THEORETICAL & APPLIED MECHANICS LETTERS **2**, 043006 (2012)

Stability analysis of the extended state observers by Popov criterion

Christiav Erazo,[a)] Fabiola Angulo, and Gerard Olivar
Universidad Nacional de Colombia-Sede Manizales-Facultad de Ingeniería y Arquitectura-Departamento de Ingeniería Eléctrica, Electrónica y Computación-Percepción y Control Inteligente-Bloque Q, Campus La Nubia, Manizales 170003, Colombia

(Received 4 June 2012; accepted 20 June 2012; published online 10 July 2012)

Abstract The analysis and design of the extended state observer (ESO) involves a continuous non-smooth structure, thus the study of the ESO dynamic requires mathematical tools of the nonlinear systems analysis. This paper establishes the sufficient conditions for absolute stability of the ESO. Based on this study, a methodology to estimate several nonlinear functions in dynamics systems is proposed. [doi:10.1063/2.1204306]

Keywords Popov criterion, extended observer, ESO

A special class of state observers, called extended state observers (ESO) was proposed by Han in Ref. 1. This observer has the ability to estimate the states and the whole system, including uncertainties, in an integrated form. This technique has been widely used in industrial applications with excellent results;[2,3] in fact, the ESO is the brain of the control technique known as ADRC (active disturbance rejection control) which detects and compensates actively the internal and external perturbations of the system.[4,5] Although ESO has been widely used in several applications, the stability analysis is still an open problem since the ESO has a non-smooth structure which makes the analysis a difficult task. However, stability studies for the second order ESO have been reported in literature. In Ref. 6 a piecewise smooth Lyapunov function is proposed in order to analyze the convergence properties for the second order ESO. Similarly, in Ref. 7 a stability study for the second order ESO using the self-stable region (SSR) approach is proposed. The trouble is that extending these studies to higher-order ESOs is a complicated task and has not yet been reported in literatures. In this paper we propose to perform a convergence analysis for the high order ESO using the Popov criterion. Basically, we propose represent the observer error system as a feedback interconnection of a linear system and a nonlinearity in order to use the Popov criterion.[8] In addition, using the above results we propose a nonlinear estimator functions based on the ESO.

A brief review of this approach will be presented below. Consider a nonlinear dynamical system described by

$$y^{(n)} = f(y, \dot{y}, \cdots, y^{(n-1)}, w(t)) + bu(t), \tag{1}$$

where $f(\cdot)$ is an uncertain nonlinear function, $w(t)$ is an unknown disturbance, $u(t)$ is the control signal and $y(t)$ is the system output which can be measured. The system (1) is augmented as

$$\begin{aligned} \dot{x}_1 &= x_2, \\ &\vdots \\ \dot{x}_n &= x_{n+1} + bu, \\ \dot{x}_{n+1} &= h_1, \end{aligned} \tag{2}$$

and $y = x_1$, where $f(x_1, x_2, \cdots, x_n, w(t))$ is handled as an extended state x_{n+1}. Here, $h_1 = \dot{f}(x_1, x_2, \cdots, x_n, w(t))$ is assumed unknown but bounded. With this consideration, the nonlinear observer design for system (2) is described by

$$\begin{aligned} \dot{z}_1 &= z_2 + \beta_1 g_1(e_1), \\ &\vdots \\ \dot{z}_n &= z_{n+1} + \beta_n g_n(e_1) + bu, \\ \dot{z}_{n+1} &= h_2 + \beta_{n+1} g_{n+1}(e_1), \end{aligned} \tag{3}$$

and $\hat{y} = z_1$, where $e_1 = y - \hat{y} = x_1 - z_1$, z_{n+1} is an estimate of uncertain function $f(\cdot)$, $\beta_i' s$ are feedback gains, and each $g_i(\cdot)$ is defined as

$$g_i(e_i, \alpha_i, \delta) = \begin{cases} |e_i|^{\alpha_i} \mathrm{sgn}(e_i), & |e_i| > \delta \\ \dfrac{e_i}{\delta^{1-\alpha_i}}, & |e_i| \le \delta \end{cases}, \tag{4}$$

where $\alpha \in (0, 1)$. In order to improve the transient response of the estimation error and decrease the observer sensitivity to model and external disturbances, the nonlinear function (4) is incorporated into the observer, such as discussed in Ref. 8. Equation (4) is a nonlinear gain function where small errors produce high gains, and large error produce small gains. The last issue prevents excessive gain which could cause high frequency.

Let $e_i = x_i - z_i$, $i = 1, 2, \cdots, n+1$. The observer estimation error is defined as

$$\begin{aligned} \dot{e}_1 &= e_2 - \beta_1 g_1(e_1, \alpha_1, \delta), \\ &\vdots \\ \dot{e}_n &= e_{n+1} - \beta_n g_n(e_1, \alpha_n, \delta), \\ \dot{e}_{n+1} &= -\beta_{n+1} g_{n+1}(e_1, \alpha_{n+1}, \delta). \end{aligned} \tag{5}$$

To relax notation, in that follows $g_i(e_1, \alpha_i, \delta)$ will be noted as g_i. Taking the same parameter values α_i for all

[a)] Corresponding author. Email: ccerazoo@unal.edu.co.

components and adding and subtracting the term $\beta_i e_1$ to the right side of Eq. (5), Eq. (5) can be expressed as $\dot{\boldsymbol{e}} = \boldsymbol{A}e + \boldsymbol{b}u$, where $u = -\varphi(e)$ is a function of error e, and $\varphi(e)$ is defined as $\varphi(e) = g_1 - e_1$, and $\boldsymbol{A}$ and $\boldsymbol{b}$ are matrix given by

$$\boldsymbol{A} = \begin{bmatrix} -\beta_1 & 1 & 0 & \cdots & 0 \\ -\beta_2 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ -\beta_n & 0 & \cdots & 0 & 1 \\ -\beta_{n+1} & 0 & \cdots & 0 & 0 \end{bmatrix},$$

$$\boldsymbol{b} = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \\ \beta_{n+1} \end{pmatrix}. \tag{6}$$

Remark 1: β_i must be computed such that $\boldsymbol{A}$ is Hurwitz and the par $(\boldsymbol{A}, \boldsymbol{b})$ is controllable.

With the aim to apply the Popov criterion, it is necessary that $\varphi(e)$ satisfies the sector condition. It is said that the function $\varphi(e)$ belongs to sector $[0\ k]$ if

$$\varphi(e)[ke - \varphi(e)] \geq 0, \quad \forall\ t \in R_+ \text{ and } \forall\ e \in \varGamma \subset R, \tag{7}$$

where $\varGamma$ is the region of the error which guarantees the stability of the observer. In this case, condition is fulfilled if $0 < \delta < 1$ and $k \geq 1/\delta^{1-\alpha} - 1$. Then, δ is taken from this range. According to the Popov criterion, the system (5) is absolutely stable if there is $\eta \geq 0$ (not corresponding to any eigenvalue of $\boldsymbol{A}$) such that

$$\frac{1}{k} + \mathrm{Re}[G(\mathrm{j}w)] - \eta w \mathrm{Im}[G(\mathrm{j}w)] > 0,\ \forall w \in R \tag{8}$$

with a suitable selection of parameters (α_i, δ) and $\beta_i' s$, the ESO will have asymptotic stability.

Consider the following dynamical system

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= f(x_1, x_2, w) + \boldsymbol{b}u, \end{aligned} \tag{9}$$

and $y = x_1$, where w is an external disturbance, j$\boldsymbol{b}$ is a constant value and $f(\cdot)$ is the dynamics of the system which is unknown. Assuming that $f(\cdot)$ is differentiable, the extended state observer is defined as

$$\begin{aligned} \dot{z}_1 &= z_2 + \beta_1 g_1(e_1), \\ \dot{z}_2 &= z_3 + \beta_2 g_2(e_1) + \boldsymbol{b}u, \\ \dot{z}_3 &= \beta_3 g_3(e_1), \end{aligned} \tag{10}$$

and $\hat{y} = z_1$, where z_3 is the estimate of the uncertain function $f(\cdot)$ and $e_1 = y - \hat{y}$. The observer estimation error is defined as

$$\begin{aligned} \dot{e}_1 &= e_2 - \beta_1 g_1(e_1), \\ \dot{e}_2 &= e_3 - \beta_2 g_2(e_1), \\ \dot{e}_3 &= -\beta_3 g_3(e_1), \end{aligned} \tag{11}$$

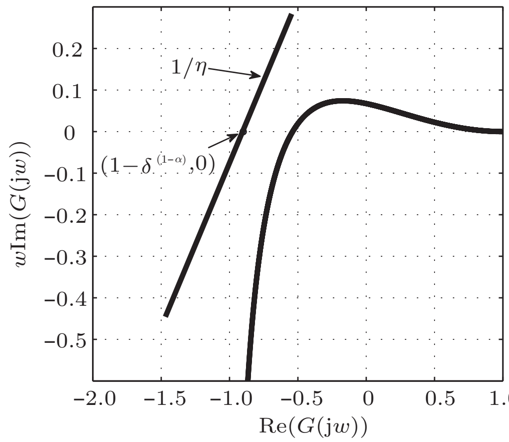

Fig. 1. Graphical interpretation of 3rd order observer stability.

and $\tilde{y} = e_1$. Now, according to the above methodology, system (11) can be rewritten as

$$\begin{bmatrix} \dot{e}_1 \\ \dot{e}_2 \\ \dot{e}_3 \end{bmatrix} = \begin{bmatrix} -\beta_1 & 1 & 0 \\ -\beta_2 & 0 & 1 \\ -\beta_3 & 0 & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} + \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} u,$$

$$\tilde{y} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}, \tag{12}$$

where $u = -\varphi_1(e_1)$ and $\varphi_1(e_1) = g_1 - e_1$. The system (11) is absolutely stable if

$$\delta^{1-\alpha} + \frac{\beta_1 w^4 - \beta_2 w^4 + \beta_2 w^2 - 2\beta_1\beta_3 w^2 + \beta_3^2}{(\beta_3 - \beta_1 w^2)^2 + (\beta_2 w - w^3)^2} - \eta w \frac{-\beta_1 w^5 + \beta_3 w^3}{(\beta_3 - \beta_1 w^2)^2 + (\beta_2 w - w^3)^2} > 0, \tag{13}$$

for the sake of simplicity the pole placement method can be used for the initial design of this observer. In this case, suppose the poles of the observer on $(S+8)^3$, the linear gains are $\beta_1 = 24$, $\beta_2 = 192$, and $\beta_3 = 512$, and the observe (10) will be globally asymptotically stable if $\delta^{1-\alpha} > 0.947\,5$. Taking $\alpha = 0.5$ and $\delta = 0.9$ absolute stability is obtained. Figure 1 shows a graphical interpretation. The plot of $\mathrm{Re}[(G(\mathrm{j}w))]$ vs. $w\mathrm{Im}[(G(\mathrm{j}w))]$ is to the right of the line with slope $1/\eta$ that intercepts the point $(-1/k, 0)$ (see Ref. 8, for a complete explanation). The limit 0.947 5 was found as it is explained below. (1) For simplicity we assume $\eta = 0$; (2) The derivative of $\mathrm{Re}[G(\mathrm{j}w)]$ is computed and equaled to zero; (3) It is verified that the condition given by Eq. (8) is satisfied for values obtained in the above (2); (4) Values of δ and α are computed.

Now, taking into account the ESO abilities to estimate nonlinear functions, a theorem for the estimation of nonlinear functions of a dynamical system is proposed. The number of nonlinear functions to estimate depends on the number of available outputs. A similar procedure to the previous case can be developed to prove the stability, using the multivariable Popov criterion. In this paper, the stability is proven when Luenberguer observer is used.

Theorem: Consider an nth order nonlinear dynamical system given by

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}u + \boldsymbol{E}f(\boldsymbol{x}), \\ \boldsymbol{y} &= \boldsymbol{C}\boldsymbol{x}, \end{aligned} \tag{14}$$

where $\boldsymbol{x} \in \boldsymbol{R}^n$ is the state vector, $\boldsymbol{y} \in \boldsymbol{R}^m$ is the output vector, $u \in \boldsymbol{R}$ is the signal control, $f(\boldsymbol{x}) : \boldsymbol{R}^n \to \boldsymbol{R}^m$ corresponds to the set of nonlinear functions, which are going to estimate, $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ and $\boldsymbol{E}$ are $n \times n$, $n \times 1$, $m \times n$, and $n \times m$ matrices, respectively. Therefore in this system it can be estimated m nonlinear functions given by $f(\boldsymbol{x})$, if the pair $(\boldsymbol{A}, \boldsymbol{C})$ is observable, the derivative of $f(\boldsymbol{x})$ is bounded, and the observer design leads to BIBO stable error dynamics.

Proof: The first step is to augment the system order, taking the nonlinear pieces as extended states, as follows

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}u + \boldsymbol{E}\hat{\boldsymbol{x}}, \\ \dot{\hat{\boldsymbol{x}}} &= \dot{f}(\boldsymbol{x}), \end{aligned} \tag{15}$$

where $\hat{\boldsymbol{x}} \in \boldsymbol{R}^m$ is the extended state corresponding to the m nonlinear functions, which are going to estimate, $\dot{f}(\boldsymbol{x})$ are the derivatives of nonlinear functions, are unknown but bounded. Now, the extended state observer can be expressed as

$$\begin{aligned} \dot{\boldsymbol{z}} &= \boldsymbol{A}\boldsymbol{z} + \boldsymbol{B}u + \boldsymbol{E}\hat{\boldsymbol{z}} + \boldsymbol{K}\boldsymbol{C}e, \\ \dot{\hat{\boldsymbol{z}}} &= h_2 + \boldsymbol{G}\boldsymbol{C}\boldsymbol{e}, \end{aligned} \tag{16}$$

where $\boldsymbol{z}$ is the estimation of the state $\boldsymbol{x}$, $\boldsymbol{e}$ is the estimation error and it is defined as $\boldsymbol{e} = \boldsymbol{x} - \boldsymbol{z}$, $\hat{\boldsymbol{z}}$ is the estimation of nonlinear functions $f(\boldsymbol{x})$, $h_2 \in \boldsymbol{R}^m$ is a function defined by the user, $\boldsymbol{K}$ and $\boldsymbol{G}$ are $n \times m$ and $m \times m$ gain matrices. Now, the dynamics of estimation errors are obtained by subtracting Eq. (16) from Eq. (15) as follows

$$\begin{bmatrix} \dot{e} \\ \dot{\hat{e}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{A} - \boldsymbol{K}\boldsymbol{C} & \boldsymbol{E} \\ -\boldsymbol{G}\boldsymbol{C} & \boldsymbol{0}_m \end{bmatrix} \begin{bmatrix} e \\ \hat{e} \end{bmatrix} + \begin{bmatrix} \boldsymbol{0}_h \\ h \end{bmatrix}, \tag{17}$$

where $h = \dot{f}(\boldsymbol{x}) - h_2$, $\boldsymbol{0}_m$ is a $m \times m$ zero matrix, $\boldsymbol{0}_h \in \boldsymbol{R}^n$ is a zero vector and $h \in \boldsymbol{R}^m$. Since the system is observable, the nonlinear functions will have a bounded estimation error (due to elements of h) if and only if all eigenvalues of $\hat{\boldsymbol{A}}$ are placed on the open left complex half plane. Where $\hat{\boldsymbol{A}}$ is a $(n+m) \times (n+m)$ matrix and it is defined by Eq. (17). The gain terms that can be chosen arbitrarily are $nm + m^2$, which correspond to terms of $\boldsymbol{K}$ and $\boldsymbol{G}$ matrices, respectively. Now, because the system is observable, the order of characteristic polynomial of $\hat{\boldsymbol{A}}$ is $n+m$ and its zeros can be placed arbitrarily, i.e., the characteristic polynomial will have $n+m$ unknowns; therefore, the system can be solved if $m\,(n+m) \geq n+m$, which is true for $m \geq 1$. If it does not have information about $\dot{f}(\boldsymbol{x})$, a good selection of h_2 is $h_2 = 0$.

On the other hand, the exponential gain functions of the nonlinear functions observer are bounded by sliding observer gain and Luenberger observer gain, therefore, these satisfy the sector theorem and it can be used the multivariable Popov criterion to determine the absolute stability as we said previously.

The previous proposed methodology was verified through numerical simulations on a third order nonlinear dynamic system with two outputs, as it is shown below.

Consider a nonlinear system described by

$$\begin{aligned} \dot{x}_1 &= x_2 - x_1 - 2{x_1}^2 + u, \\ \dot{x}_2 &= -2x_2 + 2x_1x_2, \end{aligned} \tag{18}$$

and $y_1 = x_1$ and $y_2 = x_2$. Taking into account the methodology, the first step is to augment the order of the system, considering the nonlinear pieces as extended states

$$\begin{aligned} \dot{x}_1 &= x_2 - x_1 + x_3 + u, \\ \dot{x}_2 &= -2x_2 + x_4, \\ \dot{x}_3 &= h_1, \\ \dot{x}_4 &= h_2, \end{aligned} \tag{19}$$

where f_1 and f_2 are the nonlinear functions to be estimated. For the observer design we assume $h_{1,2} = 0$. Now, ESO for system (19) is expressed as

$$\begin{aligned} \dot{z}_1 &= z_2 - z_1 + z_3 + u + \beta_{11}g_{11}(e_1), \\ \dot{z}_2 &= -2z_2 + z_4 + \beta_{22}g_{22}(e_2), \\ \dot{z}_3 &= \beta_{31}g_{31}(e_1) + \beta_{32}g_{32}(e_2), \\ \dot{z}_4 &= \beta_{41}g_{41}(e_1) + \beta_{42}g_{42}(e_2), \end{aligned} \tag{20}$$

where z_3 and z_4 are the estimation of the nonlinear functions and the signal control u corresponds to unit step. Next, the observer error dynamics are obtained by subtracting Eq. (19) from Eq. (20)

$$\boldsymbol{A}_e = \begin{bmatrix} -1+\beta_{11} & 1 & 1 & 0 \\ 0 & -2-\beta_{22} & 0 & 1 \\ -\beta_{31} & -\beta_{32} & 0 & 0 \\ -\beta_{41} & -\beta_{42} & 0 & 0 \end{bmatrix}. \tag{21}$$

The pole placement technique is used to place the observer poles in -2 and each β_i in ESO is determined. The parameters of ESO are $\alpha_i = 1, 0.25, 0.25, 0.25, 0.25$ and $\delta = 10^{-4}$. The estimation of nonlinear pieces and their errors are displayed in percent in Figs. 2 and 3, respectively.

The error percents of the nonlinear functions converge quickly and accurately to zero. It is worth noting that as the ESO does not assume the knowledge of the system dynamics, the transient of the estimation error is high; however, in steady state it converges to zero.

Consider a nonlinear system described by

$$\begin{aligned} \dot{x}_1 &= 3x_2 - x_1 + x_3 + u, \\ \dot{x}_2 &= -4x_1 - 2x_2 + {x_3}^2 \cos x_2, \\ \dot{x}_3 &= x_1 - 3x_3 - x_1x_3, \end{aligned} \tag{22}$$

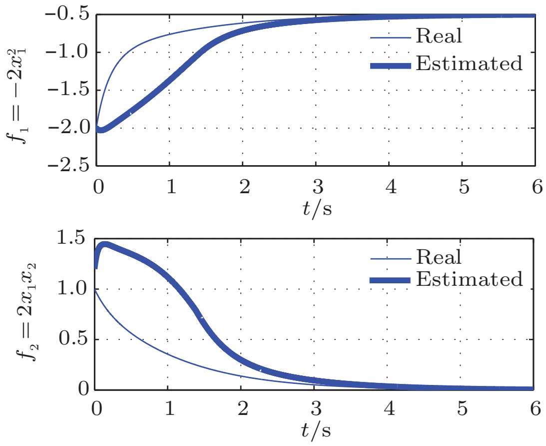

Fig. 2. Estimation of nonlinear pieces.

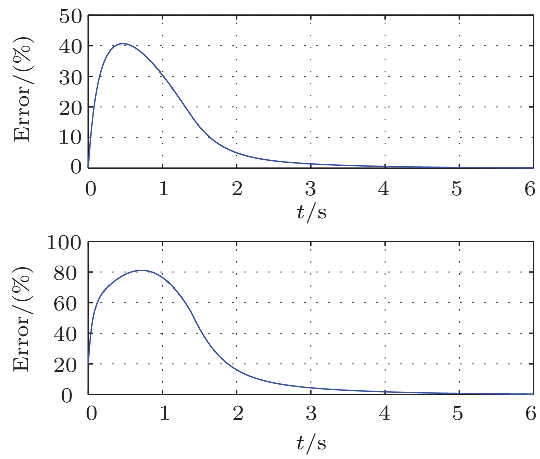

Fig. 3. Error percent of nonlinear functions.

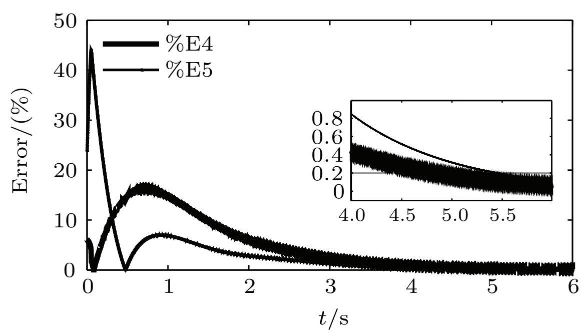

Fig. 4. Error percent of nonlinear functions.

and $y_1 = x_2$ and $y_2 = x_3$. The nonlinear parts are considered as extended states, with the nonlinear functions $f_1 = {x_3}^2 \cos x_2$ and $f_2 = -x_1x_3$ to be estimated. For the observer design we assume $h_{1,2} = 0$. Now, the ESO for the system (22) is

$$\begin{aligned}
\dot{z}_1 &= 3z_2 - z_1 + z_3 + u + \beta_{12}g_{12}(e_2) + \beta_{13}g_{13}(e_3)\\
\dot{z}_2 &= -2z_2 - 4z_1 + z_4 + \beta_{22}g_{22}(e_2),\\
\dot{z}_3 &= -3z_3 + z_1 + z_5 + \beta_{33}g_{33}(e_3),\\
\dot{z}_4 &= \beta_{42}g_{42}(e_2) + \beta_{43}g_{43}(e_3),\\
\dot{z}_5 &= \beta_{52}g_{52}(e_2) + \beta_{53}g_{53}(e_3),
\end{aligned} \tag{23}$$

where z_4 and z_5 are the estimation of the nonlinear functions and the signal control u corresponds to unit step. Following the previous procedure and applying pole placement technique for placing the poles at -2, we compute each β_i. The nonlinear functions percentage error are displayed in Fig. 4. All estimated converge quickly and accurately to zero. As the ESO does not assume the knowledge of the system dynamics the transient of the estimation error is high; however, in steady state it converges to zero.

Requirements on δ and α were found graphically to guarantee the stability of 2nd order ESO. A methodology to estimate nonlinear functions on dynamical systems based on ESO approach was proposed. The simulations carried out with different kinds of nonlinear systems have shown that the methodology is effective, assuring that the estimation of states and uncertainties can be performed in an integrated way. Nevertheless, the observer for nonlinear functions is only valid for steady state, because in the transient steady the derivative of the nonlinear functions is different from zero. Thus the estimation fails in transient part. On the other hand, for nonlinear functions whose time derivative is different from zero in steady state but bounded, the estimation error will converge to a finite value which will depend on the observer gains and the bound of nonlinear functions.

This paper was presented at the 11th Conference on Dynamical Systems—Theory and Applications, December 5-8, 2011, Lodz, Poland. This work was supported by Programa de Jovenes Investigadores e Innovadores COLCIENCIAS (DFIA-0494) and Universidad Nacional de Colombia-Manizales (12475), Vicerrectoría de Investigación, DIMA.

1. J. Han, Control and Decision **138**, 85 (1995).
2. S. E. Talole, J. P. Kolhe, and S. B. Phadke, Industrial Electronics **137**, 1411 (2010).
3. Z. Tao, W. Shu, and W. Lei, et al., Automation and Systems Engineering **137**, 558 (2009).
4. H. Chen, X. Qi, and J. Chen, et al., Information and Control **137**, 1443 (2009).
5. J. Han, and Z. Rong, J. Systems Science and Mathematical Sciences **21**, 1938 (1999).
6. Y. Huang, and J. Han, Chinese Science Bulletin, **45**, 1938 (2000).
7. H. K. Khalil, *Nonlinear Systems* (Prentice Hall, New Jersey, 1996.)
8. W. Wang, and Z. Gao, American Control Conference **6**, 4754 (2003).