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C^*-Algebra Extension Theory and Duality

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We develop a duality theory introduced by Paschke to give a simplified account of the main results of the Brown–Douglas–Fillmore extension theory and the relative K-homology theory of Baum and Douglas.

0. INTRODUCTION

The Brown–Douglas–Fillmore extension theory is concerned with C^*-algebra extensions of the form

$$0 \to \mathscr{K} \to E \to A \to 0, \tag{0.1}$$

where $\mathscr{K}$ denotes the C^*-algebra of compact operators on a separable Hilbert space. Modulo a small technical condition on (0.1), the "stable equivalence classes" of such extensions, for a fixed C^*-algebra A, may be organized into an abelian group $\mathrm{Ext}(A)$. It exhibits many remarkable homological properties [4, 8], and the purpose of this paper is to give a new treatment of the topic on the basis of an isomorphism between $\mathrm{Ext}(A)$ and the K-theory of a C^*-algebra which is "dual" to A. Our principal objective is the six term, periodic exact sequence of Ext-groups associated to a short exact sequence of C^*-algebras, which will be derived from the six term exact sequence in K-theory, but we shall also take a look at the homotopy invariance of $\mathrm{Ext}(A)$ from this point of view. In addition the dual C^*-algebras fit neatly with the theory of relative K-homology developed by Baum and Douglas [2]. We shall give a new account of their formula for the boundary map in K-homology.

The results in this note constitute Section 2 of a previous three-section article circulated by the author [7]. For no particular reason he delayed publication of that paper. Section 1 contained a long calculation in

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KK-theory which no longer seems very timely, while an account of Section 3 has in the mean time appeared elsewhere [11]. So it seemed appropriate to remove them and publish a shorter, reworked article. The author would like to express his gratitude to the referee of the previous version, whose suggestions have been incorporated here: they have greatly improved the paper. In addition the author has benefited from many discussions with John Roe on the subjects of *K*-homology and extension theory.

1. *K*-HOMOLOGY AND PASCHKE DUALITY

In this section we review Paschke's duality theory [10].[1]

We shall use the following notation: if X and Y are operators on a Hilbert space we shall write

$$X \sim Y$$

if X and Y differ by a compact operator.

Let A be a separable C^*-algebra. Following Kasparov [8] we define abelian groups $K^j(A)$ $(j=0, 1)$ by specifying "cycles" for each, along with an equivalence relation on cycles, and then defining $K^j(A)$ to be the sets of equivalence classes (which turn out to be abelian groups).

A cycle for $K^0(A)$ consists of a triple (Φ_0, Φ_1, F), where Φ_0 and Φ_1 are representations of A on separable Hilbert spaces $\mathscr{H}_0$ and $\mathscr{H}_1$, and $F: H_0 \to H_1$ is a bounded operator such that

$$\Phi_1(a)\, F \sim F\Phi_0(a), \qquad \Phi_0(a)(F^*F - I) \sim 0, \qquad \Phi_1(a)(FF^* - I) \sim 0, \quad (1.1)$$

for all $a \in A$. A cycle is *degenerate* if we can replace the occurrences of "$\sim$" in (1.1) with "$=$" signs. An *operator homotopy* of cycles is a family (Φ_0, Φ_1, F_t) of cycles (where the representations Φ_i are fixed) such that F_t varies norm-continuously with t. We generate an equivalence relation from the relations of unitary equivalence (in the obvious sense), homotopy, and direct sum (in the obvious sense) of degenerate cycles. See [3] for more details.

A cycle for $K^1(A)$ consists of a pair (Φ, F), where Φ is a representation of A on separable Hilbert space $\mathscr{H}$ and F is a bounded *self-adjoint* operator on $\mathscr{H}$ such that

$$\Phi(a)\, F \sim F\Phi(a) \qquad \text{and} \qquad \Phi(a)(F^2 - I) \sim 0, \quad \text{for all } a \in A.$$

[1] We should point out that our definitions are, in various details, a little different from Paschke's.

We define degeneracy and homotopy as for $K^0(A)$ and once again generate an equivalence relation from the relations of unitary equivalence, homotopy and direct sum of degenerate cycles. Once again, the reader unfamiliar with all this is referred to [3].

The group $K^1(A)$ identifies with the abelian group obtained by dividing the semigroup of unitary equivalence classes of extensions

$$0 \to \mathcal{H} \to E \to A \to 0$$

admitting a completely positive section $A \to E$ by the subsemigroup compromised of extensions admitting a section $A \to E$ which is a $*$-homomorphism. See [3, 9].

1.1. Definition. Let Φ be a representation of A on a separable Hilbert space $\mathcal{H}$. Denote by $D_\Phi(A)$ the *essential commutant* of $\Phi[A]$ in $\mathcal{B}(\mathcal{H})$. Thus

$$D_\Phi(A) = \{x \in \mathcal{B}(\mathcal{H}) \mid \forall a \in A, [\Phi(a), x] \sim 0\}.$$

If J is an ideal in A then we define

$$D_\Phi(A, J) = \{x \in D_\Phi(A) \mid \forall j \in J, \Phi(j)\, x \sim 0 \sim x\Phi(j)\}$$

(note that this is an ideal in $D_\Phi(A)$).

Given two representation Φ and Φ' of A, on separable Hilbert spaces $\mathcal{H}$ and $\mathcal{H}'$, let us write $\Phi \lesssim \Phi'$ if there is an isometry $V\colon \mathcal{H} \to \mathcal{H}'$ such that

$$\Phi'(a)\, V \sim V\Phi(a), \qquad \text{for all} \quad a \in A. \tag{1.2}$$

Given such an isometry, the map $\mathrm{Ad}(V)\colon x \mapsto VxV^*$ is a $*$-homomorphism from $D_\Phi(A)$ to $D_{\Phi'}(A)$. It maps $D_\Phi(A, J)$ into $D_{\Phi'}(A, J)$, for any ideal J in A.

1.2. Lemma. *If V_1 and V_2 are two isometries satisfying* (1.2) *then* $\mathrm{Ad}(V_1)$ *and* $\mathrm{Ad}(V_2)$ *induce the same map on the K-theory groups of $D_\Phi(A)$ and $D_\Phi(A, J)$.*

Proof. The maps

$$x \mapsto \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad x \mapsto \begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix},$$

taking $D_{\Phi'}(A)$ into $M_2(D_{\Phi'}(A))$, induce the same isomorphism on K-theory. So to prove the lemma for $D_\Phi(A)$ it suffices to show that the maps

$$x \mapsto \begin{pmatrix} V_1 x V_1^* & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad x \mapsto \begin{pmatrix} 0 & 0 \\ 0 & V_2 x V_2^* \end{pmatrix},$$

which take $D_\Phi(A)$ to $M_2(D_{\Phi'}(A))$, induce the same map on K-theory. But the second is obtained from the first by conjugating with the unitary

$$\begin{pmatrix} I - V_1V_1^* & V_1V_2^* \\ V_2V_1^* & I - V_2V_2^* \end{pmatrix},$$

which is an element of $M_2(D_{\Phi'}(A))$. Now use the fact that inner automorphisms act trivially on K-theory to prove the result for $K_*(D_\Phi(A))$. Exactly the same argument works for $K_*(D_\Phi(A, J))$, since the restrictions of inner automorphisms to ideals also act trivially on K-theory. ∎

The relation "$\lesssim$" makes the set of all representations[2] of A into a directed system. Using Lemma 1.2 we can form the direct limit

$$K_*(D(A)) \stackrel{\text{def}}{=} \varinjlim K_*(D_\Phi(A)). \tag{1.3}$$

Let us recall that this is constructed from the disjoint union of all $K_*(D_\Phi(A))$ by identifying $a \in K_*(D_\Phi(A))$ and $\mathrm{Ad}(V)_*(a) \in K_*(D_{\Phi'}(A))$, where $\Phi \lesssim \Phi'$ and $V: \mathscr{H} \to \mathscr{H}'$ is an isometry satisfying (1.2). Our notation for the direct limit, as the K-theory of some C^*-algebra $D(A)$, should not cause any confusion. It is in any event justified by the following well known result of Voiculescu [1, 14].

1.3. THEOREM. *Let Φ be a faithful representation of A whose image contains no compact operator. If A has a unit suppose that $\Phi(1) \neq I$. If Φ' is any representation of A at all (on a separable Hilbert space) then $\Phi' \lesssim \Phi$.*

For convenience, let us call a representation *admissible* if it satisfies the hypotheses of Voiculescu's theorem. The theorem implies that if Φ is admissible then the natural map of $K_*(D_\Phi(A))$ into $K_*(D(A))$ is an isomorphism. Although it is not absolutely necessary to do so, we shall for convenience use this result at several points below.

Let us define groups

$$K_*(D(A, J)) \stackrel{\text{def}}{=} \varinjlim K_*(D_\Phi(A, J))$$

and

$$K_*(D(A)/D(A, J)) \stackrel{\text{def}}{=} \varinjlim K_*(D_\Phi(A)/D_\Phi(A, J))$$

[2] To avoid possible set-theoretic difficulties we might limit ourselves to consideration of all representations of A on a fixed Hilbert space.

(by Voiculescu's theorem we can replace the direct limits with $K_*(D_\Phi(A, J))$ and $K_*(D_\Phi(A)/D_\Phi(A, J))$, where Φ is admissible). They organize themselves into a six term exact sequence

$$\begin{array}{ccccc} K_0(D(A,J)) & \longrightarrow & K_0(D(A)) & \longrightarrow & K_0(D(A)/D(A,J)) \\ \uparrow & & & & \downarrow \\ K_1(D(A)/D(A,J)) & \longleftarrow & K_1(D(A)) & \longleftarrow & K_1(D(A,J)). \end{array} \tag{1.4}$$

We shall study this carefully in the next section.

1.4. Definition. Define a homomorphism

$$\alpha_\Phi\colon K_i(D_\Phi(A)) \to K^{i+1}(A)$$

for $i = 0, 1 \pmod 2$ as follows.

For $i = 0$, given a projection $P \in D_\Phi(A)$ (or in some matrix algebra over $D_\Phi(A)$) associate to $[P] \in K_0(D_\Phi(A))$ the class $\alpha_\Phi[P] \in K^1(A)$ given by the cycle $(\Phi, 2P - 1)$.

For $i = 1$, given a unitary $U \in D_\Phi(A)$ (or again in some matrix algebra over $D_\Phi(A)$), associate to $[U] \in K_1(D_\Phi(A))$ the class $\alpha_\Phi[U] \in K^0(A)$ given by the cycle (Φ, Φ, U).

The definition of α_Φ is compatible with the direct limit in (1.3), and we obtain homomorphisms

$$\alpha_A\colon K_i(D(A)) \to K^{i+1}(A), \qquad i = 0, 1.$$

1.5. Theorem. (*Compare* [10].) *The maps α_A are isomorphisms.*

The proof is simply a matter of defining a map $\beta_A\colon K^i(A) \to K_{i-1}(D(A))$ and then using the equivalence relations in the definitions of K-theory and K-homology to show that α_A and β_A are inverse to each other. The following fact simplifies matters somewhat.

1.6. Lemma. *The natural map $K_*(D(A)) \to K_*(D(A)/D(A, A))$ is an isomorphism.*

Proof. In view of the long exact sequence it suffices to show that $K_*(D(A, A)) = 0$. Given a representation Φ of A, form the zero representation of A on the same Hilbert space. It suffices to show that the inclusion $D_\Phi(A, A) \to D_{\Phi\oplus 0}(A, A)$ induces the zero map on K-theory. The maps

$$X \mapsto R_t \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} R_t^*, \qquad 0 \leqslant t \leqslant \pi/2,$$

where

$$R_t = \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix},$$

form a homotopy from the given inclusion to a map into the subalgebra $D_0(A, A) \subseteq D_{\Phi \oplus 0}(A, A)$. Since $D_0(A, A) = \mathscr{B}(\mathscr{H})$ and since $K_*(\mathscr{B}(\mathscr{H})) = 0$ the result follows. ∎

Returning to the proof of Theorem 1.5, it suffices to define maps

$$\beta_A: K^i(A) \to K_{i+1}(D(A)/D(A, A)).$$

In the case $i = 1$, given a cycle (Φ, F) for $K^1(A)$, the operator $\frac{1}{2}(F+1)$ is an element of $D_\Phi(A)$ which is a projection, modulo $D_\Phi(A, A)$. We define

$$\beta_A[\Phi, F] = [\tfrac{1}{2}(F+1)].$$

In the case $i = 1$, given a cycle (Φ_0, Φ_1, F) for $K^0(A)$, if $\Phi_0 = \Phi_1$ then we define

$$\beta_A[\Phi_0, \Phi_1, F] = [F]$$

(we observe that F is a unitary element of $D_{\Phi_0}(A)$, modulo $D_{\Phi_0}(A, A)$). In the general case, construct first the equivalent cycle $(\Psi, \Psi, \bar{F})$ compromised of the natural "diagonal" representation Ψ of A on the Hilbert space

$$\mathscr{H} = \cdots \oplus \mathscr{H}_0 \oplus \mathscr{H}_0 \oplus \mathscr{H}_0 \oplus \mathscr{H}_1 \oplus \mathscr{H}_1 \oplus \cdots, \tag{1.5}$$

and the operator $\bar{F}$ shifting the summands to the right, and mapping $\mathscr{H}_0$ into $\mathscr{H}_1$ via F. We then define

$$\beta_A[\Phi_0, \Phi_1, F] = [\bar{F}].$$

Given these definitions, it is straightforward to check that β_A is well defined and inverse to α_A.

Let $f: B \to A$ be a $*$-homomorphism of separable C^*-algebras. If Φ is a representation of A then of course $\Phi \circ f$ is a representation of B. There is an inclusion $D_\Phi(A) \subseteq D_{\Phi \circ f}(B)$. It induces a map on K-theory groups and passing to direct limits we obtain a map

$$D(f)_*: K_*(D(A)) \to K_*(D(B)).$$

This definition makes $A \mapsto K_*(D(A))$ into a contravariant functor.

1.7. LEMMA. *The diagram*

$$\begin{array}{ccc} K_*(D(A)) & \xrightarrow{D(f)} & K_*(D(B)) \\ \downarrow{\scriptstyle \alpha_A} & & \downarrow{\scriptstyle \alpha_B} \\ K^{*+1}(A) & \xrightarrow{f^*} & K^{*+1}(B) \end{array}$$

commutes.

Proof. This follows immediately from the definitions. ∎

2. EXCISION

Our objective is to derive from the long exact sequence (1.4) the following result.

2.1. THEOREM. *Corresponding to any short exact sequence*

$$0 \longrightarrow J \xrightarrow{j} A \xrightarrow{p} A/J \longrightarrow 0$$

which admits a completely positive and contractive section $s\colon A/J \to A$ there is a functorial six term exact sequence

$$\begin{array}{ccccc} K^0(A/J) & \xrightarrow{p^*} & K^0(A) & \xleftarrow{j^*} & K^0(J) \\ \uparrow{\scriptstyle \partial_1} & & & & \downarrow{\scriptstyle \partial_0} \\ K^1(J) & \xleftarrow{j^*} & K^1(A) & \xleftarrow{p^*} & K^1(A/J). \end{array}$$

If Ψ is a representation of A/J then we have an inclusion

$$D_\Psi(A/J) \hookrightarrow D_{\Psi \circ p}(A, J),$$

from which we obtain a map

$$K_*(D(A/J)) \to K_*(D(A, J)). \tag{2.1}$$

2.2. LEMMA. *If the representation $p\colon A \to A/J$ admits a completely positive and contractive section $s\colon A/J \to A$ then* (2.1) *is an isomorphism.*

Proof. Let Φ be an admissible representation of A on a Hilbert space $\mathscr{H}$. By a theorem of Stinespring [12] there exists a dilation of the completely positive contraction $\Phi \circ s\colon A/J \to \mathscr{B}(\mathscr{H})$ to a representation

$$\Psi = \begin{pmatrix} \Phi_A \circ s & \Psi_{12} \\ \Psi_{21} & \Psi_{22} \end{pmatrix} : A/J \to \mathscr{B}(\mathscr{H} \oplus \mathscr{H}'),$$

where $\mathscr{H}'$ is some other Hilbert space. This is an admissible representation of A/J. Now, by Voiculescu's theorem the lemma will be proved if we can show that the composite map

$$D_{\Psi}(A/J) \to D_{\Psi \circ p}(A, J) \to D_{\Psi \circ p \oplus \Phi}(A, J) \tag{2.2}$$

induces an isomorphism on K-theory.

The inclusion of $\mathscr{H}$ into $\mathscr{H} \oplus \mathscr{H}'$ gives an inclusion

$$D_{\Phi}(A, J) \hookrightarrow D_{\Psi}(A/J). \tag{2.3}$$

Composing (2.2) with (2.3) gives the map

$$x \mapsto \begin{pmatrix} x & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

of $D_{\Phi}(A, J)$ into $D_{\Psi \circ p \oplus \Phi}(A, J)$ (note that the Hilbert space of the representation $\Psi \circ p \oplus \Phi$ is $\mathscr{H} \oplus \mathscr{H}' \oplus \mathscr{H}$; this explains the 3×3 matrix notation). Conjugating with the rotation matrices

$$\begin{pmatrix} \cos t & 0 & \sin t \\ 0 & 1 & 0 \\ -\sin t & 0 & \cos t \end{pmatrix}, \qquad 0 \leqslant t \leqslant \pi/2,$$

we obtain a homotopy to the map $D_{\Phi}(A, J) \to D_{\Psi \circ p \oplus \Phi}(A, J)$ induced from the inclusion of Φ as a subrepresentation of $\Psi \circ p \oplus \Phi$. It follows from Lemma 1.2 and Voiculescu's theorem that this is an isomorphism at the level of K-theory, which proves (2.2) is surjective at the level of K-theory.

The inclusion of $\mathscr{H} \oplus \mathscr{H}' \oplus \mathscr{H}$ into $\mathscr{H} \oplus \mathscr{H}' \oplus \mathscr{H} \oplus \mathscr{H}'$ as the first three summands gives an inclusion

$$D_{\Psi \circ p \oplus \Phi}(A, J) \hookrightarrow D_{\Psi \oplus \Psi}(A/J). \tag{2.4}$$

The composition of (2.2) with (2.4) is the map from $D_{\Psi}(A/J)$ into $D_{\Psi \oplus \Psi}(A/J)$ induced from the inclusion of Ψ as the first summand in $\Psi \oplus \Psi$. By Lemma 1.2 and Voiculescu's theorem again, this gives an isomorphism at the level of K-theory, which shows that (2.2) is injective on K-theory. ∎

The inclusion

$$D_{\Phi}(A) \hookrightarrow D_{\Phi \circ j}(J)$$

maps $D_\Phi(A, J)$ into $D_{\Phi\circ j}(J, J)$. Thus we obtain a map

$$D_\Phi(A)/D_\Phi(A, J) \to D_{\Phi\circ j}(J)/D_{\Phi\circ j}(J, J), \tag{2.5}$$

and, passing to K-theory and direct limits, a map

$$K_*(D(A)/D(A, J)) \to K_*(D(J)/D(J, J)). \tag{2.6}$$

2.3. Lemma. *The map* (2.6) *is an isomorphism.*

Proof. If Φ is an admissible representation of A then $\Phi\circ j$ is an admissible representation of J. So by Voiculescu's theorem it suffices to show that (2.5) induces an isomorphism when Φ is admissible. In fact more is true: the map (2.5) is itself an isomorphism (whether or not Φ is admissible). It is clearly injective. On the other hand, suppose $x \in D_{\Phi\circ j}(J)$. Note that

$$\Phi(J)[\Phi(A), x] \sim 0, \qquad [x, \Phi(J)] \sim 0, \qquad [\Phi(A), \Phi(J)] \subset \Phi(J).$$

It follows from the Kasparov technical theorem (see [6, 9]) that there exists a positive operator X such that

(i) $X\Phi(J) \sim 0$,

(ii) $(1 - X)[\Phi(A), x] \sim 0$,

(iii) $[x, X] \sim 0$ and $[\Phi(A), X] \sim 0$.

Since

$$[(1 - X)\, x, \Phi(a)] = (1 - X)[x, \Phi(a)] - [X, \Phi(a)]\, x$$

it follows from (ii) and (iii) that $(1 - X)\, x \in D_\Phi(A)$. In addition, it follows from (ii) that $Xx \in D_{\Phi\circ j}(J, J)$, and so the image of $(1 - X)\, x$ in $D_{\Phi\circ j}(J)/D_{\Phi\circ j}(J, J)$ coincides with the image of x. ∎

Proof of Theorem 2.1. The six term exact sequence is constructed by substituting $K_*(D(J))$ and $K_*(D(A/J))$ into (1.4), using the isomorphisms in Lemmas 2.2 and 2.3. From a commuting diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & J & \xrightarrow{j} & A & \xrightarrow{p} & A/J & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow{\scriptstyle f} & & \uparrow & & \\
0 & \longrightarrow & J' & \xrightarrow{j'} & A' & \xrightarrow{p'} & A'/J' & \longrightarrow & 0
\end{array}$$

we obtain the commuting diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & D_\Phi(A, J) & \longrightarrow & D_\Phi(A) & \longrightarrow & D_\Phi(A)/D_\Phi(A, J) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & D_{\Phi\circ f}(A', J') & \longrightarrow & D_{\Phi\circ f}(A') & \longrightarrow & D_{\Phi\circ f}(A')/D_{\Phi\circ f}(A', J') & \longrightarrow & 0.
\end{array}$$

Passing to K-theory we obtain a commuting diagram of six term exact sequences, and then passing to direct limits, and using the (functorial) isomorphisms in Lemmas 2.2 and 2.3, we get a commuting diagram of six term exact sequences in K-homology. This proves that our six term exact sequence is functorial. ∎

3. Relative K-Homology Theory

Let

$$0 \to J \to A \to A/J \to 0$$

be a short exact sequence of separable C^*-algebras, as in the previous section. Suppose that (Φ_0, Φ_1, F) is a cycle for $K^0(J)$ such that

$$F \text{ is a partial isometry} \tag{3.1}$$

and

$$\begin{aligned}&\Phi_0 \text{ and } \Phi_1 \text{ extend to representations of } A \text{ with}\\ &\Phi_1(a)\,F - F\Phi_0(a) \sim 0, \text{ for all } a \in A.\end{aligned} \tag{3.2}$$

Baum and Douglas [2] have given a useful formula for the image of the class $[\Phi_0, \Phi_1, F]$ under the K-homology boundary map

$$\partial_0 \colon K^0(J) \to K^1(A/J).$$

Let $P = I - F^*F$ and $P' = I - FF^*$. The compression of the representation Φ_0 to the range of P is a completely positive map. Compose it with the completely positive section $s\colon A/J \to A$ and dilate to a representation

$$\Psi = \begin{pmatrix} P(\Phi_0 \circ s)\,P & \Psi_{12} \\ \Psi_{21} & \Psi_{22} \end{pmatrix} : A/J \to \mathcal{B}(P\mathcal{H}_0 \oplus \mathcal{H}).$$

If we write

$$G = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

then it is easily checked that (Ψ, G) is a cycle for $K^1(A/J)$.

Starting with the compression of Φ_1 to the range of P', we build an analogous cycle (Ψ', G').

3.1. Theorem. $\partial_0[\Phi_0, \Phi_1, F] = [\Psi, G] - [\Psi', G']$.

Proof. By using the construction (1.5) if necessary, we may assume that $\Phi_0 = \Phi_1$. We shall simply write Φ for Φ_0 and Φ_1. Reviewing the previous section, we see that $\partial_0[\Phi, \Phi, F]$ is computed as follows:

(a) Associate to $[\Phi, \Phi, F]$ the dual class $[\dot{F}] \in K_1(D_\Phi(J)/D_\Phi(J, J))$ (here $\dot{F}$ denotes the image of F in the quotient $D_\Phi(J)/D_\Phi(J, J)$).

(b) Lift $[\dot{F}]$ to a class in $K_1(D_\Phi(A)/D_\Phi(A, J))$.

(c) Apply to the lifting the K-theory boundary map from $K_1(D_\Phi(A)/D_\Phi(A, J))$ to $K_0(D_\Phi(A, J))$.

(d) Lift the image to a class in $K_1(D(A/J))$ and apply Paschke duality in reverse.

In the case at hand step (b) is trivial since $\dot{F}$ already lies in $D_\Phi(A)/D_\Phi(A, J)$, thanks to (3.1). In addition, the lifting $F \in D_\Phi(A)$ of $\dot{F}$ is a partial isometry, and so a well known formula for the boundary map in K-theory (see [3.13]) tells us that

$$\partial[\dot{F}] = [P] - [P'] \in K_0(D_\Phi(A, J)).$$

If Q and Q' denote the $+1$ eigenspaces of G and G' then it is a simple matter to check that $[Q]$ and $[Q']$ map to $[P]$ and $[P']$, respectively, under the map (2.1). This completes the proof. ∎

The following is an "odd dimensional" version of this formula. The proof is quite similar to the one above and is left to the reader.

3.2. THEOREM. *Suppose that (Φ, F) is a cycle for $K^1(J)$, such that Φ is the restriction of a representation of A with $[\Phi(a), F] \sim 0$, for all $a \in A$. Then*

$$\partial_1[(\Phi, F)] = \left(\Psi, \Psi, \begin{pmatrix} e^{i\pi F} & 0 \\ 0 & 1 \end{pmatrix}\right),$$

where the representation Ψ is a dilation of $\Phi \circ s\colon A/J \to \mathscr{B}(\mathscr{H})$.

4. HOMOTOPY INVARIANCE

Our objective is the following result.[3]

[3] There is of course a similar result for $K^0(A)$, but since it can be reduced in a number of ways to the case of $K^1(A)$ we shall concentrate on Theorem 4.1.

4.1. THEOREM. *The functor K^1 is homotopy invariant, meaning that for every C*-algebra A the two maps*

$$e_0, e_1 \colon C[0,1] \otimes A \to A,$$

given by evaluation at $0 \in [0,1]$ *and* $1 \in [0,1]$, *induce the same map on* $K^1(A)$:

$$e_0^* = e_1^* \colon K^1(A) \to K^1(C[0,1] \otimes A).$$

Using the short exact sequence

$$0 \longrightarrow K^1(A) \xrightarrow{e_0^*} K^1(C[0,1] \otimes A) \longrightarrow K^1(C_0(0,1] \otimes A) \longrightarrow 0$$

(a degeneration of the six term exact sequence) the theorem reduces to showing that the map

$$e_1^* \colon K^1(A) \to K^1(C_0(0,1] \otimes A)$$

is zero, and by the six term exact sequence again it suffices to show that the boundary map

$$\partial_0 \colon K^0(C_0(0,1) \otimes A) \to K^1(A)$$

associated to the short exact sequence

$$0 \to C_0(0,1) \otimes A \to C_0(0,1] \otimes A \to \mathbb{C} \otimes A \to 0$$

is surjective. This is what we shall now prove.

4.2. LEMMA. *Let (Φ, F) be a cycle for $K^1(A)$ with $F^2 = I$ and define a $*$-homomorphism from A into $\mathscr{B}(\mathscr{H})/\mathscr{K}(\mathscr{H})$ by the formula $a \mapsto \frac{1}{2}(F+I)\,\Phi(a)$. Use this map to pull back the natural extension of $\mathscr{B}(\mathscr{H})/\mathscr{K}(\mathscr{H})$ to a short exact sequence*

$$0 \to \mathscr{K} \to E \to A \to 0.$$

The associated boundary map $\partial_0 \colon K^0(\mathscr{K}(\mathscr{H})) \to K^1(A)$ takes the generator of $K^0(\mathscr{K}(\mathscr{H})) \cong \mathbb{Z}$ to $[\Phi, F]$.

Proof. The generator of $K^0(\mathscr{K}(\mathscr{H}))$ is given by the cycle ($\phi_0 = \mathrm{id}$, $\phi_1 = 0$, $F = 0$). This cycle is of the special form considered in Theorem 3.1, and the formula there for ∂_0 gives the result. ∎

4.3. LEMMA. *Suppose given two C*-algebra extensions*

$$0 \to J \to A \to A/J \to 0$$

and

$$0 \to J' \to A' \to A'/J' \to 0.$$

Form new extensions by taking tensor products of each of the above with the constituent C^*-algebras in the other. The diagram of connecting homomorphisms

$$\begin{array}{ccc} K^1(J \otimes J') & \xrightarrow{\partial_1} & K^0(A/J \otimes J') \\ {\scriptstyle \partial_1'} \downarrow & & \downarrow {\scriptstyle \partial_0} \\ K^0(J \otimes A'/J') & \xrightarrow{\partial_0'} & K^1(A/J \otimes A'/J'), \end{array}$$

so obtained is anti-commutative.

Proof. It follows from the existence of completely positive sections that the subalgebra

$$J \otimes A' + A \otimes J' \subset A \otimes A'$$

is a C^*-algebra. So consider the commuting diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & J \otimes J' & \longrightarrow & B & \longrightarrow & J \otimes A' + A \otimes J' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & J \otimes J' & \longrightarrow & J \otimes A' + A \otimes J' & \longrightarrow & A/J \otimes J' \oplus J \otimes A'/J' & \longrightarrow & 0, \end{array}$$

where B is the pull-back C^*-algebra. Consider also the commuting diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & J \otimes A' + A \otimes J' & \longrightarrow & A \otimes A' & \longrightarrow & A/J \otimes A'/J' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A/J \otimes J' \oplus J \otimes A'/J' & \longrightarrow & C & \longrightarrow & A/J \otimes A'/J' & \longrightarrow & 0, \end{array}$$

where

$$C = \{x \oplus y \in A \otimes A'/J' \oplus A/J \otimes J' \mid x = y \quad \text{in } A/J \otimes A'/J'\}.$$

Denote by

$$\Delta_1 \colon K^1(J \otimes J') \to K^0(A/J \otimes J' \oplus J \otimes A'/J')$$

and

$$\Delta_2 \colon K^0(A/J \otimes J' \oplus J \otimes A'/J') \to K^1(A/J \otimes A'/J')$$

the connecting homomorphisms in the long exact sequences associated with the bottom two extensions in the above diagrams. By functionarility, the composition $\Delta_2\Delta_1$ is equal to the composition of the connecting homomorphisms associated with the top two extensions in our diagrams. Since the first of these is split, its connecting homomorphism is zero, and so

$$\Delta_2\Delta_1 = 0.$$

We shall deduce the lemma from this equation. Denote by

$$A/J\otimes J' \xrightarrow{\sigma} A/J\otimes J' \oplus J\otimes A'/J' \xrightarrow{\pi} A/J\otimes J'$$

the natural inclusion and projection, and define σ' and π' similarly. By functoriality of boundary maps again, we have that

$$\partial_1 = \sigma^*\Delta_1, \quad \partial'_1 = \sigma'^*\Delta_1, \quad \partial'_0 = \Delta_2\pi^*, \quad \text{and} \quad \partial_0 = \Delta_2\pi'^*.$$

Hence

$$\partial_3\partial_1 + \partial_4\partial_2 = \Delta_2(\pi^*\sigma^* + \pi'^*\sigma'^*)\,\Delta_1.$$

But

$$\pi^*\sigma^* + \pi'^*\sigma'^* = \mathrm{id}_{K_0(A/J\otimes J'\oplus A'/J')},$$

which proves the lemma. ∎

Proof of Theorem 4.1. We want show that for any A, the boundary map $\partial_0\colon K^0(\mathbb{R})\otimes A)\to K^1(A)$ associated with the extension

$$0\to C_0(\mathbb{R})\otimes A\to C_0(-\infty,\infty]\otimes A\to A\to 0$$

is surjective. Let $[\Phi, F]\in K^1(A)$. By passing to an equivalent cycle if necessary we may assume that $F^2 = I$. Let

$$0\to \mathcal{K}\to E\to A\to 0$$

be the extension in Lemma 4.2. By Lemma 4.3 the diagram of boundary maps

$$\begin{array}{ccc} K^1(C_0(\mathbb{R})\otimes\mathcal{K}) & \longrightarrow & K^0(\mathcal{K}) \\ \downarrow & & \downarrow \\ K^0(C_0(\mathbb{R})\otimes A) & \longrightarrow & K^1(A) \end{array}$$

anticommutes. Since $[\Phi, F]$ is in the image of the right hand vertical map, it suffices to show that $\partial\colon K^1(C_0(\mathbb{R})\otimes\mathcal{K})\to K^0(\mathcal{K})$ is surjective. Consider the cycle (ϕ, F) for $K^1(C_0(\mathbb{R})) \cong K^1(C_0(\mathbb{R}) \otimes \mathcal{K})$ consisting of the standard representation of $C_0(\mathbb{R})$ on $L^2(\mathbb{R})$ and the operator F whose Fourier transform is pointwise multiplication by the function $\xi(1+\xi^2)^{-1/2}$. It is easily verified that this is a cycle of the sort considered in Theorem 3.1, and a standard index computation (compare [5]) shows that $\partial[(\phi, F)]$ is the generator of $K^0(\mathbb{C})$. ∎

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