Intersections in Projective Space II: Pencils of Quadrics

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A complete classification is given of pencils of quadrics in projective space of three dimensions over a finite field, where each pencil contains at least one non-singular quadric and where the base curve is not absolutely irreducible. This leads to interesting configurations in the space such as partitions by elliptic quadrics and by lines.

1. INTRODUCTION

A pencil of quadrics in PG(3, q) is non-singular if it contains at least one non-singular quadric. The base of a pencil is a quartic curve and is reducible if over some extension of GF(q) it splits into more than one component: four lines, two lines and a conic, two conics, or a line and a twisted cubic. In order to contain no points or to contain some line over GF(q), the base is necessarily reducible.

In the main part of this paper a classification is given of non-singular pencils of quadrics in \( \Sigma = \text{PG}(3, q) \) with a reducible base. This leads to geometrically interesting configurations obtained from elliptic and hyperbolic quadrics, such as spreads and partitions of \( \Sigma \).

The remaining part of the paper deals with the case when the base is not reducible and is therefore a rational or elliptic quartic. The number of points in the elliptic case is then governed by the Hasse formula. However, we are able to derive an interesting combinatorial relationship, valid also in higher dimensions, between the number of points in the base curve and the number in the base curve of a dual pencil. This relationship has an elegant geometrical interpretation in the plane due to Cicchese [5], [6] for characteristic greater than three. The result in this paper is valid in all characteristics and in higher dimensions.

As regards Section 5, van Halteren [9] has studied pencils of quadrics in \( \text{PG}(3, q) \) with empty base. A classification over the complex numbers can be found in Baker [1, chap. 3], Todd [16, chap. 6], or Semple and Kneebone [15, chap. 13]. Pencils of plane quadrics have been classified by Dickson [7] for odd \( q \) and by Campbell [4] for even \( q \).

After various preliminaries in Sections 2–3, in Section 4 the possible base curves are classified and described. Then in Section 5 the classification of non-singular pencils with reducible base is given; this is embodied in Table 2. In Section 6 some combinatorial consequences of the classification are described. The main result of Section 7 is to give the dual pencil for each pencil in Table 2. Finally, Section 8 comments briefly on the classification when the base is irreducible, without giving specific forms.

2. PRELIMINARIES

In PG(3, q), a point with coordinate vector \( X = (x_0, x_1, x_2, x_3) \) is denoted by \( P(X) \). Let \( F \) be a quaternary, quadratic form over GF(q). Then a quadric \( \mathcal{Q} = V(F) \), where

\[
V(F) = \{ P(X) | F(X) = 0 \}
\]

For properties of quadrics in PG(3, q), see [11, chaps 15 and 16]. The classification of quadrics is summarized in the following result.
THEOREM 2.1. The quadrics $\mathcal{F} = \mathbf{V}(F)$ in $\text{PG}(3, q)$ fall into six classes under projective equivalence. For each class, Table 1 lists the symbol for $\mathcal{F}$, its rank, the dimension $d$ of its singular space, its description, its cardinality $|\mathcal{F}|$, a canonical form for $F$, and the number $n(\mathcal{F})$ in the class. The form $f$ is binary, quadratic and irreducible over $\text{GF}(q)$. Only $\mathcal{H}_3$ and $\mathcal{E}_3$ are non-singular.

TABLE 1. Quadrics in $\text{PG}(3, q)$

| Class | Rank | $d$ | Description | $|\mathcal{F}|$ | $F$ | $n(\mathcal{F})$ |
|-------|------|----|-------------|-------------|-----|----------------|
| $\mathcal{P}_0$ | 1 | 2 | plane | $q^2 + q + 1$ | $x_0^2$ | $(q^2 + 1)(q + 1)$ |
| $\mathcal{H}_1$ | 2 | 1 | plane pair | $2q^2 + q + 1$ | $x_0x_1$ | $q(q^2 + q + 1)(q^2 + 1)(q + 1)/2$ |
| $\mathcal{E}_1$ | 2 | 1 | line | $q + 1$ | $f(x_0, x_1)$ | $q(q^2 - 1)(q^2 + 1)/2$ |
| $\mathcal{P}_2$ | 3 | 0 | cone | $q^2 + q + 1$ | $x_0^2 + x_1x_2$ | $q(q - 1)(q^2 + 1)(q + 1)$ |
| $\mathcal{H}_2$ | 4 | -1 | ruled quadric | $(q + 1)^2$ | $x_0x_1 + x_2x_3$ | $q(q - 1)(q^2 + 1)/2$ |
| $\mathcal{E}_2$ | 4 | -1 | ovoid | $q^2 + 1$ | $f(x_0, x_1) + x_2x_3$ | $q(q - 1)(q^2 - 1)/2$ |

3. NON-SINGULAR PENCILS

If $F$ and $G$ are quaternary, quadratic forms over $\text{GF}(q)$, a pencil $\mathcal{B}$ of quadrics is the set

$$\{\mathbf{V}(F + tG) | t \in \text{GF}(q) \cup \{∞\}\}.$$  

The pencil is non-singular if at least one quadric in $\mathcal{B}$ is non-singular. Let $\mathcal{F}$ and $\mathcal{G}$ be the quadrics defined by $\mathcal{F}$ and $\mathcal{G}$ over $\bar{K}$, the algebraic closure of $K = \text{GF}(q)$. Then the base curve $\mathcal{C}$ of $\mathcal{B}$ is the intersection cycle $\mathcal{F} \cdot \mathcal{G}$. Let $k$ be the number of points of $\mathcal{C}$ rational over $\text{GF}(q)$.

To classify pencils of quadrics, we examine each type of quartic curve. This is considered in Section 4. First, some more information is obtained about the quadrics in a given pencil.

In a non-singular pencil $\mathcal{B}$, let the number of the respective types of quadric be as follows:

$$n_1 \quad n_2^+ \quad n_2^- \quad n_3 \quad n_4^+ \quad n_4^-$$

**LEMMA 3.1.** The integers $n_i$ satisfy the following:

(i) $n_1 + n_2^+ + n_2^- + n_3 + n_4^+ + n_4^- = q + 1$,
(ii) $n_1 + n_2^+ + n_2^- + n_3 \leq 4$,
(iii) $n_1 \leq 1$, $n_2^+ \leq 2$, $n_2^- \leq 2$,
(iv) $n_1 + (q + 1)n_2^+ − (q − 1)n_2^- + n_3 + 2n_4^+ = k$,
(v) $q(n_2^+ − n_2^-) + (n_4^+ − n_4^-) = k − (q + 1)$.

**PROOF.** (i) This equation simply says that there are $q + 1$ quadrics in a pencil.
(ii) This inequality merely expresses the fact that there are at most four singular quadrics in a non-singular pencil.
(iii) If $n_1 \geq 2$, the general quadric in the pencil can be given the form $\mathbf{V}(x_0^2 + tx_1^2)$, which is again singular.
If a pencil has two plane pairs, there are four possibilities:
(a) the four planes have no point in common,
(b) the four planes have a point in common,
(c) three of the planes have a line in common,
(d) two of the planes coincide.
For each of these cases we have the following, corresponding canonical forms:

(a) \( V(x_0x_1 + tx_2x_3) \),
(b) \( V(x_0x_1 + tx_2(x_0 + x_1 + x_2)) \),
(c) \( V(x_0x_1 + tx_2(x_0 + x_1)) \),
(d) \( V(x_0x_1 + tx_0x_2) \).

In (a), \( n^+_2 = 2 \); in (b), (c), (d), the pencils are singular.

If \( n^- > 2 \), then in the pencil considered over \( \text{GF}(q^2) \) there would be more than two plane pairs which would make the pencil singular.

(iv) If the pencil \( \mathcal{B} \) comprises the quadrics \( \mathcal{F}_0, \ldots, \mathcal{F}_q \) and \( \mathcal{X} \) is the set of rational points of the base curve \( \mathcal{C} \), then

\[
\text{PG}(3, q) = \mathcal{X} \cup (\mathcal{F}_0 \setminus \mathcal{X}) \cup \cdots \cup (\mathcal{F}_q \setminus \mathcal{X})
\]
is a partition. So

\[
(q^2 + 1)(q + 1) = k + n_1(q^2 + q + 1 - k) + n^+_2(2q^2 + q + 1 - k) + n^-_2(q + 1 - k) + n_3(q^2 + q + 1 - k)
\]
\[
+ n^+_4(q^2 + 2q + 1 - k) + n^-_4(q^2 + 1 - k).
\]

Multiplying the result of (i) by \( q^2 + 1 \) and subtracting it from this equation gives the result.

(v) This is the equation in (iv) minus that in (i).

**COROLLARY.** If the base \( \mathcal{C} \) of a pencil is an irreducible quartic or an irreducible cubic plus a line, then

(i) \( n_1 = n^+_2 = n^-_2 = 0 \),
(ii) \( n_3 \leq 4 \),
(iii) \( n_3 + 2n^+_4 = k \),
(iv) \( n^+_4 - n^-_4 = k - (q + 1) \).

**4. THE BASE CURVE OF A NON-SINGULAR PENCIL**

When the base curve \( \mathcal{C} \) is reducible, the degree may be partitioned in four ways:

\[
4 = 1 + 1 + 1 + 1 = 1 + 1 + 2 = 2 + 2 = 1 + 3.
\]

Correspondingly, \( \mathcal{C} \) is one of the following:

(1) four lines,
(2) two lines and a conic,
(3) two conics,
(4) a line and a twisted cubic.

This is a crude classification as it does not stipulate whether any of the components of \( \mathcal{C} \) coincide or over which field the components are rational. Before listing all the possibilities for \( \mathcal{C} \) which are projectively distinct over \( \text{GF}(q) \), some preliminary restrictions are obtained for each of the four cases.

In the following lemmas, we regard the components of \( \mathcal{C} \) as lying over \( \bar{K} \), an algebraic closure of \( K = \text{GF}(q) \).

**LEMMA 4.1.** If \( \mathcal{C} \) is four lines, it does not contain three skew lines or a triangle; that is, \( \mathcal{C} \) is a subset of a skew quadrilateral.

**PROOF.** A non-singular pencil for which the base has a rational linear component contains a hyperbolic quadric by definition. A hyperbolic quadric containing three skew lines is unique and does not contain a triangle.
Lemma 4.2. If \( \mathcal{C} \) is two lines and a conic, then the lines are distinct and intersect.

Proof. The lines are distinct, for if \( \mathcal{C} \) is a conic \( \mathcal{P}_2 \) and a repeated line \( l \), then each cone with base \( \mathcal{P}_2 \) and vertex on \( l \) is in the pencil, which is therefore singular.

Now, let \( \mathcal{C} \) be \( \mathcal{P}_2 \) plus the lines \( l \) and \( l' \). The lines \( l \) and \( l' \) are not in the plane of \( \mathcal{P}_2 \), as a plane meets a non-singular quadric in a curve of degree two. If \( \mathcal{P}_2 \) lies on \( \mathcal{H}_3 \), the plane \( \pi \) containing \( \mathcal{P}_2 \) meets each generator of \( \mathcal{H}_3 \) in exactly one point. Hence each generator meets \( \mathcal{P}_2 \) in exactly one point. If \( l \) and \( l' \) are skew, then, through a point \( P \) of \( \mathcal{P}_2 \) on neither \( l \) nor \( l' \), there is a transversal \( t \) of \( l \) and \( l' \). Since \( t \) contains three distinct points \( P, t \cap l, t \cap l' \) of any quadric in the pencil, it lies on such a quadric and is contained in \( \mathcal{C} \). As this is impossible, so \( l \) and \( l' \) intersect.

Lemma 4.3. If \( \mathcal{C} \) is twisted cubic \( \mathcal{T} \) and a line \( l \), then \( l \) is a chord of \( \mathcal{T} \).

Proof. Each plane meets \( \mathcal{T} \) in three points rational over \( \bar{K} \). In particular, if \( \Pi_0 \mathcal{P}_2 \) is a cone containing \( \mathcal{T} \), the planes through two generators of \( \Pi_0 \mathcal{P}_2 \) meet \( \mathcal{T} \) in three such points. This is only possible if \( \mathcal{T} \) contains the vertex \( \Pi_0 \) and each generator meets \( \mathcal{T} \) in a further point rational over \( K \). So if \( \mathcal{C} \) lies on \( \Pi_0 \mathcal{P}_2 \), then \( l \) is either a bisecant or a tangent of \( \mathcal{T} \).

If \( \mathcal{H}_3 \) is a hyperbolic quadric containing \( \mathcal{T} \), then each line of one regulus \( \mathcal{R} \) meets \( \mathcal{T} \) in two points (over \( \bar{K} \)) and each line of the other regulus \( \mathcal{R}' \) meets \( \mathcal{T} \) in one point (over \( K \)). The lines are distinct, for if \( \mathcal{C} \) is a conic \( \mathcal{P}_2 \) and a repeated line \( l' \), then each cone \( \mathcal{P}_2 \) which meets \( \mathcal{T} \) in three such points. Hence \( l \) is a chord of \( \mathcal{T} \).

The possibilities for \( \mathcal{C} \) are as follows, where, in each case, \( k \) is the number of distinct points of \( \mathcal{C} \) rational over GF\((q)\).

1. \( \mathcal{C} \) is four lines:
   (a) four distinct lines over GF\((q), k = 4q\);
   (b) two distinct skew lines over GF\((q)\) and two conjugate lines over GF\((q^2), k = 2q + 2\);
   (c) two pairs of skew conjugate lines over GF\((q^2)\) with two common points over GF\((q)\) in each pair, \( k = 2\);
   (d) two pairs of conjugate lines over GF\((q^2)\) with no common point over GF\((q), k = 0\);
   (e) four conjugate lines over GF\((q^2), k = 0\);
   (f) a repeated line and two skew lines over GF\((q), k = 3q + 1\);
   (g) a repeated line over GF\((q)\) and a conjugate skew pair over GF\((q^2), k = q + 1\);
   (h) two repeated lines over GF\((q), k = 2q + 1\);
   (i) two repeated lines over GF\((q^2)\) with intersection over GF\((q), k = 1\).
2. \( \mathcal{C} \) is a conic \( \mathcal{P}_2 \) over GF\((q)\) and two lines:
   (a) two lines over GF\((q)\) which meet on \( \mathcal{P}_2, k = 3q + 1\);
   (b) two conjugate lines over GF\((q^2)\) which meet on \( \mathcal{P}_2, k = q + 1\);
   (c) two lines over GF\((q)\) which meet off \( \mathcal{P}_2, k = 3q\);
   (d) two conjugate lines over GF\((q^2)\) which meet off \( \mathcal{P}_2, k = q + 2\).
3. \( \mathcal{C} \) is two conics:
   (a) two conics over GF\((q)\) with no points in common, \( k = 2q + 2\);
   (b) two conics over GF\((q)\) with one point in common, \( k = 2q + 1\);
   (c) two conics over GF\((q)\) with two points in common, \( k = 2q\);
   (d) two conjugate conics over GF\((q^2)\) with no points in common, \( k = 0\);
   (e) two conjugate conics over GF\((q^2)\) with one point in common, \( k = 1\);
   (f) two conjugate conics over GF\((q^2)\) with two points in common, \( k = 2\);
   (g) a repeated conic over GF\((q), k = q + 1\).
(4) \( \mathcal{C} \) is a twisted cubic \( \mathcal{F} \) and a chord \( l \):
(a) \( l \) is a bisecant of \( \mathcal{F} \), \( k = 2q \);
(b) \( l \) is a tangent of \( \mathcal{F} \), \( k = 2q + 1 \);
(c) \( l \) meets \( \mathcal{F} \) in a pair of conjugate points over \( \text{GF}(q^2) \), \( k = 2q + 2 \).
For details of the geometry of the conic see [10, chaps 7-8], and of the twisted cubic see [11, chap. 21].

**Theorem 4.4.** Each of the configurations of type 1, 2, 3(g) or 4 is projectively unique in \( \text{PG}(3, q) \). For types 3(a), 3(c), 3(d), 3(f) there are two configurations when \( q \) is odd and one when \( q \) is even; for types 3(b), 3(e) there is one configuration when \( q \) is odd and two when \( q \) is even.

**Proof.** This requires a case by case study, but each calculation is straightforward.
It should be noted that, when \( q \) is even, only one of the two configurations 3(b) and 3(e) is the base of a non-singular pencil.

### 5. The Classification of Non-Singular Pencils with Reducible Base

To give canonical forms for quadrics in pencils we take a canonical form for the base curve \( \mathcal{C} \), as allowed by Theorem 4.4. In Table 2, components of \( \mathcal{C} \) rational over \( \text{GF}(q) \) are pictured with unbroken lines, components rational over \( \text{GF}(q^2) \) with broken lines and components rational over \( \text{GF}(q^4) \) with dotted lines.

The following conventions are adopted:
(1) \( x^2 + bx + 1 \) is irreducible over \( \text{GF}(q) \) with zeros \( \delta, \delta' \) in \( \text{GF}(q^2) \);
(2) \( x^2 + x + \mu \) is irreducible over \( \text{GF}(q) \), \( q \) odd, with \( \mu \) a non-square in \( \text{GF}(q) \);
(3) \( \alpha \) is an element of \( \text{GF}(q^4) \) such that \( \{ \alpha, \alpha^e, \alpha^2, \alpha^3 \} \) is a normal base for \( \text{GF}(q^4) \) over \( \text{GF}(q) \);
(4) \( f_i = A^{q^i}x_0 - A^{q^i + 1}x_1 + A^{q^i + 2}x_2 - A^{q^i + 3}x_3 \),
where
\[
A = \begin{vmatrix}
\alpha^q & \alpha^3 & \alpha \\
\alpha^e & \alpha & \alpha^e \\
\alpha & \alpha^e & \alpha^2
\end{vmatrix};
\]
(5) \( v \) is a non-square;
(6) \( U_0 = \mathbf{P}(1, 0, 0, 0), \ U_1 = \mathbf{P}(0, 1, 0, 0), \ U_2 = \mathbf{P}(0, 0, 1, 0), \ U_3 = \mathbf{P}(0, 0, 0, 1) \);
(7) \( \mathcal{P}_2 = \mathbf{V}(x_0, x_1^2 + x_1x_3) \);
(8) \( \mathcal{F} = \{ \mathbf{P}(r^3, r^2, r, 1)|r \in \text{GF}(q) \cup \{ \infty \} \} \);

**Theorem 5.1.** The projectively distinct non-singular pencils with reducible base are given by Table 2. The successive columns give, for each \( \mathcal{B} \), a diagram of \( \mathcal{C} \), \( k \), restrictions on \( q, n_1, n_2^+ \), \( n_2^- \), \( n_3 \), \( n_4^+ \), \( n_4^- \), the forms \( F, G \) for two quadrics \( \mathcal{F} = \mathbf{V}(F), \mathcal{G} = \mathbf{V}(G) \) in \( \mathcal{B} \), and the type of the pencil.

**Corollary.** For \( q \) odd, there are 27 pencils; for \( q \) even, there are 23 pencils. They are subdivided as in Table 3.

**Remarks.** (1) There exist curves \( \mathcal{C} \) like those in the classification with a singular pencil through them. For example, when \( q \) is even and \( \mathcal{C} = \mathbf{V}(x_0x_1, x_1^2 + x_0x_3 + x_1x_3) \), the pencil \( \mathcal{B} \) has \( n_2^+ = 1, n_5 = q \); the curve \( \mathcal{C} \) is a pair of touching conics.
### Table 2.

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<td>1(c)</td>
<td>$F = x_0 x_1, x_2 + bx_1 x_3 + x_3$</td>
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<td>1(d)</td>
<td>$F = x_0 x_1, x_2 + bx_2 x_3 + x_3$</td>
</tr>
<tr>
<td>1(e)</td>
<td>$F = A^{s+1} f_1 f_2 + A^{s+1} f_1 f_3, A^{s+1} f_0 f_2 + A^{s+1} f_1 f_3$</td>
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</tbody>
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### Notes
- $s = (a, a^2, a^2, a^2)$
- $F, G$ are functions of the variables $x_0, x_1, x_2, x_3$.
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<tbody>
<tr>
<td>0</td>
<td>2q</td>
<td>odd</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>$\frac{1}{g} (g - 1)$</td>
</tr>
<tr>
<td></td>
<td>2q</td>
<td>even</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>$\frac{1}{g} (g - 1)$</td>
</tr>
<tr>
<td></td>
<td>2q</td>
<td>odd</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>$\frac{1}{g} (g - 1)$</td>
</tr>
<tr>
<td></td>
<td>2q</td>
<td>even</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>$\frac{1}{g} (g - 1)$</td>
</tr>
<tr>
<td>1</td>
<td>2q+1</td>
<td>odd</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>$\frac{1}{g} (g - 1)$</td>
</tr>
<tr>
<td></td>
<td>2q+1</td>
<td>even</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>$\frac{1}{g} (g - 1)$</td>
</tr>
</tbody>
</table>

$F, G$

- $q = \frac{x_0 x_1^2 + x_2 x_3}{x_0 x_1 + x_2 x_3} + x_1 x_3^2 + \frac{x_3}{x_0 x_1 + x_2 x_3}$
- $q + 1 = \frac{x_0 x_1^2 + x_2 x_3}{x_0 x_1 + x_2 x_3} + x_1 x_3^2 + \frac{x_3}{x_0 x_1 + x_2 x_3}$
- $g = \frac{x_0 x_1^2 + x_2 x_3}{x_0 x_1 + x_2 x_3} + x_1 x_3^2 + \frac{x_3}{x_0 x_1 + x_2 x_3}$
- $g + 1 = \frac{x_0 x_1^2 + x_2 x_3}{x_0 x_1 + x_2 x_3} + x_1 x_3^2 + \frac{x_3}{x_0 x_1 + x_2 x_3}$

$U_2$

- $U_3$

$8^1 \oplus 8^2$

- $8^3 \oplus 8^1$

- $8^3 \oplus 8^2$

$8^1 \oplus 8^2$

- $8^3 \oplus 8^1$

- $8^3 \oplus 8^2$
Intersections in projective space II

Table 3.

<table>
<thead>
<tr>
<th>Type 1: Four lines</th>
<th>Type 2: Conic + two lines</th>
<th>Type 3: Two conics</th>
<th>Type 4: Twisted cubic + line</th>
</tr>
</thead>
<tbody>
<tr>
<td>( q ) odd</td>
<td>9</td>
<td>4</td>
<td>11</td>
</tr>
<tr>
<td>( q ) even</td>
<td>9</td>
<td>4</td>
<td>7</td>
</tr>
</tbody>
</table>

(2) In the classification, when \( \mathcal{C} \) is a pair of conics the planes containing the conics meet in a line \( l_0 \) which has the same polar \( l_1 \) with respect to each non-singular quadric in \( \mathfrak{B} \). The line \( l_1 \) meets the quadrics of \( \mathfrak{B} \) in pairs of an involution \( \mathfrak{I} \). For \( q \) odd, \( \mathfrak{I} \) is either hyperbolic or elliptic [10, p. 126] and has correspondingly two or no fixed points; the two cases give \( n_3 = 2 \) or 0, and \( l_1 \) contains the vertices of the two cones in the hyperbolic case. For \( q \) even, \( \mathfrak{I} \) is necessarily parabolic and \( n_3 = 1 \).

(3) The only two pencils with the same parameters are types 1(b) and 4(c), for which \( n_4^+ = q + 1, k = 2q + 2 \).

(4) The only pencils with no singular quadrics are 1(b), 1(e) and 4(c).

The pencil that requires more exposition is 1(e), where the four lines lie in a quartic extension. If \( P_0 \) is the intersection of two lines in the base, then the others are \( P_1 = P_0^q, P_2 = P_0^{q^3}, P_3 = P_0^{q^5} \), where \( P(X)^q = P(x_0^q, x_1^q, x_2^q, x_3^q) \). As \( P_0, P_1, P_2, P_3 \) are not coplanar, if \( P_0 = P(a_0, a_1, a_2, a_3) \), the elements \( a_0, a_1, a_2, a_3 \) of \( GF(q^4) \) are linearly independent over \( GF(q) \). So by a linear transformation of \( GF(q^4) \) as a vector space over \( GF(q) \), we may transform \( a_0, a_1, a_2, a_3 \) to a normal basis \( \alpha, \alpha^q, \alpha^{q^2}, \alpha^{q^3} \). We may take

\[
\begin{align*}
P_0 &= P(\alpha, \alpha^q, \alpha^{q^2}, \alpha^{q^3}), \\
P_1 &= P(\alpha^q, \alpha^{q^2}, \alpha^{q^3}, \alpha), \\
P_2 &= P(\alpha^{q^2}, \alpha^{q^3}, \alpha, \alpha^q), \\
P_3 &= P(\alpha^{q^3}, \alpha, \alpha^q, \alpha^q).
\end{align*}
\]

Let \( \pi_0 = P_1P_2P_3, \pi_1 = P_0P_2P_3, \pi_2 = P_0P_1P_3, \pi_3 = P_0P_1P_2 \). The equation of \( \pi_0 \) is

\[
\begin{bmatrix}
x_0 & x_1 & x_2 & x_3 \\
x_0^q & x_1^q & x_2^q & x_3^q \\
x_0^{q^2} & x_1^{q^2} & x_2^{q^2} & x_3^{q^2} \\
x_0^{q^3} & x_1^{q^3} & x_2^{q^3} & x_3^{q^3}
\end{bmatrix}
= 0.
\]

Hence, with

\[
A = \begin{bmatrix}
\alpha^{q^2} & \alpha^{q^3} & \alpha \\
\alpha^{q^2} & \alpha & \alpha^q \\
\alpha & \alpha^q & \alpha^{q^2}
\end{bmatrix},
\]

\[
\pi_0 = V(Ax_0 - A^q x_1 + A^{q^2} x_2 - A^{q^3} x_3),
\]

\[
\pi_1 = V(A^q x_0 - A^{q^2} x_1 + A^{q^3} x_2 - A x_3),
\]

\[
\pi_2 = V(A^{q^2} x_0 - A^{q^3} x_1 + A x_2 - A^q x_3),
\]

\[
\pi_3 = V(A^{q^3} x_0 - A x_1 + A^q x_2 - A^{q^2} x_3).
\]
More succinctly, for \( i = 0, 1, 2, 3 \),
\[ \pi_i = V(f_i), \]
with
\[ f_i = A^d x_0 - A^{d+1} x_1 + A^{d+2} x_2 - A^{d+3} x_3. \]

The pencil \( \mathcal{B} \) in \( \text{PG}(3, q^4) \) with the four lines as base has as members the plane pairs \( \pi_0 + \pi_3 \) and \( \pi_1 + \pi_2 \). If \( \pi_0 \) contained a point \( P \) of \( \text{PG}(3, q) \), it would also lie in \( \pi_2 \), and \( \pi_1 + \pi_2 \) are singular in \( \mathcal{B} \), and the argument shows that they do not lie in \( \mathcal{B} \). As the base of \( \mathcal{B} \) is empty over \( \text{GF}(q) \), the \( q + 1 \) members of \( \mathcal{B} \) partition \( \text{PG}(3, q) \) and are therefore all elliptic quadrics \( \mathcal{E}_3 \).

This partition of the space by elliptic quadrics, although not explicitly as a pencil, has also been obtained by Ebert [8] for \( q > 2 \), extending a similar result of Kestenband [12], [13]. For \( q \) odd, it was also observed by Brouwer [2], who gave \( F \) and \( G \) in forms equivalent to
\[ F = x_0^2 - vx_1^2 + 2x_2x_3, \]
\[ G = 2dx_0^2 + 2vx_0x_1 + \mu x_2^2 + 2dx_2x_1 + x_3^2. \]

Here \( \mu \) and \( v \) are non-squares with \( \mu = d^2 - \sqrt{v} \). Then \( V(tF + G) \) is a plane pair for
\[ t = -d \pm \sqrt{\mu}. \]
This gives the two plane pairs \( \pi_0 + \pi_2 \) and \( \pi_1 + \pi_3 \) over \( \text{GF}(q^2) \) as above.

6. SOME CONSEQUENCES OF THE CLASSIFICATION

PARTITION OF PG(3, q) BY QUADRICS

Theorem 6.1. If \( \text{PG}(3, q) \) is partitioned by a pencil \( \mathcal{B} \) of quadrics, then \( \mathcal{B} \) is one of the following:

(a) (for \( q \) odd):
1(d) with \( n_2^- = 2, n_4^+ = q - 1 \);
1(e) with \( n_4^- = q + 1 \);
3(d)(i) with \( n_2^- = 1, n_3 = 2, n_4^+ = \frac{1}{2}(q - 3), n_4^- = \frac{1}{2}(q - 1) \);
3(d)(ii) with \( n_2^- = 1, n_3 = 0, n_4^+ = \frac{1}{2}(q - 1), n_4^- = \frac{1}{2}(q + 1) \);

(b) (for \( q \) even):
1(d) with \( n_2^- = 2, n_4^+ = q - 1 \);
1(e) with \( n_4^- = q + 1 \);
3(d)(ii) with \( n_2^- = 1, n_3 = 1, n_4^+ = \frac{1}{2}(q - 2), n_4^- = \frac{1}{2}q \).

Proof. If the pencil has reducible base, then the theorem follows from Table 2. If the base is irreducible, then \( \mathcal{B} \) is either an elliptic or rational quartic both of which have a positive number of points rational over \( \text{GF}(q) \).

SPREADS OF LINES IN PG(3, q)

Theorem 6.2. The pencils 1(b), 1(d), 1(g) give the following spreads:
1(b): \( (n_2^- = q + 1, k = 2q + 2) \) there is a spread comprising the two real lines of the base and the \( q - 1 \) lines on each quadric in the same regulus as these two;
1(d): \( (n_4^+ = q - 1, n_2^- = 2, k = 0) \) there are \( 2^{q-1} \) spreads comprising the lines of the two quadrics \( \Pi_i \mathcal{E}_1 \) in \( \mathcal{B} \) and one of the two reguli in each \( \mathcal{K}_3 \) in \( \mathcal{B} \) exactly two of these are regular, [11, p. 54];
1(g): \( (n_4^- = q, n_2^- = 1, k = q + 1) \) there is a spread comprising the real repeated line of the base and the \( q \) lines on each \( \mathcal{K}_3 \) in the same regulus as this line.
THEOREM 6.3. PG(3, q) may be partitioned into two lines and \( q^2 - 1 \) conics.

PROOF. This may be done by taking all the planes through an \( \Pi_i \xi_i \) of the partition \( 1(d) \) or through the \( \Pi_i \xi_i \) of the partitions \( 3(d) \).

7. DUAL PENCILS

Given any two non-singular quadrics \( 2l_1 \) and \( 2l_2 \), let \( 2l_1^* \) and \( 2l_2^* \) be their respective duals. Write \( 2l_1 \sim 2l_2 \) if \( 2l_1 \) and \( 2l_2 \) are projectively equivalent; similarly write \( 2l_1 \sim 2l_2 \) if the two pencils \( 2l_1 \) and \( 2l_2 \) are projectively equivalent. Denote by \( \mathcal{B}(2l_1, 2l_2) \) the pencil containing \( 2l_1 \) and \( 2l_2 \); so \( \mathcal{B}(2l_1^*, 2l_2^*) \) is the pencil of quadric envelopes containing \( 2l_1^* \) and \( 2l_2^* \). Thus we denote by \( \mathcal{B}^*(2l_1, 2l_2) \) the pencil whose members are the duals of the envelopes in \( \mathcal{B}(2l_1^*, 2l_2^*) \) and call it the dual pencil to \( \mathcal{B}(2l_1, 2l_2) \); that is, if \( \mathcal{B}(2l_1^*, 2l_2^*) \) has base \( \Sigma \xi_i^* \), then \( \mathcal{B}^*(2l_1, 2l_2) \) is the pencil with base \( \Sigma \xi_i \), where \( \xi_i \) is the dual of \( \xi_i^* \). More simply, having obtained the dual quadrics \( 2l_1^* \) and \( 2l_2^* \), the substitution of point coordinates for dual coordinates gives two members of the pencil \( \mathcal{B}^*(2l_1, 2l_2) \). When the characteristic of the field is not two and the matrices of \( 2l_1 \) and \( 2l_2 \) are \( A_1 \) and \( A_2 \), then the matrices of \( 2l_1^* \) and \( 2l_2^* \) are \( A_1^{-1} \) and \( A_2^{-1} \).

Let \( k \) be the number of rational points in the base of \( \mathcal{B}(2l_1, 2l_2) \) and \( k^* \) the number in the base of \( \mathcal{B}^*(2l_1, 2l_2) \). From [3, sect. 5],

\[
k^* = k \quad \text{if} \quad 2l_1 \sim 2l_2 \tag{7.1}
\]
\[
k^* = 2(q + 1) - k \quad \text{if} \quad 2l_1 \not\sim 2l_2 \tag{7.2}
\]

There are two problems:

(I) Identify the type of \( \mathcal{B}^*(2l_1, 2l_2) \) for each \( \mathcal{B}(2l_1, 2l_2) \).

(II) If \( 2l_1', 2l_2' \in \mathcal{B}(2l_1, 2l_2) \), when is \( \mathcal{B}^*(2l_1', 2l_2') \sim \mathcal{B}^*(2l_1, 2l_2) \)?

The answers to (I) depend very much on whether \( 2l_1 \) and \( 2l_2 \) are equivalent or not. Equations (7.1) and (7.2) considerably restrict the possibilities for \( \mathcal{B}^* \) in each case. Further, only when the base is a pair of conics does \( \mathcal{B} \) admit both types of quadric.

<table>
<thead>
<tr>
<th>( k )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>( q + 1 )</th>
<th>( q + 2 )</th>
<th>( 2q )</th>
<th>( 2q + 1 )</th>
<th>( 2q + 2 )</th>
<th>( 3q )</th>
<th>( 3q + 1 )</th>
<th>( 4q )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1(e)</td>
<td>1(i)</td>
<td>1(c)</td>
<td>1(g)</td>
<td>2(d)</td>
<td>3(c)(i)</td>
<td>1(h)</td>
<td>1(b)</td>
<td>2(c)</td>
<td>1(f)</td>
<td>1(a)</td>
<td></td>
</tr>
<tr>
<td>1(d)</td>
<td>3(e)(i)</td>
<td>3(f)(i)</td>
<td>2(b)</td>
<td>3(c)(ii)</td>
<td>3(b)(i)</td>
<td>3(a)(i)</td>
<td>2(a)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3(d)(i)</td>
<td>3(e)(ii)</td>
<td>3(f)(ii)</td>
<td>3(g)(i)</td>
<td>3(c)(iii)</td>
<td>3(b)(ii)</td>
<td>3(a)(ii)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3(d)(ii)</td>
<td>3(f)(iii)</td>
<td>3(g)(ii)</td>
<td>4(a)</td>
<td>4(b)</td>
<td>3(a)(iii)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3(d)(iii)</td>
<td>4(c)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The results are obtained by a case analysis. Table 4 gives the possible pencils for each admissible value of \( k \). The unsurprising results given in Theorem 7.1 are in effect proved by Table 5. For each pencil \( \mathcal{B} \) in Table 2, the forms \( F \) and \( G \) of two of the quadrics are given: Table 5 lists for each \( H_i = tF + G \) the corresponding dual \( H_i^* \), where \( u_0, u_1, u_2, u_3 \) are the dual coordinates.

As an example, let us analyse a little more closely the pencil \( 3(a)(iii) \). Here

\[
H_i = tx_0x_1 + x_1^2 + vx_1^2 + x_2^2 + bx_2x_3 + x_3^2,
\]
\[
H_i^* = (t^2 - 4v)^{-1}(vu_0^2 - tu_0u_1 + u_1^2) + (b^2 - 4)^{-1}(u_2^2 - bu_2u_3 + u_3^2),
\]
Theorem 7.1. Let \( \mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_1', \mathcal{D}_2' \) be non-singular quadrics in \( \text{PG}(3, q) \) with the base \( \mathcal{C} \) of \( \mathcal{B}(\mathcal{D}_1, \mathcal{D}_2) \) reducible. Then the following results hold.

1. If \( \mathcal{D}_1', \mathcal{D}_2' \in \mathcal{B}(\mathcal{D}_1, \mathcal{D}_2) \) with \( \mathcal{D}_1' \sim \mathcal{D}_1 \) and \( \mathcal{D}_2' \sim \mathcal{D}_2 \), then \( \mathcal{B}^*(\mathcal{D}_1', \mathcal{D}_2') \sim \mathcal{B}^*(\mathcal{D}_1, \mathcal{D}_2) \).

2. \( \mathcal{B}^*(\mathcal{D}_1', \mathcal{D}_2') \sim \mathcal{B}(\mathcal{D}_1, \mathcal{D}_2) \).

(a) for each pencil \( \mathcal{B} \) of type 1, 2, 3(g) or 4,

(b) for each pencil of type 3 when \( \mathcal{D}_1 \sim \mathcal{D}_2 \).

3. When \( \mathcal{B}(\mathcal{D}_1, \mathcal{D}_2) \) is of type 3 and \( \mathcal{D}_1 \not\sim \mathcal{D}_2 \), then \( \mathcal{B}^*(\mathcal{D}_1, \mathcal{D}_2) \) is the unique pencil of type 3 allowed by (7.2) with the subclass (i), (ii) or (iii) preserved; that is, there is an interchange of types 3(a)(x) and 3(d)(x), types 3(b)(x) and 3(e)(x), types 3(c)(x) and 3(f)(x), where \( x \) is i, ii or iii.

where \( q \) is odd, \( v \) is a non-square and \( b^2 - 4 \) is a non-square. Hence

\[
H_t^* - H_s^* = \frac{t - s}{(t^2 - 4v)(s^2 - 4v)} \{ (v(t + s)u_0^2 - (ts + 4v)u_0u_1 + (t + s)u_1^2) \}
\]

Neglecting the factor outside the brackets, the discriminant \( \Delta \) of \( H_t^* - H_s^* \) is

\[
\Delta = (t^2 - 4v)(s^2 - 4v).
\]

From this it follows that \( \mathcal{B}^* \) is of type 3(a)(iii) if \( \mathcal{V}(H_t) \) and \( \mathcal{V}(H_s) \) are quadrics of the same type and of type 3(d)(iii) if quadrics of different type.

The results are summarized in Theorem 7.1.
8. The Case of an Irreducible Base

An irreducible quartic $C$ in $PG(3, q)$ is either rational or elliptic. If it is the intersection of two quadrics, it is of type I; if it is not, it is type II. When $C$ is elliptic it is always of type I. When $C$ is rational and singular, it is of type I; when $C$ is rational and non-singular, it is of type II. In all cases, $C$ can be projected to a plane cubic from a point of itself. When $C$ is rational of type I and the point of projection is the singular point, then the image is a conic. When $C$ is rational of type II, it has trisecant lines which ensure that the projection is singular.

The number of points on a rational cubic is $A_2(1) = q + 1$ or $q + 2$, although the number of points on a non-singular model is always $q + 1$. The number $N$ of points on an elliptic curve satisfies

\[ \sqrt{q} - 1 \leq N \leq \sqrt{q} + 2; \]

see [11, appendix IV] for more details and references.

Let the number of projectively distinct cubics of genus $g$ in $PG(2, q)$ be denoted by $P_q(g)$ and the number of isomorphism classes of cubics of genus $g$ defined over $GF(q)$ be denoted by $A_q(g)$. See Schoof [14] for the connection between these numbers. For $g = 0, 1$, the values are as follows:

\[
\begin{align*}
A_q(0) & = 1, \quad P_q(0) = 4, \\
A_q(1) & = 2q + 3 + 2 \left( \frac{-3}{q} \right) + \left( \frac{-4}{q} \right), \\
P_q(1) & = 2q + 3 + \left( \frac{-3}{q} \right)^2 + 3 \left( \frac{-3}{q} \right) + \left( \frac{-4}{q} \right).
\end{align*}
\]
here

\[
\begin{align*}
\left(\frac{-4}{q}\right) &= \begin{cases} 
1, & \text{for } q \equiv 1 \pmod{4}, \\
0, & \text{for } q \equiv 0 \pmod{2}, \\
-1, & \text{for } q \equiv -1 \pmod{4}, 
\end{cases} \\
\left(\frac{-3}{q}\right) &= \begin{cases} 
1, & \text{for } q \equiv 1 \pmod{3}, \\
0, & \text{for } q \equiv 0 \pmod{3}, \\
-1, & \text{for } q \equiv -1 \pmod{3}. 
\end{cases}
\end{align*}
\]

For \(P_q(0)\), see [10, p. 267]; for \(P_q(1)\), see [10, p. 315] or [14]; for \(A_q(1)\), see [14].

Now we examine the projection more carefully to see which plane cubics give rise to a space quartic \(q\).

**Lemma 8.1.** Let \(\mathcal{F}\) be an irreducible cubic with at least two, non-singular points in \(\text{PG}(2, K)\), where \(K\) is any field. Then there exists a quartic curve \(\mathcal{C}\) in \(\text{PG}(3, K)\) such that

(i) \(\mathcal{C}\) is the intersection of two quadrics,

(ii) \(\mathcal{C}\) projects to \(\mathcal{F}\) from a point of itself,

(iii) \(\mathcal{C}\) and \(\mathcal{F}\) have the same number of \(K\)-rational points.

**Proof.** The family of conics through two points on \(\mathcal{F}\) cuts out on \(\mathcal{F}\) a linear series \(g^3\) whose image is the required curve \(\mathcal{C}\). The proof can be given by standard manipulations. Briefly, let \(\mathcal{F}\) lie in the plane \(V(x_3)\) and contain \(U_1\) and \(U_2\); then \(\mathcal{F}\) has equation

\[
\sum a_{ijk}x_ix_jx_k = 0,
\]

with \(i, j, k \in \{0, 1, 2\}, i \leq j \leq k\) and \(a_{111} = a_{222} = 0\). Then there is a bijection from \(\mathcal{C}\) to \(\mathcal{F}\), where \(\mathcal{C} = \mathbb{P}^1 \cap \mathcal{H}_3\) and

\[
\mathcal{H}_3 = V(x_0x_3 - x_1x_2),
\]

\[
\mathcal{F} = V(a_{000}x_0^2 + a_{001}x_0x_1 + a_{011}x_1^2 + a_{002}x_0x_2 + a_{022}x_2^2 + a_{112}x_1x_3 + a_{122}x_2x_3 + a_{012}x_1x_2). 
\]

The cubics which do not have two non-singular points occur only for \(q \leq 4\), [10, p. 311]; for \(q = 2\), there is a nodal cubic with only one non-singular point and, for \(q = 2, 3, 4\), there is one elliptic cubic (up to projectivity) with exactly one point.

**Theorem 8.2.** For every isomorphism class of rational or elliptic cubic \(\mathcal{F}\) with at least two non-singular points, there is a pencil of quadric surfaces with base \(\mathcal{C}\) isomorphic to \(\mathcal{F}\).

It should be noted that, if \(\mathcal{C}'\) is a quartic of type II, it can be projected to a plane cubic \(\mathcal{F}\), which is the projection of a rational quartic \(\mathcal{C}\) of type I by Lemma 8.1.

**Lemma 8.3.** For \(q > 9\), every pencil of quadrics with base an irreducible quartic has \(n^+_4 > 0\) and \(n^-_4 > 0\).

**Proof** By Lemma 3.1, corollary,

\[
n_3 + 2n^+_4 = k, \quad n^+_4 - n^-_4 = k - (q + 1), \quad n_3 \leq 4.
\]

If \(n^+_4 = 0\), then \(k = n_3 \leq 4\). As \(k \geq q + 1 - 2\sqrt{q}\), so \(q - 2\sqrt{q} - 3 \leq 0\) and \(q \leq 9\).
Similarly, if \( n_t = 0 \), then eliminating \( n^+_t \) gives
\[
2(q + 1) - k = n_3 \leq 4.
\]
So \( q + 1 + 2\sqrt{q} \geq k \geq 2(q - 1) \), whence \( q - 2\sqrt{q} - 3 \leq 0 \) and again \( q \leq 9 \).

**Theorem 8.4.** For every rational or elliptic curve \( C \) with \( N \) rational points over \( \text{GF}(q) \), \( q > 9 \), there is a curve \( C^* \) with \( N^* \) rational points such that
\[
N + N^* = 2(q + 1).
\]

**Proof.** Every rational or elliptic curve is isomorphic to a plane cubic \( \mathcal{F} \). For each such \( \mathcal{F} \) with at least two non-singular points (certainly so when \( q > 9 \)), there is a space quartic \( C \) with \( N \) points projecting to \( \mathcal{F} \), by Lemma 8.1. By Lemma 8.3, the pencil of quadrics with base \( C \) contains both elliptic and hyperbolic non-singular quadrics. As in Lemma 8.1, we may take \( C = 2 \cap \mathcal{H}_3 \) where \( \mathcal{F} \) is elliptic. Let \( C^* \) be the dual of the intersection of the duals of \( \mathcal{F} \) and \( \mathcal{H}_3 \). Then, by [3, corollary 5.3],
\[
N + N^* = 2(q + 1),
\]
as required.

**Remarks.** (1) This result was obtained by Cicchese [5], [6] in the case that the characteristic \( p > 3 \) by taking the two projectively distinct curves with the same absolute invariant when the curves are neither harmonic nor equianharmonic. A suitable pairing also exists in the harmonic and equianharmonic cases.

(2) The result easily extends to all \( q \leq 9 \); see [10, p. 312].

Theorem 8.4 can be generalized to any space of odd dimension.

**Theorem 8.5.** Let \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) be non-singular quadrics of different types in \( \text{PG}(2d + 1, q) \) intersecting in a variety \( C \). Let \( \mathcal{F}_1^* \) and \( \mathcal{F}_2^* \) be the respective dual quadrics and \( C^* \) the dual of their intersection. If \( x \) and \( y \) are the numbers of \( \text{GF}(q) \)-rational points on \( C \) and \( C^* \), respectively, then
\[
x + y = 2(q^{2d-1} + q^{2d-2} + \cdots + q + 1).
\]

**Proof.** This is theorem 5.2 in [3].

**References**

2. A. E. Brouwer, letter.
6. M. Cicchese, Sulle cubiche di un piano lineare \( S_{2n} \) con \( q \equiv 1 \) (mod 3), *Rend. Mat.* 4 (1971), 249-283.


Received 5 August 1986

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