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Heat Transfer in a Viscoelastic Fluid over a Stretching Sheet

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An analysis is carried out to study the heat transfer characteristics in a viscoelastic fluid over a stretching sheet with frictional heating and internal heat generation or absorption. Two cases are studied, namely (i) the sheet with prescribed surface temperature (PST case) and (ii) the sheet with prescribed wall heat flux (PHF case). The solutions for the temperature, the heat transfer characteristics and their asymptotic limits for large and small Prandtl numbers are obtained in terms of Kummer's and parabolic cylinder functions. For large Prandtl numbers, a boundary layer width of $\sqrt{1/\mathrm{Pr}}$ is noticed in both PST and PHF cases. Furthermore, it is shown that there is no boundary layer type of solution for small Prandtl number.

1. INTRODUCTION

Boundary layer behavior over a moving continuous solid surface is an important type of flow occurring in several engineering processes. For example, materials manufactured by extrusion processes and heat-treated materials traveling between a feed roll and a wind-up roll or on conveyor belts possess the characteristics of a moving continuous surface. In view of these applications, Sakiadis [1] initiated the study of boundary layer flow over a continuous solid surface moving with a constant speed. Due to entrainment of ambient fluid, this boundary layer flow is quite different than boundary layer flow over a semi-infinite flat plate. Erickson *et al.* [2] extended this problem to the case in which the transverse velocity at the moving surface is nonzero, with heat and mass transfer in the boundary layer being taken into account.

These investigations have a definite bearing on the problem of a polymer sheet extruded continuously from a dye. It is often tacitly assumed that the sheet is inextensible, but situations may arise in the polymer industry in

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which it is necessary to deal with a stretching plastic sheet, as pointed out by McCormack and Crane [3]. Danberg and Fansler [4] investigated the nonsimilar solution for the flow in the boundary layer past a wall that is stretched with a velocity proportional to the distance along the wall. Gupta and Gupta [5] analyzed the heat and mass transfer corresponding to the similarity solution for the boundary layer over a stretching sheet subject to suction or blowing. Recently, Chen and Char [6] investigated the effects of power-law surface temperature and power-law surface heat flux vaiation on the heat transfer characteristics of a continuous, linearly stretching sheet subject to suction or blowing.

All the above investigations are resricted to flows of Newtonian fluid. However, of late non-Newtonian fluids have become more important industrially. The laminar boundary layer on an inextensible continuous flat surface moving with a constant velocity in its own plane in a non-Newtonian fluid characterized by a power-law model (Ostwald-de Waele fluid) is studied by Fox *et al.* [7], using both exact and approximate methods. Apart from the limitations of the above power-law model, which does not exhibit any elastic properties (such as normal stress differences in shear flow), in certain polymer processing applications one deals with flow of a viscoelastic fluid over a stretching sheet. Due to this reason, Rajagopal *et al.* [8] studied the flow behavior of a viscoelastic fluid over a stretching sheet and gave an approximate solution to the flow field. They considered the incompressible second-order fluid whose constitutive equation is based on the postulate of gradually fading memory suggested by Coleman and Noll [9] as

$$\mathbf{T} = -P\mathbf{I} + \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2, \tag{1}$$

where $\mathbf{T}$ is the stress tensor, P is the pressure, μ, α_1, α_2 are material constants with $\alpha_1 < 0$, and $\mathbf{A}_1$ and $\mathbf{A}_2$ are defined as

$$\mathbf{A}_1 = (\text{grad } \mathbf{v}) + (\text{grad } \mathbf{v})^T, \tag{2}$$

$$\mathbf{A}_2 = \frac{d}{dt}\mathbf{A}_1 + \mathbf{A}_1 \cdot \text{grad } \mathbf{v} + (\text{grad } \mathbf{v})^T \cdot \mathbf{A}_1. \tag{3}$$

Recently, Troy *et al.* [10] gave the exact solution for the problem of Rajagopal *et al.* [8].

Motivated by these analyses, the present authors studied heat transfer in a viscoelastic fluid over a continuous stretching sheet with power-law surface temperature or power-law surface heat flux including the effects of viscous dissipation and internal heat generation or absorption. A series solution to the energy equation in terms of Kummer's and parabolic cylinder functions are developed and some asymptotic cases are studied. Also, several closed form analytical solutions are presented for special conditions. Further, the contributions of the elastic parameter m, the Prandtl number Pr, the frictional heating parameter (Eckert number) E, and the heat source/sink parameter α to the heat transfer characteristics are found to be quite significant.

2. Flow Analysis

Consider the flow of a second-order fluid obeying (1) past a flat sheet coinciding with the plane $y=0$, the flow being confined to $y>0$. Two equal and opposite forces are applied along the x-axis so that the wall is stretched keeping the origin fixed (see Fig. 1). The steady two-dimensional boundary layer equations for this fluid (for details see Beard and Walters [11]) in usual notation are

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{4}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2}-\lambda\left[\frac{\partial}{\partial x}\left(u\frac{\partial^2 u}{\partial y^2}\right)+\frac{\partial u}{\partial y}\frac{\partial^2 v}{\partial y^2}+v\frac{\partial^3 u}{\partial y^3}\right], \tag{5}$$

where $\nu=\mu/\rho$ and $\lambda=-\alpha_1/\rho$. In deriving these equations it was assumed that the contribution due to the normal stress is of the same order of magnitude as that due to the shear stress, in addition to the usual boundary layer approximations. Thus both ν and λ are $O(\delta^2)$, δ being the boundary layer thickness.

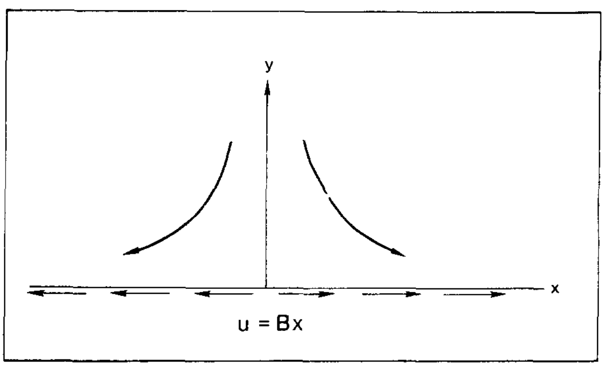

FIG. 1. A sketch of the physical model.

The appropriate boundary conditions for the problem are

$$\begin{aligned} u = Bx,\ v = 0 \quad & \text{at} \quad y = 0,\ B > 0, \\ u \to 0 \quad & \text{as} \quad y \to \infty. \end{aligned} \tag{6}$$

Here the flow is caused solely by the stretching of the sheet, since the free stream velocity is zero. Equations (4) and (5) admit a self-similar solution

$$u = Bxf'(\eta), \qquad v = -(B\nu)^{1/2} f(\eta), \tag{7}$$

$$\eta = (B/\nu)^{1/2}\, y, \tag{8}$$

where a prime denotes differentiation with respect to η. Clearly u and v defined above satisfy the continuity equation (4). Substituting (7) and (8) in (5) we get

$$(f')^2 - ff'' = f''' - \lambda_1[2f'f''' - (f'')^2 - ff^{iv}], \tag{9}$$

where $\lambda_1 = \lambda B/\nu$ is the elastic parameter. The boundary conditions (6) become

$$\begin{aligned} f' = 1, f = 0 \quad & \text{at} \quad \eta = 0, \\ f' \to 0 \quad & \text{as} \quad \eta \to \infty. \end{aligned} \tag{10}$$

In 1987, Troy *et al.* [10] obtained the exact solution

$$f = (1 - e^{-m\eta})/m, \qquad m = 1/\sqrt{1 - \lambda_1}, \tag{11}$$

for the differential equation (9) satisfying conditions (10). This gives us the velocity components

$$\begin{aligned} u &= Bxe^{-m\eta}, \\ v &= -(B\nu)^{1/2}(1 - e^{-m\eta})/m, \end{aligned} \tag{12}$$

and the dimensionless shear stress at the wall

$$\tau = (1 - \lambda_1) f''(0) = -(1 - \lambda_1)^{1/2}. \tag{13}$$

3. Heat Transfer Analysis

The governing boundary layer equation with viscous dissipation (or frictional heating) and internal heat generation or absorption is

$$\rho C_p\left(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right) = k\frac{\partial^2 T}{\partial y^2} + \mu\left(\frac{\partial u}{\partial y}\right)^2 + Q(T - T_\infty). \tag{14}$$

The thermal boundary conditions depend on the type of heating process under consideration. We consider two different heating processes, namely, (i) prescribed surface temperature and (ii) prescribed wall heat flux. The heat transfer analysis for these two processes are carried out in Sections 3.1 and 3.2.

3.1. *Prescribed Surface Temperature (PST-Case)*

For this circumstance, the boundary conditions are

$$\begin{aligned} T = T_w[= T_\infty + A(x/l)^2] \quad &\text{at} \quad y = 0, \\ T \to T_\infty \quad &\text{as} \quad y \to \infty, \end{aligned} \tag{15}$$

where l is characteristic length. Defining the nondimensional temperature

$$\theta(\eta) = (T - T_\infty)/(T_w - T_\infty), \tag{16}$$

and using the relations (7)–(8), Eq. (14) and the boundary conditions (15) can be written as

$$\theta'' + \Pr f\theta' - \Pr(2f' - \alpha)\,\theta = -\Pr E(f'')^2, \tag{17}$$

$$\begin{aligned} \theta = 1 \quad &\text{at} \quad \eta = 0, \\ \theta \to 0 \quad &\text{as} \quad \eta \to \infty, \end{aligned} \tag{18}$$

where

$$\begin{aligned} f &= (1 - e^{-m\eta})/m, && f' = e^{-m\eta}, \\ m &= 1/\sqrt{1 - \lambda_1}, && \text{elastic parameter} \\ \Pr &= \mu C_p/k, && \text{Prandtl number} \\ \alpha &= Q/B\rho C_p, && \text{heat source/sink parameter} \\ E &= Bl^2/C_p A, && \text{Eckert number,} \end{aligned} \tag{19}$$

and a prime denotes differentiation with respect to η.

Defining a new variable

$$\xi = -re^{-m\eta}\left(\text{with } r \equiv \frac{\Pr}{m^2}\right) \tag{20}$$

and substituting the solution f into Eq. (17), we get

$$\xi\theta'' + (1 - r - \xi)\,\theta' + (2 + \alpha r/\xi)\,\theta = -(\beta E/r^2)\,\xi, \tag{21}$$

where prime denotes differentiation with respect to ξ and $\beta = \mathrm{Pr}$. The boundary conditions translate to

$$\theta(-r) = 1 \qquad \text{and} \qquad \theta(0^-) = 0. \tag{22}$$

The solution of Eq. (21) satisfying the conditions (22) in terms of Kummer's functions (see [12]) is

$$\theta(\xi) = \frac{1 + \beta E(4 - 2r - \alpha r)^{-1}}{M((r+s-4)/2,\, 1+s; -r)} \left(\frac{-\xi}{r}\right)^{(r+s)/2}$$
$$\times M\left(\frac{r+s-4}{2},\, 1+s; \xi\right) - \beta E(4 - 2r - \alpha r)^{-1} \left(\frac{\xi}{r}\right)^2, \tag{23}$$

where

$$s = r\left(1 - \frac{4\alpha}{r}\right)^{1/2},$$

$$M(a, b: z) = 1 + \sum_{n=1}^{\infty} \frac{(a)_n z^n}{(b)_n n!},$$

$$(a)_n = a(a+1)(a+2) \cdots (a+n-1),$$

and

$$(b)_n = b(b+1)(b+2) \cdots (b+n-1).$$

The solution (23) can be written in terms of η as

$$\theta(\eta) = \frac{1 + \beta E(4 - 2r - \alpha r)^{-1}}{M((r+s-4)/2,\, 1+s;\, -r)} e^{-(r+s)m\eta/2} M\left(\frac{r+s-4}{2},\, 1+s;\, -re^{-m\eta}\right)$$
$$- \beta E(4 - 2r - \alpha r)^{-1} e^{-2m\eta}. \tag{24}$$

The nondimensional temperature gradient derived from (24) is

$$\theta'(0) = \frac{1 + \beta E(4 - 2r - \alpha r)^{-1}}{M((r+s-4)/2,\, 1+s;\, -r)} \left[-\frac{m}{2}(r+s)\, M\left(\frac{r+s-4}{2},\, 1+s;\, -r\right) \right.$$
$$+ \frac{rm}{2}\left(\frac{r+s-4}{1+s}\right)$$
$$\left. \times M\left(\frac{r+s-2}{2},\, 2+s;\, -r\right)\right] + 2m\beta E(4 - 2r - \alpha r)^{-1}, \tag{25}$$

and the local wall heat flux can be expressed as

$$q_w = -k\left(\frac{\partial T}{\partial y}\right)_w = -kA(B/\nu)^{1/2}\, (x/l)^2\, \theta'(0).$$

TABLE Ia

Temperature Expressions for Various r and s

r $(=\mathrm{Pr}/m^2)$	s $\left\{=\frac{\mathrm{Pr}}{m^2}\left(1-\frac{4\alpha m^2}{\mathrm{Pr}}\right)^{1/2}\right\}$	$\theta(\eta)$
5	1	$\left(\frac{1-5m^2E/12}{1-e^{-5}}\right)e^{-2m\eta}[1-\exp(-5e^{-m\eta})]+\left(\frac{5m^2E}{12}\right)e^{-2m\eta}.$
r	$r-6$	$\left(1+\frac{rm^2E}{13-5r}\right)\exp[r-(r-3)\,m\eta-re^{-m\eta}]-\frac{rm^2E}{13-5r}e^{-2m\eta}.$
r	$r-4$	$\left(1+\frac{rm^2E}{8-4r}\right)e^{-2m\eta}\frac{\gamma(r-4,\,re^{-m\eta})}{\gamma(r-4,\,r)}-\frac{rm^2E}{8-4r}e^{-2m\eta}.$

where γ is the incomplete gamma function.

TABLE Ib

Temperature Gradient Expressions for Various r and s

r $(=\mathrm{Pr}/m^2)$	s $\left\{=\frac{\mathrm{Pr}}{m^2}\left(1-\frac{4\alpha m^2}{\mathrm{Pr}}\right)^{1/2}\right\}$	$\theta'(\eta)$
5	1	$-\left(\frac{1-5m^2E/12}{1-e^{-5}}\right)(2m+3me^{-5})-\frac{5}{6}m^2E.$
r	$r-6$	$3m\left(1+\frac{rm^2E}{13-5r}\right)+\frac{2rm^3E}{13-5r}.$
r	$r-4$	$-\left(1+\frac{rm^2E}{8-4r}\right)\left[\frac{2m\gamma(r-4,\,r)+me^{-r}r^{r-3}}{\gamma(r-4,\,r)}\right]+r\frac{m^3E}{4-2r}.$

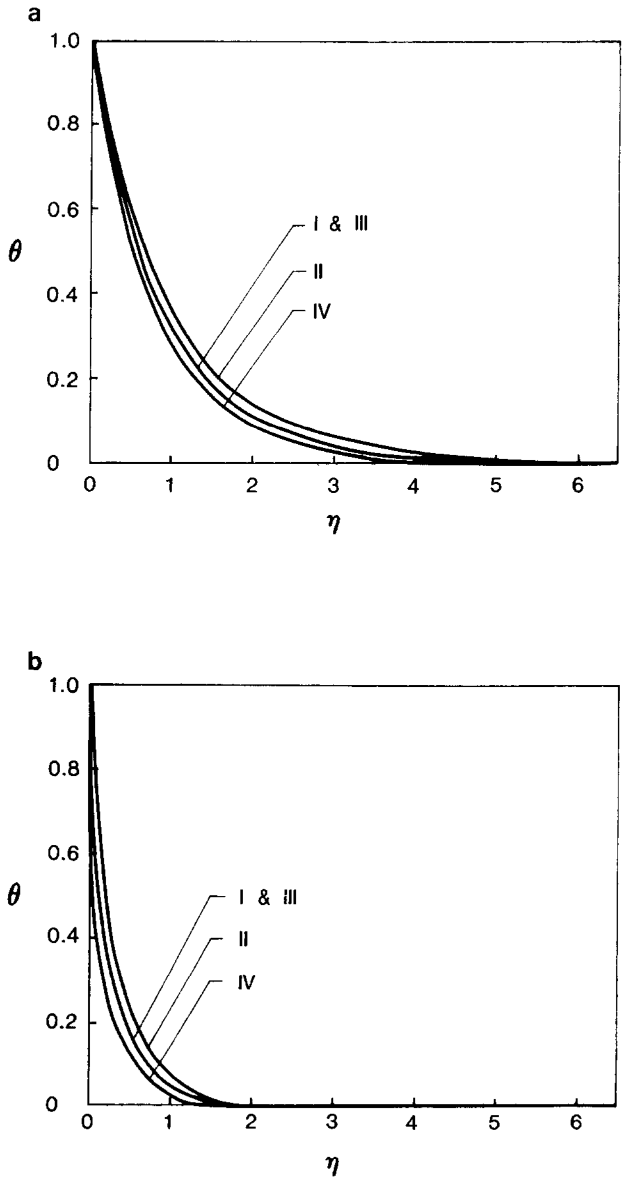

FIG. 2. Temperature profiles for $E = 0.02$ when (a) $\Pr = 1$ and (b) $\Pr = 5$.

	I	II	III	IV
m	1	1.1	1.1	1.1
α	0.1	0.1	0	−0.1

For several sets of values of $r\ (=\mathrm{Pr}/m^2)$ and $s\ [=r(1+4\alpha/r)^{1/2}]$, closed form solutions are obtained and some of the interesting results are presented in Table I. Also, the expressions in (24) and (25) are numerically evaluated for several sets of values of the parameters m, Pr, E, and α and some of the qualitatively interesting results are presented in Fig. 2 and 3.

3.2. *Prescribed Wall Heat Flux (PHF Case)*

Here, the boundary conditions are

$$-k\frac{\partial T}{\partial y}=q_w=D(x/l)^2 \qquad \text{at} \quad y=0,$$

and (26)

$$T\to T_\infty \qquad \text{as} \quad y\to\infty.$$

Defining

$$T-T_\infty=\frac{D(x/l)^2}{k}(v/B)^{1/2}g(\eta) \tag{27}$$

and substituting the relations (7) and (8) into (14) and (26), we get

$$g''+\mathrm{Pr}\,fg'-\mathrm{Pr}(2f'-\alpha)\,g=-\mathrm{Pr}\,E(f'')^2, \tag{28}$$

$$g'(0)=-1 \qquad \text{and} \qquad g(\infty)=0, \tag{29}$$

where a prime denotes differentiation with respect to η, $E=B^2l^2(B/v)^{1/2}/DC_p$ and all other parameters are as defined in Section 3.1.

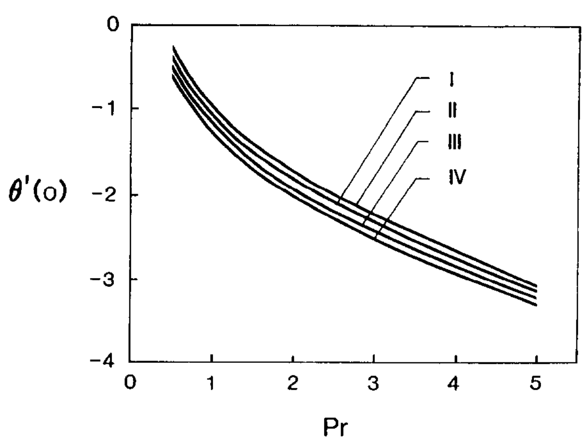

FIG. 3. Temperature gradient in the PST case for $E=0.02$; curves as in Fig. 2.

Using transformation (20), we reduce Eq. (28) and the boundary conditions (29) to

$$\xi g'' + (1 - r - \xi) g' + \left(2 + \frac{\alpha r}{\xi}\right) g = -\frac{\beta E}{r^2} \xi, \tag{30}$$

$$g'(-r) = -\frac{1}{rm} \quad \text{and} \quad g(0^-) = 0, \tag{31}$$

where a prime denotes differentiation with respect to ξ. The solution g satisfying (30) and (31) is obtained as

$$\begin{aligned} g(\xi) = &\left(\frac{2\beta E}{4 - 2r - \alpha r} + \frac{1}{m}\right)\left[\frac{r+s}{2} M\left(\frac{r+s-4}{2}, 1+s; -r\right)\right. \\ &\left. - rM'\left(\frac{r+s-4}{2}, 1+s; -r\right)\right]^{-1} \\ &\times \left(\frac{-\xi}{r}\right)^{(r+s)/2} M\left(\frac{r+s-4}{2}, 1+s; \xi\right) - \frac{\beta E}{r^2}(4 - 2r - \alpha r)^{-1} \xi^2, \end{aligned} \tag{32}$$

where $M'(a, b; z) = (a/b) M(a+1, b+1, z)$.

In terms of η, the solution can be written as

$$\begin{aligned} g(\eta) = &\left(\frac{2\beta E}{4 - 2r - \alpha r} + \frac{1}{m}\right)\left[\frac{r+s}{2} M\left(\frac{r+s-4}{2}, 1+s; -r\right)\right. \\ &\left. - rM'\left(\frac{r+s-4}{2}, 1+s; -r\right)\right]^{-1} \\ &\times e^{-(r+s) m\eta/2} M\left(\frac{r+s-4}{2}, 1+s; -re^{-m\eta}\right) \\ &- \beta E(4 - 2r - \alpha r)^{-1} e^{-2m\eta}. \end{aligned} \tag{33}$$

The wall temperature T_w is obtained from Eq. (27) as

$$T_w - T_\infty = \frac{D(x/l)^2}{k} (\nu/B)^{1/2} g(0). \tag{34}$$

As in Section 3.1, several closed form solutions are developed from Eq. (33) and are presented in Table II. Also, numerical values of $g(0)$ for several sets of values of the parameters m, Pr, E, and α are obtained and are presented in Fig. 4.

TABLE II

Temperature Expressions for Various r and s

r $(=\mathrm{Pr}/m^2)$	s $\left\{=\frac{\mathrm{Pr}}{m^2}\left(1-\frac{4\alpha m^2}{\mathrm{Pr}}\right)^{1/2}\right\}$	$g(\eta)$
5	1	$\left(\frac{c_1 e^{-2m\eta}}{5}\right)[1-\exp(-5e^{-m\eta})] + \frac{5}{12} m^2 E e^{-2m\eta}.$
r	$r-6$	$c_2 \exp[-(r-3)\, m\eta - re^{-m\eta}] - \left(\frac{rm^2 E}{13-5r}\right) e^{-2m\eta}.$
r	$r-4$	$c_3 e^{-2m\eta}\gamma(r-4, re^{-m\eta}) - \left(\frac{rm^2 E}{8-4r}\right) e^{-2m\eta}.$

where

$$c_1 = \left(\frac{1}{m} - \frac{5m^2E}{6}\right)\left[\frac{3}{5}(1-e^{-5}) - 5M'(1, 2; -5)\right]^{-1},$$

$$c_2 = -\left(\frac{2rm^2E}{13-5r} + \frac{1}{m}\right) 3e^r,$$

$$c_3 = \left(\frac{rm^2E}{4-2r} + \frac{1}{m}\right)[(r-2)\,\gamma(r-4, r) - \gamma(r-3, r)]^{-1}.$$

4. Asymptotic Limit for Large Prandtl Number

In this section we derive the asymptotic results for the temperature functions $\theta(\eta)$ and $g(\eta)$, which arise respectively in the PST and PHF cases.

4.1. *PST Case*

In this case the boundary layer equation and the boundary conditions are

$$\theta'' + \frac{\mathrm{Pr}}{m}(1-e^{-m\eta})\,\theta' - \mathrm{Pr}(2e^{-m\eta} - \alpha)\,\theta = -\mathrm{Pr}\, Em^2 e^{-mn}, \tag{35}$$

$$\theta(0) = 1 \qquad \text{and} \qquad \theta(\infty) = 0, \tag{36}$$

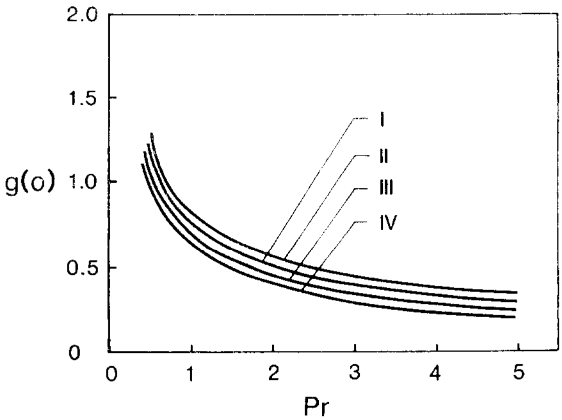

FIG. 4. Temperature at $\eta = 0$ in the PHF case for $E = 0.02$; curves as in Fig. 2.

where the prime denotes differentiation with respect to η. Letting $\varepsilon = 1/\mathrm{Pr}$ (be the small parameter) and using ε we can write (35) as

$$\varepsilon\theta'' + \frac{1}{m}(1 - e^{-m\eta})\,\theta' - (2e^{-m\eta} - \alpha)\,\theta = -Em^2 e^{-2m\eta}. \tag{37}$$

The outer solution to the zeroth-order for $O(\varepsilon^{1/2}) \leqslant \eta < \infty$ is

$$\theta_0(\eta) = \frac{Em^2}{2-\alpha} e^{-2m\eta} + C_0 e^{-\alpha m\eta}(1 - e^{-m\eta})^{2-\alpha}, \tag{38}$$

with C_0 to be determined from matching.

The above expression only satisfies the infinity condition when $\alpha > 0$. When $\alpha \leqslant 0$ the only solution to the reduced version of (37) is the zero solution. Also the form of the particular solution changes when $\alpha = 2$. We restrict our attention to $\alpha > 0$ and $\alpha \neq 2$.

To satisfy the boundary condition at zero, introduce the scaled coordinate $\tilde{\eta} = \eta/\varepsilon^{1/2}$, then for $0 \leqslant \tilde{\eta} < 1$, Eq. (37) becomes

$$\frac{d^2\theta}{d\tilde{\eta}^2} + \tilde{\eta}\frac{d\theta}{d\tilde{\eta}} + (\alpha - 2)\,\theta(\tilde{\eta}) = -Em^2. \tag{39}$$

The particular solution to (39) is simply $Em^2/(2-\alpha)$. The homogeneous version of (39) can be transformed to

$$\frac{d^2\Phi}{d\tilde{\eta}^2} + \left(\alpha - \frac{5}{2} - \frac{1}{4}\tilde{\eta}^2\right)\Phi = 0, \tag{40}$$

by the change of dependent variable

$$\theta(\tilde{\eta}) = e^{-\tilde{\eta}^2/4}\Phi(\eta). \tag{41}$$

Equation (40) has a solution in terms of parabolic cylinder functions (see [12]). The solution that matches (38) is

$$\theta_i(\eta) = \left(1 - \frac{Em^2}{2-\alpha}\right) \pi^{-1/2} 2^{(3-\alpha)/2} \Gamma\left(2 - \frac{\alpha}{2}\right) e^{-\tilde{\eta}^2/4} D_{\alpha-3}(-\tilde{\eta}) + \frac{Em^2}{2-\alpha}, \tag{42}$$

where $D_{\alpha-3}(x) = \sqrt{(2/\pi)}\, e^{x^2/2} \int_0^\infty e^{-t^2/2} t^{\alpha-3} \cos(xt - (\alpha\pi/2) + (3\pi/2))\, dt$. After matching (38) with (42), the inner and outer solutions can be combined into a single uniform asymptotic expression as

$$\theta_u(\eta) = \frac{Em^2}{2-\alpha} e^{-2m\eta} + \left(1 - \frac{Em^2}{2-d}\right) \pi^{-1/2} 2^{(3-\alpha)/2} \Gamma(2-\alpha/2)\, e^{-\alpha m\eta} \times \left(\frac{1-e^{-m\eta}}{m\eta}\right)^{2-\alpha} e^{-\eta^2/4\varepsilon} D_{\alpha-3}\left(\frac{-\eta}{\sqrt{\varepsilon}}\right). \tag{43}$$

From (43) we observe that there is a boundary layer of width $\sqrt{\varepsilon}$.

4.2. *PHF case*

The analysis in Section 4.1 can be applied to *PHF* case. The uniform asymptotic expression in this case is

$$g_u(\eta) = \frac{m^2}{2-\alpha} e^{-2m\eta} + \pi^{-1/2} 2^{(2-\alpha)/2} \varepsilon^{1/2} \Gamma\left(\frac{3}{2} - \frac{\alpha}{2}\right) \times e^{-\alpha m\eta} \left(\frac{1-e^{-m\eta}}{m\eta}\right)^{2-\alpha} \times e^{-\eta^2/4\varepsilon} D_{\alpha-3}\left(\frac{-\eta}{\sqrt{\varepsilon}}\right). \tag{44}$$

As in the *PST* case, the boundary layer width is also $\sqrt{\varepsilon}$.

5. Asymptotic Limit for Small Prandtl Number

As in Section 4, we derive the asymptotic results for the temperature functions $\theta(\eta)$ and $g(\eta)$. It is not possible to find matched asymptotic expansions for small Prandtl number as we obtained for the large Prandtl number case. This is due to the solution changing by $O(1)$ on a length scale of order 1/Pr, which is arbitrarly large for small Pr. Hence, a perturbation expansion would require all terms. However, we give the exact solution for small Prandtl numer.

5.1. *PST Case*

Letting $\varepsilon = \Pr$ in (35), we get

$$\theta'' + \frac{\varepsilon}{m}(1 - e^{-m\eta})\,\theta' - \varepsilon(2e^{-m\eta} - \alpha)\,\theta = -\varepsilon m^2 e^{-2m\eta}. \tag{45}$$

The appropriate boundary conditions are

$$\theta(0) = 1 \qquad \text{and} \qquad \theta(\infty) = 0. \tag{46}$$

The solution satisfying Eq. (45) and conditions (46) can be obtained as

$$\begin{aligned}\theta(\eta) = {} & \frac{1 + \varepsilon E(4 - (\varepsilon/m^2) - (\varepsilon\alpha/m^2))^{-1}}{M(\varepsilon a - 2,\, 1 + \varepsilon b;\, -\varepsilon/m^2)}\, e^{-am\eta} \\ & \times M\left(\varepsilon a - 2,\, 1 + \varepsilon b;\, \frac{-\varepsilon}{m^2} e^{-m\eta}\right) \\ & - \frac{\varepsilon}{m^2} E\left(4 - \frac{\varepsilon}{m^2} - \frac{\alpha\varepsilon}{m^2}\right)^{-1} e^{-2m\eta},\end{aligned} \tag{47}$$

where

$$\begin{aligned} a &= \frac{1}{2m^2}\left(1 + \sqrt{1 - \frac{4\alpha m^2}{\varepsilon}}\right). \\ b &= \frac{1}{m^2}\sqrt{1 - \frac{4\alpha m^2}{\varepsilon}}. \end{aligned} \tag{48}$$

The term $e^{-\varepsilon am\eta}$ shows the slow exponential decay. A boundary layer type of solution is not possible here because of this term.

5.2. *PHF-Case*

Here, $g(\eta)$ turns out to be

$$\begin{aligned} g(\eta) = {} & \left(\frac{2\varepsilon E}{4 - 2r - \alpha r} + \frac{1}{m}\right)[(\varepsilon a)\, M(\varepsilon a - 2,\, 1 + \varepsilon b;\, -\varepsilon/m^2) \\ & - \varepsilon/m^2 M'(\varepsilon a - 2,\, 1 + \varepsilon b;\, -\varepsilon/m^2)]^{-1} \\ & \times e^{-\varepsilon am\eta} M\left(\varepsilon a - 2,\, 1 + \varepsilon b;\, -\frac{\varepsilon}{m^2} e^{-m\eta}\right), \\ & - \frac{e}{m^2} E\left(4 - \frac{\varepsilon}{m^2} - \frac{\alpha\varepsilon}{m^2}\right)^{-1} e^{-2m\eta}, \end{aligned} \tag{49}$$

where a and b are as in (48). Again, the solution decays slowly due to the term $e^{-\varepsilon am\eta}$.

6. Discussion of the Results

In Fig. 2a, for $\mathrm{Pr}=1$ and $E=0.02$, we have plotted the temperature distribution $\theta(\eta)$ for several sets of values of m and α. Similar results are plotted in Fig. 2b with $\mathrm{Pr}=5$. From Fig. 2 it is evident that the temperature θ increases with an increase in the viscoelastic parameter m. This phenomenon is true even with the heat source/sink parameter α. Further, it can be seen that the temperature at a point in a given second-order fluid decreases with an increase in the Prandtl number Pr. This is consistent with the fact that the thermal boundary layer thickness decreases with increasing Prandtl number.

For $E=0.02$, the wall terperature gradient $\theta'(0)$ as a function of Pr for several sets of values of m and α is shown in Fig. 3. For given m, α, and E, the larger the Pr, the larger (in an absolute sense) the magnitude of the wall temperature gradient. In addition, the magnitude (in absolute sense) of the wall temperature gradient increases as m increases. This phenomenon is true even with the heat source/sink parameter α. From Fig. 3 it is evident that the temperature gradient $\theta'(0)$ is negative for all values of the Prandtl number. Physically it means that there is heat flow only from the wall.

The behavior of the wall temperature $g(0)$ with changes in m, α, and Pr is shown in Fig. 4 for $E=0.02$. From this figure it is clear that the wall temperature decreases rapidly as Pr increases from 0.5 to 1 and then slowly decreases with an increase in Pr. Furthermore, the effects of m and α are to increase the wall temperature $g(0)$. Finally it should be mentioned that the effect of frictional heating parameter E on the heat transfer characteristics is to augment the values.

References

1. B. C. Sakiadis, *AIChEJ.* **7** (1961), 26.
2. L. E. Erickson, L. T. Fan, and V. G. Fox, *Ind. Eng. Chem. Fundam.* **5** (1966), 19.
3. P. D. McCormack and L. Crane, "Physical Fluid Dynamics," Academic Press, New York, 1973.
4. J. E. Danberg and K. S. Fansler, *Quart. Appl. Math.* **34** (1976), 305.
5. P. S. Gupta and A. S. Gupta, *Canad. J. Chem. Eng.* **55** (1977), 744.
6. C. K. Chen and M. Char, *J. Math. Anal. Appl.* **135** (1988), 568.
7. V. G. Fox, L. E. Erickson, and L. T. Fan, *AIChE J.* **15** (1969), 327.
8. K. R. Rajagopal, T. Y. Na, and A. S. Gupta, *Rheol. Acta* **23** (1984), 213.
9. B. D. Coleman and W. Noll, *Arch. Rational Mech. Anal.* **6** (1960) 355.
10. W. C. Troy, E. A. Overman II, G. B. Ermen-Trout, and J. P. Keener, *Quart. Appl. Math.* **44** (1987), 753.
11. D. W. Beard and K. Walters, *Proc. Cambridge Philos. Soc.* **60** (1964), 667.
12. M. Abramowitz and L. A. Stegun, "Handbook of Mathematical Functions," Vol. 55, National Bureau of Standards/Amer. Math. Soc., Providence, RI, 1972.