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NOTE

On Number of Circles Intersected by a Line[1]

Lu Yang and Jingzhong Zhang

Laboratory of Automatic Reasoning, CICA, Academia Sinica Chengdu, Sichuan 610041, People's Republic of China

and

Weinian Zhang[2]

Department of Mathematics, Sichuan University Chengdu, Sichuan 610064, People's Republic of China

Consider a set U of circles in the plane such that any line intersects at least one of those circles. For a given natural number m, is there a line in the plane intersecting at least m circles in U? In this paper this problem is solved. Our result is also generalized to compact convex subsets and to higher dimensional cases. © 2002 Elsevier Science (USA)

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1. INTRODUCTION

In combinatorial geometry the Sylvester-type problems are very important and attractive. A century ago Sylvester [6] posed a question: For a finite set of points in the plane such that the line through any two of them passes through a third point of the set, must all the points lie on one line? Later it resurfaced as a conjecture by Paul Erdös [2]: If a finite set of points in the plane is not on one line then there is a line through exactly two of the points. Since then there has appeared a substantial literature (seen in [1]) on the problem and its generalizations. For example, in [5] Motzkin considered n points in the plane, not all on a line and not all on a circle, and showed that there is either a circle or a line containing exactly

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[2] To whom correspondence should be addressed. E-mail: wnzhang@scu.edu.cn

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three of the points. Herzog and Kelly [4] also proved that for given n pairwise disjoint compact sets in $\mathbf{R}^d$, which are not all contained in a line and at least one of which contains infinitely many points, there is a line intersecting exactly two of them.

Related to the above mentioned, another problem says the following: *For a set U of some unit circles in the plane $\mathbf{R}^2$ such that any line in $\mathbf{R}^2$ intersects at least one of those circles, given natural number m, is there a line in $\mathbf{R}^2$ which intersects at least m circles in U?* Although it has been proposed for a long time and known extensively, no published answer is found yet. It is too hard to search where this was stated originally but one version states that it was once raised in a personal letter of P. Erdös to Y. Q. Yin. A closer result is Proposition 93 in [3], which tells that if a collection of mutually congruent convex bodies is not "extremely sparsely distributed" then for any natural number m there is a line which intersects more than m bodies of the family. However, it does not give a full answer to the problem because it requires a different assumption that the collection of circles are not extremely sparsely distributed. No matter where it comes from, this problem is interesting and has been puzzling us since we heard of it.

In this paper this problem is solved by reducing to divergent series, the same idea as used for Proposition 93 in [3]. We state the main result in next section. By lemmas given in Section 3, we prove the result in Section 4. The basic result is generalized in Section 5 to compact convex subsets and to higher dimensional cases. For convenience, let $\mathbf{R}^2$ and $\mathbf{N}$ denote the plane (2-dimensional Euclidean space) and the set of natural numbers, respectively. Let $C(Q, r)$ denote the circle of radius r centered at Q. Let $|PQ|$ represent the distance between P and Q in Euclidean spaces if P, Q stand for points and $|B|$ represents the area of B if B stands for a set.

2. MAIN RESULT

THEOREM 1. *Let U be a set of circles in the plane $\mathbf{R}^2$ such that any line in $\mathbf{R}^2$ intersects at least a circle in U. Then for any $m \in \mathbf{N}$ and any point $P \in \mathbf{R}^2$ there exists a line in $\mathbf{R}^2$ through P intersecting at least m circles in U.*

This result is stronger than the original problem hoped since we do not require the circles of U to be unitary. Moreover, different from [3] we do not require the collection of circles to be congruent. Consider $U = \{C(O, 2^k) : k \in \mathbf{N}\}$. As in [3], let $\mathcal{N}(R)$ denote the number of circles in U which lie entirely inside the disk of radius R about the reference point O. Then $\liminf_{R \to \infty} \mathcal{N}(R)/R = 0$. Therefore U is extremely sparsely distributed and as in the case of Proposition 93 in [3] it cannot give an answer to our problem. However our theorem works in this case; actually, each line through a point $P \in \mathbf{R}^2$ intersects infinitely many circles in this U.

For another example, consider U to be a set U consisting of a circle centered at the origin and some small circles along the hyperbolas $x^2 - y^2 = \pm 1$ such that along the same branch of the hyperbolas each circle is tangent to its consecutive two circles; two circles respectively along different branchs do not intersect, and no circle along a branch intersects the circle centered at the origin. Such U satisfies the condition of Theorem 1. Notice that no line in the plane intersects infinitely many circles in such a U.

Solving this problem would be easier if the set U contains more circles. We mainly prove Theorem 1 when U is countable. The case of uncountable U is simple and its proof is a standard argument. In the following, we suppose that U is countable; i.e., $U = \{C(A_k, r_k) : A_k \in \mathbf{R}^2, r_k > 0, k = 1, 2, ...\}$. Let $d_k = |A_k O|$, where O is the origin of $\mathbf{R}^2$. We only need to consider the case that

$$r_k / d_k < 1, \qquad \forall k \geqslant k_0, \tag{2.1}$$

for some $k_0 > 0$; otherwise there is a subsequence k_i such that $r_{k_i} / d_{k_i} \geqslant 1$ $(i = 1, 2, ...)$ and thus Theorem 1 holds naturally because every line through O intersects all circles $C(Q_{k_i}, r_{k_i})$, $i = 1, 2, ...$

Lemma 1. *Suppose* (2.1) *holds. If* $\sum_{k=k_0}^{+\infty} r_k / d_k$ *diverges, then for any* $m \in \mathbf{N}$ *there is a line through* O *intersecting at least* m *circles in* U.

Proof. When $r_k / d_k < 1$, the origin O is outside the circle $C(A_k, r_k)$. Let ϕ_k denote the scope-angle of O to $C(A_k, r_k)$, namely, the angle between the two tangents from O to $C(A_k, r_k)$. Clearly, $\phi_k = 2 \arcsin(r_k / d_k) > 2r_k / d_k$, so $\sum_{k=k_0}^{\infty} \phi_k$ also diverges. Thus for any natural number m there exists a natural number N such that $\sum_{k=k_0}^{N} \phi_k > 2m\pi$. By the drawer principle, there is at least a line through O intersecting at least m circles in U. ▌

Lemma 1 gives a way to reduce our problem to divergence of a series.

3. SOME LEMMAS

For a line l in $\mathbf{R}^2$, let F_l denote the intersection point of l with its normal through O. Obviously, $F_l = 0$ if l is through O. F_l is unique and $F_l \neq F_{l'}$ if neither l nor l' is through O and $l \neq l'$. Let

$$\Omega_k := \{F_l : l \cap C(A_k, r_k) \neq \varnothing\}. \tag{3.2}$$

Under the condition in Theorem 1,

$$\mathbf{R}^2 = \bigcup_{k=1}^{\infty} \Omega_k. \tag{3.3}$$

In fact, for each $P \in \mathbf{R}^2$ there is a line l such that $F_l = P$. It is assumed in Theorem 1 that l intersects a circle in U, so $F_l \in \Omega_k$ for some $k \in \mathbf{N}$.

We still need more geometric properties of Ω_k. Let S_k be the circle of diameter $|OA_k|$ through 0 and A_k and let Q be the center of S_k. Then $S_k \subset \Omega_k$ since $S_k = \{F_l\colon A_k \in l\}$. For $A' \in S_k$, let

$$\theta(A') := \angle A'A_kO \tag{3.4}$$

and let $C(A')$ denote the circle of radius $r_k \sin\theta(A')$ centered at A'.

LEMMA 2. $\Omega_k \subset \bigcup_{A' \in S_k} C(A')$.

Proof. Let the line l satisfy that $l \cap C(A_k, r_k) \neq \varnothing$ and let A^* be one of two intersection points of l and $C(A_k, r_k)$ arbitrarily fixed. Take $A \in S_k$ such that the rectangular triangle ΔOAA_k is similar to ΔOF_lA^* with the same orientation, as in Fig 1. Clearly A exists uniquely. Thus $\angle A^*OF_l = \angle A_kOA$ and $|OA|/|OF_l| = |OA_k|/|OA^*|$. It follows that $\angle AOF_l = \angle A^*OF_l + \angle AOA^* = \angle A_kOA + \angle AOA^* = \angle A_kOA^*$ and $|OA|/|OA_k| = |OF_l|/|OA^*|$. Hence $\Delta OAF_l \sim \Delta OA_kA^*$ and thus

$$|AF_l| = |A_kA^*| \cdot |OA|/|OA_k| = r_k \sin\theta(A), \tag{3.5}$$

where $\theta(A) = \angle AA_kO$. Therefore, $F_l \in C(A)$. Since $A \in S_k$, we get $F_l \in \bigcup_{A' \in S_k} C(A')$. ∎

Let $|\Omega_k|$ represent the area (or out measure) of Ω_k, which clearly has its area. By Lemma 2, each Ω_k is covered by an annular region between a

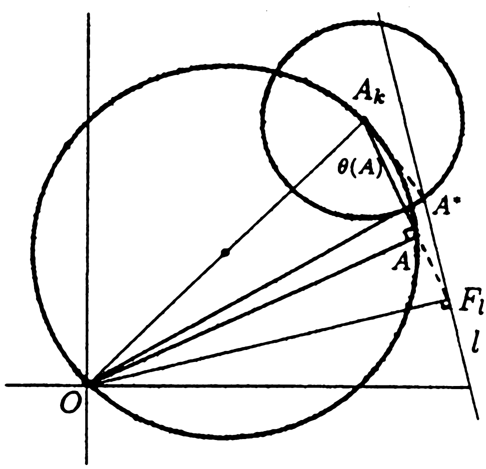

FIG. 1. $\Delta OAF_l \sim \Delta OA_kA^*$

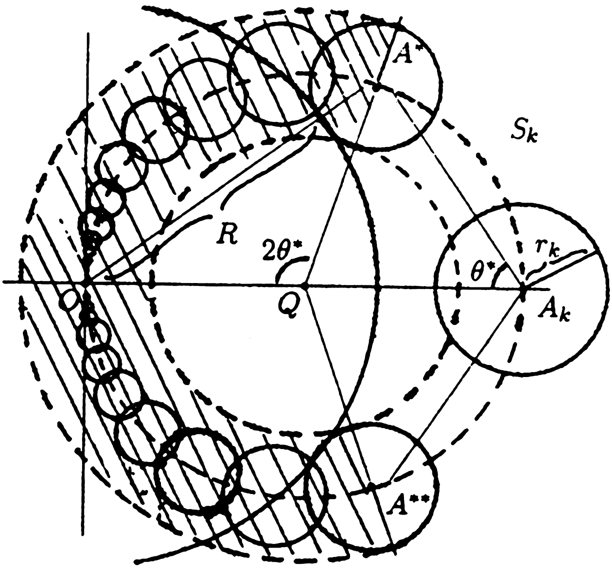

FIG. 2. What is $\Omega_{k,R}$ covered by?

circle of radius $d_k/2+r_k$ and a circle of radius $d_k/2-r_k$, which are both centered at Q. Thus

$$|\Omega_k| \leqslant \pi\left(\frac{d_k}{2}+r_k\right)^2 - \pi\left(\frac{d_k}{2}-r_k\right)^2 = 2\pi r_k\, d_k, \tag{3.6}$$

if $d_k \geqslant 2r_k$.

For a given $R>0$, let $B(O, R)$ be the open disk of radius R centered at O and let

$$\Omega_{k,R} := \Omega_k \cap \bar{B}(O, R), \qquad k=1, 2, ..., \tag{3.7}$$

where $\bar{B}(O, R)$ is the closure of $B(O, R)$ (see Fig. 2).

Lemma 3. *If $d_k \geqslant \max\{R+r_k, 2r_k\}$, then $|\Omega_{k,R}| \leqslant 8\pi R^2 r_k/d_k$.*

Proof. For $A \in S_k$, $C(A) \cap B(O, R) = \varnothing$ if and only if

$$|OA| > R + r_k \sin\theta(A). \tag{3.8}$$

Note that $|OA|$ varies continuously from 0 to $\mathrm{d_k}$ when A goes from O to A_k. Thus we can take $A^* \in S_k$ appropriately such that

$$|OA^*| = R + r_k \sin\theta(A^*); \tag{3.9}$$

i.e., $C(A^*)$ is externally tangent to $B(O, R)$. Substituting $|OA| = d_k \sin\theta(A)$ in (3.8) and (3.9) separately we obtain

$$\sin\theta(A) > \frac{R}{d_k - r_k} = \sin\theta(A^*). \tag{3.10}$$

Hence $C(A) \cap \Omega_{k,R} = \varnothing$ when $\sin\theta(A) > \sin\theta(A^*)$. Furthermore, let A^{**} be a symmetric point of A^* to the line OA_k. Then in the angular region centered at Q, the middle of OA_k, and faced by the arc $A^*\widehat{A_k}A^{**}$ there is no point of $\Omega_{k,R}$ because the distances between Q and points on $C(A)$ are greater than $d_k/2 - r_k \sin\theta(A^*)$ when $\sin\theta(A) < \sin\theta(A^*)$. Therefore, $\Omega_{k,R}$ is covered by both the closed annulus of width $2r_k \sin\theta(A^*)$ along S_k and the angular region of angle $4\theta(A^*)$ centered at Q and faced by the arc $A^*\hat{O}A^{**}$. Notice that Q is the center of S_k. Applying the known inequalities $d_k - r_k \geqslant d_k/2$ and $|\theta| \leqslant |\frac{\pi}{2}\sin\theta|$ for $|\theta| \leqslant \frac{\pi}{2}$ and applying (3.10), we obtain

$$\begin{aligned}
|\Omega_{k,R}| &\leqslant 4\theta(A^*)\cdot\frac{d_k}{2}\cdot 2r_k \sin\theta(A^*) \\
&\leqslant \left(4 \arcsin\frac{R}{d_k - r_k}\right)\cdot\frac{d_k}{2}\cdot 2r_k\cdot\frac{R}{d_k - r_k} \\
&\leqslant 4\cdot\frac{\pi}{2}\cdot\frac{R}{d_k - r_k}\cdot\frac{Rd_k r_k}{d_k - r_k} \leqslant \frac{8\pi R^2 r_k}{d_k}.
\end{aligned} \tag{3.11}$$

∎

4. PROOF OF THEOREM 1

If there is a subsequence $\{k_i, k_{i+1}, ...\}$ such that $r_j/d_j > 1/2$, $j = k_i, k_{i+1}, ...$, then

$$\sum_{j=1}^{\infty} \frac{r_j}{d_j} > \sum_{i=0}^{\infty} \frac{r_{k_i}}{d_{k_i}} > \frac{1}{2} + \frac{1}{2} + \cdots = \infty, \tag{4.12}$$

implying the result of Theorem 1 by Lemma 1. Thus in what follows we suppose that there exists a natural number K such that

$$\frac{r_k}{d_k} \leqslant \frac{1}{2}, \qquad \forall k > K. \tag{4.13}$$

Take $R > 0$ large enough such that

$$R > 2 \max_{0 \leqslant k \leqslant K} \{d_k + r_k\}. \tag{4.14}$$

By (3.3) we have

$$\left(\bigcup_{k=1}^{\infty} \Omega_k \right) \cap \mathscr{A} = \mathscr{A}, \tag{4.15}$$

where $\mathscr{A}$ is the annulus between two circles both centered at O and respectively of radii R and $R/2$. The definition of $\mathscr{A}$ implies that $(\bigcup_{d_k + r_k < R/2} \Omega_k) \cap \mathscr{A} = \varnothing$. Hence $(\bigcup_{d_k + r_k \geqslant R/2} \Omega_k) \cap \mathscr{A} = ((\bigcup_{d_k + r_k \geqslant R/2} \Omega_k) \cup (\bigcup_{d_k + r_k < R/2} \Omega_k)) \cap \mathscr{A} = \mathscr{A}$; that is,

$$\begin{aligned}
&\left(\bigcup_{R/2 - r_k \leqslant d_k \leqslant R + r_k} \Omega_k \right) \cup \left\{ \left(\bigcup_{d_k > R + r_k} \Omega_k \right) \cap \mathscr{A} \right\} \\
&\quad \supset \left\{ \left(\bigcup_{R/2 - r_k \leqslant d_k \leqslant R + r_k} \Omega_k \right) \cap \mathscr{A} \right\} \cup \left\{ \left(\bigcup_{d_k > R + r_k} \Omega_k \right) \cap \mathscr{A} \right\} \\
&\quad = \left(\bigcup_{R/2 - r_k \leqslant d_k} \Omega_k \right) \cap \mathscr{A} = \mathscr{A}.
\end{aligned} \tag{4.16}$$

Estimating areas for both sides of (4.16) we get

$$\sum_{R/2 - r_k \leqslant d_k \leqslant R + r_k} |\Omega_k| + \sum_{d_k > R + r_k} |\Omega_{k,R}| > \frac{3}{4} \pi R^2. \tag{4.17}$$

It follows from (3.6) and Lemma 3 that

$$\frac{2}{R^2} \sum_{R/2 - r_k \leqslant d_k \leqslant R + r_k} r_k d_k + 8 \sum_{d_k > R + r_k} \frac{r_k}{d_k} > \frac{3}{4}. \tag{4.18}$$

Now we claim that (4.18) implies the divergence of $\sum_{k=1}^{\infty} r_k / d_k$.

Assume this series converges. Then the second sum in (4.18) is arbitrarily small as R is large enough; i.e., there is a constant $M > 0$ such that for all $R > M$,

$$\frac{2}{R^2} \sum_{R/2 - r_k \leqslant d_k \leqslant R + r_k} r_k d_k > \frac{1}{2}. \tag{4.19}$$

For sufficiently large R such that

$$R > M^* := \max\{M, 2 \max_{0, \leqslant k \leqslant K} \{d_k + r_k\}\}, \tag{4.20}$$

by (4.14), all those k satisfying $d_k + r_k \geqslant R/2$ must be greater than K. From (4.13) we see that $d_k \geqslant 2r_k$ holds for all k such that $d_k + r_k \geqslant R/2$. Thus the condition $R/2 - r_k \leqslant d_k \leqslant R + r_k$ of summation in (4.19) implies that

$$\frac{d_k}{2} \leqslant d_k - r_k \leqslant R. \tag{4.21}$$

It follows that $d_k/R \leqslant 2$ and $r_k d_k/R^2 \leqslant 4r_k/d_k$. Therefore, from (4.19) we get

$$8 \sum_{R/2 - r_k \leqslant d_k \leqslant R + r_k} \frac{r_k}{d_k} \geqslant \frac{2}{R^2} \sum_{R/2 - r_k \leqslant d_k \leqslant R + r_k} r_k d_k > \frac{1}{2}, \qquad \forall\, R > M^*. \tag{4.22}$$

Clearly, for arbitrarily large K in (4.13) we can take a correspondingly large R. Thus

$$\sum_{k=K+1}^{\infty} \frac{r_k}{d_k} \geqslant \sum_{R/2 - r_k \leqslant d_k \leqslant R + r_k} \frac{r_k}{d_k} > \frac{1}{16}. \tag{4.23}$$

This contradicts to the assumption of convergence of $\sum_{k=1}^{\infty} r_k/d_k$. By Lemma 1, we obtain the conclusion of Theorem 1. ∎

5. GENERALIZATION

COROLLARY 1. *Let U and U^* be sets of circles in $\mathbf{R}^2$ and let there be a mapping $f\colon U^* \to U$ defined by $C(A^*_\alpha, r^*_\alpha) \mapsto C(A_\alpha, r_\alpha)$ such that*

$$\frac{r_\alpha + |A^*_\alpha A_\alpha|}{r^*_\alpha} < L, \tag{5.24}$$

where L is a positive constant independent of α. If any line in $\mathbf{R}^2$ intersects at least one circle in U, then for any $m \in \mathbf{N}$ and any point $P \in \mathbf{R}^2$ there exists a line in $\mathbf{R}^2$ through P intersecting at least m circles in U^*.

The proof is simple. In fact, under the assumption (5.24) the divergence of $\sum_{k=1}^{\infty} r_k/d_k$ implies the divergence of $\sum_{k=1}^{\infty} r^*_k/d^*_k$, where $d^*_k = |OA^*_k|$. Applying Corollary 1, we can generalize Theorem 1 further to compact convex subsets in the plane. We refer to the ratio between the diameter of the minimum circle containing a compact convex subset V and the diameter of the maximum circle contained in the V as the *rectangular ratio* of the V.

COROLLARY 2. *Let M consist of some compact convex subsets in $\mathbf{R}^2$ such that their rectangular ratios have a uniform upper bound $L > 0$. If any line in $\mathbf{R}^2$ intersects at least a compact convex subset in M, then for any $m \in \mathbf{N}$ and any point $P \in \mathbf{R}^2$ there exists a line in $\mathbf{R}^2$ through P intersecting at least m compact convex subsets in M.*

Proof. Let U^* consist of all those circles, each of which is the maximum circle contained in a compact convex subset in M. Similarly, let U consist of all those circles, each of which is the minimum circle containing a compact convex subset in M. Define a mapping $f: U^* \to U$ such that the image of a circle C^* in U^* is that one in U which corresponds to the same compact convex subset as C^* does. Because of the uniform boundedness of rectangular ratios, f satisfies (5.24). Thus our result can be deduced directly from Corollary 1. ∎

The uniform boundedness of rectangular ratios in Corollary 2 is indispensible. Consider a hyperbola C in $\mathbf{R}^2$ and take a sequence of different points $\{P_k: k = 0, \pm 1, \pm 2, \ldots\}$ on the same branch of C such that $|P_k P_{k-1}| = 1$. Similarly take another sequence $\{A_k: k = 0, \pm 1, \pm 2, \ldots\}$ on the other branch of C such that $|A_k A_{k-1}| = 1$. Let $B(P_k P_{k-1})$ denote the closed region surrounded by the chord $\overline{P_k P_{k-1}}$ and the arc $P_k \widehat{P_{k-1}}$. Moreover, let B_0 be the closed unit disk centered at the center of C. Let $M := \{B_0, B(P_k P_{k-1}), B(A_k A_{k-1}) : k = 0, \pm 1, \pm 2, \ldots\}$, which clearly consists of compact convex subsets but does not possess uniform boundedness of

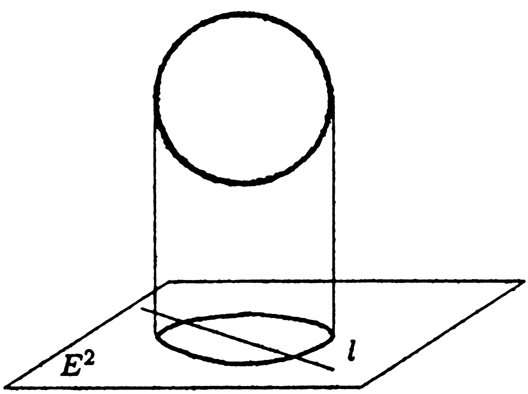

FIG. 3. Projection ΥS.

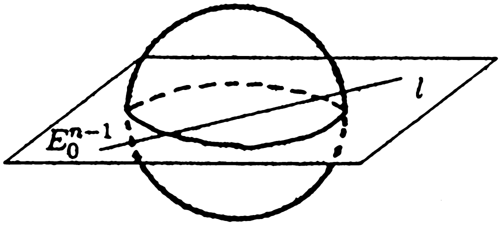

FIG. 4. $U^{n\text{-}1}$ on $E_0^{n\text{-}1}$.

rectangular ratios. Obviously, every line in $\mathbf{R}^2$ intersects at least one but at most five in M.

We can also generalize our result to $\mathbf{R}^n$.

Remark 1. Let $n \geqslant 3$ and U consist of $(n-1)$-dimensional superspheres in $\mathbf{R}^n$ such that any $(n-1)$-dimensional superplane in $\mathbf{R}^n$ intersects at least one supersphere in U. Then for any given $m \in \mathbf{N}$, any plane E^2, and any $P \in E^2$ there exists a superplane E^{n-1} in $\mathbf{R}^n$ which is through P, orthogonal to E^2, and intersects at least m superspheres in U. This can be shown easily with the orthogonal projection $\Upsilon: \mathbf{R}^n \to E^2$ as in Fig. 3 since $\Upsilon U := \{\Upsilon S \colon S \in U\}$ is a set of some circles on the plane E^2 and Theorem 1 can be applied.

Remark 2. Let $n \geqslant 3$ and U consist of $(n-1)$-dimensional superspheres in $\mathbf{R}^n$. For given $k \in \mathbf{N}$ with $1 \leqslant k < n$, if any k-dimensional superplane in $\mathbf{R}^n$ intersects at least one supersphere in U, then for any $m \in \mathbf{N}$ and any $P \in \mathbf{R}^n$ there exists a k-dimensional superplane E^k in $\mathbf{R}^n$ through P intersecting at least m superspheres in U (Fig. 4). In fact, by Remark 1, it suffices to discuss the case of $k \leqslant n-2$. For $k = n-2$, take a $(n-1)$-dimensional superplane E_0^{n-1} in $\mathbf{R}^n$ through O and P. Let $\mathscr{E}_{n-1}^{n-2}$ be the set of all $(n-2)$-dimensional superplanes in E_0^{n-1} and let $U^{n-1} := \{B_n \cap E_0^{n-1} : B_n \in U\}$. For any $l \in \mathscr{E}_{n-1}^{n-2}$ there is a supersphere $B_n \in U$ such that $l \cap B_n \neq \varnothing$. Thus $B_n \cap E_0^{n-1} \neq \varnothing$ and $l \cap (B_n \cap E_0^{n-1}) \neq \varnothing$ since $l \subset E_0^{n-1}$. This means that every $(n-2)$-dimensional superplane in E_0^{n-1} intersects at least one $(n-2)$-dimensional supersphere of U^{n-1}. By Remark 1 there exists a $(n-2)$ dimensional superplane l^* in E_0^{n-1} through P intersecting $m(n-2)$-dimensional superspheres $B_n^1 \cap E_0^{n-1}, B_n^2 \cap E_0^{n-1}, \ldots, B_n^m \cap E_0^{n-1}$ in U^{n-1}. Of course, l^* intersects all $B_n^1, B_n^2, \ldots, B_n^m$.

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