An algebraic characterization of stability groups

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Abstract

In this paper, necessary and sufficient conditions are established which characterize the stability groups within the full homeomorphism group of a manifold. Other results apply to more general homeomorphism groups and uncover properties of subgroups which are transitive on the complement of themselves in their coset space.

Key words: Homeomorphism group; Manifold; Stability group

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1. Introduction

The goal of this paper is to establish necessary and sufficient conditions which imply that a subgroup of certain homeomorphism groups is the stability group of a point in the underlying space. The spaces under consideration will primarily be connected topological manifolds, where a topological manifold is a Hausdorff, second countable, locally Euclidean space, and if M is such a space, its full homeomorphism group, \( G = \mathcal{H}(M) \), will generally be the group under consideration. A “manifold” will mean a “topological manifold”. By the stability group \( G_x \) of \( x \), we mean \( G_x = \{ h \in G \mid h(x) = x \} \). In particular, if \( M \) is a connected \( n \)-dimensional manifold without boundary, compact or not, and if \( G \) is the full homeomorphism group of \( M \) we will show the following:

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Theorem 1.1. A subgroup $H$ of $G$ is the stability group of a point in $M$ if and only if the following conditions apply to $H$:

1. $H$ is closed,
2. $H$ contains no nontrivial subgroups which are normal in $G$, and
3. $H$ has exactly two double cosets in $G$.

Condition (3) means of course that there are exactly two equivalence classes in $G$ under the relation $f \sim g$ if and only if $HfH = HgH$. The important consequence of this condition is that given any $f$ not in $H$, $G/H$ decomposes into exactly two orbits under multiplication by $H$, namely $H(fH)$ and $H$ itself. This fact will be used repeatedly.

Unless otherwise noted, the underlying topological space $M$ will be as defined for the main Theorem 1.1. The topology on $G$ is the compact-open topology in which case Gleason and Palais [6] showed in 1957 (citing results from Arens [2]) that $G$ is a separable and completely metrizable topological group. In 1986, Kallman [8] showed that this topology is the only topology in which $G$ is a complete separable metric group. Since $M$ is a metric space, this topology coincides with the topology of compact convergence, and in case $M$ is compact, both topologies are equivalent to the sup metric topology. (See Munkres [9].)

In view of condition (1) of Theorem 1.1, it is appropriate to ask in what sense this constitutes an “algebraic” characterization. Among the closed subgroups of $G$, Theorem 1.1 certainly provides an algebraic characterization of the stability groups. But, in another sense, closedness is itself an algebraic property as Kallman [8] showed in 1986.

Proposition 1.2 (Kallman). Let $K$ be a complete separable metric group and let $\psi : G \to K$ be an abstract group isomorphism. Then $\psi$ is a topological isomorphism.

In particular, this proposition implies that the three properties described in Theorem 1.1 are preserved by isomorphisms and in that sense represent algebraic characteristics.

Conditions (1) and (3) prove to be extremely strong conditions, but they are not quite sufficient for the conclusion of Theorem 1.1. In Fisher [5], it is shown that if $n \leq 3$, then the connected component of the identity in the group of homeomorphisms of the sphere $S^n$ is of index 2. It also follows from results in that paper that the connected component of the identity is transitive on $S^n$. Therefore, conditions (1) and (3) apply to this group while condition (2) certainly does not, and of course a transitive group of homeomorphisms leaves no point fixed.

It is known that if $x$ is an element of $M$, then $G/G_x$ is homeomorphic to the orbit of $x$ under $G$ (see Effros [4, Proposition, p. 6]), this orbit being all of $M$ in case $M$ is connected and without boundary. To see one simple corollary to Theorem 1.1, suppose for the moment that $K$ is a complete separable metric group and that $\psi : G \to K$ is a group isomorphism. Then given Kallman’s results, it is a straightforward exercise to show that $G/G_x$ and $K/\psi(G_x)$ are homeomorphic. If we also assume $K$ to be the homeomorphism group of a connected manifold $N$
without boundary, then Theorem 1.1 and Kallman’s result imply that $\psi(G_y) = K_y$ for some $y \in N$. Consequently, $M$ and $N$ are homeomorphic.

**Definition.** An open subset $U$ of $M$ is called an open $n$-cell if there exists a homeomorphism $\phi$ of $U$ onto the open unit ball in $\mathbb{R}^n$. A closed $n$-cell is defined similarly. A closed $n$-cell $A$ is called internal if there is an open $n$-cell $U$ in $M$ which contains $A$. An open $n$-cell will be called internal if it is contained in the interior of a closed internal $n$-cell.

The identity element of $G$ will be denoted by $e$.

**Definition.** If $A \subseteq M$, $S(A)$ will denote the group of all $h \in G$ such that $h|_{A^c} = e|_{A^c}$. If $h \in S(A)$, we will say that $h$ is supported on $A$.

$S(A)$ is a closed subgroup of $G$, and it is easily verified that if $h \in G$, then

$$S(h(A)) = hS(A)h^{-1}.$$  

This important identity will be used frequently.

Other useful and easily obtained facts are that $A \subseteq B$ implies $S(A) \subseteq S(B)$, and that $f \in S(A)$ and $g \in S(B)$ imply $fg \in S(A \cup B)$. Furthermore, if $A \cap B = \emptyset$, then the above $f$ and $g$ commute.

If we set

$$\mathcal{S} = \bigcup \{S(A) \mid A \text{ is an internal closed } n\text{-cell}\},$$

and we denote the intersection of all nontrivial normal subgroups of $G$ by $Q$, then Fisher [5] was able to show, using techniques found in Anderson [1], that $Q = \langle \mathcal{S} \rangle$, where $\langle \mathcal{S} \rangle$ denotes the group generated by $\mathcal{S}$. If $h \in G$ and $A$ is an internal closed $n$-cell, then $h(A)$ is again an internal closed $n$-cell, and since

$$hS(A)h^{-1} = S(h(A)) \subseteq \mathcal{S},$$

it follows that $\langle \mathcal{S} \rangle$ is certainly normal. It is important then to note that condition (2) of Theorem 1.1 implies that a subgroup $H$ of $G$ with condition (2) of Theorem 1.1 cannot contain $\mathcal{S}$.

In 1962, Whittaker [10] added the following result concerning $Q$ which will be crucial in proving Theorem 1.1.

**Proposition 1.3** (Whittaker). Let $J$ be a subgroup of $G$. Suppose that for every $x \in M$ there is a neighborhood $U$ of $x$ such that $S(U) \subseteq J$. Then $Q \subseteq J$.

This says, among other things, that given any open cover $\mathcal{U}$ of $M$, $Q$ is contained in the group generated by $\{S(U) \mid U \in \mathcal{U}\}$. Once again, given a subgroup $H$ with condition (2) of Theorem 1.1, it follows that $H$ cannot contain such a collection $\{S(U) \mid U \in \mathcal{U}\}$, for otherwise it would contain the normal subgroup $Q$.

Completeness plays an important role in the following results. $G$ is complete and separable, but the upcoming lemmas depend on the completeness (and
separability) of $G/H$ where $H$ satisfies the conditions of Theorem 1.1. That $G/H$ is complete when $G$ is may not be as obvious as it might first appear. That it is in general true is a result due to Hausdorff [7]. The more obvious fact is that since $H$ is closed, it is complete; moreover, $H$ is separable.

Given that both $H$ and $G/H$ are complete and separable and that $H$ acts as a transformation group on the space $G/H$, we will use the following result proved in Effros [4]. In the following notation, $f = fH$ and $H_f$ is the subgroup of $H$ which leaves $f$ fixed under left multiplication.

**Proposition 1.4** (Effros). *For any $fH \in G/H$, the orbit $H(fH) \subseteq G/H$ is second category if and only if the map $hH_f \to h(fH)$ is a homeomorphism.*

In order to show the sufficiency of the conditions in Theorem 1.1, we will show that $S(U) \subseteq H$ for some open set $U \subseteq M$. Then given condition (2) and Whittaker's proposition, it will follow that the union of all such $U$ is not equal to $M$. It will in fact follow from the subsequent lemmas and their corollaries that the union of all such $U$ is the complement of a point in $M$. It will then be shown that this point must be left fixed by $H$, and the conclusion of Theorem 1.1 will quickly follow.

2. Main results

Several of the following results apply to spaces more general than manifolds. We will, therefore, state the following lemmas and corollaries in as much generality as possible, gradually strengthening the assumptions for the later results. In all cases, the hypotheses are satisfied by the manifold $M$ as described before the statement of the main Theorem 1.1.

The following lemma exploits the transitivity provided by assuming condition (3) of Theorem 1.1. In this lemma, the fundamental tools for the results of this paper will be developed. We will call a space topologically complete if it is homeomorphic to a complete metric space.

**Lemma 2.1.** Let $M$ be a locally compact, locally connected, separable metric space such that for each open subset $U$ of $M$, $S(U)$ is a nontrivial subgroup of $G = \mathcal{X}(M)$. Suppose $H$ is a closed subgroup of $G$ with exactly two double cosets. Then for every open subset $U$ of $M$, $S(U) \cap H$ is a nontrivial subgroup of $G$.

**Proof.** Under these assumptions on $M$, Gleason and Palais [6] showed that $G$ is a complete separable metric homeomorphism group in the compact-open topology. Also, since $G$ is metrizable, it is first countable and hence it is known that $G$ possesses a right invariant topology $\rho$ compatible with its topology. (Such a metric is not in general complete.) Since $G$ is a metrizable group and $H$ is closed, it follows that the quotient space $G/H$ is metrizable by a "quotient metric" $D$ derived from the metric $\rho$ on $G$. Furthermore, Hausdorff [7] in 1934 proved that the metrizable image of a topologically complete space under a continuous open
map is again topologically complete. In particular, $G$ is certainly topologically complete, the quotient map from $G$ to $G/H$ is open and continuous, and therefore $G/H$ possesses a complete metric compatible with its quotient topology.

Now since $H$ is closed, it is complete and separable, and it acts as a transformation group on the complete separable metric space $G/H$. Furthermore, condition (3) (two double cosets) of Theorem 1.1 implies that $H$ is transitive on the left cosets in $G/H$ which are different from $H$. Thus, for any $f \not\in H$, the orbit of $fH$ under multiplication by $H$ is the complement of a point, namely $H$ itself, and therefore the orbit $H(fH)$ is open in $G/H$. From a result proved in Effros [4] and stated in the introduction, there is then a homeomorphism

$$\varphi : H/H_f \to H(\tilde{f}) \quad \text{given by} \quad \varphi(hH_f) = h\tilde{f},$$  

where

$$\tilde{f} = fH \in G/H \quad \text{and} \quad H_f = \{ h \in H \mid h(\tilde{f}) = \tilde{f} \}.$$ 

This homeomorphism and the preceding observations will be crucial in the following arguments. Let $d$ be a metric on $M$.

Let $U$ be open in $M$ and choose open $V \subseteq U$ such that $\overline{V}$ is compact and $\overline{V} \subseteq U$. Let $\varepsilon > 0$ be small enough that the $\varepsilon$-neighborhood of $V$ is contained in $U$. Call this $\varepsilon$-neighborhood $W$. Now choose open $V_1 \subseteq V$ such that $\overline{V}_1 \subseteq V$ and fix $f \in S(V_1)$ with $f \not= e$. If $f \in H$, the lemma holds because $S(V_1) \subseteq S(U)$, so we assume $f \not\in H$. Let $x \in V \setminus \overline{V}_1$ and set $\varepsilon' = d(x, \overline{V}_1)$. We then find a compact neighborhood $A \subseteq V \setminus \overline{V}_1$ containing $x$ which is small enough to satisfy three conditions, which we now describe.

Noting first that $H^e$ is an open neighborhood of $f^{-1}$ and that multiplication in $G$ by $f^{-1}$ is continuous, it follows that if an element $g$ of $G$ is close enough to the identity, then $f^{-1}g$ is in $H^e$. Since the topology on $G$ is equivalent to the topology of compact convergence (see Munkres [9]), it follows that $S(A)$ may be made arbitrarily close to the identity by making the diameter of $A$ sufficiently small. Thus, for the first condition, we first choose $A$ such that $f^{-1}S(A) \subseteq H^e$. Having done this, it of course follows that for all $g \in S(A)$, $f^{-1}gH \neq H$.

Second, shrink $A$ if necessary so that $d(A, \overline{V}_1) > \varepsilon'/2$. Let

$$F_1 = \{ a \in G \mid \sup\{d(a(x), x) \mid x \in \overline{V} \} < \varepsilon \}$$

and

$$F_2 = \{ a \in G \mid \sup\{d(a(x), x) \mid x \in \overline{V}_1 \} < \frac{\varepsilon'}{2} \}.$$ 

Set $F = F_1 \cap F_2$. From the definition of the topology of compact convergence, $F$ is a basic open set in $G$ containing $e$. Also, note that from the definitions of $F_1$ and $F_2$,

$$a \in F_1 \implies a(\overline{V}) \subseteq W \subseteq U,$$

and

$$a \in F_2 \implies a(\overline{V}_1) \cap A = \emptyset.$$ 

These facts will be recalled shortly.
Now choose $\delta$ so that $\rho(h, e) < \delta$ implies that $h \in F$. (We may note that there are nonidentity elements of $H$ arbitrarily close to the identity since otherwise, $H$ would be discrete hence countable, and condition (3) of Theorem 1.1 would then imply that $G$ is countable, which it is not.) Restricting $\rho$ to $H$, one defines the quotient metric $D$ on $H/H_{f^{-1}}$ to be

$$D(h_1H_{f^{-1}}, h_2H_{f^{-1}}) = \inf\{\rho(h_1a, h_2b) \mid a, b \in H_{f^{-1}}\}.$$ 

Then noting again that there is a homeomorphism $\varphi: H/H_{f^{-1}} \rightarrow H_{f^{-1}}$ as defined in (*), we may find a neighborhood $N$ of $f^{-1}$ in $H(f^{-1})$ (which, since $H(f^{-1})$ is open, is also a neighborhood of $f^{-1}$ in $G/H$) so that

$$\varphi^{-1}(N) \subseteq B(H_{f^{-1}}, \delta) = \{hH_{f^{-1}} \mid D(hH_{f^{-1}}, H_{f^{-1}}) < \delta\}.$$

Finally, we further shrink $A$ so that $g \in S(A)$ implies that $f^{-1}gH \subseteq N$. This is possible since the map

$$g \mapsto f^{-1}g \mapsto f^{-1}gH$$

is a composition of continuous maps.

If any nonidentity element $g \in S(A)$ is also in $H$, the lemma holds, so again, assume that this is not the case. Then it follows that $f^{-1}H \neq f^{-1}gH$ for any $g \in S(A)$ (where $g \neq e$, and since $A$ was chosen so that $f^{-1}S(A) \subseteq H^c$, it is also true that $f^{-1}gH \neq H$ for any $g \in S(A)$. In other words, for any $g \in S(A)$, $f^{-1}gH$ is in the orbit $H(f^{-1}H)$ since $H$ is transitive on the complement of $\{H\}$ in $G/H$.

Next, fix $g \in S(A)$ with $g \neq e$. Since $H$ is transitive on its cosets that differ from $H$, there exists an $h_0 \in H$ such that $h_0(f^{-1}H) = f^{-1}gH$. Since $f^{-1}gH \subseteq N$,

$$h_0H_{f^{-1}} = f^{-1}(h_0f^{-1}H) = f^{-1}(f^{-1}gH) \in B(H_{f^{-1}}, \delta)$$

which implies that

$$D(h_0H_{f^{-1}}, H_{f^{-1}}) < \delta.$$

Since $D(h_0H_{f^{-1}}, H_{f^{-1}}) = \inf\{\rho(h_0a, b) \mid a, b \in H_{f^{-1}}\}$, and since $\rho$ is right invariant, this distance is equal to $\inf\{\rho(h_0ab^{-1}, e) \mid a, b \in H_{f^{-1}}\}$. Thus there is some coset representative of $h_0H_{f^{-1}}$ whose distance from $e$ is less than $\delta$, and since all elements of $h_0H_{f^{-1}}$ “move” $f^{-1}$ to $f^{-1}g$, we will assume that $h_0$ was chosen to be within $\delta$ of $e$, and thus we have $h_0$ within the set $F$ described earlier in the lemma.

Now since $h_0f^{-1}H = f^{-1}gH$, it follows that

$$h_0 \in h_0(H \cap f^{-1}Hf) = H \cap (h_0f^{-1}H)f = H \cap f^{-1}gHf.$$ 

Thus,

$$h_0 = f^{-1}gh_1f \quad \text{for some } h_1 \in H,$$ 

and of course $h_0h_1^{-1} \in H$. We will in fact show that $h_0h_1^{-1} \in H \cap S(U)$. Solving the preceding equation for $h_1^{-1}$ produces

$$h_1^{-1} = fh_0^{-1}f^{-1}g,$$ 

(5)
so that
\[ h_0 h_1^{-1} = h_0 f h_0^{-1} f^{-1} g = \left( h_0 f h_0^{-1} \right) (f^{-1} g). \]

Consider this grouping of the factors of \( h_0 h_1^{-1} \). Recalling that \( f \subset S(V) \subseteq S(V) \), it follows that \( h_0 h_0^{-1} \in h_0 S(V) h_0^{-1} = S(h_0(V)) \). But \( \rho(h_0, e) < \delta \) implies that \( h_0 \in F_1 \) which, as noted in (1), then implies that \( h_0(V) \subseteq W \). Hence,

\[ h_0 f h_0^{-1} \in S(h_0(V)) \subseteq S(W) \subseteq S(U). \]

Clearly, \( f^{-1} g \in S(V) \subseteq S(V) \) and therefore,

\[ h_0 h_1^{-1} = \left( h_0 f h_0^{-1} \right) (f^{-1} g) \in S(U \cup V) = S(U). \]

To see that \( h_0 h_1^{-1} \) is not the identity, write \( h_0 h_1^{-1} = [h_0 f h_0^{-1} f^{-1}] g \) and recall once more that \( f \in S(V) \). Then \( h_0 f h_0^{-1} \in h_0 S(V) h_0^{-1} = S(h_0(V)) \) and \( h_0 f h_0^{-1} \) is supported on \( h_0(V) \). But \( \rho(h_0, e) < \delta \) implies that \( h_0 \in F_2 \) and as noted in (2), it then follows that \( h_0(V) \cap A = \emptyset \). Consequently,

\[ h_0 f h_0^{-1} \mid_A = e \mid_A. \]

By the choice of \( A \), \( V_1 \cap A = \emptyset \), and thus \( f^{-1} \subseteq S(V) \) implies that \( f^{-1} \mid_A = e \mid_A \). Consequently,

\[ (h_0 f h_0^{-1}) f^{-1} \mid_A = e \mid_A \quad \text{and} \quad h_0 f h_0^{-1} f^{-1} g \mid_A = g \mid_A = e \mid_A, \]

and the proof is complete. \( \square \)

We have shown that if \( V \) is an open subset of \( M \), \( H \cap S(V) \) is a nontrivial subgroup of \( G \). Now we will use arguments similar to those in Lemma 2.1 to uncover more about this intersection. \( M \) and \( G \) will still be as defined for the previous lemma.

**Lemma 2.2.** Suppose \( H \) is a closed subgroup of \( G \) with exactly two double cosets. Let \( V \) be a nonempty open subset of \( M \) with compact closure such that \( V \neq M \). Then either

1. \( S(V) \subseteq H \), or
2. for every \( x \notin \overline{V} \), there exists a neighborhood \( A \) of \( x \) such that \( H \) contains a nontrivial closed, normal subgroup of \( S(A) \). Moreover, any neighborhood \( A \) of \( x \) with sufficiently small diameter has this property.

**Proof.** Assuming \( S(V) \subseteq H \), we choose \( f \in S(V) \) with \( f \neq e \) and \( f \notin H \). Let \( x \in \overline{V}^c \) and set \( \varepsilon = d(x, \overline{V}) \). As in the previous lemma, we will find a compact neighborhood \( A \) of \( x \) small enough to satisfy several conditions. First, find \( A \) small enough that \( f^{-1} S(A) \subseteq H^c \) and so that \( d(V, A) > \varepsilon / 2 \). Let

\[ F = \left\{ a \in G \mid \sup \{ d(a(x), x) \mid x \in \overline{V} \} < \frac{\varepsilon}{2} \right\}. \]

As a consequence of this definition,

\[ h \in F \quad \Rightarrow \quad h(V) \cap A = \emptyset. \]
Choose \( \delta \) so that \( \rho(h, e) < \delta \) implies that \( h \in F \). Using a homeomorphism \( \varphi \) as defined by \((*)\) in the previous lemma, find a neighborhood \( N \) of \( f^{-1}H \) in \( G/H \) such that \( \varphi^{-1}(N) \subseteq B(H \setminus \{e\}, \delta) \). Then shrink \( A \) if necessary so that for every \( g \in S(A) \), we have \( f^{-1}gH \in N \).

Fix \( g \in S(A) \). By the previous lemma, there is some \( h \in H \cap S(A) \) with \( h \neq e \). We will show that \( ghg^{-1} \) is in \( H \cap S(A) \). If \( g \in H \cap S(A) \), then, obviously, \( ghg^{-1} \in H \cap S(A) \) and the preceding statement is true. If \( g \notin H \), then it follows that \( f^{-1}gH \neq f^{-1}H \); furthermore, since neither \( f^{-1}g \) nor \( f^{-1}H \) is in \( H \), it follows that neither of their cosets is equal to \( H \). In particular, the transitivity referred to before implies that \( f^{-1}gH \) is in the orbit of \( f^{-1}H \) under multiplication by \( H \). Thus, there is some \( h_0 \in H \) such that \( h_0 f^{-1}H = f^{-1}gH \) and since \( f^{-1}gH \in N \), we may as in the previous lemma assume that \( \rho(h_0, e) < \delta \). (See (3) and the subsequent comments.)

Again,
\[
h_0 = h_0(H \cap f^{-1}Hf) = H \cap f^{-1}gHf,
\]
and therefore,
\[
h_0 = f^{-1}gh_1f \quad \text{for some} \quad h_1 \in H.
\]
Solving for \( h_1^{-1} \) as in (5), we have
\[
h_0h_1^{-1} = \left[ h_0, f^{-1}g \right] g \in H.
\]
But \( h_0f^{-1}g \in h_0S(V)h_0^{-1} = S(h_0(V)) \) and \( \rho(h_0, e) < \delta \) implies that \( h_0 \in F \). Thus \( h_0(V) \cap A = \emptyset \) and since \( h_0f^{-1}g \) is supported on \( h_0(V) \), we have as a consequence that \( h_0f^{-1}g \mid A = e \mid A \). Then \( f \in S(V) \) and \( V \cap A = \emptyset \) imply that \( f \mid A = e \mid A \), and thus \( (h_0f^{-1}g)^{-1}f^{-1} \mid A = e \mid A \). As a result, \( (h_0f^{-1}g)^{-1}f^{-1} \) commutes with both \( g \) and \( h \) since both of these are supported on \( A \), and we have that
\[
\left[ (h_0f^{-1}g)h_0f^{-1}g \right]^{-1} = ghg^{-1} \in H \cap S(A).
\]
The above inclusion holds since the left side of the equality is a product of elements of \( H \) while the right side is a product of elements of \( S(A) \). It has just been shown that for all \( g \in S(A) \), \( ghg^{-1} \in H \cap S(A) \).

Let \( S = \{ghg^{-1} \mid g \in S(A)\} \) and let \( \langle S \rangle \) denote the group generated by \( S \). Clearly \( \langle S \rangle \) is normal in \( S(A) \) and \( \langle S \rangle \subseteq H \cap S(A) \). Since \( H \cap S(A) \) is closed,
\[
\langle S \rangle \subseteq H \cap S(A),
\]
and \( \langle S \rangle \) is a normal subgroup of \( S(A) \) which contains more than the identity.

Finally, we note that once the neighborhood \( A \) of \( x \) was chosen to have small enough diameter, it satisfied the conclusion of the lemma. Consequently, any neighborhood of \( x \) with diameter less than or equal to that of \( A \) would also satisfy the conclusion of the lemma. \( \square \)

For the following corollary, we again assume the same hypotheses on \( M \) as in the previous lemmas. As before, \( G = \mathcal{H}(M) \).
Corollary 2.3. Suppose $H$ is closed with exactly two double cosets in $G$. If $x$ is not a fixed point of $H$, then there exists a neighborhood $U$ of $x$ such that $H$ contains a nontrivial closed, normal subgroup of $S(U)$.

Proof. Supposing that the conclusion is false, it follows in particular that for each neighborhood $U$ of $x$, $S(U)$ is not contained in $H$. Therefore, if $U$ is a compact neighborhood of $x$, and $y$ is not in $U$, Lemma 2.2 implies that $y$ has a neighborhood $V$ so that $H$ contains a nontrivial closed, normal subgroup of $S(V)$. Furthermore, we may take the diameter of $V$ to be arbitrarily small. Since each $y \neq x$ has a neighborhood $V_y$ whose closure misses some compact neighborhood of $x$, it must be that each $y \neq x$ has a neighborhood $V_y$ (whose closure misses $x$) so that $H$ contains a nontrivial closed, normal subgroup of $S(V_y)$.

Now if $x$ is not left fixed by $H$, there is some $h \in H$ and $y \neq x$ so that $h(x) = y$. From the remarks above, there is then a neighborhood $V_y$ of $y$ such that $H$ contains a nontrivial closed, normal subgroup $J$ of $S(V_y)$. That is, $J \subseteq S(V_y) \cap H$. Since $x = h^{-1}(y)$, $h^{-1}(V_y)$ is a neighborhood of $x$, and

$$h^{-1}Jh \subseteq h^{-1}S(V_y)h \cap H = S(h^{-1}(V_y)) \cap H.$$ 

Suppose now that

$$f \in S(h^{-1}(V_y)) = h^{-1}S(V_y)h, \tag{15}$$

and write

$$f = h^{-1}gh \quad \text{where } g \in S(V_y). \tag{16}$$

Then

$$f(h^{-1}Jh) f^{-1} = (h^{-1}gh)(h^{-1}Jh)(h^{-1}g^{-1}h)$$

$$= h^{-1}gLg^{-1}h$$

$$= h^{-1}Jh$$

where the last equality holds since $J$ is normal in $S(V_y)$. Thus $h^{-1}Jh$ is normal in $S(h^{-1}(V_y))$. Recalling that $J$ was closed, it follows that $h^{-1}Jh$ is closed, and recalling that we are assuming that $H$ contains no closed, normal subgroup of $S(h^{-1}(V_y))$ (since $h^{-1}(V_y)$ is a neighborhood of $x$), this is a contradiction, and thus it must be that $x$ is in fact left fixed by $H$. $\square$

Before proceeding, we note that if $A$ is an internal open $n$-cell in a manifold $M$, then $S(A)$ is transitive on $A$, and as a consequence any $J \subseteq S(A)$ which is a nontrivial normal subgroup of $S(A)$ cannot fix any point $x \in A$. For suppose $g(x) = x$ for some $x \in A$ and for all $g \in J$. Then pick an arbitrary $y \in A$ and find $f \in S(A)$ such that $f(x) = y$. Then $J = ff^{-1}$, but if $g \in J$, $fg^{-1}(y) = y$ and $J$ then fixes $y$ as well as $x$. Since $y$ was arbitrary, $J$ must be the identity on $A$, and since $J \subseteq S(A)$, $J$ is already the identity on $A^c$. Thus $J = \{e\}$. These remarks apply, of course, to any set $A$ on which $S(A)$ is transitive. This observation will be useful in
the following corollaries and in Lemma 2.6 which provides the last major tool needed for the proof of the main Theorem 1.1.

For the following corollaries, we will need one additional property of manifolds. For Corollaries 2.4 and 2.5, suppose that \( M \) is a locally compact, locally connected, separable metric space each point of which has arbitrarily small neighborhoods \( A \) such that \( S(A) \) is nontrivial and transitive on \( A \).

**Corollary 2.4.** If \( H \) is closed with exactly two double cosets in \( G \), then \( H \) has at most one fixed point.

**Proof.** Suppose that \( x \in M \) is left fixed by \( H \). Then, given the current hypothesis on \( M \), it must be that \( H \) contains no \( S(U) \) where \( U \) is a neighborhood of \( x \) since \( S(U) \) does not leave \( x \) fixed. Thus, as was shown in Lemma 2.2, for each \( y \neq x \) all neighborhoods \( V \) of \( x \) with sufficiently small diameter have the property that \( H \) contains a nontrivial closed, normal subgroup \( J \) of \( S(V) \). Since we are also assuming that \( y \) has arbitrarily small neighborhoods \( V \) on which \( S(V) \) is transitive, we may choose \( V \) so that \( S(V) \) is transitive on its support and so that \( H \) contains a nontrivial closed, normal subgroup \( J \) of \( S(V) \). From the observations preceding this corollary, it follows that \( J \), and hence \( H \), cannot leave \( y \) fixed. Therefore, \( x \) can be the only fixed point of \( H \). \( \square \)

**Corollary 2.5.** Given any \( x \) and \( y \) in \( M \) with \( x \neq y \), at least one of the points has a neighborhood \( U \) such that \( H \) contains a nontrivial closed, normal subgroup of \( S(U) \).

**Proof.** From Corollary 2.4, \( H \) must move one of \( x \) or \( y \). The conclusion then follows from Corollary 2.3. \( \square \)

The existence of the closed, normal subgroups of Lemma 2.2 and its corollaries is of considerable importance. The next lemma shows one consequence of that existence and it also suggests an important connection between the collections \( S(U) \), where \( U \) is a closed internal \( n \)-cell in a manifold, and normal subgroups of \( G \) in general. For the following lemma, we will need the locally Euclidean properties of a manifold.

**Lemma 2.6.** Let \( M \) be a connected manifold without boundary, and suppose that \( H \) is closed with exactly two double cosets in \( G \). If \( H \) does not leave a point \( x \in M \) fixed, then there is a neighborhood \( U \) of \( x \) such that \( S(U) \subseteq H \).

**Proof.** If \( x \) is not a fixed point of \( H \), then by Corollary 2.3, there is some neighborhood \( V \) of \( x \) so that \( H \) contains a nontrivial closed, normal subgroup \( J \) of \( S(V) \). Let \( V \) be such a set and from the remarks prior to this lemma, \( J \) can fix no point of \( V \). In particular, we fix an \( h \in J \subseteq H \cap S(V) \) such that \( h(x) \neq x \). Consequently, there exists a closed internal \( n \)-cell \( U \) with \( x \) in its interior such that

(i) \( U \subseteq V \),
(ii) \( h(U) \cap U = \emptyset \), and
(iii) \( h(U) \subseteq V \).
By the definition of a closed internal n-cell, there is some open n-cell \( O \) containing \( h(U) \), since \( h(U) \) is also a closed internal n-cell, and evidently \( U \) and \( O \) may be found so that \( U \cap O = \emptyset \) and so that \( O \subseteq V \).

Let \( t \in S(U) \). Then

\[
ht^{-1}h^{-1} \in hS(U)h^{-1} = S(h(U)).
\]

Since \( h(U) \) is closed and in the interior of \( O \), for any \( n \geq 1 \) there is a \( g_n \in S(O) \) so that \( g_n(h(U)) \) has diameter less than \( 1/n \). Let \( n \) and such a \( g_n \) be given and note first that since \( U \cap O = \emptyset \), \( g_n \) and \( t \) commute. Then note that

\[
tht^{-1}h^{-1} \in J
\]

since \( t \in S(U) \subseteq S(V) \), \( h \in J \), and \( J \) is normal in \( S(V) \). Also

\[
g_n(ht^{-1}h^{-1})g_n^{-1} \in J
\]

since \( g_n \in S(O) \subseteq S(V) \). Recalling that \( g_n \) and \( t \) commute, it follows that

\[
t(g_nht^{-1}h^{-1}g_n^{-1}) = g_n(tht^{-1}h^{-1})g_n^{-1} \in J.
\]

Now since \( ht^{-1}h^{-1} \in S(h(U)) \), we have that

\[
g_n(h^{-1}h^{-1})g_n^{-1} \in g_n(S(h(U)))g_n^{-1} = S(g_n(h(U))).
\]

If we write \( s_n = g_nht^{-1}h^{-1}g_n^{-1} \), then \( s_n \) is supported on \( g_n(h(U)) \) and \( ts_n \in J \). Since \( n \geq 1 \) was arbitrary, it follows that there is a sequence \( \{s_n\}_{n=1}^{\infty} \) with \( s_n \) supported on \( g_n(h(U)) \) where \( g_n(h(U)) \) has diameter less than \( 1/n \), and such that for each \( n \), \( ts_n \in J \). Now \( s_n \to e \) and therefore \( ts_n \to t \), but \( J \) is closed from which it follows that \( t \in J \). Since \( t \in S(U) \) was arbitrary, \( S(U) \subseteq J \subseteq H \cap S(V) \). \( \square \)

More useful in the proof of Theorem 1.1 is the contrapositive of Lemma 2.6. That is, if \( x \) has no neighborhood \( U \) with \( S(U) \subseteq H \), then \( H \) must leave \( x \) fixed. And if this is the case, then Corollary 2.4 implies that \( x \) is the only point left fixed by \( H \).

This will be very useful in the following proof.

**Proof of Theorem 1.1.** The necessity of the conditions is fairly straightforward.

Suppose \( H \) is the stability group of a point. Then \( H \) is closed.

Suppose that \( H = G_x \) is the stability group of \( x \) in \( M \). If \( g \notin H \), then \( g(x) = y \) for some \( y \) different from \( x \). For such a \( g \), \( g^{-1}(y) = x \), and thus for any \( h \in H \),

\[
ghg^{-1}(y) = gh(x) = g(x) = y
\]

implies that \( ghg^{-1} \) fixes \( y \). Since \( M \) is homogeneous, it follows that some conjugate of \( H \) will fix any given point of \( M \). Therefore, if \( H \) contains any subgroup \( J \) which is normal in \( G \), then \( J \) must simultaneously fix all points of \( M \). Consequently, it must follow that \( J = \{e\} \) and thus \( H \) contains no nontrivial subgroups which are normal in \( G \).

To establish the third condition, again suppose that \( H - G_x \) is the stability group of the point \( x \in M \). Let \( f \notin H \), let \( g \) be any other element of \( H^c \), and
suppose \( f(x) = y \) and \( g(x) = z \). Now \( H \) is transitive on the complement of its fixed point, for if \( M \) has dimension 1, then \( M \) is either a circle or a line in which case \( H \) is certainly transitive on the complement of its fixed point. Then if \( M \) has dimension greater than or equal to 2 and \( u \) and \( v \) are any points which differ from \( x \), they may be joined by a finite chain of \( n \)-cells which miss \( x \). Consequently, a composition \( a = a_n \circ a_{n-1} \circ \cdots \circ a_1 \), where each \( a_i \) is supported on one of the \( n \)-cells, exists such that \( a(u) = v \). Then \( a \) is supported on the union of this chain, which misses \( x \), and \( a(x) = x \). Therefore, there exists an \( h \in H \) with \( h(z) = y \).

Then \( f^{-1}hg(x) = x \) implies that \( f^{-1}hg \in H \), and thus \( g \in h^{-1}fH \subseteq HfH \). Since \( g \) was arbitrary, \( HfH \) contains \( H^c \) (in fact, \( HfH = H^c \)), and the third condition of the theorem holds.

Now assume the conditions listed in the theorem. Since \( H \) has only two double cosets, it certainly contains more than the identity. Consequently, there is some \( y \in M \) which is not left fixed by all of \( H \). Then, given Lemma 2.6, there is a neighborhood \( V \) of \( y \) with \( S(V) \subseteq H \). Now Whittaker's proposition stated in the introduction implies that not all points in \( M \) can have such a neighborhood, for if that were the case, \( Q \) would be contained in \( H \) which is impossible given condition (2) of the theorem. That is, there exists an \( x \in M \) such that \( x \) has no neighborhood \( U \) with \( S(U) \subseteq H \). Then, following from Lemma 2.6, it must be that \( H \) leaves \( x \) fixed. That is, \( H \subseteq G_x \).

Finally, we note that condition (3) also implies that \( H \) is a maximal subgroup of \( G \), for suppose \( J \) is a subgroup of \( G \) which properly contains \( H \). Let \( f \in J \setminus H \). Then \( J \) contains \( HfH \), and since \( J \) also contains \( H \), condition (3) implies that \( J = G \). Therefore, since \( H \) is maximal and \( H \subseteq G_x \), it must be that \( H = G_x \).

The strength of the first and third conditions in Theorem 1.1 is now evident. In fact, much of the proof of Theorem 1.1 relies solely on the presence of these conditions. After some simple observations, we will conclude in a corollary to the main Theorem 1.1 that subgroups of \( G \) to which conditions (1) and (3) apply fall into exactly two categories—those which are transitive on \( M \) and those which are the stability groups of some point in \( M \).

Suppose that a subgroup \( H \) of \( G \) is not transitive on \( M \). Then since \( Q \) is certainly transitive on \( M \), it follows that \( Q \) is not contained in \( H \). Fisher showed that \( Q \) was contained in all subgroups which are normal in \( G \), and therefore \( Q \notin H \) implies that \( H \) contains no subgroups which are normal in \( G \). Thus, if in addition to nontransitivity, we suppose \( H \) to have conditions (1) and (3) from Theorem 1.1, then \( H \) in fact has the three conditions which guarantee that it is the stability group of a point.

**Corollary 2.7.** Let \( H \) be a closed subgroup of \( G \) and suppose that \( H \) has exactly two double cosets. Then one and only one of the following holds:

1. \( H \) is transitive on \( M \), or
2. \( H \) has a unique fixed point and \( H \) is the stability group of that point.
3. References