = 1$.

Proof. Let p be an odd prime and $K = O_p(G_{F(a)})$. By 6.5, $C(K\langle a \rangle)^{F(a)}$ is transitive and then covers $L_0^{F(a)}$. It follows that t centralizes $\mathrm{Fit}(G_{F(a)}))$ and then $G_{F(a)}$.

We now derive a contradiction proving:

THEOREM 8.9. $a \notin Z^*(H)$.

For let $L = \langle t^{C(a)} \rangle'$. As $[t, G_{F(a)}] = 1$, 8.4 implies $L \simeq L_2(q)$. Let R be the subgroup of order q in $L \cap H$. Then $QR \trianglelefteq QC_H(a) = H$ and R is regular on $F(a) - \alpha$, so QR is regular on $\Omega - \alpha$. This contradicts [13].

9. THE CASE FIT$(H) \neq 1$

It follows from 8.9 that $Z^*(H)$ has odd order. In particular $m(H) > 1$, so U is dihedral. In this section we assume $O(H) \neq 1$ and derive a contradiction. Define A and Q as in Section 8.

LEMMA 9.1. $C_A(u) \neq 1$ *for each involution* $u \in U$.

Proof. If u inverts A then $Q\langle u \rangle \trianglelefteq H$ and then $u \in Z^*(H)$.

LEMMA 9.2. *There exists a 4-group* $X = \langle a, a_2 \rangle \leqslant U$ *with* $X^{\#}$ *fused in* H *and* $X_{F(a)} = \langle a \rangle$.

Proof. As $a \notin Z^*(H)$ and U is dihedral, there exists a 4-group $X = \langle a, a_2 \rangle \leqslant U$ with $X^{\#}$ fused in H. If $X \leqslant G_{F(a)}$ then $C_A(a) = C_A(a_2) = C_A(aa_2) = A$, contradicting 7.6.

LEMMA 9.3. *If* $C(a)^{F(a)}$ *has a* RNS *then* $|F(a)| \neq |F(X)|^2$.

Proof. Assume $|F(a)| = |F(X)|^2 = m^2$. Then by 7.13 and 5.7, $m = 2$ or 4, and $C_Q(a)$ is of order $p = 3$ or 5. Then $Q = \prod_{X^{\#}} C_Q(x)$ is elementary or order p^3. By 7.6, Q is self centralizing. Also $N_H(X)$ acts irreducibly on Q, so $Q = C_Q(O_p(H))$ and then $O_p(H) = Q$. So $H/Q = H/C_H(Q)$ acts as a subgroup of $GL_3(p)$ with dihedral Sylow 2-groups with no normal 2-compliment, and with $O_p(H/Q) = 1$. Therefore by 2.8, $A_4 \leqslant H/Q \leqslant S_4$, or $p = 5$ and $A_5 \leqslant H/Q \leqslant S_5$.

By [13] Q is not regular on $\Omega - \alpha$, so if $H/Q \leqslant S_4$ then $D = U$ and $n - 1 = | H: D | = 3p^3$. As $n \equiv 0 \bmod 4$, $p = 5$ and $n = 376$. So $n \equiv 8 \bmod 16$, impossible as $|F(a)| = 16$. Then $H/Q \leqslant S_5$, $p = 5$, and $D = U$. So $n = 3.5^4 + 1 \equiv 4 \bmod 8$, again impossible as $|F(a)| = 16$.

For the remainder of this section let $L_1 = \langle C_Q(a), C_Q(a)^t \rangle$. If $L_1/Z(L_1) \cong L_2(q)$ then let $Y = O(L_1 \cap D)$ and $Y_1 = YO(C(XL_1))$. If $| U | > 4$, let B be the cyclic subgroup of order 4 in U. Choose X as in 9.2, and if possible choose X so that $X^{\#}$ is not fused in D.

LEMMA 9.4. *Assume $C_Q(X) = 1$ and $| U | > 4$. Then either*

(1) $L_1/Z(L_1) \cong L_2(q)$, $3 < q \equiv -1 \bmod 4$, *or*

(2) $L_1 \leqslant S(|F(v)|)$, $C_Q(a) = C_Q(v)$, *and* $|F(v)| > 4$.

Proof. Let $u = a_2 v$ and $W = \langle u, a \rangle$. Set $m = | C_Q(a)|$, $k = | C_Q(v)|$, $w = | C_Q(W)|$, and $r = | C_Q(u)|$. Notice ua is conjugate to u.

If $L_1/Z(L_1) \cong L_2(3)$, then $Q = C_Q(a)\, C_Q(a_2)\, C_Q(aa_2)$ is elementary of order 27. Now by 2.8, $H/Q \cong S_4$, and $n = 82 \equiv 2 \bmod 4$, a contradiction. So if $L_1/Z(L_1) \cong L_2(q)$, then $q > 3$.

Now $m = kw$, and $| Q | = m^3$, since $C_Q(X) = 1$ and $X^{\#}$ is fused in H. As au is conjugate to u, $| Q | = r^2 m/w^2 = r^2 k/w$. Therefore $r = kw^2 = mw$. So if $u \in a^G$ then $m = r$ and hence $w = 1$. Thus $m = k$, and we appeal to 7.14.

So we may assume $u \notin a^G$ and $w > 1$. $u^H \cap U = u^U$, so by 2.1 and 7.12, $C(u)^{F(u)}$ is 2-transitive. $W = C_U(u)$, so $| U^{F(u)} | = 2$. Also as $w > 1$, $|F(W)| > 2$. Let $L = \langle C_Q(u)^{C(u)} \rangle$. Then minimality of G and the remarks above imply $L^{F(u)}$ has a RNS or $L^{F(u)} \cong R(q)$.

In the first case $C_Q(W)$ contains a normal subgroup Z of prime order p, and Z is normal in $C_Q(u)$ and $C_Q(ua)$. As a_2 inverts $C_Q(a)$, Z is normal in $C_Q(a)$. Hence $Z \trianglelefteq Q$. Let A be a minimal normal subgroup of H containing Z. $\Phi(A) = 1$, so $Z = C_A(u) = C_A(ua)$, since $C(u)^{F(u)} \leqslant S(|F(u)|)$. Hence $A = C_A(u)\, C_A(ua)\, C_A(a) = ZC_A(a) = C_A(a)$, contradicting 7.6.

So $L^{F(u)} \cong R(q)$. Then $w = q$ and $r = q^2$ or q^3. If $k = 1$ then $L_1 \leqslant C(u)$, and we are in (1), so we may take $k > 1$. Hence $r = kw^2 = kq^2 > q^2$, so $r = q^3$ and $k = q$. $|F(BW)| = 2$, so by 7.14, $C(v)^{F(v)}$ is an extension of $L_2(q)$. Also a inverts $Z(C_Q(u))$ and the second center $Z_2(C_Q(u))$ of $C_Q(u)$ is $C_Q(W)\, Z(C_Q(u))$.

As $a \notin Z^*(H)$, a does not invert $Z(Q)$. But $C_Q(a) = C_Q(W)\, C_Q(v)$ and

$C_Q(W) \cap Z(C_Q(u)) = 1$, and $N_H(v)$ acts irreducibly on $C_Q(v)$, so $Z(Q) = Z(C_Q(u))Z(C_Q(ua))\, C_Q(v)$. Further $Z_2(Q) = Z(Q)\, C_Q(W)$. So a centralizes $Z_2(Q)/Z(Q)$. Thus as $X^{\#}$ is fused in H, X centralizes $Z_2(Q)/Z(Q)$. So $C_Q(X) \neq 1$, a contradiction.

THEOREM 9.5. *Let* $L = \langle C_Q(a)^{C(a)} \rangle$. *Then*

(1) $L/Z(L) \cong L_2(q)$, $3 < q$.

(2) $|F(Y\langle a \rangle)| = 2$.

(3) $C_H(a) = C_Q(a)\, N_H(Y\langle a \rangle)$.

(4) $Y \trianglelefteq N_G(X)$.

The proof of Theorem 9.5 involves a series of lemmas.

LEMMA 9.6. *If* $L_1/Z(L_1) \cong L_2(q)$, $3 < q \equiv -1 \bmod 4$ *and* $|F(X)| \leqslant 4$, *then* 9.5 *holds.*

Proof. Let $W = C_U(L_1)$. U/W centralizes Y, so $[U, Y] = 1$. Further Y is inverted in L_1, so by 5.3, $C_Q(Y) = 1$. $Y_1 \leqslant D$ and Y_1 acts on $F(X)$, which has order 2 or 4 by hypothesis. Hence Y_1 fixes $F(X)$ pointwise.

There exists $\alpha \neq \alpha^g = \gamma \in F(X)$ such that a_2 is in the center of a Sylow 2-group of H_γ containing X. Let $L_2 = \langle C_Q(a_2), C_{Q_g}(a_2) \rangle$, and let P be a subgroup of prime order in Y. P acts on L_2 and semiregularly on $C_Q(a_2)$, so $P \leqslant L_2 C(L_2)$ (e.g., Lemma 2.7 in [3]). As this holds for each prime divisor of $|Y|$ and $[Y_1, P] = 1$ it follows that $Y_1 = O(C(XL_2))(L_2)_{\alpha\gamma}$. Therefore $Y_1 \trianglelefteq N_G(X)$. Now $Y = [Y_1, t]$ is cyclic. Assume $|F(X)| = 4$. Then $N(X)^{F(X)} \cong A_4$ or S_4, so by 2.5, $[Y_1, t] = 1$, a contradiction.

So $|F(X)| = 2$. Thus $X^{\#}$ is fused in D, so $C(a)^{F(a)}$ is 2-transitive and then $L_1 = \langle C_Q(a)^{C(a)} \rangle = L$. This yields (1)–(3) of 9.5. Also $Y = [Y_1, t] = O(L_2 \cap D)$, so $Y \trianglelefteq N_G(X)$.

LEMMA 9.7. *If* $C(a)^{F(a)}$ *is* 2-*transitive then* 9.5 *holds.*

Proof. This follows from 7.9, 9.3, and 9.6.

Given 9.7 we may assume $C(X)^{F(X)}$ is not 2-transitive.

LEMMA 9.8. *If* $|U| = 4$ *then* 9.5 *holds.*

Proof. Assume $|U| = 4$. Then $U = X$. $C(X)^{F(X)}$ is not 2-transitive, so by 7.8, 7.10, and 7.13, $L^{F(a)}$ satisfies Hypothesis 5.4. Then 5.5, either $L^{F(a)}$ has a RNS or $L^{F(a)} \cong L_2(8)$ or $5L_2(32)$. By 9.3, it must be the latter. Then $C_Q(a)$ is cyclic of order 3, 9, 11, or 33. Take A to be minimal normal in H. Then $|A| = p^3$, $p = 3$ or 11.

Suppose $p = 11$. $C_H(X)$ contains an element w inducing an outer automorphism of order 5 on L with $C_L(w)^{F(\langle a,w\rangle)} \cong S_3$. Now Q is abelian of order 11^3 or 33^3, and in the latter case as w centralizes an element of order 3 in $L^h \cap H$ for each $a^h \in X$, w centralizes $O_3(Q)$, contradicting 7.6. So $Q = A$. Now $C_L(\langle a, w\rangle)$ acts irreducibly on $[a, A]$ of order 121, so w has scalar action on $[a, A]$. Indeed this holds for each member a of $X^\#$, so w has scalar action on A. Hence $H = AC_H(w)$. $C_H(w)$ is a subgroup of $GL_3(11)$ whose Sylow 2-group U is a 4-group fused in $C_H(w)$ and containing an element of order 6. It follows that $C_H(w) \cong Z_5 \times L_2(11)$ and $D = U$. Then $n = (11^4 \cdot 5^2 \cdot 3) + 1 \equiv 12 \bmod 16$. But $|F(a)| = 496 \equiv 0 \bmod 16$, a contradiction.

So $p = 3$. Then by 2.8, $A_4 \leqslant H/Q \leqslant S_4$ and $D = U$. So $C_Q(a)$ has order 9 and Q has order 9^3. Then $n = 3^7 + 1 \equiv 4 \bmod 7$, so $|G| = n|H|$ is not divisible by 7. But the order of L is divisible by 7.

Given 9.8 we may assume $|U| > 4$. Recall B is the cyclic subgroup of order 4 in U.

LEMMA 9.9. *If* $N(B)^{F(B)}$ *has a* RNS *then* 9.5 *holds.*

Proof. Let $W = U_{F(B)}$. As in 7.4, $|U: W| \leqslant 2$. Suppose $|F(BX)| = 2$. Then as $|F(X)| = |F(BX)|^2$ and $C(X)^{F(X)}$ is not 2-transitive, $C_Q(X) = 1$. Now appeal to 9.4 and 9.6.

So we may assume $|F(BX)| > 2$. As $C(a)^{F(a)}$ is not 2-transitive, $F(a) \neq F(B)$. Hence $BX^{F(a)}$ is a 4-group in $C(a)^{F(a)}$, and by 7.13, $C(a)^{F(a)}$ satisfies the hypothesis of Lemma 5.7. Thus choosing P as in 5.7, $[C_A(a), P] = 1$. By symmetry, $[C_A(x), P] = 1$ for each $x \in X^\#$, so $P \leqslant C(A) \leqslant Q$. Now arguing as in the last paragraph of 5.7, P is the unique subgroup of order p in $C_Q(x)$, each $x \in X^\#$, so P is even unique in Q. This contradicts 7.6.

LEMMA 9.10. *If* $N(B)^{F(B)}$ *is the extension of* $L_2(q)$, *then* 9.5 *holds.*

Proof. $|F(BX)| = 2$, so $|F(X)| = 4$ and $C_Q(X) = 1$. Now appeal to 9.4 and 9.6.

Notice 9.8–9.10 and 7.14 imply Theorem 9.5. We can now complete this section proving:

THEOREM 9.11. $O(H) = 1$.

Let $K = \Gamma_{1,X}(H)$. By 9.5, $K = Q(K \cap D)$. Now if $QD = H$ then Q is regular on $\Omega - \{\alpha\}$, contradicting [13]. So $K \leqslant QD \neq H$. But by 9.5, $C(a)^{F(a)}$ is 2-transitive, so $a^H \cap D = a^D$. So $a^H \cap QD = a^{QD}$, and QD has one class of involutions. Thus QD is strongly embedded in H. Therefore $H/O(H) \cong A_5$ and $\bar{D} = D/O(D) \cong A_4$. Also $n - 1 = |Q|\,|H: K| = 5q^3$.

Now t acts on $\bar{D}$ and centralizes U, so we may choose t to centralize $\bar{D}$. So $tU^{\#}$ is fused in D. Let $U^* \leqslant S \in \mathrm{Syl}_2(C(a))$. Then $S = \langle a \rangle \times T$, where $S \cap L \leqslant T$ is nonabelian dihedral. Let z be the involution in $Z(T)$. By 2.9, $z \notin a^G$. Further $uz \in u^T$ for each $a \neq u \in U$, so $| a^G \cap zU | > 1$. Thus $z = t$ is FPF. Further $[t, C_D(u)] = 1$ for each $u \in U^{\#}$, so $[t, D] = 1$.

Define s and e as in 3.6. Then $s = | a^D |$, and $e = | at^D | = s$, so by 3.6, $n = |F(a)|^2 = (q + 1)^2$. But $(q + 1)^2 \neq 5q^3 + 1$.

10. The Case $E(H) \neq 1$

By 9.11, $O(H) = 1$, so by 7.7, $L = E(H) \neq 1$, and $H \leqslant \mathrm{Aut}(L)$. As H has dihedral Sylow 2-subgroups it follows from [11] that $L \simeq A_7$ or $L_2(q)$, $q > 3$ odd, and that L is of index at most 2 in UL with $UL \simeq PGL_2(q)$ if $UL \neq L$.

LEMMA 10.1. *$H \not\simeq A_7$.*

Proof. Assume $H \simeq A_7$. We consider the various possibilities for D.

Assume first D is solvable. If X is a nilpotent subgroup of odd order in H with $| H: N_H(X)|$ odd, then X is of order 1 or 3. Thus we may choose $X = O(D) \leqslant C(a)$. Suppose X has order 3. Then $UX \leqslant D \leqslant N_H(X)$ of order 72, so D has order 24 or 72. Then $n - 1 = 105$ or 35. As $n \equiv 0 \bmod 4$, $n = 36$. Let N be the number of pairs (a^h, γ), $\alpha \neq \gamma \in F(a^h)$. Let $m = | F(a)|$. Then $35.21 = (n - 1) \mid a^H \cap D \mid = N = | a^H | (m - 1) = 105(m - 1)$. So $m = 8$. But $C(a)^{F(a)}$ is transitive, so if $S \in \mathrm{Syl}_2(C(a))$ then $8 = | S: U |$. But $| G: H | = n = 36 \not\equiv 0 \bmod 8$, a contradiction.

So $X = 1$, and either $D \leqslant C(a)$ or $D \simeq S_4$. If $D \simeq S_4$ then $n - 1 = 105$ and $n \equiv 2 \bmod 4$. So $D \leqslant C(a)$ and then $D = U$ and $n - 1 = 315$. Calculating as above we find $m = 16$, whereas $n \not\equiv 0 \bmod 8$, a contradiction.

So D is not solvable. As $U \leqslant D$, $D \simeq A_6$, S_5, or $L_2(7)$. In the first case $G \simeq A_8$. In the second $n = 22 \not\equiv 0 \bmod 4$. So $D \simeq L_2(7)$ and $n = 16$. Calculating we find $m = 4$. As D is transitive on its involutions, 2.1 implies $C(a)^{F(a)}$ is 2-transitive. $C_D(a)$ is maximal in D, so $\langle a \rangle = G_{F(a)}$. Then minimality of G implies $C(a)^{F(a)} \simeq S_4$. As $m = 4$ and $n = 16$, we may choose t to be FPF. As $G_{F(a)} = \langle a \rangle$, t centralizes U and t is the unique FPF involution in U^*. t acts on $D \simeq L_2(7)$ and centralizes U, so t or ta centralizes D. $ta \in a^G$ and $C(a)$ is solvable, so $[t, D] = 1$ and t is the unique FPF involution in D^*. Now 2.4 implies G has a RNS.

LEMMA 10.2. *One of the following holds*:

(1) *$L \cap D \leqslant C(a)$ and $C(a)^{F(a)}$ has rank* 3 *or* 4 *for $U \nleqslant L$ or $U \leqslant L$, respectively.*

(2) *$L \cap D \cong PGL_2(q_0)$, some odd $q_0 \geqslant 3$, $U \leqslant L$, and $C(a)^{F(a)}$ has rank* 3.

(3) *$L \cap D \cong L_2(q_0)$, some odd $q_0 \geqslant 3$, and $C(a)^{F(a)}$ is 2-transitive.*

Proof. By the opening remarks in this section and 10.1, $H \leqslant \mathrm{Aut}(L)$, with $L \cong L_2(q)$. As D has dihedral Sylow 2-group U, $L \cap D$ has one of the forms claimed. By 2.1 and 7.12, $C(a)^{F(a)}$ is transitive of the stated rank.

LEMMA 10.3. *$L \ncong L_2(5)$ or $L_2(7)$ and if $L \cong L_2(27)$ then $D \leqslant LU$. If $H \cong L_2(9)$ then $D \ncong S_4$.*

Proof. The arguments in 7.8 show $L \ncong L_2(5)$. If $L \cong L_2(7)$, then $D = U$ or $D \cong S_4$. In the first case $n = 22 \equiv 2 \bmod 4$. In the second case $n = 8$, and it is easy to show, using 2.4, that G has a RNS. Similarly if $H \cong L_2(9)$ then $D \ncong S_4$.

So assume $L \cong L_2(27)$ but $D \nleqslant LU$. Then D contains an element w of order 3 inducing a field automorphism on L. Let $\langle w \rangle = W \leqslant P \in \mathrm{Syl}_3(D)$. If $P \neq W$ then $D = N_H(U \cap L)$ and $n = 7.9.13 + 1 \equiv 4 \bmod 8$. But $|F(a)| = 8$, a contradiction. So $P = W$ and then by 2.1, $w^G \cap H = w^H$. Further $n \equiv 1 \bmod 3$, so H contains a Sylow p subgroup of G. Then as $w^G \cap H = w^H$ and W has a normal complement in H, W has a normal compliment in G, contradicting 7.2.

LEMMA 10.4. *Let Y be the cyclic subgroup of index 2 in $C_{LU}(a)$ containing a. Assume $C(a)^{F(a)}$ is not 2-transitive and let X be a 4-group in U used in H but not in D. Then*

(1) $\langle a \rangle = G_{F(a)}$.

(2) *Either $Y \cap D = \langle a \rangle$ or $F(Y \cap D) = \alpha \cup \beta^{C_H(a)}$ is a set of imprimitivity for $C(a)^{F(a)}$ and $|F(X)| = 4$.*

Proof. Let $X = \langle a, x \rangle$ and $h \in N_H(X)$ with $a^h = x$. Then as x is not fused to a in D, $Y_{\beta\beta h} = \langle a \rangle$. Thus $Y_{F(a)} = \langle a \rangle$. Indeed if $Y \cap D \geqslant Y_1 \neq \langle a \rangle$ then Y_1 is weakly closed in D with respect to H, so $N_H(Y_1) \leqslant C_H(a)$ is transitive on $F(Y_1) - \alpha$. Further $Y_1 \trianglelefteq C_H(a)$.

Then $[G_{F(a)}, Y] \leqslant Y_{F(a)} = \langle a \rangle$. So $G_{F(a)}$ centralizes Y unless possibly $|Y| = 4$ and $G_{F(a)} = \langle a, u \rangle$ is a 4-group. In the latter case $G_{F(a)} \trianglelefteq C(a)$, so $q = 5$, 7 or 9, since Y is self-centralizing for $q > 5$. By 10.3, $q = 9$. Then $n = 46 \equiv 2 \bmod 4$, a contradiction. This yields (1).

Assume $Y \cap D \neq \langle a \rangle$. Then as $Y \cap D \trianglelefteq C_H(a)$ and $C_H(a)$ is transitive on $F(Y \cap D) - \alpha$, $\beta^{C_H(a)} = F(Y \cap D) - \alpha$ is an orbit of $C_H(a)$ on $F(a) - \alpha$.

$Y \cap D$ is weakly closed in $C_H(a)$ with respect to $C_G(a)$, so $F(Y \cap D)$ is a set of imprimitivity for $C(a)^{F(a)}$. Then $F(Y \cap D) \cap F(X)$ is a set of imprimitivity for $C(X)^{F(X)}$, so by 7.13, $|F(X)| = 4$.

LEMMA 10.5. *Define Y as in* 10.4. *Assume $C(a)^{F(a)}$ is not 2-transitive and $Y \cap D \neq \langle a \rangle$. Then either*

(1) $D = C_H(a)$ *and* $|F(Y)| = 2$, *or*

(2) $DY = C_H(a)$, $|Y: Y \cap D| = 3$, *and* $|F(Y \cap)| = 4$.

Proof. By 10.4, $F(Y \cap D) = \alpha \cup \beta^{C_H(a)}$ is a set of imprimitivity for $C(a)^{F(a)}$ and $|F(X)| = 4$. Then $N(Y \cap D)^{F(Y \cap D)}$ is 2-transitive with $|U^{F(Y \cap D}| = 2$. As $|F(X)| = 4$, $|F(X(Y \cap D))| = 2$. Finally $U^{F(Y \cap D)} \not\leqslant C(Y \cap D)^{F(Y \cap D)}$.

With these facts in mind, minimality of G implies either $|F(Y \cap D)| = 2$ or $N(Y \cap D)^{F(X \cap D)}$ is an extension of $L_2(q_1)$ with $q_1 \equiv -1 \bmod 4$.

Now if $|F(Y \cap D)| = 2$ then as $Y \cap D \trianglelefteq C_H(a)$, $C_H(a) \leqslant D$. By 10.3, $q > 7$ and if $H \cong L_2(9)$ then $D \not\cong S_4$, so $C_H(a)$ is maximal in H, and $D = C_H(a)$.

On the other hand if $Y \leqslant D$ then as $YU \trianglelefteq C_H(a)$, $F(Y) \subseteq F(X)$ and then $|F(Y)| = 2$. So we may assume $Y \not\leqslant D$. Then $Y^{F(Y \cap D)}$ is a normal cyclic subgroup of $H^{F(Y \cap D)}$, so $q_1 = |Y^{F(Y \cap D)}| = |Y: Y \cap D|$ is prime. Further $C(Y \cap D)^{F(Y \cap D)}$ covers the socle of $N(Y \cap D)^{F(Y \cap D)}$, so $C_D(Y \cap D)^{F(Y \cap D)}$ contains a cyclic subgroup $W^{F(Y \cap D)}$ of order $(q_1 - 1)/2$ acting semiregularly on $Y^{F(Y \cap D)}$.

Assume $q_1 > 3$. Then as $W \leqslant C(Y \cap D)$ acts semiregularly on $Y/(Y \cap D)$, we conclude W is of prime order p. Then $q = q_2{}^p$ and $q_1 = (q_2{}^p - \epsilon)/(q_2 - \epsilon)$ where $\pm 1 = \epsilon \equiv q \bmod 4$. But $(q_1 - 1)/2 = p$, so we must have $q = 27$ and $p = 3$, contradicting 10.3.

Thus $q_1 = 3$, and it remains to show $D \leqslant C(a)$. So assume not. Then by 10.2, $L \cap D \cong PGL_2(q_0)$. Now either $q = q_0{}^e$ or $q_0 = 3$ or 5. As $3 \neq (q_0{}^e - \epsilon)/(q_0 - \epsilon)$, $q_0 = 3$ or 5. Thus $|Y \cap D| = 4$ and $q = 4q_1 \pm 1 = 12 \pm 1 = 11$ or 13. But then $|U \cap L| = 4$, a contradiction.

LEMMA 10.6. *Define Y as in* 10.3 *and assume $C(a)^{F(a)}$ is not 2-transitive. Then $Y \cap D = \langle a \rangle$.*

Proof. Assume $Y \cap D \neq \langle a \rangle$. Then by 10.4 and 10.5, $F(Y \cap D)$ is a set of imprimitivity for $C(a)^{F(a)}$, and is of order 2 or 4. Let θ be the set of conjugates of $F(Y \cap D)$ under $C(a)$. Let $m = |F(a)|$, and $s = |a^H \cap D|$. By 10.5, $|F(Y \cap D)| = 2$ or 4 and $s = 1 + (q - \epsilon)/2$ or $1 + (q - \epsilon)/6$, respectively, for $\epsilon = \pm 1 \equiv q \bmod 4$.

Now by 10.4, $F(Y \cap D) - \alpha = \beta^{C_H(a)}$ and $a \in Z(D)$. So by 2.1,

$|F(Y \cap D)| = 1 + (m - 1)/s$. Then $m = 2 + (q - \epsilon)/2$ or $4 + (q - \epsilon)/2$ for $|F(Y \cap D)| = 2$, or 4, respectively.

Next, each Δ in θ distinct from $F(Y \cap D)$ corresponds to a unique 4-group, X in $C_L(a)$ fixing 2 points of Δ. Suppose $B = \langle b \rangle$ is a cyclic subgroup of order 4 in U. Then B normalizes each 4-group X in $C_L(a)$ and then also $F(X) - F(Y \cap D) = F(X) \cap \Delta$. So B is in the kernal of the action of $C(a)$ on θ. As $B\langle t \rangle$ is the weak closure of B in the stabilizer of $F(Y \cap D)$ we conclude $B\langle t \rangle \trianglelefteq C(a)$. B and $\langle bt \rangle$ are the conjugates of B in $B\langle t \rangle$, so $C(a)$ acts on $F(B) \cup F(bt)$. Then $F(a) = F(B) \cup F(bt)$ is of order $2\,|F(Y \cap D)|$ so $(q - \epsilon)/2 = |F(Y \cap D)|$ and either $q = 5$, or $q = 7$ or 9 and $|Y: Y \cap D| = 3$. By 10.9, $q = 9$, so Y is a 2-group and $|Y: Y \cap D| \neq 3$.

So $|U| = 4$. Now $m \equiv 0 \bmod 4$, so if $|F(Y \cap D)| = 4$ then $q \equiv \epsilon \bmod 8$ and $|U| > 4$, while if $|F(Y \cap D)| = 2$ then $q \not\equiv \epsilon \bmod 8$. So $|F(Y)| = 2$ and $q \not\equiv \epsilon \bmod 8$. Then $Y/\langle a \rangle$ acts regularly on the $(q - \epsilon)/4$ 4-groups in $C_L(a)$ and then also on $\theta - F(Y)$. So $C(a)^\theta$ is 2-transitive and the stabilizer of $F(Y)$ has a normal cyclic subgroup $Y/\langle a \rangle$ regular on $\theta - F(Y)$. It follows that $C(a)$ either has a RNS or is an extension of $L_2(q_1)$, $q_1 = (q - \epsilon)/4$. In either case $C(X)$ is 2-transitive on the fixed points of X on θ, so as X fixes $F(Y)$ pointwise, any member of θ fixed by X is fixed pointwise, so as $|F(X)| = 4$, X fixes exactly 2 points of θ. Thus $C(a)^\theta$ is an extension of $L_2(q_1)$.

Then $C_D(a)$ contains a subgroup W of order $(q_1 - 1)/2\delta$, $\delta = 1$ or 2, acting semiregularly on $Y\langle a \rangle$. W must induce field automorphisms on L.

If $W = 1$ then $q_1 = 3$ or 5 and $q = 11, 13$, or 19 and $n = 1 + |H: D| = 56, 92$, or 172. If $q_1 = 5$, then $m = 12 \equiv n \bmod 8$, so a Sylow 2-group S of $C(a)$ is an abelian Sylow group for G, contradicting 3.4. Similarly $q_1 \neq 3$.

So $W \neq 1$. Then as W acts semiregularly on $Y/\langle a \rangle$, W is of prime order p, and $q = 3^p$ or 5^p. If $q = 5^p$ then $p = (q_1 - 1)/2\delta = ((5^p - 1)/4 - 1)/2\delta = 5(5^{p-1} - 1)/8\delta$. So $p = 5$. But $5^4 - 1 \neq 8$ or 16. So $q = 3^p$ and as above $p = 3$, contradicting 10.3.

Lemma 10.7. *$C(a)^{F(a)}$ is 2-transitive.*

Proof. Assume $C(a)^{F(a)}$ is not 2-transitive. Then by 10.6, $LU \cap D = U$ is of order 4. By 7.13, $C(U)^{F(U)}$ has a RNS $T_0^{F(U)}$ and if $T \in \mathrm{Syl}_2(T_0)$, then T is elementary.

Next $D = UK$, where K is a cyclic group inducing field automorphisms on L. By 2.5, $[K_{F(U)}, T] = 1$, so we may choose $t \in T$ with $[D, t] = 1$.

By 7.8, $q \neq 5$, so $1 \neq 0(C_L(a)) = Q \trianglelefteq C_H(a)$ acts semiregularly on $F(a) - \alpha$. Thus $\langle Q^{C(a)} \rangle^{F(a)}$ satisfies Hypothesis 5.4, so by 5.5, either $C(a)^{F(a)}$ has a RNS or $C(a)^{F(a)} \cong L_2(8)$ or $5L_2(32)$.

Now $L_2(2^i)$ has no FPF involutions so if $C(a)^{F(a)} \cong L_2(2^i)$ then all involutions in G fix points of Ω and hence are conjugate to a. Let $S \in \mathrm{Syl}_2(C(a))$. Then $S \in \mathrm{Syl}_2(G)$. We find S abelian contradicting 3.4.

So $C(a)^{F(a)}$ has a RNS $E_0^{F(a)}$. Let $E \in \mathrm{Syl}_2(E_0)$ and $S = EU \in \mathrm{Syl}_2(C(a))$. As T is elementary E is elementary. $S' \leqslant Z(S)$ so by 2.9, $S' \cap a^G$ is empty. So we may choose t to be FPF. Let $U = \langle u, a\rangle$. Then $ut \in u^E$ and u and ua are conjugate to a, so $a^G \cap D^* - D = \{at, ut, aut\}$. We conclude from 3.6 that $n = |F(a)|^2$. Then by 2.4, G has a RNS.

We are now almost in position to derive a contradiction and establish the theorem.

Let Y be the cyclic subgroup of index 2 in $C_{LU}(a)$. Then $Y \cap D \trianglelefteq C_H(a)$ so as $C(a)^{F(a)}$ is 2-transitive, $Y \cap D \leqslant G_{F(a)}$. $Y \cap D$ is weakly closed in $C_H(a)$ with respect to $C_G(a)$, so $Y \cap D \trianglelefteq C(a)$.

By 7.9, $C(a)^{F(a)}$ has a RNS or is an extension of $L_2(r)$, $r \equiv 1 \bmod 4$. As $Y \cap D$ is a cyclic normal subgroup of $C(a)$ contained in $G_{F(a)}$ it follows from 2.5, that we may choose t to centralize $Y \cap D$.

Next by 10.2, $L \cap D \cong L_2(q_0)$ for some odd $q_0 \geqslant 3$. Then as t centralizes $Y \cap D$ and U and acts on $L \cap D$, t induces an inner automorphism on $L \cap D$ and we may choose t to centralize $L \cap D$. Indeed we may take $[D, t] \leqslant O(D)$.

Now $D = K(UL \cap D)$ where K is a cyclic group of odd order inducing field automorphisms in L. Further $O(D) \leqslant K$. As above, t centralizes $O(D)_{F(a)}$.

Let $\langle d\rangle \in \mathrm{Syl}_p(O(D))$. Then t either inverts or centralizes d. Assume the former. If $|F(d^i a)| > 2$ then by 7.9, t centralizes $d^i a$. So if $d^i \neq 1$ then $C_H(ad^i) \leqslant D$. As d induces a field automorphism on L, it follows that $q = q_0{}^p$, d has order p, and $H = LU\langle d\rangle$. By 7.9, $C(a)^{F(a)}$ is an extension of $L_2(r)$, $r \equiv -1 \bmod 4$, so $p = |[D^{F(a)}, t]| = (r-1)/2$. Then $2p + 1 = r = |F(a)| - 1 = |C_H(a): C_D(a)| = (q_0{}^p - \epsilon)/(q_0 - \epsilon) > q_0^{p-2}(q_0 - 1)$. So $q = 27$, contradicting 10.3.

Thus we have shown that:

LEMMA 10.8. $[D, t] = 1$.

It follows from 7.9 that:

LEMMA 10.9. *$C(a)^{F(a)}$ has a* RNS $T_0^{F(a)}$.

As $q > 7$, $C_L(a)$ is maximal in L. Thus as $L \cap D \nleqslant C_L(a)$ and $L \nleqslant D$, $Y \nleqslant D$.

Suppose $|F(a)| = 4$. Then $|Y: Y \cap D| = 3$. Recall $L \cap D \cong L_2(q_0)$. Suppose $q = q_0{}^r$. Then $3 < (q - \epsilon)/(q_0 - \epsilon) = |Y: Y \cap D|$, a contradiction. So $q_0 = 3$ or 5, and $U = C_{L \cap D}(a)$. Thus $q - \epsilon = 6$, so $q \leqslant 7$ contradicting 10.3.

So $|F(a)| > 4$. Let $T \in \mathrm{Syl}_2(T_0)$ and $S = UT \in \mathrm{Syl}_2(C(a))$. By 7.9, $T = V \times E$, where $E = [T, N_H(T) \cap C(V)]$. Let $a \neq u = a^h \in U$. Then $uC_E(u) \subseteq u^E$ and $C_E(u) = [C_E(u), N_H(T) \cap C(U)] \leqslant E^h$. So $uE^h \subseteq a^G$ and then $aE \subseteq a^G$.

Next T is the unique abelian subgroup of index 2 in S, so T is characteristic in S. Further if $V \neq \langle a \rangle$ then $\langle a \rangle = \Omega_1(\mho^1(T))$ is characteristic in S, contradicting 2.9. Thus $V = \langle a \rangle$ and $S' = C_E(u)$. Also $U \leqslant L$ so H has one class of involutions. By 2.9, $a^G \cap S'$ is empty. So $E^{\#}$ consists of FPF involutions.

Suppose $t = a^g$. Then $U = [U, D] \leqslant [S^g, D] \leqslant E^g$, impossible as $E^{\#}$ consists of FPF involutions. So t is the unique FPF involution with cycle (α, β). Further defining e and s as in 3.6, $s = |a^D| = |(at)^D| = e$. So by 3.6, $n = |F(a)|^2$. Now by 2.4, G has a RNS.

This completes the proof of Theorem 2.

References

1. M. Aschbacher, On doubly transitive groups of degree $n \equiv 2 \bmod 4$, *Illinois J. Math.* **16** (1972), 276–279.
2. M. Aschbacher, Finite groups with a proper 2-generated core, *Trans. A.M.S.* **197** (1974), 87–112.
3. M. Aschbacher, F-sets and permutation groups, *J. Algebra* **30** (1974), 400–416.
4. R. Baer, Engelsche Elemente noetherscher Gruppen, *Math. Ann.* **133** (1957), 256–270.
5. H. Bender, Endliche Zweifach transitive Permutations gruppen, deren involutions beine Fixpunkte haben, *Math. Z.* **104** (1968), 175–204.
6. H. Bender, Transitive Gruppen gerader Ordung in denen jede Involution genan einere Punkt festalt, *J. Algebra* **17** (1971), 527–554.
7. C. Curtis, W. Kantor, and G. Seitz, The 2-transitive permutation representations of the finite Chevally groups, to appear.
8. D. Gorenstein, "Finite Groups," Harper and Row, New York, 1968.
9. D. Gorenstein and K. Harrada, On finite groups with Sylow 2-subgroups of type $\hat{A}_n$, $n = 8, 9, 10$, and, *J. Algebra* **19** (1971), 185–227.
10. D. Gorenstein and K. Harrada, Finite groups whose Sylow 2-subgroups are the direct product of two dihedral groups, *Amer. Math.* **95** (1972), 1–54.
11. D. Gorenstein and J. Walter, The characterization of finite groups with dihedral Sylow 2-subgroups, I, II, III, *J. Algebra* **2** (1964), 85–151, 218–270, 334–393.
12. C. Hering, On subgroups with trivial normalizer intersection, *J. Algebra* **20** (1972), 622–629.
13. C. Hering, W. Kantor, and G. Seitz, Finite groups with a split BN-pair of rank 1, *J. Algebra* **20** (1972), 435–475.
14. G. Higman, Suzuki 2-groups, *Illinois J. Math.* **7** (1963), 79–96.
15. B. Huppert, Zweifach, transitive, auffasbare Permutationsgruppen, *Math. Z.* **60** (1954), 409–434.

16. W. Manning, The order of primitive groups, III, *Trans. Amer. Math. Soc.* **19** (1918), 127–142.
17. M. O'Nan, A characterization of $L_n(q)$ as a permutation group, *Math. Z.* **127** (1972), 301–314.
18. D. Shaw, The Sylow 2-subgroups of finite soluble groups with a single class of involutions, *J. Algebra* **16** (1970), 14–26.
19. E. Shult, On the fusion of an involution in its centralizer, to appear.
20. A. Wagner, On finite affine line transitive planes, *Math. Z.* **89** (1965), 1–11.
21. D. Gorenstein and J. Walter, Centralizers of involutions in balanced groups, *J. Algebra* **20** (1972), 284–319.
22. Z. Janko, A characterization of the Mathieu simple groups, II, *J. Algebra* **9** (1968), 20–41.
23. D. Goldschmidt, 2-fusion in finite groups, *Ann. Math.* **99** (1974), 70–117.