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Group classification of systems of non-linear reaction–diffusion equations with general diffusion matrix. I. Generalized Ginzburg–Landau equations

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Abstract

A group classification of the generalized complex Ginzburg–Landau equations is presented. An approach to group classification of systems of reaction–diffusion equations with general diffusion matrix is formulated.

Keywords: Ginzburg–Landau equation; Group classification; Galilei invariance

1. Introduction

A group classification of differential equations is one of the central problems of group analysis. It specifies non-equivalent classes of equations and opens the way to applications of symmetry tools such as constructing and group generation of exact solutions, separation of variables, etc. One of the goals of group classification is a priori description of mathematical models with a desired symmetry (e.g., relativistic invariance).

The first (and very impressive) achievements in group classification belong to S. Lie who had classified second order ordinary differential equations and specified all cases when such equations can be integrated in quadratures [1]. Lie had presented also a group classification of an entire class of partial differential equations, namely, linear equations with two independent

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variables. In particular, it was Lie who for the first time described group properties of the linear heat equation [2].

The next step in classification of heat equations was made by L.V. Ovsiannikov in 1959 [3] who had classified non-linear heat equations

$$u_t - \big(f(u)u_x\big)_x = 0,$$

where $f(u)$ is an arbitrary function of u and the subscripts denote derivations w.r.t. the corresponding variables. Then Dorodnitsyn [4] classified non-linear diffusion equations[1]

$$u_t - u_{xx} = f(u). \tag{1}$$

This result was extended by Fushchych and Serov [5] and Clarkson and Mansfield [6] to the case of non-classical (conditional) symmetries.

The results of group classification of Eq. (1) play an important role in constructing of their exact solutions and qualitative analysis of the non-linear heat equation, refer, e.g., to [7].

In the present paper we perform the group classification of systems of the non-linear reaction–diffusion equations

$$\begin{aligned} &u_t - \Delta(au - v) = f^1(u, v),\\ &v_t - \Delta(u + av) = f^2(u, v), \end{aligned} \tag{2}$$

where u and v are the functions of $t, x_1, x_2, \dots, x_m$, a is a real constant and Δ is the Laplace operator in R^m. We shall write (2) also in the matrix form:

$$U_t - A\Delta U = f(U), \tag{3}$$

where A is a matrix whose elements are $A^{11} = A^{22} = a$, $A^{21} = -A^{12} = 1$, $U = \binom{u}{v}$ and $f = \binom{f^1}{f^2}$.

Mathematical models based on Eqs. (2) are widely used in mathematical physics, biology, chemistry, etc. Here we present only two significant examples:

- The non-linear Schrödinger (NS) equation in m-dimensional space:

  $$(i\partial_t + \Delta)\psi = F(\psi, \psi^*) \tag{4}$$

  is a particular case of (2). If we denote $\psi = u + iv$, $F = f_1 + if_2$ then (4) reduces to the form (3) with $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

  Equations (4) with various non-linearities F are used in non-linear optics, non-linear quantum mechanics [8], they serve as one of basic models of inverse scattering problem [9]. The most popular models are connected with the following non-linearities [10]:

  $$F = F(\psi^*\psi)\psi, \qquad F = (\psi^*\psi)^k\psi, \qquad F = (\psi^*\psi)^{\frac{2}{m}}\psi, \qquad F = \ln(\psi^*\psi)\psi.$$

  One more interesting particular case of the NS equation corresponds to

  $$(i\partial_t + \Delta)\psi = (\psi - \psi^*)^2;$$

  in this case (4) is a potential equation for the Boussinesq equation for the function $V = \partial_t(\psi - \psi^*)$.

  Group classification of the NS equation has been performed in paper [11].

[1] In paper [4] a more general equations with non-linear diffusion were classified which include (1) as a particular case.

- Generalized complex Ginzburg–Landau (CGL) equation

$$W_\tau - (1 + i\beta)\Delta W = F(W, W^*) \tag{5}$$

also can be treated as a particular case of system (3). Indeed, representing W and F as $W = (u + iv)$, $F = \beta(f^1 + if^2)$ and changing independent variable $\tau \to t = \beta\tau$, we transform (5) to the form (3) with $A = \left(\begin{smallmatrix} \beta^{-1} & -1 \\ 1 & \beta^{-1} \end{smallmatrix}\right)$. The standard CGL equation corresponds to the case $F = W - (1 + i\alpha)W|W|^2$.

Thus the symmetry analysis of Eqs. (2) has a large application value and can be used, e.g., to construct exact solutions for a very extended class of physical and biological systems. The comprehensive group analysis of systems (2) is also a nice "internal" problem of the Lie theory which admits exact general solution for the case of *arbitrary* number of independent variables $x_1, x_2, \ldots, x_m$.

We notice that group classification of Eqs. (2) by no means is a standard problem of group analysis of partial differential equations which can be solved with direct application of well-known algorithms. Because of presence of two arbitrary elements, i.e., f^1 and f^2, this classification needs a rather non-trivial generalization of the approach [4] used for classification of Eq. (1).

Equations (3) with arbitrary invertible matrix A were classified in paper [12]. To our great a pity, mainly due to typographical errors (made during the editing procedure), presentation of results in [12] was not satisfactory.[2]

The present paper is the first from the series in which we present the completed group classification of coupled reaction–diffusion equations (3) with *arbitrary* (i.e., invertible or singular) matrix A. Moreover, we present a straightforward and easily verified procedure of solution of the determining equations which guarantees the completeness of the obtained results.

We also indicate clearly the equivalence relations used in the classification procedure, i.e., present explicitly the equivalence groups for all classified equations. In addition, we extend the results obtained in [12] to the important case of *non-invertible* matrix A and more general equations including both the first and second order derivatives with respect to spatial variables.

Let us note that there are three ad hoc non-equivalent classes of Eqs. (3) corresponding to the following forms of matrices A:

$$\text{(I)}\quad A = \begin{pmatrix} a & -1 \\ 1 & a \end{pmatrix}; \qquad \text{(II)}\quad A = \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}; \qquad \text{(III)}\quad A = \begin{pmatrix} a & 0 \\ 1 & a \end{pmatrix}, \tag{6}$$

where a is an arbitrary parameter. Moreover, any 2×2 matrix A can be reduced to one of the forms (6) using linear transformations of dependent variables and scaling independent variables in (3).

The NS and CGL equations correspond to matrices A of form (I). The general equations (3) with such matrices (i.e., generalized CGL equations) are the main subject of group classification carried out in the present paper while the cases (II) and (III) will be considered in the following publications. Nevertheless till an appropriate moment we will consider equations with all types of matrices A enumerated in (6).

[2] The tables presenting the results of group classification have been deformed and cut off. It is necessary to stress that it was the authors fault, one of whom signed the paper proofs without careful reading.

2. Determining equations and equivalence transformations

In the first stage we restrict ourselves to group classification of Eqs. (3) with invertible matrix A. Moreover, till an appropriate moment we consider Eqs. (3) with arbitrary number n of dependent variables.

Using the standard Lie algorithm [13] (or its specific version proposed in [12]), one can find determining equations for the functions η, ξ^a and π^a which specify generator X of the symmetry group admitted by Eq. (3):

$$X = \eta\frac{\partial}{\partial t} + \xi^\nu\frac{\partial}{\partial x_\nu} - \pi^b\frac{\partial}{\partial u_b} \equiv \eta\partial_t + \xi^\nu\partial_{x_\nu} - \pi^b\partial_{u_b}, \tag{7}$$

where a summation from 1 to m and from 1 to 2 is assumed over repeated indices ν and b respectively, and a temporary notation $u = u_1$, $v = u_2$ is used. In a more general case of n dependent variables $U = \text{column}(u_1, u_2, \dots, u_n)$ the repeated indices b run over the values $1, 2, \dots, n$.

We shall not reproduce the deduction of the determining equations here (refer to [12]) but present them directly.

Dependence of η, ξ^ν and π^b on U is defined by the following relations:

$$\eta_{u_a} = 0, \qquad \xi^\nu_{u_b} = 0, \qquad \pi^a_{u_c u_b} = 0. \tag{8}$$

So from (8), η and ξ^ν are the functions of t and x_μ and π^ν is linear in u_a. Thus,

$$\pi^a = N^{ab}u_b + B^a, \tag{9}$$

where N^{ab}, B^a are the functions of t and x_ν only. The remaining equations are [12]:

$$2A\xi^\nu_{x_\mu} = -\delta^{\mu\nu}\big(\eta_t A + [A, N]\big), \quad \eta_{x_\nu t} = 0, \tag{10}$$

$$\xi^\nu_t - 2AN_{x_\nu} - A\Delta\xi^\nu = 0,$$

$$\eta_t f^k + N^{kb}f^b + \big(N^{kb}_t - \Delta A^{ks}N^{sb}\big)u_b + B^k_t - \Delta A^{kc}B^c = \big(B^a + N^{ab}u_b\big)f^k_{u_a}. \tag{11}$$

Here N and A are the matrices whose elements are N^{ab} and A^{ab}, δ^{ab} is the Kronecker symbol.

In accordance with (9)–(11) the general form of the related generator (7) is [12]:

$$X = \lambda K + \sigma_\mu G_\mu + \omega_\mu \hat{G}_\mu + \mu D - \big(C^{ab}u_b + B^a\big)\partial_{u_a} + \Psi^{\mu\nu}x_\mu\partial_{x_\nu} + \nu\partial_t + \rho_\mu\partial_{x_\mu}, \tag{12}$$

where the Greek letters denote arbitrary constants, B^a are the functions of t, x, and C^{ab} are the functions of t satisfying

$$C^{ab}A^{bk} - A^{ab}C^{bk} = 0 \tag{13}$$

and

$$\begin{aligned} K &= 2t(t\partial_t + x_\mu\partial_{x_\mu}) - \frac{x^2}{2}\big(A^{-1}\big)^{ab}u_b\partial_{u_a} - tmu_a\partial_{u_a},\\ G_\mu &= t\partial_{x_\mu} + \frac{1}{2}x_\mu\big(A^{-1}\big)^{ab}u_b\partial_{u_a},\\ \hat{G}_\mu &= e^{\gamma t}\left(\partial_{x_\mu} + \frac{1}{2}\gamma x_\mu\big(A^{-1}\big)^{ab}u_b\partial_{u_a}\right),\\ D &= t\partial_t + \frac{1}{2}x_\mu\partial_{x_\mu}. \end{aligned} \tag{14}$$

Here A^{ab} and $(A^{-1})^{ab}$ are the elements of matrix A and matrix inverse to A, respectively.

In accordance with (11), Eq. (3) admits symmetry operator (12) iff the following classifying equations for f^1 and f^2 are satisfied:

$$\begin{aligned}&\big(2\lambda t(m+4)+\mu\big)f^a+\big(\lambda x^2+\sigma_\mu x_\mu+\gamma e^{\gamma t}\omega_\mu x_\mu\big)\big(A^{-1}\big)^{ab}f^b+C^{ab}f^b+C_t^{ab}u_b\\&\qquad+B_t^a-\Delta A^{ab}B^b\\&\quad=\big(B^s+C^{sb}u_b+\lambda mtu_s+\big(\lambda x^2+\sigma_\mu x_\mu+\gamma e^{\gamma t}\omega_\mu x_\mu\big)\big(A^{-1}\big)^{sk}u_k\big)f^a_{u_s}.\end{aligned}\tag{15}$$

Thus the group classification of Eqs. (3) with a non-singular matrix A reduces to solving Eq. (15) where $\lambda, \mu, \sigma_\mu, \omega_\mu, \gamma$ are arbitrary parameters, B^a and C^{ab} are the functions of (t, x) and t, respectively. Moreover, matrix C with elements C^{ab} should commute with A.

We notice that relations (12)–(15) are valid for group classification of systems (3) of coupled reaction–diffusion equations including *arbitrary* number n of dependent variables $U = (u_1, u_2, \dots, u_n)$ provided the related $n \times n$ matrix A be invertible [12]. In this case indices a, b, s, k in (12)–(15) run over the values $1, 2, \dots, n$.

We will solve classifying equations (15) up to equivalence transformations $U \to \tilde{U} = G(U, t, x)$, $t \to \tilde{t} = T(U, t, x)$, $x \to \tilde{x} = X(U, t, x)$ and $f \to \tilde{f} = F(U, t, x, f)$ which keep the general form of Eqs. (3) but can change functions f^1 and f^2. The group of equivalence transformations for Eq. (3) can be found using the classical Lie approach and treating f^1 and f^2 as additional dependent variables. In addition to the obvious symmetry transformations

$$t \to t' = t + a, \qquad x_\mu \to x'_\mu = R_{\mu\nu}x_\nu + b_\mu,\tag{16}$$

where a, b_μ and $R_{\mu\nu}$ are arbitrary parameters satisfying $R_{\mu\nu}R_{\mu\lambda} = \delta_{\mu\lambda}$, this group includes the following transformations:

$$\begin{aligned}&u_a \to K^{ab}u_b + b_a, \qquad f^a \to \lambda^2 K^{ab} f^b,\\&t \to \lambda^{-2}t, \qquad x_a \to \lambda^{-1}x_a,\end{aligned}\tag{17}$$

where K^{ab} are the elements of an invertible constant matrix K commuting with A, $\lambda \neq 0$ and b_a are arbitrary constants.

For the case when $n = 2$ and matrix A belongs to type (I) (6) the transformation matrix has the following form:

$$K = \begin{pmatrix} K_1 & -K_2 \\ K_2 & K_1 \end{pmatrix}.\tag{18}$$

It is possible to show that there are no more extended equivalence relations valid for arbitrary non-linearities f^1 and f^2. However, if functions f^1, f^2 are specified, the invariance group can be more extended. In addition to transformations (17) it includes symmetry transformations generated by infinitesimal operator (12), and can include additional equivalence transformations (AET). We will specify AET in the following.

3. Basic, main and extended symmetries

Thus to describe Lie symmetries of Eq. (3) (whose generators have the general form (12)) it is necessary to find all non-linearities f^1 and f^2 which satisfy the corresponding classifying equations (15). To solve these rather complicated equations, we use the main algebraic property of the related symmetries, i.e., the fact that they should form a Lie algebra. In other words, instead of going throw all non-equivalent possibilities arising via separation of variables in the classifying equations, we first specify all non-equivalent realizations of the invariance algebra for our equations. Then using the one-to-one correspondence between these algebras and classifying equations (15), we easily solve the group classification problems for Eqs. (3).

Equation (15) does not include parameters $\Psi^{\mu\nu}$, ν and ρ_ν present in (12) thus for any f^1 and f^2 Eq. (3) admits symmetries generated by the following operators:

$$P_0 = \partial_t, \qquad P_\lambda = \partial_{x_\lambda}, \qquad J_{\mu\nu} = x_\mu \partial_{x_\nu} - x_\nu \partial_{x_\mu}. \tag{19}$$

Infinitesimal operators (19) generate the evident symmetry transformations (16) which form the kernel of invariance groups of Eq. (3). For some classes of non-linearities f^1 and f^2 the invariance algebra of Eq. (3) is more extended but includes (19) as a subalgebra. We will refer to (19) as to the *basic symmetries*.

Let us specify one more subclass of symmetries of Eq. (3) which we call *main symmetries*. The related generator $\tilde{X}$ has the form (12) with $\Psi^{\mu\nu} = \nu = \rho_\nu = \sigma_\nu = \omega_\nu = 0$, i.e.,

$$\tilde{X} = \mu D + C^{ab} u_b \partial_{u_a} + B^a \partial_{u_a}. \tag{20}$$

The classifying equation for symmetries (20) can be obtained from (15) by setting $\mu = \sigma^a = \omega^a = 0$. As a result we obtain

$$\left(\mu\delta^{ab} + C^{ab}\right) f^b + C_t^{ab} u_b + B_t^a - \Delta A^{ab} B^b = \left(C^{nb} u_b + B^n\right) f^a_{u_n}. \tag{21}$$

It is easily verified that operators (20) and (19) form a Lie algebra which is a subalgebra of symmetries for Eq. (3) (this algebra can be either finite of infinite dimensional).

On the other hand, if Eq. (3) admits a more general symmetry (12) with $\sigma_a \neq 0$ or (and) $\lambda \neq 0, \omega^\mu \neq 0$ then it has to admit symmetry (20) also. To prove this, we calculate multiple commutators of (12) with the basic symmetries (19) and use the fact that such commutators have to belong to the symmetries of Eq. (3), i.e., generate their own classifying equation (15).

Let Eq. (3) admits symmetry (12) with $\sigma_\mu \neq 0$, $\Psi^{\mu\nu} = \rho_\mu = \nu = \lambda = \omega^k = 0$, i.e.,

$$X = \sigma_\nu G_\nu + \mu D + \left(C^{ab} u_b + B^a\right)\partial_{u_b}. \tag{22}$$

Commuting Y with P_μ, we obtain one more symmetry

$$Y_\mu = -\frac{\sigma_\mu}{2}\left(A^{-1}\right)^{ab} u_b \partial_{u_a} + B^a_\mu \partial_{u_a} + \mu P_\mu. \tag{23}$$

The last term belongs to the basic symmetry algebra (19) and so can be omitted. The remaining terms are of the type (20).

Thus supposing the extended symmetry (22) is admissible, we conclude that Eq. (3) has to admit the main symmetry also.

Commuting (23) with P_0 and P_λ, we come to the following symmetries:

$$Y_{\mu\nu} = B^a_{\mu\nu} \partial_{u_a}, \qquad Y_{\mu t} = B^a_{\mu t} \partial_{u_a}. \tag{24}$$

Any symmetry (22)–(24) generates its own system (15) of classifying equations. After straightforward but rather cumbersome calculations we conclude that all these systems are compatible provided the following condition is satisfied:

$$\left(A^{-1}\right)^{ab} f^b = \left(A^{-1}\right)^{nb} u_b f^a_{u_n}. \tag{25}$$

If (25) is satisfied, Eq. (3) admits symmetry (22) with $\mu = C^{ab} = B^a = 0$, i.e., Galilei generators G_ν of (14).

Analogously, supposing that Eq. (3) admits extended symmetry (12) with $\lambda \neq 0$ and $\omega^a = 0$, we conclude that it has to admit symmetry (22) with $\mu \neq 0$ and $\sigma_\nu \neq 0$ also. The related functions f^1 and f^2 should satisfy relations (25) and (21). Moreover, analyzing possible dependence of C^{ab} and B^a in the corresponding relations (15) on t, we conclude that they should be either scalars or linear in t, i.e., $C^{ab} = \mu^{ab} t + \nu^{ab}$. Moreover, up to equivalence transformations (17) we can choose $B^a = 0$ and reduce the related equation system (15), (21) to the following equations:

$$\begin{aligned}
&(m+4) f^a + \mu^{ab} f^b = \left(\mu^{kb} u_b + m u_k\right) \frac{\partial f^a}{\partial u_k}, \\
&\nu^{ab} f^b + \mu^{ab} u_b = \nu^{kb} u_b \frac{\partial f^a}{\partial u_k},
\end{aligned} \tag{26}$$

where constants ν^{ab} and μ^{ab} are non-trivial in the case of the diagonal diffusion matrix only.

Finally, for general symmetry (12) it is not difficult to show that the condition $\omega^a \neq 0$ leads to the following equation for f^a:

$$\left(A^{-1}\right)^{kb}\left(f^b + \gamma u^b\right) = \left(A^{-1}\right)^{ab} u_b f^k_{u_a}. \tag{27}$$

We notice that relations (25) and (27) are particular cases of (21) for $\mu = 0$, $C^{ab} = (A^{-1})^{ab}$ and $\mu = 0$, $C^{ab} = e^{\gamma t}(A^{-1})^{ab}$, respectively. Thus if relation (25) is valid then, in addition to G_μ (14), Eq. (3) admits the symmetry

$$X = \left(A^{-1}\right)^{ab} u_b \partial_{u_a}. \tag{28}$$

Alternatively, if (27) is satisfied, Eq. (2) admits the symmetry $\hat{G}_\mu$ (2.6) and also the following one:

$$X = e^{\gamma t}\left(A^{-1}\right)^{ab} u_b \partial_{u_a}, \quad \gamma \neq 0. \tag{29}$$

Thus it is reasonable first to classify Eqs. (3) which admit main symmetries (20) and then specify all cases when these symmetries can be extended.

The conditions when system (3) admits extended symmetries are given by relations (25)–(27).

The results of Section 3 are valid for Eqs. (3) with arbitrary invertible diffusion matrix. In the following we restrict ourselves to the case when $n = 2$ and matrix A has the form (I) given in (6), i.e., to the case of generalized CLG equations. In other words, we will classify the following systems of coupled reaction–diffusion equations:

$$u_t - \Delta(au - v) = f^1(u, v), \qquad v_t - \Delta(av + u) = f^2(u, v), \tag{30}$$

which are particular cases of general systems (3) when matrix A has the form (I) (6).

4. Algebras of main symmetries

In accordance with the plan outlined above, we start with investigation of main symmetries (20) admitted by Eq. (30).

The first step of our analysis is to describe non-equivalent Lie algebras $\mathcal{A}$ of operators (20) which can be admitted by this equation. We shall consider consequently one-, two-, ..., n-dimensional algebras $\mathcal{A}$.

For any type of matrix A enumerated in (6) we specify all non-equivalent "tails" of operators (20), i.e., the terms

$$N = C^{ab} u_b \partial_{u_a} + B^a \partial_{u_a}. \tag{31}$$

These terms can either be a constituent part of a more general symmetry (20) or represent a particular case of (20) corresponding to $\mu = 0$. Thus the problem of classification of algebras $\mathcal{A}$ includes a subproblem of classification of algebras of operators (31).

Let Eq. (3) admits a one-dimensional invariance algebra whose basis element has the form (31), and does not admit a more extended algebra of the main symmetries. Then commutators of N with the basic symmetries P_0 and P_a should be equal to a linear combination of N and operators (19). It is easily verified that there are three possibilities:

$$\begin{aligned}
&(1)\quad C^{ab} = \mu^{ab}, \qquad B^a = \mu^a,\\
&(2)\quad C^{ab} = e^{\lambda t}\mu^{ab}, \qquad B^a = e^{\lambda t}\mu^a,\\
&(3)\quad C^{ab} = 0, \qquad B^a = e^{\lambda t + \omega\cdot x}\mu^a,
\end{aligned} \tag{32}$$

where μ^{ab}, μ^a, λ, and ω are the constants, and the matrix with elements μ^{ab} should commute with A. In the case when A is of the form (I), Eq. (6), constants μ^{ab} are restricted by the following relations: $\mu^{11} = \mu^{22}$, $\mu^{12} = -\mu^{21}$.

To specify all non-equivalent operators (31), (32), we use the isomorphism of (31) with 3×3 matrices of the following form:

$$g = \begin{pmatrix} 0 & 0 & 0 \\ B^1 & C^{11} & C^{12} \\ B^2 & C^{21} & C^{12} \end{pmatrix} \sim \begin{pmatrix} 0 & 0 & 0 \\ \mu^1 & \mu^{11} & \mu^{12} \\ \mu^2 & -\mu^{12} & \mu^{11} \end{pmatrix}. \tag{33}$$

Equations (30) admit equivalence transformations (17) which change the term N (31) and can be used to simplify it. The corresponding transformation for matrix (33) can be represented as

$$g \to g' = UgU^{-1}, \tag{34}$$

where U is a 3×3 matrix of the following special form:

$$U = \begin{pmatrix} 1 & 0 & 0 \\ b^1 & K^1 & K^2 \\ b^2 & -K^2 & K^1 \end{pmatrix}. \tag{35}$$

Up to equivalence transformations (34), (35) there exist three matrices g, namely

$$g_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad g_2 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad g_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \alpha & -1 \\ 0 & 1 & \alpha \end{pmatrix}. \tag{36}$$

In accordance with (31), (32) the related symmetry operator can be represented in one of the following forms:

$$X_1 = \mu D - 2(g_a)_{bc}\tilde{u}_c \partial_{u_b}, \qquad X_2 = e^{\lambda t}(g_a)_{bc}\tilde{u}_c \partial_{u_b} \tag{37}$$

or

$$X_3 = e^{\lambda t + \omega\cdot x}\partial_{u_2}. \tag{38}$$

Here $(g_a)_{bc}$ are the elements of a chosen matrix (36), $b, c = 0, 1, 2$, $\tilde{u} = \text{column}(1, u, v)$.

Formulae (37) and (38) give the principal description of one-dimension algebras A for Eq. (30).

To describe two-dimension algebras $\mathcal{A}$, we classify the matrices g (33) forming two-dimension Lie algebras. Choosing one of the basis elements in the forms given in (36) and the other element in the general form (33), we find that up to equivalence transformations (34) there exist three two-dimension algebras of matrices g,

$$A_{2,1} = \{g_1, g_3\}, \qquad A_{2,2} = \{g_2, g_4\}, \qquad A_{2,3} = \{g_1, g_2\}, \tag{39}$$

where g_1, g_2, g_3 are matrices (36), and

$$g_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Algebras $A_{2,1}$ and $A_{2,2}$ are Abelian while the basis elements of $A_{2,3}$ satisfy $[g_1, g_2] = g_2$.

Using (39), we easily find the pairs of operators (20) forming two-dimension Lie algebras. Denoting

$$\hat{e}_\alpha = (e_\alpha)_{ab}\tilde{u}_b\partial_{u_a}, \quad \alpha = 1, 2,$$

we represent them as follows:

$$\begin{aligned} &\langle \mu D + \hat{e}_1 + \nu t\hat{e}_2, \hat{e}_2\rangle, \quad \langle \mu D + \hat{e}_2 + \nu t\hat{e}_1, \hat{e}_1\rangle, \\ &\langle \mu D - \hat{e}_1, \nu D - \hat{e}_2\rangle, \quad \langle F_1\hat{e}_1 + G_1\hat{e}_2, F_2\hat{e}_1 + G_2\hat{e}_2\rangle \end{aligned} \tag{40}$$

for e_1, e_2 belonging to algebras $A_{2,1}$, $A_{2,2}$, and

$$\langle \mu D - \hat{e}_1, \hat{e}_2\rangle, \quad \langle \mu D + \hat{e}_1 + \nu t\hat{e}_2, \hat{e}_2\rangle \tag{41}$$

for e_1 and e_2 belonging to $A_{2,3}$.

Here $\{F_1, G_1\}$ and $\{F_2, G_2\}$ are fundamental solutions of the following system:

$$F_t = \lambda F + \nu G, \qquad G_t = \sigma F + \gamma G \tag{42}$$

with arbitrary parameters λ, ν, σ, γ.

The list (40)–(41) does not include algebras of the types

$$\langle F\hat{e}, G\hat{e}\rangle \quad \text{and} \quad \left\langle \mu D + e^{\nu t+\omega\cdot x}\hat{e}, e^{\nu t+\omega\cdot x}\hat{e}\right\rangle \tag{43}$$

(with F, G satisfying (42)) which are either incompatible with classifying equations (15) or reduce to one-dimension algebras. All the other two-dimension algebras $\mathcal{A}$ can be reduced to the one form given in (40), (41) using equivalence transformations (17).

In analogous way we can find three- and four-dimensional algebras of operators (20). Thus we have two three-dimension algebras of matrices (33)

$$\begin{aligned} &A_{3,1}\colon \quad e_1 = g_1, \quad e_2 = g_2, \quad e_3 = g_4; \\ &A_{3,2}\colon \quad e_1 = g_2, \quad e_2 = g_3, \quad e_3 = g_4 \end{aligned} \tag{44}$$

and the only four-dimension algebra:

$$A_{4,1}\colon \quad e_1 = g_1, \quad e_2 = g_3, \quad e_3 = \tilde{g}_4, \quad e_4 = g_2. \tag{45}$$

Algebras (44) can be generalized to the following algebras $\mathcal{A}$:

$$\langle \mu D - 2\hat{e}_1, \hat{e}_2, \hat{e}_3\rangle, \quad \langle D + 2\hat{e}_1 + 2\nu t\hat{e}_2, \hat{e}_2, \hat{e}_3\rangle,$$
$$\langle D + 2\hat{e}_1 + 2\nu t\hat{e}_3, \hat{e}_3, \hat{e}_2\rangle, \quad \langle \hat{e}_1, F_1\hat{e}_2 + G_1\hat{e}_3, F_2\hat{e}_2 + G_2\hat{e}_3\rangle \tag{46}$$

while (44) generate the following algebras:

$$\langle \mu D - 2\hat{e}_1, \hat{e}_2, \hat{e}_3\rangle, \quad \langle \hat{e}_1, D + 2\hat{e}_2 + 2\mu t\hat{e}_3, \hat{e}_3\rangle. \tag{47}$$

Finally, $A_{4,2}$ generates the following algebras $\mathcal{A}$:

$$\langle \mu D - 2\hat{e}_1, \nu D - 2\hat{e}_2, \hat{e}_3, \hat{e}_4\rangle, \quad \big\langle \hat{e}_1, \hat{e}_2, \hat{e}^{\mu t + \nu\cdot x}\hat{e}_3, \hat{e}^{\mu t+\nu\cdot x}\hat{e}_4\big\rangle. \tag{48}$$

The list (46)–(48) does not include algebras which have subalgebras (43) (which are incompatible with classifying equations (21)).

5. Classification results

Using the results presented in the previous section we easily perform group classification of Eqs. (30). To do this, we solve the classifying equations (21) with their known coefficients C^{ab} and B^a which are defined comparing (20) with the found realizations of algebras $\mathcal{A}$.

If Eq. (30) admits one-dimensional algebra $\mathcal{A}$, i.e., one of algebras (37), (38) the related functions f^1 and f^2 have to satisfy the corresponding determining equation (21) which define f^1 and f^2 up to arbitrary functions. In the case of two-dimensional algebras whose realizations are given by relations (40), (41) we have a system of two determining equations corresponding to two basis elements which usually define f^1 and f^2 up to arbitrary parameters. In addition, we control the cases when Eq. (30) admits extending symmetries G_μ, $\hat{G}_\mu$ and K (14), i.e., when the found functions f^1 and f^2 satisfy conditions (25), (27) and (26), respectively.

The next important step is to find additional equivalence transformations admitted by Eqs. (3) with specified non-linearities f^1 and f^2. These transformations are relatively easy calculated using the standard Lie algorithm and treating f^1 and f^2 as additional variables.

We will not reproduce here the related routine calculations but present their results in the following tables. In the fifth and sixth columns of Table 1 the extending symmetries and additional equivalence transformations are specified. The list of possible AETs is present in the following formulae:

$$\begin{aligned}
&(1) \quad u \to \exp(\omega t)u, \quad v \to \exp(\omega t)v,\\
&(2) \quad u \to u\cos\omega t - v\sin\omega t, \quad v \to v\cos\omega t + u\sin\omega t,\\
&(3) \quad u \to \exp(\omega t)u, \quad v \to v + \omega\frac{t^2}{2},\\
&(4) \quad u \to u + \omega t, \quad v \to v,\\
&(5) \quad u \to u, \quad v \to v + \omega t,\\
&(6) \quad u \to \exp(\nu\omega t)\big(u\cos(\sigma\omega t) + v\sin(\sigma\omega t)\big),\\
&\qquad v \to \exp(\nu\omega t)\big(v\cos(\sigma\omega t) - u\sin(\sigma\omega t)\big),\\
&(7) \quad u \to \exp(2\omega t)\big(u\cos\big(\sigma\omega t^2\big) - v\sin\big(\sigma\omega t^2\big)\big),\\
&\qquad v \to \exp(2\omega t)\big(v\cos\big(\sigma\omega t^2\big) + u\sin\big(\sigma\omega t^2\big)\big),\\
&(8) \quad u \to \exp\big(\lambda\omega t^2\big)\big(u\cos(2\omega t) + v\sin(2\omega t)\big),\\
&\qquad v \to \exp\big(\lambda\omega t^2\big)\big(v\cos(2\omega t) - u\sin(2\omega t)\big).
\end{aligned} \tag{49}$$

Table 1
Non-linearities and extendible symmetries for Eqs. (2)

<table>
<tr><th>No.</th><th>Non-linear terms</th><th>Arguments of F_1, F_2</th><th>Main symmetries</th><th>Additional symmetries</th><th>AET (49)</th></tr>
<tr><td rowspan="2">1</td><td rowspan="2">$f^1 = uF_1 + vF_2 - \nu z(\mu u + v)$,
$f^2 = vF_1 - uF_2 + \nu z(u - \mu v)$,
$R = (u^2 + v^2)^{1/2}$,
$z = \tan^{-1}\left(\frac{v}{u}\right)$</td><td rowspan="2">$\mathrm{R}e^{\mu z}$</td><td rowspan="2">$e^{\nu t}(\mu R\partial_R - \partial_z)$</td><td>$\hat{G}_\alpha$ if
$\mu = a$,
$\nu \neq 0$;</td><td rowspan="2">2 if
$\mu = \nu = 0$</td></tr>
<tr><td>G_α if
$\mu = a$,
$\nu = 0$</td></tr>
<tr><td>2</td><td>$f^1 = e^{\mu z}R^\nu(\lambda u - \sigma v)$,
$f^2 = e^{\mu z}R^\nu(\lambda v + \sigma u)$</td><td></td><td>$\nu D - u\partial_u - v\partial_v$,
$\mu D - u\partial_v + v\partial_u$</td><td>$G_\alpha$ if
$\mu = a\nu$
and K if
$\nu = \frac{4}{m}$</td><td>6 if
$\nu = 0$</td></tr>
</table>

Table 2
Non-linearities and non-extendible symmetries for Eqs. (2)

No.	Non-linear terms	Arguments of F_α	Symmetries	AET (49)
1	$f^1 = u^{\nu+1}F_1$, $f^2 = u^{\nu+1}F_2$	$\frac{v}{u}$	$\nu D - u\partial_u - v\partial_v$	1 if $\nu = 0$
2	$f^1 = \nu u + F_1$, $f^2 = -\mu u + F_2$	v	$e^{(\nu+a\mu)t}\Psi_\mu(x)\partial_u$	4 if $\mu = \nu = 0$
3	$f^1 = e^{\nu v}F_1$, $f^2 = e^{\nu v}F_2$	u	$\nu D - \partial_v$	
4	$f^1 = u(F_1 + \varepsilon \ln u)$, $f^2 = v(F_2 + \varepsilon \ln u)$	$\frac{v}{u}$	$e^{\varepsilon t}(u\partial_u + v\partial_v)$	
5	$f^1 = e^{\lambda z}(uF_1 + vF_2)$, $f^2 = e^{\lambda z}(vF_1 - uF_2)$	$\mathrm{R}e^{\nu z}$	$\lambda D + R\partial_R - \partial_z$	6, $\sigma = 1$ if $\lambda = 0$
6	$f^1 = \lambda$, $f^2 = \varepsilon \ln u$		$D + u\partial_u + v\partial_v + \varepsilon t\partial_v$, $\Psi_0(x)\partial_v$	3, 5
7	$f^1 = \lambda v^{\nu+1}$, $f^2 = \sigma v^{\nu+1}$		$\nu D - u\partial_u - v\partial_v$, $\Psi_0(x)\partial_u$	4
8	$f^1 = \lambda e^v$, $f^2 = \sigma e^v$		$D - \partial_v$, $\Psi_0(x)\partial_u$	4
9	$f^1 = \mu \ln v$, $f^2 = \nu \ln v$		$\Psi_0(x)\partial_u$, $D + u\partial_u + v\partial_v + \left((\mu - \nu a)t - \frac{\nu}{2m}x^2\right)\partial_u$	4

where the Greek letters denote parameters whose values will be specified in the tables. In contrast with (16), (17) these transformations are valid only for equations with some special non-linearities f^1, f^2.

Table 1 presents non-equivalent equations which admit the main or the main and extended symmetries. The conditions for non-linearities which extend the symmetries are specified in the fifth column.

In Table 2 the non-linearities are collected which correspond to the main symmetries only. In Table 3 symmetries of a subclass of Eqs. (30) are specified.

Table 3
Symmetries of Eqs. (2) with non-linearities $f^1 = (\mu u - \sigma v)\ln R + z(\lambda u - \nu v)$, $f^2 = (\mu v + \sigma u)\ln R + z(\lambda v + \nu u)$

No.	Conditions for coefficients	Main symmetries and AET (49)	Additional symmetries
1	$\lambda = 0$, $\mu = \nu$	$e^{\mu t}\partial_z$, $e^{\mu t}(R\partial_R + \sigma t v\partial_v)$ [AET 7 if $\mu = 0$]	$\hat{G}_\alpha$ if $a\sigma = 0, \mu \neq 0$; G_α if $a = \nu = 0, \sigma \neq 0$
2	$\lambda = 0$, $\mu \neq \nu$	$e^{\nu t}\partial_z$, $e^{\mu t}(\sigma\partial_z + (\mu - \nu)R\partial_R)$ [AET 6 if $\mu\nu = 0$]	$\hat{G}_\alpha$ if $a\sigma = \nu - \mu,\ \mu \neq 0$ or $a = 0,\ \mu \neq 0$; G_α if $a\sigma = \nu,\ \mu = 0$
3	$\delta = \lambda\sigma + \frac{1}{4}(\mu - \nu)^2 = 0$, $\mu + \nu = 2\omega_0$, $\lambda \neq 0$	$X_3 = e^{\omega_0 t}(2\lambda R\partial_R + (\nu - \mu)\ \partial_z)$, $2e^{\omega_0 t}\partial_z + tX_3$ [AET 6 if $\mu + \nu = 0$; AET 8 and 1 if $\mu = \nu = 0$]	$\hat{G}_\alpha$ if $\omega_0 \neq 0$ $a(\mu - \nu) = 2\lambda$; G_α if $a\nu = -\lambda,\ \omega_0 = 0$
4	$\lambda \neq 0$, $\delta = 1$, $\omega_\pm = \omega_0 \pm 1$	$e^{\omega_+ t}(\lambda R\partial_R + (\omega_+ - \mu)\partial_z)$, $e^{\omega_- t}(\lambda R\partial_R + (\omega_- - \mu)\partial_z)$ [AET 6 if $\mu\nu = \lambda\sigma$ and 1 if $\mu = \sigma = 0$]	$\hat{G}_\alpha$ if $\mu\nu \neq \lambda\sigma$, $\lambda = a(\mu - \nu + a\sigma)$; G_α if $\nu\mu = \lambda\sigma$, $\lambda = a\mu$
5	$\delta = -1$	$\exp(\omega_0 t)[2\lambda\cos tR\partial_R + ((\nu - \mu)\cos t - 2\sin t)\partial_z]$, $\exp(\omega_0 t)[2\lambda\sin tR\partial_R + ((\nu - \mu)\sin t + 2\cos t)\partial_z]$	none

Here G_μ, $\hat{G}_\mu$ and K are operators (14) where A is matrix (I), Eq. (6), $\Psi_\mu(x)$ and $\psi_\mu = \psi_\mu(t, x)$ are arbitrary solutions of the Laplace and heat equations: $\Delta\Psi_\mu = \mu\Psi_\mu$, $\Psi_\mu(x)$ is a function satisfying $\Delta\Psi_\mu(x) = \mu\Psi_\mu(x)$.

We notice that parameters μ and ν in Tables 1 and 2 can take arbitrary real values including zero ones. Parameter ε is non-zero; moreover up to equivalence transformations (17) we can restrict ourselves to $\varepsilon = \pm 1$.

6. Discussion

We perform group classification of generalized CLG equations (30), i.e., find all non-equivalent equations of the considered type and describe their symmetries. The obtained results can be used, e.g., to construct exact solutions for equations which admit sufficiently extended symmetries, using the standard Lie algorithm [13].

In Tables 1–3 all non-equivalent generalized CLG equations are listed together with their symmetries and additional equivalence transformations. More exactly, we specify here only extensions of the basic symmetries (19) and do not consider linear equations.

First we notice that the usual CGL equation appears as a particular case of the classification procedure. The related non-linearities and symmetries are present in Table 1, item 2 when $\mu = 0$. In addition to the basic symmetries (19) this equation admits the dilatation symmetry and symmetry $u\partial_v - v\partial_u$, which correspond to scaling of dependent and independent variables and multiplying solutions of the CLG equation by a phase factor.

In accordance with our classification there exist eight types of generalized CLG equations defined up to arbitrary functions F_1 and F_2 depending on variables indicated in the third column of Table 1, item 1 and Table 2, items 1–5. Nonlinearities corresponding to the most extended symmetries are collected in Table 1. In particular, there are non-linearities corresponding to Galilei-invariant equations (30) (refer to Table 1, item 1 for $a = \mu$, $\nu = 0$):

$$u_t - \Delta(au - v) = uF_1 + vF_2,$$
$$v_t - \Delta(av + u) = vF_1 - uF_2,$$

where F_1 and F_2 are arbitrary functions of Re^{az}. This system can be rewritten as a single equation for a complex function $W = u + iv$:

$$W_t - (a + i)\Delta W = \mathcal{F}W, \tag{50}$$

where $\mathcal{F} = F_1 - iF_2$ is a complex function of real variable $\xi = \ln|W| + az$ with z being a phase of W.

The standard CLG equation does not belong to the class (50) and so is not Galilei invariant. On the other hand, setting in (50) $a = 0$, we come to the Galilei-invariant subclass of the NS equations (4).

The non-linearities enumerated in Table 3, item 2 of Table 1 and items 6–9 of Table 2 are defined up to arbitrary parameters. The most extended symmetry is indicated in item 2 of Table 1 and corresponds to the following equation for complex function $W = u + iv$:

$$W_t - (a + i)\Delta W = \alpha\left(e^{az}|W|\right)^{\rho} W, \tag{51}$$

where $\alpha = \lambda + i\sigma$ is a complex parameter and $\rho = 4/m$.

In accordance with the above Eq. (51) admits Lie algebra of the Schrödinger group including operators P_μ, $J_{\mu\nu}$ (19) and also generators of dilatation D, Galilean transformations G_μ and conformal transformations K (14). Setting in (51) $a = 0$, we reduce (51) to the very popular NS equation with critical power $4/m$ non-linearity.

If $\rho \neq 4/m$ then Eq. (51) admits all the above mentioned symmetries except the generator K of conformal transformations.

In general we indicate six classes of Eqs. (30) which admit symmetries G_μ and so are invariant with respect to Galilei group. Namely, in addition to (50), (51) there are the following Galilei-invariant equations:

$$iW_t + \Delta W = -\sigma W \ln|W|, \tag{52}$$

which correspond to item 1, $\nu = a = 0$ of Table 3, and

$$W_t - (a + i)\Delta W = cW\left(\ln|W| + az\right), \tag{53}$$

where c is a complex number equal to $i\sigma$, $\sigma(i - a)$ or $\mu + i\sigma$ for versions indicated in items 2, 3 or 4 of Table 3 correspondingly.

Non-linearities collected in Table 2 correspond to Eqs. (30) which admit the main and basic symmetries only.

Thus we present completed group classification of systems of reaction–diffusion equations (3) with square diffusion matrix of type (I) (6).

The additional aim of this paper is to present an effective approach for solving classifying equations (15). It was demonstrated in Sections 3 and 4 that the problem of group classification of Eqs. (3) can be effectively reduced to searching for the main symmetries (3). We also make a priori specification of these symmetries using the fact that they should form a basis of a Lie algebra. The idea of such a specification was proposed in papers [14,15].

In our following publications we use the approach described here to classify reaction–diffusion equations with diagonal and triangular diffusion matrix. In other words, we plan to complete classification of Eqs. (3) with general diffusion matrix whose non-equivalent versions are given by formulae (6).

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