Henriksen’s contributions to residue class rings of analytic and entire functions

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The author surveys, summarizes and generalizes results of Golasiński and Henriksen, and of others, concerning certain residue class rings. Let $\mathcal{A}(\mathbb{R})$ denote the ring of analytic functions over reals $\mathbb{R}$ and $\mathcal{E}(\mathbb{K})$ the ring of entire functions over $\mathbb{R}$ or complex numbers $\mathbb{C}$. It is shown that if $m$ is a maximal ideal of $\mathcal{A}(\mathbb{R})$, then $\mathcal{A}(\mathbb{R})/m$ is isomorphic either to the reals or a real-closed field that is $\eta_1$-set, while if $m$ is a maximal ideal of $\mathcal{E}(\mathbb{K})$, then $\mathcal{E}(\mathbb{K})/m$ is isomorphic to one of these latter two fields or to complex numbers.

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0. Introduction

Let $\mathbb{K}$ denote the field $\mathbb{R}$ of reals or complex numbers $\mathbb{C}$. Write $\mathcal{A}(\mathbb{K})$ and $\mathcal{E}(\mathbb{K})$ for the ring of all analytic functions and the ring of entire functions over $\mathbb{K}$, respectively. That is, $\mathcal{A}(\mathbb{K})$ is the set of all $\mathbb{K}$-valued functions $f$ such that for each $x_0 \in \mathbb{K}$, there exists an open neighborhood $V$ of $x_0$ such that for all $x \in V$, the value $f(x)$ is the sum of an absolutely convergent power series in powers of $x - x_0$. However, $\mathcal{E}(\mathbb{K})$ consists of functions given by power series $\sum_{n=0}^{\infty} a_n x^n$ with $a_n \in \mathbb{K}$. Clearly, any $f \in \mathcal{E}(\mathbb{R})$ extends uniquely to an entire function over $\mathbb{C}$, so there is an inclusion $\mathcal{E}(\mathbb{R}) \subseteq \mathcal{E}(\mathbb{C})$. As is well-known, $\mathcal{A}(\mathbb{C})$ and $\mathcal{E}(\mathbb{C})$ coincide, while $\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}$ is in $\mathcal{A}(\mathbb{R}) \setminus \mathcal{E}(\mathbb{R})$.

The aim of this survey note is to summarize, based mainly on [7,8,11,12] and [13], results on the structure of residue class rings of $\mathcal{A}(\mathbb{K})/I$ and $\mathcal{E}(\mathbb{K})/I$ for $I$ a prime or maximal ideal. Because most of the results which appear in the paper are in the literature, no proofs are given. After reviewing what is known about the ideal structure of $\mathcal{A}(\mathbb{R})$, a description of the maximal ideals of this ring and the corresponding residue class fields is given in Section 1. In [11], the maximal ideals of $\mathcal{E}(\mathbb{C})$ were described, and it was shown that $\mathcal{E}(\mathbb{C})/m$ is always isomorphic as a ring to $\mathbb{C}$, even though for some maximal ideals $m$, the field $\mathcal{E}(\mathbb{C})/m$ is infinite-dimensional as an algebra over $\mathbb{R}$.

Section 2, based mainly on [7] and [8], uses techniques developed in [12] to carry out a similar program to study $\mathcal{E}(\mathbb{R})$. Here, we use a characterization of algebraically closed fields proved by E. Steinitz in 1909 that is still not as well-known as it should be. It states in [18] that an algebraically closed field is determined uniquely by its prime field and its degree of transcendence over it. As a byproduct, we deduce that the algebraic closure of the fields of quotients: $\mathcal{A}(\mathbb{R})_{(0)}$, $\mathcal{E}(\mathbb{R})_{(0)}$, and $\mathcal{E}(\mathbb{C})_{(0)}$ (the field of meromorphic functions on $\mathbb{C}$) are isomorphic to $\mathbb{C}$. Use is made also of techniques described by Gillman and Jerison in [6].
Section 3, based on [7] and [12], is concerned with prime ideals of $\mathcal{A}(\mathbb{R})$ and $\mathcal{E}(\mathbb{R})$. The results of [14] are applied to show that the Krull dimension of any of those rings is at least $2^{\text{cont}}$.

It was a great pleasure to investigate together with Mel residue class rings of $\mathcal{A}(\mathbb{R})$ and $\mathcal{E}(\mathbb{R})$. Our work [7] was based on papers which derived from his Ph.D. thesis at the University Wisconsin under the direction of R.H. Bruck and W.F. Eberlein.

1. Maximal ideals of $\mathcal{A}(\mathbb{R})$

The following proposition from [5] is useful in the sequel.

**Proposition 1.1.** Let $\mathbb{K}$ be one of the fields $\mathbb{R}$ or $\mathbb{C}$. Whenever $\{x_n\}$ is a sequence of elements in $\mathbb{K}$ such that $\lim_{n \to \infty} |x_n| = \infty$ and $\{w_{nk}\}$ is a double sequence of elements of $\mathbb{K}$, there is an $f \in \mathcal{E}(\mathbb{K})$ such that $f^{(k)}(x_n) = w_{nk}$ for $n = 1, 2, \ldots$ and $k = 0, 1, 2, \ldots$, where $f^{(k)}$ denotes the $k$th derivative of $f$.

In particular, because $\mathcal{E}(\mathbb{R}) \subseteq \mathcal{A}(\mathbb{R})$, there is also such an $f \in \mathcal{A}(\mathbb{R})$ for such sequences $\{x_n\}$ and $\{w_{nk}\}$ of reals.

Now, we present a brief summary of the properties of $\mathcal{A}(\mathbb{R})$ given in [2].

In what follows, all rings considered are assumed to be commutative and to have an identity element. Such a ring is said to be a Bézout domain if it is also an integral domain. Recall that an integral domain $\mathbb{R}$ with quotient field $\mathbb{K}$ is said to be completely integrally closed if for any $x \in \mathbb{K}$, there exists a finitely generated $\mathbb{R}$-submodule $\mathcal{M}$ of $\mathbb{K}$ such that $\mathbb{R}[x] \subseteq \mathcal{M}$. By [2, Theorem 1.19], $\mathcal{A}(\mathbb{R})$ is a Bézout domain that is completely integrally closed as well. On the other hand, as is noted in [9], the sequence of functions $\{\sin(\frac{1}{n})\}x$: $n = 1, 2, \ldots$ generates an infinite ascending chain of ideals that fails to be principal, so $\mathcal{A}(\mathbb{R})$ is not a Noetherian ring.

Clearly, $\mathcal{A}(\mathbb{R})$ is a subring of the ring $\mathcal{C}(\mathbb{R})$ of continuous real valued functions $f : \mathbb{R} \to \mathbb{R}$ with the usual pointwise operations.

For a non-zero $f \in \mathcal{A}(\mathbb{R})$ or $f \in \mathcal{E}(\mathbb{K})$, its zeroseat $Z_k(f) = \{x \in \mathbb{K} : f(x) = 0\}$ is a closed discrete subset of $\mathbb{K}$ and hence a countable as well.

In view of [7, Lemma 1.2], given a proper ideal $I \subseteq \mathcal{A}(\mathbb{R})$, we have $Z_k(f) \neq \emptyset$ for any $f \in I$. We recall that an ideal $I$ is called fixed if $\bigcap_{f \in I} Z_k(f) \neq \emptyset$. Otherwise $I$ is called a free ideal. Suppose that $x_0 \in \bigcap_{f \in I} Z_k(f)$. Then $I$ is a principal ideal generated by the function $id_{x_0} - x_0$ and there is an isomorphism

$$\mathcal{A}(\mathbb{R})/I \xrightarrow{\cong} \mathbb{R}$$

provided that $I$ is maximal.

It follows that an element $f \in \mathcal{A}(\mathbb{R})$ is in a maximal ideal if and only if $Z_k(f) \neq \emptyset$. Because $\mathcal{A}(\mathbb{R}) \subseteq \mathcal{C}(\mathbb{R})$, each maximal ideal of $\mathcal{A}(\mathbb{R})$ is contained in a maximal ideal of $\mathcal{C}(\mathbb{R})$. By a theorem of Gelfand and Kolmogoroff [6, 7.3], any maximal ideal of $\mathcal{C}(\mathbb{R})$ is determined by a point of the Stone–Čech compactification $\beta\mathbb{R}$ of the real line $\mathbb{R}$. It states that the maximal ideals of $\mathcal{C}(\mathbb{R})$ are precisely those of the form $\mathbb{M}^x = \{f \in \mathcal{C}(\mathbb{R}) : x \in \text{cl}_{\beta\mathbb{R}} Z_k(f)\}$ for some unique $x \in \beta\mathbb{R}$. Clearly $\mathbb{M}^x$ is fixed if and only if $x \in \mathbb{R}$. Because the zero-set of a real analytic function is closed and discrete, we need only consider a restricted subset of $\beta\mathbb{R}$ as follows.

A point $x \in \beta\mathbb{R} \setminus \mathbb{R}$ in the $\beta\mathbb{R}$-closure of a closed discrete subspace of $\mathbb{R}$ is said to be close to $\mathbb{R}$. Otherwise, $x$ is said to be far from $\mathbb{R}$. For an example of a far point, see [6, 4U]. Recall that a totally ordered set $(L, \prec)$ is called an $\eta_1$-set if whenever $A$ and $B$ are countable subsets of $L$ such that $A \prec B$ (i.e., $a \in A$ and $b \in B$ imply $a \prec b$), then there is an $x \in L$ such that $A \prec x \prec B$, and a field such that any of its algebraic extensions is algebraically closed is said to be real-closed. Such a field is totally ordered, its positive elements have square roots, and polynomial equations of odd degree have a root in it (see [6, Chapter 13]).

Now for future reference, we remind the reader of the result from [7, Theorem 2.2].

**Theorem 1.2.** If $m$ is a maximal ideal of $\mathcal{A}(\mathbb{R})$, then the residue class field $\mathcal{A}(\mathbb{R})/m$ has cardinality $2^{\text{cont}}$, and is:

1. real if and only if $m$ is fixed;
2. a real-closed (totally ordered) $\eta_1$-field if and only if $m$ is free.

The proof of the above yields also:

**Corollary 1.3.** If $m$ is a maximal ideal of $\mathcal{A}(\mathbb{R})$ that is contained in $\mathbb{M}^x$ for some $x \in \beta\mathbb{R} \setminus \mathbb{R}$ that is close to $\mathbb{R}$, then $\mathcal{A}(\mathbb{R})/m$ is a real-closed $\eta_1$-field.

An algebraically closed field of characteristic zero is uniquely determined by its cardinal number. Of course this does not hold for a real-closed field $K$. An additional invariant is the order type of $K$, and for uncountable $K$ this may be sufficient. The authors of [4] answer this affirmatively if the order type of $K$ is an $\eta_0$-set: no subset of power $< \aleph_1$ in $K$ is cofinal, and if $A, B$ are sets in $K$ of power $< \aleph_1$, with $A < B$ then an element can be interpolated. The investigation was motivated by the
desire to classify the real-closed fields that show up as \( C(X)/m \), where \( C(X) \) is the ring of all continuous real functions on a completely regular space \( X \), and \( m \) a maximal ideal in \( C(X) \). Such fields are called hyper-real if they differ from the real numbers. Sample results from [6]: assuming the continuum hypothesis (CH), all hyper-real fields of power of the continuum are isomorphic; the hyper-real fields associated with the discrete space with the power of the continuum are not all isomorphic.

The paper [4] not only motivated but also established a preliminary version of [3] in which a real-closed \( \eta_1 \)-field is called an \( H \)-field and it is shown that all \( H \)-fields of power \( 2^\mathfrak{c} \) are isomorphic if and only if (CH) holds. Thus, if (CH) holds, then \( \mathcal{A}(\mathbb{R})/m \) is isomorphic to one of only two possible fields. Note that every \( \eta_1 \)-field is non-Archimedean since it must contain elements larger than any integral multiple of the identity element.

Recall (see e.g., [1, Chapter 4]) that an ideal \( I \subseteq \mathcal{R} \) of a commutative ring \( \mathcal{R} \) is called formally real if \( r_1^2 + \cdots + r_n^2 \in I \) implies \( r_1, \ldots, r_n \in I \). Thus, Theorem 1.2 yields [7, Corollary 2.4]:

**Corollary 1.4.** Let \( m \subseteq \mathcal{A}(\mathbb{R}) \) be a maximal ideal. Then:

1. the ideal \( m \) is formally real;
2. for any function \( f \in \mathcal{A}(\mathbb{R}) \), there is \( g \in \mathcal{A}(\mathbb{R}) \) such that \( g^2 + f \in m \) or \( g^2 - f \in m \).

### 2. Properties of \( \mathcal{E}(\mathbb{K}) \)

The first thorough study of the ideal structure of \( \mathcal{E}(\mathbb{K}) \) was made by O. Helmer in [9]. Indeed, he studied the ring of entire functions with everywhere convergent power series with coefficients in any subfield \( \mathbb{K} \subseteq \mathbb{C} \). The most striking result in this paper is that \( \mathcal{E}(\mathbb{K}) \) is a Bézout domain. What is surprising is that this result for \( \mathbb{K} = \mathbb{C} \) was obtained in 1915 by J.M.H. Wedderburn [19]. Building on this latter result, M. Henriksen showed [11, Theorems 3 and 6] that the residue class ring of every maximal ideal \( m \subseteq \mathcal{E}(\mathbb{C}) \) is isomorphic to \( \mathbb{C} \), even though the fact that \( \mathcal{E}(\mathbb{C}) \) contains all polynomials with complex coefficients shows that \( \mathcal{E}(\mathbb{C})/m \) is infinite dimensional as an algebra over \( \mathbb{R} \). The proof relies on the Steinitz Theorem cited above [18].

A ring \( \mathcal{R} \) is called an elementary divisor ring if whenever \( A \) is a matrix with entries from \( \mathcal{R} \), there are invertible matrices \( P, Q \) of appropriate size such that \( PAQ \) is a diagonal matrix. A ring \( \mathcal{R} \) is said to be adequate [10] if \( \mathcal{R} \) is Bézout and for \( a, b \in \mathcal{R} \) with \( a \neq 0 \), there exists \( r, s, t \in \mathcal{R} \) such that \( a = rs \), \( (r, b) = 1 \), and if a non-unit \( s' \) divides \( s \), then \( (s', b) \neq 1 \).

It is known that every elementary divisor ring is a Bézout ring and that a Bézout domain that is an adequate ring is an elementary divisor ring as well. It is shown in [10, Theorem 4], that in an adequate domain every non-zero prime ideal is contained in a unique maximal ideal. For proofs of other assertions in this paragraph, see [2, Theorems 3.18 and 3.19] and [15].

As was observed in [10], in the light of [13], the ring \( \mathcal{E}(\mathbb{K}) \) is adequate. Those arguments, with a minor modification, show that \( \mathcal{A}(\mathbb{R}) \) is also an adequate ring. Whence, \( \mathcal{A}(\mathbb{R}) \) and \( \mathcal{E}(\mathbb{K}) \), being adequate Bézout domains, are elementary divisor rings as well. Actually, that \( \mathcal{E}(\mathbb{C}) \) is an elementary division ring was shown first in [19].

We say that an ideal \( I \) of a ring \( \mathcal{R} \) is formally complex if \( r^2 + 1 \in I \) for some \( r \in \mathcal{R} \). If \( \mathcal{R} \) contains an element \( j \) such that \( j^2 = -1 \), then \( (j - jr)^2 + 1 \in I \) provided that \( r \in I \). So every ideal of \( \mathcal{R} \) is formally complex. In particular, any ideal of a subring \( \mathcal{R} \subseteq \mathcal{E}(\mathbb{C}) \) that contains the constant function \( 1 \) is formally complex.

Given \( f(x) = \sum_{k=0}^{\infty} a_k x^k \in \mathcal{E}(\mathbb{R}) \), let \( f(z) = \sum_{k=0}^{\infty} a_k z^k \) denote its extension to a function in \( \mathcal{E}(\mathbb{C}) \). The lemmas below, proved in [7, Lemmas 3.1 and 3.2], show that any maximal ideal \( m \subseteq \mathcal{E}(\mathbb{R}) \) is either formally real or formally complex.

**Lemma 2.1.** Let \( m \subseteq \mathcal{E}(\mathbb{R}) \) be a maximal ideal. Then, the following properties are equivalent:

1. for each \( f \in m \), the zero set \( Z_\mathcal{R}(f) \neq \emptyset \);
2. for any maximal ideal \( m' \subseteq \mathcal{A}(\mathbb{R}) \) containing \( m \), the inclusion map \( \mathcal{E}(\mathbb{R})/m \hookrightarrow \mathcal{A}(\mathbb{R})/m' \) of residue class fields is an isomorphism;
3. the ideal \( m \) is formally real.

**Lemma 2.2.** Let \( m \) be a maximal ideal of \( \mathcal{E}(\mathbb{R}) \). Then the following properties are equivalent:

1. there is an isomorphism \( \mathcal{E}(\mathbb{R})/m \overset{\sim}{\longrightarrow} \mathbb{C} \) of fields;
2. the ideal \( m \) is formally complex;
3. there is \( f \in m \) with \( Z_\mathcal{R}(f) = \emptyset \).

In the light of Theorem 1.2, Lemmas 2.1 and 2.2, we summarize what we know from [7, Theorem 3.3] about residue classes of maximal ideals of \( \mathcal{E}(\mathbb{R}) \) in:

**Theorem 2.3.** Let \( m \subseteq \mathcal{E}(\mathbb{R}) \) be a maximal ideal. Then:

1. there is an isomorphic \( \mathcal{E}(\mathbb{R})/m \overset{\sim}{\longrightarrow} \mathbb{C} \) if and only if the maximal ideal \( m \) is formally complex;
(2) there is an isomorphic $E(\mathbb{R})/m \xrightarrow{\cong} \mathbb{R}$ if and only if $m$ is fixed;
(3) the residue class field $E(\mathbb{R})/m$ is real-closed otherwise. Furthermore, $E(\mathbb{R})/m$ is an $\eta_1$-field provided $m$ is a formally real maximal free ideal and two such residue class fields are isomorphic if and only if the continuum hypothesis holds.

Now let $f \in E(\mathbb{R})$ be a non-negative function. Because real zeros of $f$ have even multiplicity, we get $f = gh^2$ for some $g, h \in E(\mathbb{R})$ with $Z_{\mathbb{R}}(g) = \emptyset$, where $h$ is an entire function such that $Z_{\mathbb{C}}(h) \subseteq \mathbb{R}$. Next, take the decomposition $g = g_1(g_1')^2 + (g_2')^2$ presented in the proof of Lemma 2.1 for some $g_1', g_2' \in E(\mathbb{R})$ with $g'$ invertible. Consequently, we have redeveloped, as in [7, Corollary 3.5], the following result proved in [16].

**Corollary 2.4.** Any non-negative function $f \in E(\mathbb{R})$ is a sum of two squares,

$$f = (f')^2 + (f'')^2$$

for some $f', f'' \in E(\mathbb{R})$.

We close this section with a discussion presented in [8] of special maximal ideals $m \subseteq C(\mathbb{R})$. First, given a maximal ideal $m \subseteq C(\mathbb{R})$, consider the inclusion maps

$$E(\mathbb{R})/(m \cap E(\mathbb{R})) \hookrightarrow A(\mathbb{R})/(m \cap A(\mathbb{R})) \hookrightarrow C(\mathbb{R})/m.$$  

If $m \cap E(\mathbb{R}) \neq (0)$ then by methods presented in Section 2, those maps are isomorphisms. On the other hand, by [6, 4F, p. 61] there is a $z$-ultrafilter $F$ on $\mathbb{R}$ containing only sets of infinite measure. Let $m_F \subseteq C(\mathbb{R})$ be the corresponding maximal ideal. Because the zero set of any function in $E(\mathbb{R})$ is discrete, $m_F \cap E(\mathbb{R}) = (0)$. Hence, the canonical map $E(\mathbb{R}) \rightarrow C(\mathbb{R})/m_F$ is a monomorphism and implies the inclusion maps

$$E(\mathbb{R})_0 \hookrightarrow A(\mathbb{R})_0 \hookrightarrow C(\mathbb{R})/m_F,$$

where $A(\mathbb{R})_0$ and $E(\mathbb{R})_0$ denote the fields of quotients of $A(\mathbb{R})$ and $E(\mathbb{R})$, respectively.

If now $M(C) = E(\mathbb{C})_0$ denotes the field of quotients of $E(\mathbb{C})$ (the field of meromorphic functions on the complex plane $\mathbb{C}$), then the isomorphism $E(\mathbb{R})(j) \xrightarrow{\cong} E(\mathbb{C})$ yields $(E(\mathbb{R})_0)(j) \xrightarrow{\cong} M(C)$ and consequently, we derive an inclusion map

$$M(C) \hookrightarrow (C(\mathbb{R})/m_F)(j),$$

where $j^2 = -1$. By [6], the field $C(\mathbb{R})/m_F$ is real-closed and so the Steinitz Theorem [18] leads to an isomorphism $(C(\mathbb{R})/m_F)(j) \xrightarrow{\cong} \mathbb{C}$. Hence, we have shown in [8, Theorem 1]:

**Proposition 2.5.**

1. The field $C(\mathbb{R})/m_F$ coincides with the real closure of the formally real fields $A(\mathbb{R})_0$ and $E(\mathbb{R})_0$;
2. the algebraic closures of the fields $A(\mathbb{R})_0$, $E(\mathbb{R})_0$, and $M(C)$ coincide, being isomorphic to the field $\mathbb{C}$.

### 3. Krull dimension of $A(\mathbb{R})$ and $E(\mathbb{K})$

Recall that the Krull dimension K-dim $\mathcal{R}$ of a commutative ring $\mathcal{R}$ is the supremum of the lengths of a chain of (proper) prime ideals.

In [17] Schilling claimed to have shown that $K$-dim $E(\mathbb{C}) = 1$, but in 1952 Kaplansky showed that it is at least 2; then Henriksen proved [12] that it is at least $2^{\aleph_0}$ and also discussed the nature of residue class rings $E(\mathbb{C})/p$, where $p$ is a prime ideal of $E(\mathbb{C})$.

For $f$ in $A(\mathbb{R})$ or $E(\mathbb{R})$ that is neither 0 or a unit, let $m(f)$ denote the maximum of the multiplicity of a zero of $f$ if this is finite, and let $m(f) = \infty$ otherwise. By Proposition 1.1, it is clear that any maximal ideal of these rings must contain an element $f$ such that $m(f) = 1$. While much of what follows apes what is written in [12], it will be placed in a more general context in which it is evident that it applies to $A(\mathbb{R})$ or $E(\mathbb{R})$ as well as to $E(\mathbb{C})$.

Suppose $\mathcal{R}$ is an adequate domain that satisfies:

1. if $m$ is a maximal ideal of $\mathcal{R}$ then its powers $m^n$ for $n = 1, 2, \ldots$ are distinct;
2. if a non-maximal prime ideal $p \subseteq \mathcal{R}$ is contained in a maximal ideal $m$, then $p \subseteq p^* = \bigcap_{n=1}^{\infty} m^n$. We call such a ring nearly analytic.

**Remark 3.1.**

1. It follows easily that $p^*$ is a prime ideal and hence it is the largest non-maximal prime ideal contained in the maximal ideal $m$. 

Recall that a valuation ring is a commutative ring $R$ such that for non-zero $s, t \in R$ either $s \mid t$ or $t \mid s$. In [7, Proposition 3.2 and Theorem 3.3], we have shown:

**Proposition 3.2.** Let $p$ be a prime ideal of a nearly analytic ring $R$. Then, $R/p$ is:

1. a valuation ring;
2. Noetherian if and only if $p = \mathfrak{p}^*$.

By means of the above, arguments given in the proof of [11, Theorem 8] and results of [14], we summarize some properties of the rings $\mathcal{A}(\mathbb{R})$ and $\mathcal{E}(\mathbb{K})$ for $\mathbb{K} = \mathbb{R}, \mathbb{C}$.

**Theorem 3.3.** Let $R$ denote either $\mathcal{A}(\mathbb{R})$ or $\mathcal{E}(\mathbb{K})$. Then:

1. there is a non-maximal, prime ideal $p \subseteq R$;
2. a necessary and sufficient condition that a prime ideal $p \subseteq R$ is a non-maximal, is that $m(f) = \infty$ for every $f \in p$;
3. the residue class ring $R/p$ is formally real and formally complex prime ideals of $R$ is formal real if and only if the quotient field of the residue class ring $R/p$ is formally real.
4. the localization $R_p$ is a valuation ring for any prime ideal $p \subseteq R$;
5. the ring $R$ is an elementary divisor domain in which every non-zero prime ideal is contained in a unique maximal ideal;
6. for any maximal ideal $m$ such that $p^* \neq (0)$, the ring $\mathcal{R}/\mathfrak{p}^*$ is isomorphic to the ring $(\mathcal{R}/\mathfrak{m})[X]/[X]$ of formal power series over the field $\mathcal{R}/\mathfrak{m}$.

By [1, Chapter 4], a prime ideal $p \subseteq \mathcal{E}(\mathbb{R})$ is formally real if and only if the quotient field of the residue class ring $\mathcal{E}(\mathbb{R})/p$ is formally real. Furthermore, given a prime non-maximal ideal $p \subseteq \mathcal{E}(\mathbb{R})$, in the light of the integral extension $\mathcal{E}(\mathbb{R}) \subseteq \mathcal{E}(\mathbb{C})$, there is a non-maximal prime ideal $p' \subseteq \mathcal{E}(\mathbb{C})$ with $p = \mathcal{E}(\mathbb{R}) \cap p'$.

We have shown in [7] that proofs of Lemmas 2.1 and 2.2, and Proposition 3.2 lead *mutatis mutandis* to a characterization of formally real and formally complex prime ideals of $\mathcal{E}(\mathbb{R})$. In particular, it follows that also any prime ideal of $\mathcal{E}(\mathbb{R})$ is either formally real or formally complex. As usual, if $R$ is a ring, $R[X]$ denotes the ring of polynomials with coefficients in $R$. But first, we state [7, Proposition 4.5]:

**Proposition 3.4.** Let $p$ be a prime ideal of $\mathcal{E}(\mathbb{R})$. Then the following properties are equivalent:

1. the ideal $p$ is formally complex;
2. for any prime ideal $p' \subseteq \mathcal{E}(\mathbb{C})$ with $p = \mathcal{E}(\mathbb{R}) \cap p'$, the inclusion map $\mathcal{E}(\mathbb{R})/p \hookrightarrow \mathcal{E}(\mathbb{C})/p'$ of residue class rings is an isomorphism;
3. there is $f \in p$ with $Z_\mathbb{R}(f) = \emptyset$.

To characterize formally real prime ideals of $\mathcal{E}(\mathbb{R})$, we have proceeded in [7] by the following construction.

If $p(X) \in (\mathcal{E}(\mathbb{R})/p)[X]$ then there are $f, g \in \mathcal{E}(\mathbb{R})$ and $q(X) \in (\mathcal{E}(\mathbb{R})/p)[X]$ such that $p(X) = q(X)(X^2 + 1) + (f + p) + (g + p)X$. Then map

$$(\mathcal{E}(\mathbb{R})/p)[X] \to \mathcal{E}(\mathbb{C})/p'$$

that sends $p(X)$ to $(f + gi) + p'$ may be regarded as a homomorphism onto $\mathcal{E}(\mathbb{C})/p'$ whose kernel contains the principal ideal $(X^2 + 1)$ of $(\mathcal{E}(\mathbb{R})/p)[X]$. This yields a map

$$\eta : (\mathcal{E}(\mathbb{R})/p)[X]/(X^2 + 1) \to \mathcal{E}(\mathbb{C})/p'.$$

Then, we have shown [7, Proposition 4.6]:

**Proposition 3.5.** Let $p \subseteq \mathcal{E}(\mathbb{R})$ be a prime ideal. Then the following properties are equivalent:

1. the prime ideal $p$ is formally real;
2. for each $f \in p$, the zeroset $Z_\mathbb{R}(f) \neq \emptyset$;
3. for any prime ideal $p' \subseteq \mathcal{E}(\mathbb{C})$ with $p = \mathcal{E}(\mathbb{R}) \cap p'$, the map

$$\eta : (\mathcal{E}(\mathbb{R})/p)[X]/(X^2 + 1) \to \mathcal{E}(\mathbb{C})/p'$$

is a ring isomorphism.

Furthermore, any prime ideal of $\mathcal{A}(\mathbb{R})$ is formally real.
Methods of [12] and [14] applied to the ring $\mathcal{R}$ being $\mathcal{A}(\mathbb{R})$ or $\mathcal{E}(\mathbb{K})$ lead to the following generalization of [12, Theorem 5] stated in [7, Theorem 4.7]:

**Theorem 3.6.** If $p_1 \subsetneq p_2 \subsetneq p_3$ are prime ideals of $\mathcal{R}$ then there exists a chain of $2^{\omega_1}$ prime ideals between $p_1$ and $p_3$. In particular, the Krull dimension $\text{K-dim } \mathcal{R} \geq 2^{\omega_1}$.

The structure of the residue class ring $\mathcal{C}(\mathcal{R})/I$ for $I$ a prime or maximal ideal has been intensively studied [6]. Write $\mathcal{C}^{(n)}(\mathbb{R}) \subseteq \mathcal{C}(\mathcal{R})$ for the subring given by $\mathcal{C}^{n}$-functions $\mathbb{R} \to \mathbb{R}$ with $n = 1, \ldots, \infty$. We close this note with the problem suggested to me by M. Henriksen:

**Problem 3.7.** What about the residue class ring of $\mathcal{C}^{(n)}(\mathcal{R})/I$ when $I$ is a prime or maximal ideal of $\mathcal{C}^{(n)}(\mathbb{R})$?

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