Bonferroni Inequalities and Negative Cycles in Large Complete Signed Graphs

Dragoş Popescu and Ioan Tomescu

In this paper the problem of characterizing extremal graphs $K_n$ relatively to the number of negative $p$-cycles, when the number of negative edges is fixed, is solved for large $n$. This number can be expressed as an alternating sum for which the Bonferroni inequalities hold.

Finally, the asymptotic value of the probability that a $p$-cycle of $K_n$ is negative is found as $n \to \infty$, if the negative edges induce a subgraph the components of which are paths or cycles.

1. Introduction

A signed graph $G$ based on $F$ is an ordinary graph $F$ with each edge marked as positive or negative. In this paper we shall consider only the case in which $F$ is the complete graph $K_n$. A cycle of $K_n$ is said to be negative if it contains an odd number of negative edges; otherwise, it is positive [2]. Let us denote by $K_{p,q}$ the complete bipartite graph the partite sets of which contain $p$ and $q$ vertices respectively, by $sK_2$ the graph on $2s$ vertices consisting of $s$ vertex-disjoint edges, by $G(K_n; H)$ the complete signed graph $K_n$ the negative edges of which induce a subgraph isomorphic to $H$, and by $C_p(G)$ the number of negative $p$-cycles contained by the signed graph $G$. If signed (or ordinary) graphs $G$ and $H$ are isomorphic, we shall denote this by $G \equiv H$.

A signed graph is called balanced if each of its cycles is positive. It is easy to show that a signed graph $G$ based on $K_n$ is balanced iff $G \equiv G(K_n; K_{p,q})$, where $p + q = n$ and $p, q \geq 0$. Psychologists are sometimes interested in the smallest number $d = d(G)$ such that a signed graph $G$ may be converted into a balanced graph by changing the signs of $d$ edges. It is easy to see that, for a signed graph $G$ having $d(G) = k$, there exists a cocycle $\omega$ of $K_n$ such that the signed graph obtained from $G$ by changing the signs of all edges of $\omega$ has exactly $k$ negative edges.

2. Bonferroni Inequalities for the Symmetric Difference

It is well known that for $s$ sets $A_1, A_2, \ldots, A_s$, their symmetric difference, denoted $\bigtriangleup_{i=1}^s A_i$, is the set of elements in $\bigcup_{i=1}^s A_i$ that belong to an odd number of sets $A_1, \ldots, A_s$.

The following property is an analogue of the inclusion–exclusion principle for the symmetric difference of $s$ sets.

Lemma 1. The following equality holds:

$$\left| \bigtriangleup_{i=1}^s A_i \right| = \sum_{i=1}^s (-2)^{i-1} \sum_{K \subseteq \{1, \ldots, s\} \setminus \{i\}} \left| \bigcap_{p \in K} A_p \right|.$$
PROOF. By Jordan’s sieve formula, we deduce that

\[
\left| \hat{\Delta}_i A_i \right| = \sum_{k \geq 1} \sum_{k \text{ odd}} (-1)^{\nu(k)} \binom{k}{i} \sum_{|p=K|} \left| p \cap K \right| A_p
\]

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\]

\[
\left| \hat{\Delta}_i A_i \right| = 2^{i+1} \sum_{i=1}^s (-2)^{i-1} \sum_{K \subseteq \{1, \ldots, s\}} \left| p \cap K \right| A_p
\]

for every \(0 \leq t \leq \frac{s-1}{2}\);

\[
\left| \hat{\Delta}_i A_i \right| = 2^i \sum_{i=1}^s (-2)^{i-1} \sum_{K \subseteq \{1, \ldots, s\}} \left| p \cap K \right| A_p
\]

for every \(1 \leq t \leq \frac{s}{2}\).

The proof is similar to that of the Bonferroni inequalities for the inclusion–exclusion principle (see [3, 4]), since \(\left| \hat{\Delta}_{i=1} A_i \right| = |\Delta_{i=1} A_i| + |A_{i+1}| - 2 |\Delta_{i=1} (A_i \cap A_{i+1})|\).

3. Extremal Numbers of Negative p-Cycles in Large Complete Signed Graphs with a Given Number of Negative Edges

We need the following auxiliary result, the proof of which can be set out using standard methods [1].

LEMMA 3. Suppose that the vertex-disjoint paths \(P_1, \ldots, P_s\) containing together \(t+s\) vertices are induced subgraphs of \(K_n\). The number of \(p\)-cycles of \(K_n\) that contain all paths \(P_1, \ldots, P_s\) is equal to \(2^{s-1}(p-1)! \binom{n-p}{n-p}^s\).

THEOREM 1. Let \(G\) be a complete graph of order \(n\) containing \(s\) negative edges.

(a) If \(n \geq \max(s+1, 2p-2)\), then \(C_p(G) \geq C_p^-(G(K_n; K_{1,s}))\) and the equality holds iff: \(G \equiv G(K_n; K_{1,s})\) for \(s \neq 3\) and \(p \geq 3\) or \(s = 3\) and \(p = 3\); \(G \equiv G(K_n; K_{1,3})\) or \(G \equiv G(K_n; K_3)\) for \(s = 3\) and \(p \geq 4\).

(b) If \(n \geq 2s\) for \(p = 3\) and \(n \geq 2p-2+2(\frac{s}{2})\) for \(p \geq 4\), then \(C_p^-(G) \leq C_p^-(G(K_n; sK_2))\) and the equality holds iff \(G \equiv G(K_n; sK_2)\).

PROOF. (a) Suppose that the negative edges of \(G(K_n; K_{1,s})\) are denoted by \(e_1, \ldots, e_s\), and let \(A_i\) be the set of \(p\)-cycles of \(K_n\) containing edge \(e_i\) for \(i = 1, \ldots, s\). Then

\[
C_p^-(G(K_n; K_{1,s})) = \left| \hat{\Delta}_{i=1} A_i \right| = \sum_{i=1}^s |A_i| - \sum_{1 \leq i < j \leq s} |A_i \cap A_j|
\]

\[
= \binom{s}{1}(p-2)! \binom{n-2}{n-p} - 2 \binom{s}{2}(p-3)! \binom{n-3}{n-p}.
\]
If $G \not= G(K_n^+; K_n^+)$, let $f_1, \ldots, f_s$ be the negative edges of $G$ and let $B_i$ be the set of $p$-cycles of $K_n$ that contain edge $f_i$ for $i = 1, \ldots, s$. By the Bonferroni inequalities we deduce that

$$C_p^r(G) = \left| \bigcup_{i=1}^s B_i \right| \geq \sum_{i=1}^s |B_i| - \sum_{i<j<s} |B_i \cap B_j|.$$  

(2)

If the edges $f_i$ and $f_j$ have no common vertex, then $|B_i \cap B_j| = 2(p-3)! \binom{n-3}{p-3}$ by Lemma 3; otherwise, $|B_i \cap B_j| = (p-3)! \binom{n-3}{p-3}$ and $2(p-4)! \binom{n-3}{p-4}$ is equivalent to $n \geq 2p - 2$. For $s = 1$ the property is obvious and for $s \geq 2$ and $s \neq 3$, since $G \not= G(K_n^+; K_n^+)$, there exist two edges $f_i$ and $f_j$ having no common extremity; hence $|B_i \cap B_j| < (p-3)! \binom{n-3}{p-3}$. Since, for every $i$, $|B_i| = (p-2)! \binom{n-2}{p-2}$, from (1) and (2) it follows that $C_p^r(G) > C_p^r(G(K_n^+; K_n^+))$.

If $s = 3$, $G \not= G(K_n^+; K_n^+)$ and any pair of edges among $f_1, f_2$ and $f_3$ have a common extremity, then $G \equiv G(K_n^+; K_3)$ and in this case we obtain that $C_p^r(G) > C_p^r(G(K_n^+; K_3))$ and the equality sign occurs iff $p \geq 4$.

(b) By the Bonferroni inequalities we deduce, as above, that

$$C_p^r(G(K_n^+; sK_2)) = \left( \begin{array}{c} s \\ 1 \end{array} \right) (p-2)! \binom{n-2}{p-2} - 2 \left( \begin{array}{c} s \\ 2 \end{array} \right) (p-3)! \binom{n-3}{p-3}.$$  

(3)

Suppose that $G \not= G(K_n^+; sK_2)$. With the notation introduced above we find that $|B_i \cap B_j \cap B_k|$ is equal to: (i) $2^2(p-4)! \binom{n-4}{p-4}$ if $H_{i,j,k} \equiv 3K_2$, where $H_{i,j,k}$ denotes the subgraph induced by $f_i, f_j$ and $f_k$; (ii) $2(p-4)! \binom{n-4}{p-4}$ if $H_{i,j,k}$ is isomorphic to a graph of order 5 consisting of a path of length 2 and an edge; (iii) $0$ for $p \geq 4$ and 1 for $p = 3$ if $H_{i,j,k} \equiv K_5$; (iv) $(p-4)! \binom{n-4}{p-4}$ if $H_{i,j,k}$ is isomorphic to a path of length 3; (v) 0 if $H_{i,j,k} \equiv K_{1,3}$. Now the proof follows in a similar way as above, by considering the cases $p \geq 4$ and $p = 3$.

It is clear that $d(G(K_n^+; K_{1,3})) = d(G(K_n^+; sK_2)) = s$ for every $n \geq 2s$. Hence, for large $n$, from Theorem 1 we can deduce the structure of signed graphs based on $K_n$ such that $d(G) = s$ having a minimum (resp. maximum) number of negative $p$-cycles.

4. **NEGATIVE $p$-CYCLES IN COMPLETE SIGNED GRAPHS THE NEGATIVE EDGES OF WHICH INDUCE PATHS OR CYCLES**

**Lemma 4.** Let $k$ and $i$ be natural fixed numbers and let $G$ be a graph with $s$ edges such that $d(x) \leq k$ for every $x \in V(G)$. If $M_i(G)$ denotes the number of matchings of $G$ containing $i$ edges, the following equality holds:

$$\lim_{s \to \infty} \frac{M_i(G)}{s!} = \frac{1}{i!}.$$  

**Proof.** Since the number of ordered selections of $i$ pairwise non-adjacent edges of $G$ is greater than or equal to $(s - (2k - 1)) \cdots (s - (i - 1)(2k - 1))$, it follows that $s(s - (2k - 1)) \cdots (s - (i - 1)(2k - 1))/i! \leq M_i(G) \leq \binom{s}{i}$ and the result follows.

**Theorem 2.** Let $G = G(K_n^+; H)$ be a complete signed graph such that the negative edges span a subgraph $H$ containing $s$ edges. If $\lim_{n \to \infty} s/n = \lambda$, $\lim_{n \to \infty} p/n = \mu$ and all components of $H$ are paths or cycles, then

$$\lim_{n \to \infty} \frac{2pC_p^r(G)}{(n)_p} = \frac{1}{2}(1 - e^{-4\lambda \mu}).$$
PROOF. Suppose that the negative edges of $G$ are $e_1, \ldots, e_s$ and let $A_i$ be the set of $p$-cycles of $K_n$ containing $e_i$, for $i = 1, \ldots, s$. We can write:

$$C_p(G) = \sum_{i=1}^{s} \left( -2 \right)^{i-1} \sum_{K \subseteq \{1, \ldots, s\}} \left| \bigcap_{k \in K} A_k \right|.$$  

We shall prove that, for any fixed $i$, we have

$$\lim_{n \to \infty} 2p \sum_{K \subseteq \{1, \ldots, s\}, |K| = i} \left| \bigcap_{k \in K} A_k \right| / (n)_p = \frac{(2\lambda \mu)^i}{i!}.$$  

Suppose that the $s$ negative edges of $G$ induce $r$ components $C_1, \ldots, C_r$, that are paths or cycles, having $a_1, \ldots, a_r$ vertices, respectively. The number of the selections of $i$ edges from the set of the edges of a path $P$ of length $a-1$, such that these $i$ edges generate exactly $j$ connected components on $P$, is equal to $\binom{a-1}{i} \binom{a-j}{i-j}$ [5] and the number of these selections for a cycle $C$ of length $a$ is equal to $\binom{a-1}{i} \binom{a-j}{i-j}$ [1].

By Lemma 3, we deduce that

$$\lim_{n \to \infty} 2p \sum_{K \subseteq \{1, \ldots, s\}, |K| = i} \left| \bigcap_{k \in K} A_k \right| / (n)_p = \lim_{n \to \infty} 2p M(n, p, i, a_1, \ldots, a_r) / (n)_p,$$

where

$$M(n, p, i, a_1, \ldots, a_r) = (p - i - 1)! \sum_{i_1 + \cdots + i_r = i} \sum_{j_1, \ldots, j_r} \prod_{r=1}^{i} a(a_r, i_r, j_r) 2^{i-1} \times \binom{n - i - j}{n - p} = \sum_{i_1 + \cdots + i_r = i} \prod_{r=1}^{i} a(a_r, i_r, j_r) \times \binom{n - i - j}{n - p}$$

and

$$a(a_r, i_r, j_r) = \binom{i_r - 1}{j_r - 1} \binom{a_r - i_r}{j_r}$$

if $C_i$ is a path and

$$a(a_r, i_r, j_r) = \binom{a_r}{j_r} \binom{i_r - 1}{j_r - 1} \binom{a_r - i_r - 1}{j_r}$$

if $C_i$ is a cycle. But

$$a(a_r, i_r, j_r) = \binom{i_r - 1}{j_r - 1}$$

which implies that

$$\prod_{r=1}^{i} a(a_r, i_r, j_r) = s^i \prod_{r=1}^{i} \binom{i_r - 1}{j_r - 1}.$$  

Let

$$N(n, p, s, i, j) = 2p(p - i - 1)! \binom{n - i - j}{n - p - i - j} / (n)_p = 2^s s! (p^i + O(p^{i-1})) / (n^{i+j} + O(n^{i+j-1})).$$

For $j < i$ we have $\lim_{n \to \infty} N(n, p, s, i, j) = 0$. If $j = i$, or $j_k = i_k$ for $k = 1, \ldots, r$, the $i_k$ negative edges in $C_k$ are pairwise non-adjacent for every $k = 1, \ldots, r$, i.e., they are the edges of a matching of cardinality $i$ of $H$. By Lemma 4 we obtain that

$$\lim_{s \to \infty} \frac{M_i(H)}{s^i} = \frac{1}{i!}.$$
and if \( j_k = i_k \) for \( k = 1, \ldots, r \), then the corresponding part of the sum \( M(n, p, i, a_1, \ldots, a_r) \) is equal to \( (p - i - 1)! M_i(H) 2^{i-1}(p - 2) \). It follows that

\[
\lim_{n \to \infty} 2pM(n, p, i, a_1, \ldots, a_r)/\binom{n}{p} = \frac{1}{i!} \lim_{n \to \infty} N(n, p, s, i, i) = \frac{(2\lambda \mu)^i}{i!}
\]

which proves (5). Since the alternating sum (4) satisfies the Bonferroni inequalities, it follows, by standard techniques, that

\[
\lim_{n \to \infty} \frac{2pC^r_p(G)}{\binom{n}{p}} = \sum_{i=1}^{\infty} (-2)^{i-1} \frac{(2\lambda \mu)^i}{i!} = \frac{1}{2}(1 - e^{-2\lambda \mu}).
\]

**Note added in proof:** The authors proved that the conclusion of the Theorem 2 holds also if we suppose that the degrees of vertices of \( H \) are bounded above by an absolute constant \( C \).

**References**


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D. POPESCU AND I. TOMESCU

Department of Mathematics, University of Bucharest,
Str. Academiei, 14, R-70109 Bucharest, Romania